Why is physics better than mathematics? (Some sort of introduction to physical mathematics)

Why?

i) Easy to explain what we do (Big Bang Theory)

ii) Happy to claim that obviously many statements are right, such as:

\[ 1 + 2 + 3 + \ldots = -\frac{1}{12} \]

iii) Physicists are always right!

\[ \Rightarrow \text{we don't need to bother with proofs, life is much simpler!} \]

Defining: A physicist is a machine that produces conjectures.

More seriously:

Physics \rightarrow Mathematics

Newton etc.: "phyhematics" or "maths"

Einstein etc.: use math as a tool, a language to do physics (mathematical physics)

Last 30 years: use physics to do math (physical mathematics)
How do we do math?

- we need ideas
- theoretical physics, in particular string theory is a huge pool of ideas for math

Source: "physical dualities"

→ two mathematical models describing the same physics

⇒ deep new connections between different fields of mathematics.
Enumerative Geometry

Mathematicians like to count things.

"How many geometric structures of a given type satisfy a given set of constraints?"

Example 1:

"How many points in the plane lie on each of two given lines?"

\[ \begin{align*}
1 & \quad \text{not a good question. We want to reformulate the question so that the answer is unique.} \quad \text{(1)} \\
0, \infty & \\
\end{align*} \]

How can we do that?

Geometry \longrightarrow \text{ algebra}

How many solutions to the system of equations?

\[ \begin{align*}
a x + by &= c \\
dx + ey &= f
\end{align*} \]

We will come back to this?
Example 2: 

"How many roots does a polynomial of degree n have?"

\[ f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \]

Consider \( n=1 \): \[ f(x) = a_0 + a_1x \] \( x = -\frac{a_0}{a_1} \) if \( a_1 \neq 0 \) \[ \text{[one root]} \]

but what if \( a_1 = 0 \)?

- if \( a_0 \neq 0 \), there is no root [it has "gone to infinity"]
- if \( a_0 = 0 \), there are infinitely many roots

\( n=2 \): \[ f(x) = a_0 + a_1x + a_2x^2 \]

\[ x = -\frac{a_1}{2a_2} \pm \frac{1}{2a_2} \sqrt{a_1^2 - 4a_0a_2} \] if \( a_2 \neq 0 \).

1) \( a_2 \neq 0 \); let \( D = a_1^2 - 4a_0a_2 \).
   a) 2 roots if \( D > 0 \)
   b) 1 root if \( D = 0 \)
   c) no root if \( D < 0 \)

2) \( a_2 = 0 \); same as case \( n=1 \)

(No solution, \( \infty \)-many, or one solution) \[ \rightarrow \text{Way too complicated?} \]
We want a unique answer. How can we reformulate the enumerative problem?

- read numbers $\rightarrow$ complex numbers
- count solutions with multiplicity
- "include infinity."

$n=2$;

$$f(x) = a_0 + a_1 x + a_2 x^2$$

1) then, if $a_2 \neq 0$;
   a) two real roots if $D > 0$ \hfill (2)
   b) one real root with multiplicity two if $D = 0$ \hfill (2)
   c) two complex conjugate roots if $D < 0$ \hfill (2)

But how do we handle case (2) — roots gone off to infinity?

We need to "compactify the problem."

\[ \text{C} \rightarrow \text{C} \rightarrow \\infty \rightarrow \]
**Definition:** The complex projective line \( \mathbb{CP}^1 \) is

\[
\{ (x, y) \in \mathbb{C}^2 \mid (x, y) \neq (0, 0) \}
\]

where we consider two pairs \((x, y)\) and \((x', y')\) to be the same if they are multiple of each other: \((x, y) = \lambda(x', y')\) for some \(\lambda \in \mathbb{C}^\times\).

For instance, \((1, 1), (2, 2), (1+i, 4+4i)\) are all the same point in \(\mathbb{CP}^1\).

We can think of \(\mathbb{CP}^1\) as the set of equivalence classes under the relation \((x, y) \sim \lambda(x, y)\) with \(\mathbb{C}^\times\) acting as \((x, y) \sim \lambda(x, y)\).

The point: \(\mathbb{CP}^1\) is the union of \(\mathbb{C}\) and a point at infinity: \([\infty]\).

Let \(U_0 = \{ (x_0, x_1) \in \mathbb{CP}^1 \mid x_0 \neq 0 \}\) be open.

Then \(\phi_0 : U_0 \to \mathbb{C}\)

\[
(x_0, x_1) \mapsto \frac{x_1}{x_0}
\]

is a bijection so \(U_0 \cong \mathbb{C}\).

The complement of \(U_0\) in \(\mathbb{CP}^1\) is set of all points with \(x_0 = 0 \to (0, 1)\) "point at infinity" and the map:

\[
\left[ \begin{array}{ccc}
\phi_0 & : & \mathbb{C} \to U_0 \\
2 & \mapsto & (1, 2)
\end{array} \right]
\]
How do we define polynomials in $\mathbb{CP}^1$? An equation $F(x_0, x_1) = 0$ only makes sense in $\mathbb{CP}^1$ if it is homogeneous

$$F(x_0, x_1) = \sum_{i=0}^{d} a_i x_0^i x_1^{d-i}$$

**Example 2:** "How many roots does a polynomial of deg. $d$ have?" [now in $\mathbb{CP}^1$]

$d = 1$; $F(x_0, x_1) = a_0 x_1 + a_1 x_0$

**Answer:** $(x_0, x_1) = (-a_0, a_1) \in \mathbb{CP}^1$ (as long as $F$ is not identically equal to zero)

$d = 2$; $F(x_0, x_1) = a_0 x_1^2 + a_1 x_0 x_1 + a_2 x_0^2$

→ exactly two roots (with multiplicity)

e.g. if $a_0 = 0$; $x_0 (a_1 x_1 + a_2 x_0) = 0 \rightarrow (-a_1, a_2)$ and $(0, 1)$
Theorem: Any non-zero homogeneous polynomial $F(x_0, x_1)$ in $\mathbb{CP}^1$ of degree $d$ has exactly $d$ roots (incl. multiplicity).

Proof: follows from the fundamental theorem of algebra.

Example 1: "How many points in the plane lie on each of two given lines?"

Definition: $\mathbb{CP}^n$ is $\left\{ x = (x_0, ..., x_{n+1}) \in \mathbb{C}^{n+1} \left| \sum x_i = 0 \right. \right\}$ with the identification $x = \lambda \bar{x}$ for all $\lambda \in \mathbb{C}^*$.

You can think of $\mathbb{CP}^n$ as an "enlarged $\mathbb{C}^n$".

E.g., $\mathbb{CP}^2$; $U_0 = \left\{ (x_0, x_1, x_2) \in \mathbb{CP}^2 \mid x_0 \neq 0 \right\}$

$\phi_0 : U_0 \rightarrow \mathbb{C}^2$

$(x_0, x_1, x_2) \mapsto \left( \frac{x_1}{x_0}, \frac{x_2}{x_0} \right)$ bijection

$\phi_0^{-1} : \mathbb{C}^2 \rightarrow U_0$

$(x, y) \mapsto (1, x, y)$

Complement of $U_0$ in $\mathbb{C}^2$ is $(0, x_1, x_2) \sim$ copy of $\mathbb{CP}^1$. 
How do we define a "line" in $\mathbb{CP}^2$?

line in $\mathbb{R}^2 \iff$ solution of $a + bx + cy = 0$ \[\text{deg. 1 polynomial in } (x, y)\]

line in $\mathbb{CP}^2 \iff$ solution of $F(x_0, x_1, x_2) = 0$

$F$ homogeneous of deg. 1:

$F(x_0, x_1, x_2) = a_0 x_2 + a_1 x_1 + a_2 x_0$

Now here's something cool:

- Parallel lines in $\mathbb{R}^2$ actually meet in $\mathbb{CP}^2$!

\[\text{e.g. } x + y = 1, \quad x + y = 2 \implies x_0 + x_1 = x_2, \quad x_0 + x_1 = 2x_2\]

Those two equations have the common solution $(x_0, -x_0, 0)$ which is a point in $\mathbb{CP}^2$!

$\implies$ "How many points in $\mathbb{CP}^2$ lie on each of two given lines?"

Assuming distinct lines, the answer is always 1!

If we use the non-zero polynomial assumption $\implies$ "excess intersection theory."
What does any of this have to do with physics?

- we consider a slightly different enumerative geometric problem: enumeration of curves in a target space.

\[ \text{Recall: line in } \mathbb{CP}^2 \leftrightarrow \text{sol}^{10} \quad \text{of } F(x_0, x_1, x_2) = 0 \quad \text{& hom. of deg. 1.} \]

\[ \text{deg. d curve in } \mathbb{CP}^2 \leftrightarrow \text{sol}^{10} \quad \text{of } F(x_0, x_1, x_2) = 0 \quad \text{& hom. of deg. d.} \]

Now consider the map:

\[ \Phi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^2 \]

\[ (x_0, x_1) \mapsto (x_0^2, x_0 x_1, x_1^2) = (y_0, y_1, y_2) \]

then \( y_0 + y_1 - y_2 = 0 \) \( \Rightarrow \) line in \( \mathbb{CP}^2 \).

\[ \Phi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^2 \]

\[ (x_0, x_1) \mapsto (x_0^2, x_0 x_1, x_1^2) = (y_0, y_1, y_2) ; \] then \( y_0 y_2 - y_1^2 = 0 \)

\( \Rightarrow \) deg. 2 curve in \( \mathbb{CP}^2 \).
What we are doing is "embedding a copy of $\mathbb{CP}^1$ in $\mathbb{CP}^2"."

\[ \mathbb{CP}^1 \rightarrow \mathbb{CP}^2 \]

**Definition:** Let $f_0(x_0, x_1), \ldots, f_n(x_0, x_1)$ be a collection of homogeneous polynomials of the same degree $d$ without common factors. Then the map:

\[ \phi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^n 
\]

\[ (x_0, x_1) \mapsto (f_0(x_0, x_1), \ldots, f_n(x_0, x_1)) \]

is well defined, and its image is an irreducible rational curve in $\mathbb{CP}^n$ of deg. $d$.

\[ \uparrow \text{ (some subtlety here in defining the degree)} \]
Now we can ask all kinds of fun questions about curves in $\mathbb{CP}^n$!

E.g.: How many lines in $\mathbb{CP}^2$ contain each of two distinct points $p_1$ and $p_2$? \( (1) \)

""" Conics in $\mathbb{CP}^2$ pass through each of five given points in general position? \( (2) \)
""" Conics in $\mathbb{CP}^2$ pass through four general points and are tangent to a fixed line? \( (2) \)
""" Lines in $\mathbb{CP}^3$ intersect each of four given lines? \( (2) \)

→ for each of these problems, we must make sure that solutions do not "go off to infinity".

→ we must define the problem carefully so that the "parameter space" (moduli space) is "compact".
Our problem: counting (rational) curves in a target space

Definition: A quintic threefold $X$ is the zero locus of a degree 5 homogeneous polynomial in $\mathbb{CP}^4$.

[It is a three complex-dimensional manifold].

We want to enumerate rational curves of any degree in $X$.

\[
\begin{array}{ccc}
& \text{deg. } d & \rightarrow \\
\circ & \rightarrow & \bigtriangleup \\
& \circ & X
\end{array}
\]

\[f: \mathbb{CP}^1 \rightarrow X\]

How do we define this map?

1. \[f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^4 \]
\[
(x_0, x_1) \mapsto (f_0(x_0, x_1), \ldots, f_4(x_0, x_1))
\]

2. $X$ is defined by $F(x_0, \ldots, x_4) = 0$ so we require that

\[F(f_0(x_0, x_1), \ldots, f_4(x_0, x_1)) = 0\]

so that the curve lies on $X \subset \mathbb{CP}^4$.\]
Why should we expect a finite number of rational curves in \( X \)?

Let's do some counting:

- \( f_0, \ldots, f_k \) are degree \( d \) hom. polynomials in \( \mathbb{CP}^1 \):
  \[
  f_i(x_0, x_1) = a_{i,0}x_0^d + a_{i,1}x_1^{d-1}x_0 + \cdots + a_{i,d}x_1^0 \Rightarrow d + 1 \text{ coefficients}.
  \]
  \( \Rightarrow \) \( f_0, \ldots, f_k \) give \( 5(d+1) \) coefficients.

We must require that \( F(f_0(x_0, x_1), \ldots, f_k(x_0, x_1)) = 0 \)

\[
\text{deg} F \Rightarrow \sum_{k=0}^{5d} h_k(a_{i,j}) x_0^k x_1^{5d-k} = 0
\]

\( \Rightarrow \) \( h_k(a_{i,j}) = 0 \), \( k = 0, \ldots, 5d \) \( \Rightarrow \) \( 5d + 1 \) equations.

So we have \( 5(d+1) - (5d+1) = 4 \) free variables in the solution.

But, two maps \( f \) may give rise to the same curve. There is a 4-dim. space of "parameterizations", hence we get a 4-dim. space of solutions

\( \Rightarrow \) finite number of curves!
Clemens conjecture: let $X \subset \mathbb{P}^4$ be a “general” quintic threefold, and let $d$ be a positive integer. Then there are finitely many rational curves of degree $d$ in $X$.

$\Rightarrow$ still unresolved! $[\text{proven for } d \leq 9]$

**Question:** How many rational curves of each degree $d$ are contained in a general quintic threefold $X$?

For $d=1$ (lines): $2875$ (19th century)
$d=2$ (conics): $609250$ (1985)
$d=3$ (cubics): $317206375$ (1991)

$\Rightarrow$ Very difficult to compute these numbers directly for $d>3$

$[\text{space of rational curves in } X \text{ is not compact...}]$
Physics: 1) Provides a natural compactification for the moduli space (space of stable maps - Kontsevich) → Gromov-Witten theory

2) The number of rational curves in X of all degrees can be computed (amazingly)!!! [1991]
Instead of considering \( f : \mathbb{C} \mathbb{P}^1 \rightarrow \mathbb{C} \mathbb{P}^4 \), we allow maps from "trees" (degenerate curves with nodes). The moduli space of genus 0 deg. 2 stable maps to \( \mathbb{C} \mathbb{P}^4 \) is compact (it is a "stack"!)

Gromov-Witten theory.
2) How does physics calculate these numbers?

What is string theory?

Particle physics:

String theory:

A-model top. string theory

\[
\int D\Phi e^{-S[\Phi]} \quad \Rightarrow \quad \sum N_d q^d \quad \text{number of curves of deg. } d \text{ in } X
\]

(very roughly speaking, more precisely defined in terms of Gromov-Witten invariants)
Key point: computes all $N_d$ at once! (all degrees).

But... how do we compute this series?

\[ \sum_i N_d q^d = F_0 \]

\[ \tilde{F}_0 \leftarrow \text{mirror map} \]

* $\tilde{F}_0$ can be obtained by solving a simple ordinary differential equation

\[ \Rightarrow \text{all numbers of rational curves in quintic in } \mathbb{CP}^4 ! \]
One may ask:

What about higher genus curves?

\[ g = \text{genus} = 2 \, . \]

A-model

\[ F_g \]

B-model

\[ \widetilde{F}_g \]

\[ \text{for some choices of } X \text{ ("toric"); } \]

\[ \widetilde{F}_g \] is computed by a \underline{topological}

\underline{recursion} (for all \( g \)).
physics, especially string theory, leads to fascinating new results in mathematics (e.g. in enumerative geometry.)

"the unreasonable effectiveness of physics in mathematics"

REFERENCE: "Enumerative Geometry and String Theory" by Sheldon Katz