ON THE GAMMA FILTRATION FOR A SEVERI-BRAUER VARIETY

EOIN MACKALL

Abstract. We introduce techniques for uniformly studying the gamma filtration of a Severi-Brauer variety. These techniques are utilized in our main result, which computes the associated graded groups for the gamma filtration of a Severi-Brauer variety in (homological) degrees smaller or equal to two less than the smallest prime factor dividing one plus the dimension of the Severi-Brauer variety. In particular we show these groups are torsion free.

Notation and Conventions. We fix a field $k$ throughout. All of our objects are defined over $k$ unless stated otherwise.

If $p$ is a prime, then $v_p$ is the $p$-adic valuation.

$\#A$ denotes the cardinality of the set $A$.

1. INTRODUCTION

Chow rings of Severi-Brauer varieties have been the subject of a number of articles over the years. One attempt at studying these rings that has been particularly fruitful is Karpenko’s use of the $\gamma$-filtration and the coniveau filtration on the Grothendieck ring. Much of the material in this article lends itself to the ideas contained in this work, particularly [Kar17a, Kar95b, Kar98].

The organization is as follows: sections 2 and 3 recall some background information on the Grothendieck groups we study. Section 4 is more involved and I’ve decided to give it a certain amount more of attention than it might deserve. There we introduce the notion of a $\tau$-functorial replacement for a Severi-Brauer variety $X$. This object is another Severi-Brauer variety, over possibly a different field, that computes the $\gamma$-filtration of $X$ but has the enjoyable property the $\gamma$ and coniveau filtrations of $X$ agree. The existence of this object was known before but a proof in the general case is not in the literature.

Still in section 4, we use our $\tau$-functorial replacements to give some functorial statements about the $\gamma$-filtration which were only known to hold for the $\tau$-filtration. In particular, we show how one can reduce certain results about the $\gamma$-filtration of a Severi-Brauer variety of a central simple algebra to the Severi-Brauer varieties of the primary components of the underlying division algebra. I expect this idea could be used for a number of more general varieties, specifically when there is a decomposition of the Grothendieck ring of the variety into a sum of Grothendieck rings of central simple algebras (or, when there is a decomposition of the motive of this variety into a sum of motives of separable algebras in the sense of [Mer05]). This line of thought isn’t pursued here, however.

Sections 5 and 6 are computational. The main result of these sections is that the associated graded ring of the $\gamma$ filtration is torsion free for Severi-Brauer varieties associated to $p$-primary central simple algebras, for a prime $p$, in (homological) degrees less than or equal $p - 2$.

Date: November 25, 2018.

2010 Mathematics Subject Classification. 19E20.

Key words and phrases. Chern classes; K-theory.
2. Grothendieck groups of Severi-Brauer varieties

Throughout this section we fix a central simple algebra $A$ of degree $n$ and let $X = \text{SB}(A)$ be the Severi-Brauer variety of $A$ of dimension $n - 1$. We write $\zeta_X$ for the tautological sheaf on $X$. For any $k$-algebra $R$ and any point $x$ of $X(R)$ corresponding to a right ideal $I \subset A \otimes_k R$, the sheaf $x^*\zeta_X$ is canonically identified with $I$; in particular, $\zeta_X$ is a right module over the constant sheaf $A$.

By $K(X)$ we mean the Grothendieck ring of locally free sheaves on $X$. By $G(X)$ we mean the Grothendieck ring of coherent sheaves on $X$. The two groups are canonically isomorphic via the morphism sending the class of a locally free sheaf in $K(X)$ to the class of itself in $G(X)$. These groups have been computed, in the following sense:

**Theorem 2.1** ([Qui73, §8, Theorem 4.1]). The homomorphism of $K$-groups

$$
\bigoplus_{i=0}^{\deg(A)-1} K(A^{\otimes i}) \to K(X)
$$

sending the class of a left $A^{\otimes i}$-module $M$ to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

In particular, $K(X)$ is free of rank $\deg(A)$ generated additively by the classes of

$$
\zeta_X(i) := \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i
$$

as $0 \leq i < \deg(A)$ and for a simple $A^{\otimes i}$-module $M_i$. For any splitting field $F$ of $A$, the extension of scalars map $K(X) \to K(X_F)$ is injective, and identifies $K(X)$ as a subring of $K(X_F)$. More precisely, we have:

**Theorem 2.2.** In the setting above, let $\xi$ denote the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$. There is a ring isomorphism

$$
\mathbb{Z}[x]/(1 - x)^n \xrightarrow{\sim} K(X_F)
$$

sending $x$ to $\xi$.

Under this isomorphisms $K(X)$ identifies with the subring of $\mathbb{Z}[x]/(1 - x)^n$ generated by $\text{ind}(A^{\otimes i})x^i$.

**Proof.** The isomorphism is well-known, see [Man69]. Finally, we use that $\zeta_X \otimes_k F$ has class $\deg(A)\xi$ in $K(X_F)$ to get the remaining claim by computing the ranks of the $\zeta_X(i)$.

When working with $K(X)$, it's often more helpful to work with a covering of this ring (e.g. this is done in [Kar98, Section 4]).

**Lemma 2.3.** Consider the subring $S \subset \mathbb{Z}[x]$ generated by the elements

$$
\text{ind}(A^{\otimes i})x^i \text{ for all } 1 \leq i \leq \exp(A).
$$

Then the image of $S$ in $\mathbb{Z}[x]/(1 - x)^n$ is isomorphic with $K(X)$.

In particular we'll need the following lemma from [Kar98, Lemma 4.5]. The proof is short and goes by induction on the coefficients.

**Lemma 2.4.** Let $f, g, h$ be polynomials in $\mathbb{Z}[x]$ and assume $g(0) = \pm 1$. Assume both $f = gh$ and $f$ is contained in $S$. Then $h$ is also contained in $S$.

We include here as well the following formulas. The first is just the binomial theorem, and the second follows from the first by a change of coordinates.

**Lemma 2.5.** In any commutative ring there are equalities

$$
(1 - x)^i = \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} x^j \quad \text{and} \quad x^i = \sum_{j=0}^{i} (-1)^{j} \binom{i}{j} (1 - x)^j.
$$

2
In this section we recall some results on the $\gamma$-filtration of $K(X)$ and of the coniveau (or topological or Chow) filtration on $G(X)$ for a smooth variety $X$.

For the first, recall there are $\gamma$-operations defined on $K(X)$ as follows. The $i$th-exterior power operation induces a well-defined map $\lambda^i : K(X) \to K(X)$ which is uniquely determined by sending the class of a locally free sheaf $F$ to the class of $\wedge^i F$. The $i$th $\gamma$ operation $\gamma^i : K(X) \to K(X)$ is defined by sending an element $x$ to the coefficient of $t^i$ in the formal series

$$\gamma_i(x) = \sum_{j \geq 0} \lambda^j(x) \left( \frac{t}{1-t} \right)^j.$$  

The $\gamma$-filtration on $K(X)$ is defined as $\gamma^0 = K(X)$, $\gamma^1 = \ker(\mathrm{rk})$ where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the map sending the class of a locally free sheaf $F$ to its rank, and $\gamma^i$ for $i \geq 0$ is generated by monomials $\gamma^{i_1}(x_1) \cdots \gamma^{i_r}(x_r)$ for any $r \geq 0$, $i_1 + \cdots + i_r \geq i$ and $x_1, \ldots, x_r$ elements of $\gamma^1$. We use the notation

$$\mathrm{gr}_i K(X) := \gamma^{i+1} / \gamma^i \quad \text{and} \quad \mathrm{gr}_{\gamma} K(X) := \bigoplus_{i \geq 0} \mathrm{gr}_i K(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration respectively. When we need to be precise about which variety the $\gamma$-filtration is being considered for, we will specify by writing $\gamma^i(X)$ to mean the $i$th piece of the $\gamma$-filtration for the variety $X$. For further properties of these operations we refer to the references [Man69, MR071].

For the second, recall the coniveau filtration on $G(X)$ is defined by setting $\tau^i$, for any $i \geq 0$, to be the ideal generated by

$$\tau^i := \sum_{x \in X^{(j)}} \ker (G(X) \to G(X \setminus x))$$

where $j \geq i$, $X^{(j)}$ denotes the set of codimension $j$ points of $X$, and the arrows are flat pullbacks with respect to the inclusion. We use the notation

$$\mathrm{gr}_\tau G(X) := \tau^{i+1} / \tau^i \quad \text{and} \quad \mathrm{gr}_{\tau} G(X) := \bigoplus_{i \geq 0} \mathrm{gr}_i G(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration respectively. Sometimes when more precision is needed, we include the variety in our notation for the coniveau filtration, i.e. $\tau^i(X)$ for the $i$th piece of the coniveau filtration of $X$.

The two filtrations are related:

**Theorem 3.1.** We identify $K(X)$ with its image in $G(X)$ under the canonical isomorphism. For any $i \geq 0$ we have $\gamma^i \subset \tau^i$. Moreover, if the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes then the two filtrations are equal, i.e. $\gamma^i = \tau^i$ for all $i \geq 0$.

**Proof.** For the first claim, see [Man69]. The second claim originally appears in [Kar98] and is updated in [KM18b, Proposition 3.3]. \qed

**Remark 3.2.** Slightly more generally, if the canonical morphism $B(X) \to \mathrm{gr}_{\tau} G(X)$ is a surjection, then there is also an equality $\gamma^i = \tau^i$ for all $i \geq 0$. Here $B(X)$ is the universal source of Chern classes on $X$ constructed in [Mac18].

4. REDUCTIONS

We specialize to the case $A$ is a central simple algebra and $X = SB(A)$. The main purpose of this section is to provide a way to reduce to computations of the associated graded for the $\gamma$-filtration to the case $A$ is a $p$-primary division algebra. In this regard we utilize heavily the motivic techniques of Karpenko (e.g. [Kar95a, Corollary 1.3.2],[Kar17a, Lemma 3.5]). The reason we can use these
results is due to an observation (also Karpenko’s) that for any Severi-Brauer variety $X$ associated to an algebra $A$, there is a Severi-Brauer variety $Y$ so that the $\gamma$-filtrations of $X$ and $Y$ are equal and the $\gamma$-filtration and coniveau filtration for this $Y$ are also equal. This allows us to prove results about $X$ by first replacing it with the functorially-nicer $Y$ and then reducing to previously known results. This observation seems nice enough to name it.

**Definition 4.1.** Let $X$ be an arbitrary Severi-Brauer variety associated to $A$. We say that a Severi-Brauer variety $Y$ associated to a central simple algebra $B$ is a $\tau$-functorial replacement for $X$ if the following conditions hold:

1. $\deg(A) = \deg(B)$
2. for every prime $p$, the $p$-behavior of $A, B$ are the same $Beh(p, A) = Beh(p, B)$
3. the filtration comparison map $gr_\gamma K(X) \to gr_\tau G(Y)$ is an isomorphism.

Here we’re using the definition:

**Definition 4.2.** For an arbitrary central simple algebra $A$ with primary decomposition $A = M_n(k) \otimes \left( \bigotimes_{p \text{ prime}} A_p \right)$, the behavior of $A$ is the sequence

$$Beh(A) = \left( \text{ind}(A), \text{ind}(A^{\otimes 2}), \ldots, \text{ind}(A^{\otimes \text{exp}(A)}) \right).$$

The $p$-behavior is the sequence

$$Beh(p, A) = \left( \text{ind}(A_p), \text{ind}(A_p^{\otimes p}), \ldots, \text{ind}(A_p^{\otimes \text{exp}(A_p)}) \right).$$

The reduced $p$-behavior of $A$ is the sequence

$$rBeh(p, A) = \left( \nu_p \text{ind}(A_p), \nu_p \text{ind}(A_p^{\otimes p}), \ldots, \nu_p \text{ind}(A_p^{\otimes \text{exp}(A_p)}) \right).$$

If $A$ is a $p$-primary algebra then we will call the reduced $p$-behavior simply the reduced behavior of $A$, and write $rBeh(A)$ for the reduced behavior.

**Remark 4.3.** Note that a $\tau$-functorial replacement doesn’t necessarily need to exist over the same base field. In fact, it often doesn’t.

**Remark 4.4.** The reduced behavior is a strictly descending sequence ending in 0. Conversely, for every prime $p$ and for every strictly descending sequence ending in 0 there is a $p$-primary algebra with reduced behavior the given sequence, see [Kar98, Lemma 3.10]. Note that it’s possible to reconstruct the behavior of $A$ from the $p$-behavior (or the reduced $p$-behavior) as $p$ ranges over all primes.

The first two conditions our $\tau$-functorial replacements are required to have insure that we haven’t changed the $\gamma$-filtration on replacement.

**Lemma 4.5.** The ring $gr_\gamma K(X)$ depends only on the integers $\text{ind}(A^{\otimes i})$ for $0 \leq i < \deg(A)$. In particular, if $B$ is another central simple algebra of the same degree as $A$ with Severi-Brauer variety $Y = SB(B)$ and if there are equalities

$$\text{ind}(A^{\otimes i}) = \text{ind}(B^{\otimes i})$$

for all $i \geq 0$ then the rings $gr_\gamma K(X)$ and $gr_\gamma K(Y)$ are isomorphic (but maybe not naturally).

**Proof.** This is the content of [IK99, Theorem 1.1 and Corollary 1.2].

Modifying a proof from [Kar98], it’s possible to show a $\tau$-functorial replacement exists for any Severi-Brauer variety of a division algebra.
Proposition 4.6. If $A$ is a division algebra then there exists a division algebra $B$, possibly over a different field than $k$, with $\text{ind}(A^\otimes i) = \text{ind}(B^\otimes i)$ for all $i \geq 0$ satisfying the property that the canonical morphism comparing the $\gamma$ and coniveau filtrations for $Y = SB(B)$ is an isomorphism, i.e. $\text{gr}_\gamma K(Y) = \text{gr}_r G(Y)$.

Proof. The construction of $B$ is given in [Kar98, Lemma 3.10] for $A$ of $p$-primary index. The proof that $\text{gr}_\gamma K(Y) = \text{gr}_r G(Y)$ in this case follows from [Kar98, Theorem 3.7]. The following proof is a simple generalization of these two references.

We first construct $B$. One can find a field $F$ and a division algebra $B_0$ with $\text{ind}(A) = \text{ind}(B_0) = \exp(B_0)$. Let $B_0 = \bigotimes_{p \text{ prime}} B_{0,p}$ be a $p$-primary decomposition of $B_0$. Let $q$ be the smallest prime appearing among the indices of these factors. We consider the reduced $q$-behavior $\text{rBeh}(q,A) = (n_0,n_1,\ldots,0)$.

Set $\tilde{B}_0 = B_0^{q^{n_1}}$ with Severi-Brauer variety $Y_1 = SB(\tilde{B}_0)$ and $B_1 = B_{0,F(Y_1)}$. Then using index reduction formulas, see [SVdB92, Theorem 1.3], we find $\text{rBeh}(q,B_1) = (n_0,n_1,n_1-1,\ldots,0)$.

By repeating this process finitely many times, we can construct an algebra $B$ with the same reduced $q$-Behavior as $A$. Doing this procedure for the other primes allows us to find $B$ satisfying the restriction on its indices as in the proposition statement.

It remains to show the $\gamma$-filtration and the coniveau filtration agree for $Y = SB(B)$. For the Severi-Brauer variety of any algebra with equal index and exponent, for example $B_0$, these two filtrations coincide, [Kar98, Corollary 3.6]. In the general case, we note that $X \times Y_1$ is a projective bundle over $X$ since $\tilde{B}_0$ is in the subgroup generated by $B_0$. The commuting diagram

$$
\begin{array}{ccc}
B(X \times Y_1) & \longrightarrow & B(X_{F(Y_1)}) \\
\downarrow & & \downarrow \\
\text{gr}_r G(X \times Y_1) & \longrightarrow & \text{gr}_r G(X_{F(Y_1)})
\end{array}
$$

has surjective left vertical arrow by Lemma 4.7 below. The bottom horizontal arrow is also surjective by localization. Hence the right vertical arrow is surjective, which implies the $\gamma$ and coniveau filtrations coincide for $X_{F(Y_1)}$ by Remark 3.2. Continuing this process for each modification of $B_i$ shows $B$ has the specified properties.

\[\square\]

Lemma 4.7. Suppose $X$ is a variety (a scheme essentially smooth and essentially of finite type over $k$) with $\gamma^i = \tau^i$ for all $i \geq 0$. Then, for any chain of morphisms

$$Y_r \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 = X$$

with each $Y_j$ the projective bundle of some vector bundle over $Y_{j-1}$, the $\gamma$ and coniveau filtrations for $Y_r$ coincide.

Proof. It suffices to assume $r = 1$ and work by induction. As there is equality of the $\gamma$ and coniveau filtrations for $X$, we get a commuting diagram like the one below, using the universal maps from $B$ and the filtration comparison map.

$$
\begin{array}{ccc}
B(X) & \longrightarrow & B(X) \\
\downarrow \text{gr}_\gamma K(X) & & \downarrow \text{gr}_r G(X) \\
\end{array}
$$

As the left diagonal map is always a surjection since $\text{gr}_\gamma K(X)$ is generated by Chern classes, it follows that the right diagonal map is also a surjection.
Now for any projective bundle \( Y_1 \to X \), the Grothendieck ring \( K(Y_1) \) is generated as a \( K(X) \)-algebra by \( K(X) \) and a single element \( t \) which is the class of a rank one locally free sheaf on \( Y_1 \). In particular, \( B(Y_1) \) is generated as an algebra by the image of \( B(X) \) under the pullback of the projection and powers of the first Chern class of \( t \). Since \( \text{gr}_r G(Y_1) \) is generated as an algebra by \( \text{gr}_r G(X) \) and the Chern classes of \( t \), it follows the canonical map \( B(Y_1) \to \text{gr}_r G(Y_1) \) is a surjection. We conclude by Remark 3.2.

The remainder of this section is devoted to showing the use in \( \tau \)-functorial replacements. First, we show that the associated graded for the \( \gamma \)-filtration depends only on the underlying division algebra of \( A \).

**Proposition 4.8.** Let \( A \) be an arbitrary central simple algebra. If \( D \) is the underlying division algebra of \( A \) with \( X_D = \text{SB}(D) \), then the morphism

\[
\bigoplus_{i=1}^{\deg(A)/\deg(D)} \text{gr}_\gamma K(X_D) \to \text{gr}_\gamma K(X)
\]

taking the element \((x_1, \ldots, x_r)\) to \( x_1 + x_2c + \cdots + x_rc^{r-1} \), where \( c \) is the top Chern class of \( \zeta_X(1) \), is an isomorphism.

We'll need some lemmas.

**Lemma 4.9.** If \( D \) is the underlying division algebra of \( A \) with \( X_D = \text{SB}(D) \), then the map induced by the pullback of the inclusion

\[
\text{gr}_r K(X) \to \text{gr}_r K(X_D)
\]

is an isomorphism in degrees where both groups are nonzero.

**Proof.** As the morphism of the lemma statement depends only on the behavior of \( A \) and the degree of \( A \), we can first make a \( \tau \)-functorial replacement, Proposition 4.6, to assume the \( \gamma \) and coniveau filtrations agree for \( X_D \). Now the diagram

\[
\begin{CD}
\text{gr}_\gamma K(X) @>>> \text{gr}_\gamma K(X_D) \\
@VVV @VVV \\
\text{gr}_r G(X) @>>> \text{gr}_r G(X_D)
\end{CD}
\]

is commutative, with vertical arrows the comparison maps and horizontal arrows the pullbacks. The top horizontal arrow is surjective because the pullback \( K(X) \to K(X_D) \) is surjective and the associated graded for the \( \gamma \)-filtration is generated by Chern classes, cf. [Mac18, Lemma 2.3]. The bottom horizontal arrow is an isomorphism in degrees where both groups are nonzero by [Kar95a, Corollary 1.3.2]. And the right vertical arrow is an isomorphism because of our replacement.

Now from the commutative ladder with exact rows below

\[
\begin{array}{cccccc}
0 & \to & \gamma^i+1 & \to & \gamma^i & \to & \gamma^{i+1} \to 0 \\
& | \quad f_{i+1} & | \quad f_i & | \quad f_{i+1} & | \quad f_i & \\
0 & \to & \tau^i+1 & \to & \tau^i & \to & \tau^{i+1} \to 0
\end{array}
\]

we get short exact sequences (using the snake lemma)

\[
0 \to \ker(f_{i+1}) \to \coker(f_{i+1}) \to \coker(f_i) \to 0
\]

for all \( 0 \leq i < n = \text{ind}(A) \). To complete the proof, it suffices to show \( f_n/f_{n+1} \) is a surjection and \( \coker(f_n) = 0 \). These are both shown in the next lemma. \( \square \)
Lemma 4.10. If $A$ is a central simple algebra and $\text{ind}(A) = n$, then $\text{gr}_n^\gamma K(X)$ is torsion free and there are equalities
\[ \gamma^n = \tau^n = (\xi - 1)^n K(X) \quad \text{and} \quad \text{gr}_n^\gamma K(X) = \text{gr}_n^\gamma G(X) = (\xi - 1)^n \mathbb{Z}. \]

Proof. Let $F$ be a splitting field for $A$ and identify $K(X)$ with its image in $K(X_F)$ under the extension of scalars map. We set $\xi$ to be the class of $\mathcal{O}_{X_F}(-1)$. Our first goal is to show the inclusions
\[ (\xi - 1)^n K(X) \subset \gamma^n \subset \tau^n \subset (\xi - 1)^n G(X) \]
which will imply equalities hold throughout. Note that the left of these is immediate, as we have $(\xi - 1)^n = \gamma^n(n(\xi - 1))$.

As in Lemma 2.2, let $S \subset \mathbb{Z}[x]$ be the subring generated as an algebra by the elements $\text{ind}(A^\otimes i)x^i$. The preimage of $\tau^i$ in $S$ under the surjection
\[ S \to \mathbb{Z}[x] \to K(X) = G(X) \]
is always composed of polynomials in $(1 - x)$ of degree greater or equal $i$; this is because, if $F$ were a splitting field for $X$ then $\tau^i(X) \subset \tau^i(X_F) \subset G(X_F)$ and the preimage of $\tau^i(X_F)$ is the ideal $(1 - x)^i \subset \mathbb{Z}[x]$. We know, from the inclusion $\gamma^n \subset \tau^n$ that the preimage of $\tau^n$ contains $x^n$. We want to show that this preimage is actually also contained in the ideal $S \cap (x^n) \subset S$. It would then follow $\tau^n \subset (1 - \xi)^n G(X) = \gamma^n$ and this would complete this part of the proof.

To proceed, suppose $f$ is a polynomial in the preimage of $\tau^n$. Note this implies $f$ is in $S$. Assuming $f \neq 0$, we can write $f = (1 - x)^g$ for some polynomial $g$ of $\mathbb{Z}[x]$. It suffices to check that $g$ is in $S$ as well and this is true by Lemma 2.4.

Next we compute the quotients. The rank map $\text{rk} : K(X) \to \mathbb{Z}$ is surjective and provides a splitting $K(X) = \gamma^1 \oplus \mathbb{Z}$ given by $x \mapsto (x - \text{rk}(x), \text{rk}(x))$. We have a commuting diagram of free abelian groups
\[
\begin{array}{ccc}
\gamma^1 \oplus \mathbb{Z} & \xrightarrow{\cdot(\xi - 1)^n} & (\xi - 1)^n \mathbb{Z} \\
\downarrow & & \downarrow \pi \\
\gamma^n & \xrightarrow{\cdot(\xi - 1)^n} & \gamma^n
\end{array}
\]
with the canonical inclusion $\pi : \gamma^{n+1} \oplus (\xi - 1)^n \mathbb{Z} \subset \gamma^n$. The bottom row of this diagram is surjective by the description of $\gamma^n$. Hence $\pi$ must also be a surjection which, since both domain and target are free abelian groups of the same rank (see Lemma 4.11), must be an isomorphism. This shows $\text{gr}_n^\gamma K(X) = (\xi - 1)^n \mathbb{Z}$ which happens to be the same as $\text{gr}_n^\gamma G(X)$ by the same proof replacing everywhere $\gamma$ appears with $\tau$. \hfill \Box

Lemma 4.11. The groups $\gamma^i \subset K(X)$ and $\tau^i \subset G(X)$ are free abelian of rank $n - i$ for any $0 \leq i \leq n = \text{deg}(A)$.

Proof. For any such $i$, $\gamma^i$ (resp. $\tau^i$) is a free abelian group as its a subgroup of $K(X)$ (resp. $G(X)$). For the claim on the rank we go by induction. For any $i$ there is an exact sequence
\[ 0 \to \gamma^{i+1} \to \gamma^i \to \gamma^{i+1} \to 0 \]
and $\gamma^{i+1}$ has rank 1 by one variant of the Riemann-Roch theorem (resp. with $\tau^i$’s). For large enough $i$ we have $\gamma^{i+1} = 0, \gamma^i \neq 0$ and $\gamma^{i+1}$ of rank 1 (resp. with $\tau^i$’s). \hfill \Box

Proof of Proposition 4.8. Using Proposition 4.6 on $D$, one can find a $\tau$-functorial replacement $B$. In particular,
\[ \text{Beh}(p, A) = \text{Beh}(p, A) \]
for all primes $p$ satisfying the property the coniveau filtration and $\gamma$-filtration agree for the Severi-Brauer variety of $B$. Set $C = M_r(B)$, where $r = \deg(A)/\deg(D)$, $Z = \text{SB}(C)$, and $Z_B = \text{SB}(B)$.  

7
There’s a canonical morphism

\[
\bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} K(Z_B) \to \gr_{\gamma} K(Z).
\]

To see it, label a basis of the left sum by \(e_i, 1 \leq i \leq \deg(C)/\deg(B)\). The canonical morphism is the map that sends \(xe_i\) to \(xc^{i-1}\) where \(c\) is the top Chern class of \(\zeta_X(1)\); here \(x\) is considered in \(\gr_{\gamma} K(Z)\) via the isomorphism of Lemma 4.9. We compose this morphism with the maps

\[
\bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} K(Z_B) \to \bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} G(Z_B)
\]

where the right arrow is an the inverse of the isomorphism

\[
\bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} G(Z_B) \to \gr_{\gamma} G(Z)
\]

appearing from [Kar95a, Corollary 1.3.2].

The composition

\[
\bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} K(Z_B) \to \bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} G(Z_B)
\]

is an isomorphism due to our choice of \(Z_B\). Hence there is a surjection

\[
\gr_{\gamma} K(Z) \to \gr_{\gamma} G(Z).
\]

The filtration comparison map has the nice property that surjectivity implies injectivity, [KM18b, Proposition 3.3 (2)], so it’s an isomorphism here. Thus the map

\[
\bigoplus_{i=1}^{\deg(C)/\deg(B)} \gr_{\gamma} K(Z_B) \to \gr_{\gamma} K(Z)
\]

is both injective and surjective. Since these rings are isomorphic when replacing \(Z_B\) by \(X_D\) and \(Z\) by \(X\) the claim follows. \(\square\)

As a corollary to the above proof we get:

**Theorem 4.12.** For any arbitrary central simple algebra \(A\), there exists a \(\tau\)-functorial replacement of \(X = SB(A)\).

**Proof.** Let \(D\) be the underlying division algebra of \(A\). There is a \(\tau\)-functorial replacement \(B\) of \(D\). The proof of Proposition 4.8 shows that taking a matrix ring over \(B\) with the same degree of \(A\) satisfies all the required properties of a \(\tau\)-functorial replacement of \(A\). \(\square\)

A \(\tau\)-functorial replacement also allows us to characterize the torsion in the associated graded for the \(\gamma\)-filtration of the Severi-Brauer variety of a central simple algebra in terms of the Severi-Brauer variety associated to its underlying division algebra.

**Lemma 4.13.** If \(A\) is a central simple algebra of \(p\)-primary index for some prime \(p\), then \(\gr_{\gamma} K(X)\) and \(\gr_{\gamma} G(X)\) contain only \(p\)-primary torsion.

Additionally, for every finite field extension \(F/k\) of degree prime-to-\(p\), the extension of scalars map \(\gr_{\gamma} K(X) \to \gr_{\gamma} K(X_F)\) is an isomorphism.
Proof. The first claim is known for the associated graded of the coniveau filtration where it follows from a restriction-corestriction argument. The first claim for associated graded of the $\gamma$-filtration then follows from the existence of a $\tau$-functorial replacement of $X$.

For the second claim, it suffices to note the extension of scalars map along a prime-to-$p$ induces a natural isomorphism between $K(X)$ and $K(X_{\overline{p}})$ and then apply Lemma 4.5.

Lemma 4.14. For an arbitrary central simple algebra $A$, we write $A = \bigotimes_{p \text{ prime}} A_p \otimes M_r(k)$ for a decomposition of $A$ into $p$-primary division algebras $A_p$ and a matrix ring $M_r(k)$. Then, for any prime $p$, for any integer $0 \leq j < \deg(A)$, and for $j'$ the remainder after dividing $j$ by $\ind(A_p)$, there are isomorphisms

$$\gr^j_k K(X) \otimes \mathbb{Z}(p) \cong \gr^{j'}_k K(X) \otimes \mathbb{Z}(p) \quad \text{and} \quad \gr^j_k G(X) \otimes \mathbb{Z}(p) \cong \gr^{j'}_k G(X) \otimes \mathbb{Z}(p).$$

Proof. We use [Kar17a, Lemma 3.5] to get the claim involving the coniveau filtration and Theorem 4.12 to get the claim for the $\gamma$-filtration.

5. Generating the $\gamma$-Filtration

Again $A$ is a central simple algebra, and $X = \text{SB}(A)$ its Severi-Brauer variety. In this section we describe the $\gamma$-filtration for $X$ when $A$ is a $p$-primary division algebra. The most distinguishing property for this purpose is the $p$-level of $A$.

Definition 5.1. The $p$-level of $A$ is defined to be the level of $A_p$, where $A_p$ is the $p$-primary division algebra occurring as a factor of $A$. We write $\lev(p, A)$ for the $p$-level of $A$. Recall that the level of a $p$-primary algebra $A$, written $\lev(A)$, as defined in [KM18a], is the largest number of distinct integers $1 \leq i_1, \ldots, i_t \leq \exp(A)$ with

$$v_p \ind(A \otimes p^{i_k}) < v_p \ind(A \otimes p^{i_{k-1}}) - 1$$

for every $1 \leq k \leq l$. In other words, the level of $A$ is the number of places the reduced behavior decreases by more than one from one position to the next.

Lemma 5.2. Let $A$ be a central simple algebra with $p$-primary index for a prime $p$. We assume also that $\lev(A) = r$, and that $i_1, \ldots, i_r$ are the distinct integers satisfying

$$v_p \ind(A \otimes p^{i_k}) < v_p \ind(A \otimes p^{i_{k-1}}) - 1$$

for all $1 \leq k \leq r$. Then $\gamma^i \subset K(X)$ is generated additively by products

$$\gamma^{j_1}(x_1 - \rk(x_1)) \cdots \gamma^{j_r}(x_r - \rk(x_r))$$

where $j_1 + \cdots + j_r \geq i$ and $x_1, \ldots, x_r$ are elements of $\{\zeta_X(p^{i_k})\}_{k=1}^r$.

Proof. In this setting, the ring $\gr_k K(X)$ is generated as an algebra by the Chern classes of the $\zeta_X(p^i)$. This is because: $K(X)$ is generated as a $\lambda$-ring by $\zeta_X(p^i)$, see [KM18a, Lemma A.6], Chern classes of $\lambda$-operations of a vector bundle are polynomials in the Chern class of this bundle, see [Mac18, Lemma 3.7], and Chern classes of the dual of a bundle are Chern classes of this bundle up to a sign [Mac18, Example 3.6].

Since these Chern classes in $\gr_k K(X)$ are defined as

$$c^i_j(x) = \gamma^i(\rk(x) - x^i) \mod \gamma^{i+1},$$

it follows that $\gamma^i$ is generated by the lifts of monomials of degree $i$ in these Chern classes and $\gamma^{i+1}$. By induction we can assume $\gamma^{i+1}$ is generated by similarly defined elements but of degree $i + 1$ and $\gamma^{i+2}$. Eventually, for large enough $d$, we have $\gamma^d = 0$ and it follows $\gamma^i$ is generated by the lifts of these monomials of degree $i$ or larger.
To complete the claim then, we only need to show that $\gamma^i(x)$ is a polynomial in the $\gamma^i(-x)$. This follows as

$$\gamma_t(x) = \frac{1}{\gamma_t(-x)}$$

and the right hand side is a series in $t$ with coefficients polynomials in the $\gamma$-operations of $-x$. \hfill \Box

**Lemma 5.3.** Let $A$ be a central simple algebra with $p$-primary index for some prime $p$. Assume $A$ has reduced behavior $r\text{Beh}(A) = (n_0, \ldots, n_m)$. Fix a splitting field $F$ of $A$ and identify $K(X)$ with its image in $K(X_F)$ under the extension of scalars map. Let $\xi$ be the class of $O_{X_F}(-1)$.

Then

$$\gamma^i(\xi^p \cdot (p^j - p^{n_j})) = \binom{p^{n_j}}{i} (\xi^p - 1)^i.$$

**Proof.** This is computed in [Kar98]. It’s done by

$$\gamma_t(p^{n_j} \xi^p \cdot (p^j - p^{n_j})) = \gamma_t(p^{n_j} (\xi^p - 1)) = \gamma_t(\xi^p - 1)p^{n_j} = (1 + (\xi^p - 1)t)p^{n_j}.$$

\hfill \Box

For future reference we provide the formula below.

**Lemma 5.4.**

$$x^n - 1 = \sum_{i=1}^{n} (-1)^i \binom{n}{i} (1 - x)^i$$

**Proof.** Note $x^n - 1 = (x - 1)(1 + x + \cdots + x^{n-1})$. Now apply Lemma 2.5 to the latter sum and combine. \hfill \Box

### 6. Comparison Between the $\gamma$, $\tau$, and $\eta$ Filtrations

For this section, fix a central division algebra $A$ of index $p^n$ for some prime $p$ and $n \geq 0$. We write $X = \text{SB}(A)$ as before. We also assume that $A$ is chosen so that $\gamma^i = \tau^i$ for all $i \geq 0$, applying Theorem 4.12 if needed (and possibly renaming our base field).

We’re going to compute the $\gamma$-filtration on $X$ in degrees greater $p^n - p$. In some ways, this computation is facilitated by the fact that most of the terms in an element of the $\gamma$-filtration start to vanish in these large degrees. The restriction to degree greater $p^n - p$ in particular means we’ll be doing computations with polynomials that can be written as sums of monomials of length at most $p - 1$. After making this observation, it only takes some rudimentary approximations on the divisibility properties of these sums to get our main result:

**Theorem 6.1.** For an arbitrary central simple algebra $A$ with $\text{ind}(A) = p^n$ and $X = \text{SB}(A)$ we have

$$\text{gr}_\gamma p^{n-i}K(X) = p^n(\xi - 1)p^{n-i}\mathbb{Z}$$

for all $1 \leq i \leq p - 1$.

In the above we’re identifying $K(X)$ with its image in $K(X_F)$ for some splitting field $F$ of $A$ and we are setting $\xi$ to be the class of $O_{X_F}(-1)$ in $K(X_F)$.

Before giving the proof, we give some lower bounds on the size of the $\gamma^i$. Strictly speaking these bounds aren’t needed and the interested reader can go straight to the proof of Theorem 6.1. We take the time to work through these bounds because it was consideration of these bounds that led to the description of the $\gamma$-filtration in these degrees.

So, we introduce a new filtration on $K(X)$ using the equality $\gamma^i = \tau^i$. Up to making a prime-to-$p$ extension of the base field, we can assume there are finite field extensions $k \subset L_0 \subset L_1 \subset \cdots \subset L_n$
with \([L_i : L_{i+1}] = p\) and \(\text{ind}(A_{L_i}) = p^{n-i}\) for each \(i\). For any \(j\) with \((j, p^n) = p^{n-i}\) Consider the composition
\[
\gamma^j(X_{L_i}) \to \tau^j(X_{L_i}) \xrightarrow{N_{L_i/k}} \tau^j(X) = \gamma^j(X).
\]
The leftmost of these groups is equal (over \(\bar{k}\)) to the ideal in \(K(X)\) generated by \((\xi - 1)^j\) by Lemma 4.10. The image of this element under the composition is equal \([L_i : k](\xi - 1)^j = a_j\bar{p}^j(\xi - 1)^j\) for some \(a_j\) coprime to \(p\).

**Definition 6.2.** We define \(\eta^i \subset \gamma^i \subset K(X)\) to be the group generated by the elements
\[
p^{n-v_{p}(j)}(\xi - 1)^j \quad \text{for all } j \geq i.
\]

That these elements exist inside of \(\gamma^i\) follows because
\[
\gamma^1(\xi X(1) - p^n)^j = p^{nj}(\xi - 1)^j
\]
is an element of \(\gamma^i\) and \((a_j p^{n-v_{p}(j)}, p^n) = p^{n-v_{p}(j)}\).

Alternatively, \(\eta^i\) can be described as the ideal generated by the degree \(j\) products of \(\gamma\)-operations of \(p^n(\xi - 1)\) for all \(j \geq i\).

We denote by \(\eta^{i/i+1} := \eta^i/\eta^{i+1}\) and \(\text{gr}_p K(X)\) for the associated graded pieces and the associated graded ring respectively. The following corollary comes from the existence of the \(\eta\)-filtration; it can also be deduced, at least when \(i = 0\), from \([\text{Kar95b}, \text{Proposition 2 and Lemma 3}]\).

**Corollary 6.3.** There’s an inequality
\[
\# \text{Tors} \bigoplus_{j=i}^{p^n-1} \text{gr}_i^j K(X) \leq \prod_{j=i}^{p^n-1} p^{n-v_{p}(j)}
\]
for any \(0 \leq i \leq p^n - 1\). When \(i = 0\) or \(i = 1\) we have
\[
\prod_{j=1}^{p^n} p^{n-v_{p}(j)} = np^n - (p^{n-1} + p^{n-2} + \cdots + 1).
\]

**Proof.** The ladder below is commuting and has exact rows for every \(i \geq 0\).
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \eta^{i+1} & \longrightarrow & \eta^i & \longrightarrow & \eta^{i/i+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \gamma^{i+1} & \longrightarrow & \gamma^i & \longrightarrow & \gamma^{i/i+1} & \longrightarrow & 0
\end{array}
\]
As \(\eta^{i/i+1}\) is torsion free, all of the vertical arrows are injections. Since \(\eta^i\) and \(\gamma^i\) have the same rank for every \(j\), it follows the cokernels of these vertical arrows are torsion. Using the snake lemma we get short exact sequences
\[
0 \to \gamma^{i+1}/\eta^{i+1} \to \gamma^i/\eta^i \to \gamma^{i/i+1}/\eta^{i/i+1} \to 0.
\]
Setting \(A = \text{Tors} \gamma^{i/i+1}\) we can write
\[
\gamma^{i/i+1} = A \oplus \gamma^{i/i+1}/A \quad \text{and} \quad \gamma^{i/i+1}/\eta^{i/i+1} = A \oplus (\gamma^{i/i+1}/A)/\eta^{i/i+1}.
\]
Now
\[
\# \text{Tors} \bigoplus_{j=i}^{p^n-1} \text{gr}_i^j K(X) \leq \prod_{i \geq 1} \gamma^{i/i+1}/\eta^{i/i+1} = \frac{\prod_{i \geq 1} \# \gamma^i/\eta^i}{\prod_{i \geq i+1} \# \gamma^i/\eta^i} = \# \gamma^i/\eta^i.
\]
Considering the natural inclusions of free abelian groups \( \eta^i \subset \gamma^i \subset K(X) \) we get inequalities

\[
\# \text{Tors } \gamma^i / \eta^i \leq \# \text{Tors } K(X) / \eta^i = \prod_{j=i}^{p^n-1} p^{n-v_p(j)}
\]

which proves the corollary. \( \square \)

Our main theorem says the bound above is far from sharp. The remainder of the section is devoted to this proof.

**Proof of Theorem 6.1.** It suffices by Lemma 4.8 to assume \( A \) is a division algebra. Our proof works by showing \( p^n \) divides the coefficient of every element of \( \gamma^{p^n-p+1} \supset \gamma^{p^n-1} \) when each of these elements is written as polynomial in \( 1 - \xi \). Note since there are inclusions

\[
\gamma^{p^n-p+1}(X) \subset \tau^{p^n-p+1}(X) \subset \tau^{p^n-p+1}(X_F) = (1-\xi)^{p^n-p+1} K(X_F),
\]

we can write every element \( y \) of \( \gamma^{p^n-p+1} \) as a sum

\[
y = \sum_{j=p^n-p+1}^{p^n-1} a_j (1-\xi)^j
\]

for some integers \( a_j \). After we show \( p^n \) divides each of these \( a_j \), it follows that we have inclusions

\[
\gamma^{p^n-p+1} \subset \gamma^{p^n-p+1} \subset \eta^{p^n-p+1}
\]

and this will end the proof.

Suppose then

\[
y = \gamma^{j_1}(x_1 - \text{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \text{rk}(x_r))
\]

is an arbitrary monomial generating \( \gamma^{p^n-p+1} \) like those described in Lemma 5.2. We can work in the two cases: each of \( x_1, \ldots, x_k \) equal \( \zeta_X(1) \) for some \( 1 \leq k \leq r \) (since \( v_p(j) \geq \min\{v_p(j_1), \ldots, v_p(j_k)\} \) and \( v_p(p^n) = n - v_p(i) \), we can even assume \( k = 1 \) or \( \zeta_X(1) \) does not appear among the \( x_1, \ldots, x_r \).

Assuming we’re in the former case, we can expand \( y \) as

\[
y = \left( \frac{p^n}{j_1} \right) (\xi - 1)^{j_1} \left( \frac{p^{n-t_2}}{j_2} \right) (\xi^{p^{n-t_2}} - 1)^{j_2} \cdots \left( \frac{p^{n-t_r}}{j_r} \right) (\xi^{p^{n-t_r}} - 1)^{j_r}
\]

\[
= \left( \frac{p^n}{j_1} \right) (p^{n-t_2}) \cdots \left( \frac{p^{n-t_r}}{j_r} \right) (\xi - 1)^{j_1} (\xi^{p^{n-t_2}} - 1)^{j_2} \cdots (\xi^{p^{n-t_r}} - 1)^{j_r}.
\]

Note also that \( s_2, \ldots, s_r \geq 1 \).

Now by Lemma 5.4, there is an expansion, for each \( 2 \leq l \leq r \),

\[
\xi^{p^{n_l}} - 1 = \sum_{i=1}^{p^{n_l}} (-1)^{j_l} \left( \frac{p^{n_l}}{i} \right) (1 - x)^i.
\]

We set \( x_{low}(l) = \sum_{i=1}^{p^{n_l}-1} (-1)^{j_l} \left( \frac{p^{n_l}}{i} \right) (1 - x)^i \) and \( x_{high}(l) = \sum_{i=p^{n_l}}^{p^{n_l}-1} (-1)^{j_l} \left( \frac{p^{n_l}}{i} \right) (1 - x)^i \). Note that \( p \) divides \( x_{low}(l) \) for every \( 2 \leq l \leq r \). Rewriting \( y \) in terms of \( x_{low} \)'s and \( x_{high} \)'s gives

\[
y = \left( \frac{p^n}{j_1} \right) (\xi - 1)^{j_1} (x_{low}(2) + x_{high}(2))^{j_2} \cdots (x_{low}(r) + x_{high}(r))^{j_r}.
\]

The lowest degree of any \( x_{high} \) is \( p \), while the lowest degree of any \( x_{low} \) is 1. This means, applying the binomial theorem and expanding, the lowest degree of \( (1-\xi) \) in any monomial containing an \( x_{high} \) is \( j_1 + j_2 + \cdots + j_r - 1 + p \geq p^n - p + 1 - 1 + p \geq p^n \). Hence all of these summands are 0.
Thus we find
\[
y = \binom{p^n}{j_1} (\xi - 1)^{j_1} \cdot \frac{x_{\text{low}}(2)^{j_2} \ldots x_{\text{low}}(r)^{j_r}}{p}.
\]

Rearranging, we check
\[
\binom{p^n}{j_1} p^{j_2 + \cdots + j_r} (\xi - 1)^{j_1} \left( \frac{x_{\text{low}}(2)}{p} \right)^{j_2} \ldots \left( \frac{x_{\text{low}}(r)}{p} \right)^{j_r}
\]

since each \(x_{\text{low}}\) is divisible by \(p\).

The \(p\)-adic valuation of the coefficient leading this product is exactly \(n - v_p(j_1) + j_2 + \cdots + j_r\).

We finish by showing \(n - v_p(j_1) + j_2 + \cdots + j_r \geq n\) for all possible \(j_1, \ldots, j_r\) or, equivalently, assuming \(j_1 + \cdots + j_r = p^n - i\) with \(0 < i < p\) we finish by showing
\[
p^n - i \geq j_1 + v_p(j_1).
\]

Assuming \(i\) is largest possible we can also show \(p^n - p + 1 \geq j_1 + v_p(j_1)\). We can assume \(v_p(j_1) > 0\) as otherwise \(p^n\) divides \(\binom{p^n}{j_1}\). Hence we can assume \(j_1 = a_1p^{n-1} + \cdots + a_{n-r}p^r\) with \(0 \leq a_1, \ldots, a_{n-r} < p\) and some minimal \(r \geq 1\). This inequality becomes
\[
p^n - p + 1 \geq a_1p^{n-1} + \cdots + a_{n-r}p^r + r.
\]

We make one last approximation, and assume all \(a_1, \ldots, a_{n-r}\) are equal \((p - 1)\), as this is the largest they can be. We’re left checking
\[
p^n - p + 1 \geq a_1p^{n-1} + \cdots + a_{n-r}p^r + r = p^n - p^r + r.
\]

Rearranging, we check
\[
p^r - p \geq r - 1
\]

which is clear if \(r = 1\) and is the same as
\[
\frac{p^r - p}{r - 1} \geq 1
\]

for \(r > 1\). Using the mean value theorem, the left of this inequality equals \(f'(c)\) for some \(c\) in the interval \([1, r]\) and \(f(x) = p^x\). Since \(f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1\) we’ve completed this case.

We still need to check the second case, when \(\zeta_X(1)\) is not a part of the \(\gamma\)-operations of our monomial. Following the same process as before, we’re left to check the inequality \(p^n - i \geq n\) for \(0 < i < p\). But this is also readily checked to be true: we can assume we want to show \(p^n - p + 1 \geq n\); and \(p^n - p \geq n - 1\) is the same (ignoring the \(n = 1\) case which is trivial) as \(\frac{p^n - p}{n - 1} \geq 1\) which by the mean value theorem equals \(f'(c)\) for some \(c\) in the interval \([1, n]\) and \(f(x) = p^x\); for all such \(c\) we have \(f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1\).

We end with some more general statements that can be obtained from Theorem 6.1.

**Corollary 6.4.** Let \(B\) be a central simple algebra, and \(Y\) the Severi-Brauer variety of \(B\). Suppose \(\text{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}\) is a prime factorization of \(B\) with \(p_1 < \cdots < p_r\). Then for all \(1 \leq i \leq p_1 - 1\)
\[
\text{gr}_{\gamma_d}^{d-i} K(Y) = d(1 - \xi)^{d-i} \mathbb{Z},
\]

where \(\xi\) is the class of \(O_{X_F}(-1)\) when identifying \(K(X) \subset K(X_F)\) for a splitting field \(F\) of \(X\).

**Proof.** Apply Lemma 4.14.

**Corollary 6.5.** Suppose \(B\) is generic central simple algebra of index \(p^n\) and exponent \(p^n\) in the sense of [Kar17a, Example 2.2] and set \(X = \text{SB}(B)\). Then
\[
\text{CH}_j(X) = p^n \mathbb{Z} \quad \text{for all} \quad 0 \leq j \leq p - 2.
\]
More generally, suppose $B$ is a central simple algebra with $\text{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ a prime factorization of $d$ ordered like $p_1 < \cdots < p_r$. Suppose the $p_i$-level of $B$ is less or equal 1 for all $1 \leq i \leq r$ and suppose $\text{CH}(X)$ is generated by Chern classes where $X = \text{SB}(B)$. Then

$$\text{CH}_j(X) = d\mathbb{Z} \quad \text{for all } j \leq p_k - 2$$

where $k$ is the smallest number with $\text{lev}(p_k, B) = 1$; if no $k$ exists then $\text{CH}(X)$ is torsion free.

Proof. In the former case, the rings $\text{CH}(X)$ and $\text{gr}_* K(X)$ are isomorphic, [Kar17b, Theorem 3.1]. In the latter case, we use the same fact as before but for these algebras [KM18a, Theorem A.15]. □

References


Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA

E-mail address: mackall at ualberta.ca
URL: www.ualberta.ca/~mackall

14