Multicomponent seismic data reconstruction using the quaternion Fourier transform and POCS
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SUMMARY
The quaternion Fourier transform is used to obtain a Fourier domain representation of a multicomponent seismic record. Projection Onto Convex Sets (POCS) is used as a means to reconstruct all three components at once. The results of this process are compared with standard component-by-component reconstruction. The algorithm is applied to 2D and 3D 3C synthetic data but it could be easily extended to a higher number of dimensions and components.

INTRODUCTION
Multicomponent seismic acquisition samples the full 3D wavefield. There is a great deal of information which can be garnered from such data, but this requires careful processing to be carried out to preserve any subtle relationships which may be present (Stewart et al., 2002; Zhu et al., 1999). Reconstruction of noisy data or missing traces is commonly done by treating each component separately which could be damaging to any subtle features. This paper introduces a method to reconstruct all components at once using the quaternion Fourier transform and Projection Onto Convex Sets (POCS). Quaternions have been used for other applications in seismic data processing such as the computation of spectral attributes (Bihan and Mars, 2001), time lapse analysis and boundary detection (Witten and Shragge, 2006), and multicomponent velocity analysis (Grandi et al., 2007). Quaternions were first introduced by Sir William Rowan Hamilton while investigating how to extend the complex numbers into three dimensions. He knew how to add and multiply three dimensional numbers, but was struggling to find a way to divide them. In 1843 Hamilton discovered that to allow division a fourth dimension is necessary (Hamilton, 1866). Each component of a multicomponent sample can be placed into the arguments of a quaternion which allows for operations on all components to be carried out simultaneously. The quaternions are transformed to the $F - K_x - K_y$ domain using the quaternion Fourier transform and a single amplitude spectrum for all three spectra is defined using a polar representation of quaternions (Sangwine and Ell, 1999). Reconstruction of missing traces is carried out using Projection Onto Convex Sets (Abma and Kabir, 2006). The method has the distinct advantages that the orthogonality of the input components is maintained (the signals are not mixed), and the subtle variations between the components (whether they be large or small) serve to improve the quality of the reconstruction.

THEORY
Quaternions
A quaternion is defined as

$$q = a + bi + cj + dk$$

where, $i^2 = j^2 = k^2 = ijk = -1$. A quaternion is defined as pure when $a = 0$. In the case of a pure quaternion the elements of $(b,c,d)$ could be used to represent a 3C multicomponent sample.

It is often useful in the Fourier domain to use a polar representation of data in terms of amplitude and phase. The polar representation of quaternions is given by

$$q = |q| e^{i\mu\phi}$$

where,

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2},$$

$$\mu = \frac{bi + cj + dk}{\sqrt{b^2 + c^2 + d^2}},$$

$$\phi = \frac{a}{|q|},$$

and $\mu$ is a pure unit quaternion ($|\mu| = 1$) which is referred to as the quaternion’s eigenaxis. Another representation of equation (2) is (Ell and Sangwine, 2007)

$$q = |q|(\cos\phi + \mu\sin\phi).$$

This equation is important because it allows for the amplitude of all elements of the quaternion to be related in a single term, $|q|$. Care has to be taken when carrying out algebra with quaternions. Multiplication is not commutative, and as a result, multiplication by an inverse function, or division, has distinct left and right sided representations. This also implies that $e^{ib+c} \neq e^{ib}e^{jc}$ (Sangwine, 1998). The forward and reverse continuous quaternion fourier transforms are given by (Ell, 1992, 1993)

$$Q(\omega, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\omega t} q(t,x) e^{-jkv x} dt dx$$

(7)

$$q(t,x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\omega t} Q(\omega, v) e^{jkv x} d\omega dv .$$

(8)

where $\omega$ is temporal frequency and $v$ is the spatial wavenumber.

The formulation for the forward transform can be generalized to (Sangwine and Ell, 2000)

$$Q(\omega, v) = \frac{1}{\sqrt{MN}} \sum_{n=0}^{M-1} \sum_{m=0}^{N-1} e^{-\mu 2\pi \left( \frac{n}{M} + \frac{m}{N} \right)} q(t,x).$$

(9)

Here $\mu$ is any unit pure quaternion. The standard complex Fourier transform is a special case of this transform and occurs when $\mu = i$ and the function being transformed is complex (Ell and Sangwine, 2007).

If $\mu = (i + j + k)/\sqrt{3}$ the quaternion $q = a + bi + cj + dk$ can be decomposed into it’s symplectic form

$$q' = (a' + b'\mu_1) + (c' + d'\mu_1)\mu_2$$

(10)
Multicomponent seismic data reconstruction

where \(d', b', c', a'\) can be found using a change of basis

\[
\begin{pmatrix}
 b' \\
 c' \\
 d'
\end{pmatrix} = \begin{pmatrix}
 \mu_1 & \ldots & \mu_3 \\
 \ldots & \mu_1 & \ldots \\
 \ldots & \ldots & \mu_3
\end{pmatrix} \begin{pmatrix}
 b \\
 c \\
 d
\end{pmatrix}
\] (11)

and \(a' = a\) (Ell and Sangwine, 2007). The rows of the change of basis operator in (11) are defined by the three pure basis quaternions \(\mu_1, \mu_2, \mu_3\) written as row vectors. The choice of \(\mu_2\) is arbitrary, but must not be parallel to \(\mu_1, \mu_3\) can be calculated by taking the cross product of \(\mu_1\) and \(\mu_2\) and setting it’s modulus to one. This allows for the quaternion to be written as

\[
Q(\omega, v) = Q_1(\omega, v) + Q_2(\omega, v)\mu_2
\] (12)

where,

\[
Q_1(\omega, v) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{t=0}^{N-1} e^{-\mu_1 x |t|^\mu} T_m q_1(t, x)
\] (13)

for \(i \in \{1, 2\}\). This allows for existing FFT codes to be used in calculating the QFT (Ell and Sangwine, 2007). There is also an excellent Matlab toolbox for quaternions by Sangwine and Bihan (2005). The following are the steps to carry out the 2D quaternion Fourier transform:

1. Begin with quaternion \(q = a + bi + cj + dk\), where \(a\) can be set to zero, and \(b, c, d\) can be set to the values of the \(x, y, z\) components of a multicomponent seismic sample.
2. Apply a change of basis to the quaternions by taking the scalar product of each entry of \(q\) with the three basis quaternions \((\mu_1, \mu_2, \mu_3)\) to get \(q' = d' + b'i + c'j + d'k\).
3. Write \(q_1' = d' + b'i\) and \(q_2' = c' + d'k\).
4. Apply two 2D complex FFT’s of \(q_1'\) and \(q_2'\) giving \(Q_1' = Q' + B'i\) and \(Q_2' = C' + D'k\).
5. Construct the quaternion in the Fourier domain: \(Q' = A' + B'i + C'j + D'k\).
6. Reverse the change of basis by taking the scalar product of each entry of \(Q'\) with \((\mu_1, \mu_2, \mu_3)^{-1}\) to give \(Q = A + Bi + Cj + Dk\). The inverse QFT is done in a similar manner.

The Fourier transform of a quaternion is itself a quaternion which can be written in the polar representation (equation 2). An important distinction between the \(F - K\) spectra of quaternion data versus real data is that the usual conjugate symmetry of the upper and lower halves does not exist for the quaternion spectra. This implies that any processing needs to be applied to both halves of the spectrum. The amplitude and phase diagrams of the 2D QFT closely resemble the summation of the 2D FT of each of the individual components, but are not equivalent to this. A distinction of the 2D QFT versus if one were to take the 2D FT across components is that the 2D QFT maintains the orthogonality of the individual components whereas the 2D FT across components would be merging all components onto a single complex plane (Ell, 1993).

Projection Onto Convex Sets (POCS)

Projection onto Convex Sets (POCS) can be used to reconstruct seismic data by iteratively thresholding the data’s spectrum while using a reinsertion operator to control which traces are reconstructed. It is shown to be an effective method for seismic data reconstruction (Abma and Kabir, 2006; Gao et al., 2010; Wang et al., 2010; Stein et al., 2010), but typically requires many iterations to achieve good results. A data driven thresholding schedule has been shown to give a significant improvement in the number of iterations required while still achieving good results (Gao and Sacchi, 2011). A modification to the reinsertion operator allows for the algorithm to also be used for denoising of seismic data (Gao and Sacchi, 2011).

In this case POCS was applied to the quaternion \(F - K\) amplitude spectra, \(|Q|\), in the 2D-3C case, and to the \(F - K_s - K_2\) amplitude spectra, \(|Q_s|\), in the 3D-3C case, but the method could be applied to \(N\) spatial dimensions with minor modification, and already supports up to four components. It is also possible to process higher numbers of components using the symplectic representation (equation 10) to include more components of data, but in this case we restrict ourselves to 3D-3C data. For a given temporal frequency, \(\omega\), the quaternion in the \(f - x - y\) domain at the \(k^{th}\) iteration of POCS is given by

\[
Q^k = \alpha Q^{obs} + (1 - \alpha S) Q_0 T Q_0^{-1}, k = 1, \ldots, N,
\] (14)

where \(Q^{obs}\) is the quaternion representation of the original data with missing traces, and \(Q_0\) and \(Q_0^{-1}\) are the forward and reverse 2D quaternion Fourier transforms respectively. In this notation \(Q^k(\omega, k_s, k_g) = F_Q Q^k(\omega, x, y)\), and \(T\) is an iteration dependent quaternion threshold operator. \(S\) is the sampling operator and is equal to one for points with existing traces and zero for points with unrecorded observations. The scaling factor \(\alpha \leq 1\) can be used to simultaneously denoise the data. A choice of \(\alpha = 1\) reinserts the noisy original data at each iteration, whereas a lower value of \(\alpha\) will denoise the volume by taking an average of the original and reconstructed data. A detailed discussion of the iteration dependent threshold operator, \(T\), can be found in Gao and Sacchi (2011).

Examples

Figure 1 shows an example of three linear signals with differing dips. Figure 2 shows the \(F - K\) amplitude spectra of the signals seen in Figure 1, and (d) shows the quaternion \(F - K\) amplitude spectrum of all three signals. It is important to note that while it may appear that all three signals are mixed in (d), they remain orthogonal due to the definition of the eigenaxis \(\mu_1, \mu_2, \mu_3\).

An example of the reconstruction of 3D-2C synthetic data which has been decimated to 40% of the original number of traces is shown in Figure 3. The middle row shows the reconstruction using a traditional component-by-component approach whereas the bottom row shows the application of Quaternion POCS (QPOCS). The reconstruction using QPOCS has a lower level of noise and suffers from fewer artifacts.

Figure 4 shows an example of the reconstruction of 3D-3C gathers which have been decimated to 40% of the original
Multicomponent seismic data reconstruction

Figure 1: XT plots of three linear signals of various dips. Each maps to two points in their $F-K$ amplitude spectra. In the quaternion $F-K$ amplitude spectrum all three signals can be identified.

Figure 2: $F-K$ amplitude spectra of the signals pictured in Figure 1(a), (b), (c), and the Quaternion $F-K$ amplitude spectrum of all three components, (d). All three signals can be identified in the quaternion amplitude spectrum.

The results are plotted for both traditional component by component POCS (dashed lines), and QPOCS (solid lines). The quality of the reconstructions for all components is noticeably improved through the use of QPOCS.

CONCLUSIONS

We have studied a method to reconstruct all components of a multicomponent seismic volume simultaneously using the quaternion Fourier transform and POCS. The algorithm improves the quality of the reconstruction while preserving the orthogonality of the input components. The algorithm can also be used to denoise seismic volumes. The method works on 3D-3C volumes but could be modified to handle higher spatial dimensions and more components. A potential problem of future research is the extension of other Fourier reconstruction methods such as MWNI (Liu and Sacchi, 2004) to the multicomponent case by replacing the DFT with the QDFT.

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\[ R - \text{SNR} = 10 \log \frac{||d_o||^2}{||d - d_o||^2} (dB). \]
Multicomponent seismic data reconstruction

Figure 3: Interpolation of 2D-3C synthetic records decimated to 40% of their original number of traces. The records have random noise and two crossing linear events of differing amplitude and frequency for each component. The central row shows the result of reconstruction of individual components using POCS. The bottom row shows the result of reconstruction using Quaternion POCS. The Quaternion POCS reconstructions show a lower level of noise and fewer artifacts.

Figure 4: Time slices showing the reconstruction of a synthetic 3D-3C gather using component by component POCS (middle row), and reconstruction using all components at once using Quaternion POCS (bottom row). The input data (top row), has been randomly decimated to 40% of its original number of traces. Note that the input gathers X, Y and Z have different amplitudes, frequencies, moveouts, and times of zero incidence for both of their events.
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