Stockpiling and Shortages*
(the "Toilet Paper Paper")

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Abstract

Consumer stockpiling of everyday household items such as toilet paper occurred in the wake of the COVID-19 pandemic, resulting in shortages of these goods in stores. Both phenomena reinforce each other: The expectation of shortages causes stockpiling behavior, which amplifies the shortages experienced by consumers, which in turn encourages more stockpiling. In this paper, we examine this feedback loop. When aggregate supply is insufficient to meet aggregate demand but prices cannot adjust to clear the market, there can be multiple equilibria featuring stockpiling to different degrees. Moreover, even when the fundamental supply-demand imbalance is small, the degree of equilibrium stockpiling can be significant. Stockpiling reduces welfare, and this welfare loss can be particularly severe in the transitional phase following a supply shock, during which households build up inventories.

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There is enough in the whole country for the coming ten years. We can all poop for ten years.

Dutch prime minister Mark Rutte on the toilet paper supply in March 2020

1 Introduction

This paper examines a dynamic environment in which households want to consume a constant amount of a storable good in every period and the per-period supply of that good may fall below the aggregate demand due to an exogenous shock. In this environment, individual households may attempt to smooth consumption by accumulating inventories. However, stockpiling behavior has an externality, as it exacerbates the shortages experienced by other households. This, in turn, increases their incentive to stockpile, which reduces the availability of the good even further. The objective of this paper is to examine this feedback loop—that is, we examine how underlying supply shortages can be magnified via consumers' rational inventory responses.

The aforementioned market conditions were present, for example, during the early days of the COVID-19 pandemic (see, e.g., Wang *et al.* 2020; Micalizzi *et al.* 2021). As supply chains were being disrupted due to travel restrictions and hygiene measures at factories and logistics facilities, a stable supply of everyday household items was no longer guaranteed. Sensing that they may not be able to acquire these goods in the future, consumers started buying up available quantities of storable products such as dried pasta, canned goods, and toilet paper, resulting in rows of empty supermarket shelves.[1] Similarly, in May 2021 a cyberattack on Colonial Pipeline Corporation unexpectedly reduced the supply of transportation fuels in the Eastern United States and caused long queues at gas stations.[2] The emergence of these queues indicated that drivers were filling up their vehicles before their tanks were near empty, as drivers would normally do. Other examples include the iodized table salt shortage in China after the 2011 Fukushima nuclear disaster and the recent baby formula shortage in the United States.[3] In the situations discussed above, underlying supply disruptions were likely magnified by consumers' stockpiling responses. That is, the shortages consumers actually experienced could have been much less severe had consumers refrained from stockpiling.

[1] In addition, shortages can also originate on the demand side. The model developed here is expressed in terms of supply reductions instead of demand increases, but can be recast to apply to the latter case as well (see Section 6.1).

[2] See www.reuters.com/technology/colonial-pipeline-halts-all-pipeline-operations-after-cybersecurity-attack-2021-05-08 (retrieved December 10, 2024).

[3] See www.wsj.com/articles/BL-CJB-13504 (retrieved December 10, 2024); www.wsj.com/articles/baby-formula-shortage-stuns-states-including-tennessee-kansas-and-delaware-11652526002 (retrieved December 10, 2024).

To examine this two-way relationship between shortages and stockpiling behavior, we develop a dynamic model with a continuum of households, each wanting to consume one unit of a good per period, the price of which is fixed. Households can store, at a cost, up to a certain maximum quantity of the good but cannot resell the good to other households. The aggregate per-period supply of the good is fixed, and if it is less than the aggregate consumption requirement some households will be rationed. The prospect of rationing induces households to maintain inventories. A decision rule describes a household's optimal inventory behavior in each period, and a symmetric equilibrium is a decision rule that is optimal for a household if all other households adopt the same decision rule. Stockpiling arises if, in equilibrium, households buy and store more than the single unit consumed per period.

We show that, if there is no underlying supply shortage—that is, if the aggregate supply in each period is sufficient to meet the aggregate consumption requirement of one unit per household—the unique equilibrium is for every household to obtain exactly one unit of the good per period. Thus, stockpiling cannot merely be a "self-fulfilling prophecy"—it requires an underlying, fundamental supply-demand imbalance. If such a fundamental imbalance exists, however, equilibria may emerge in which households store more than one unit, and may, in fact, store up to their capacity limit. These outcomes can arise even if the fundamental supply shortage is negligible. We show that the overall degree of stockpiling in any equilibrium can be decomposed into a fundamental component, which is driven by the underlying supply shortage; and an excess component caused by the reinforcing effect. The magnitude of the second component can be multiple times that of the first component. In these stockpiling equilibria, households experience shortages with a much higher likelihood than what is indicated by fundamentals, and will buy large quantities whenever they are not rationed.

We give a full characterization of all symmetric, stationary equilibria in what we call z-storage rules. A z-storage rule is a decision rule under which the household tries to maintain a target inventory level of z units; if the actual inventory falls below this threshold the household attempts to restock to an inventory of z units. Any z-storage rule with $z > 1$ involves stockpiling of units not immediately needed for consumption. Stationary equilibria are generally not unique, and the entire range of possible z-storage rules can be equilibria for generic parameter values. Therefore, the aforementioned positive reinforcing effect, whereby stockpiling behavior creates additional storage incentives, results in equilibrium indeterminacy. We also provide a limited characterization of what we term transitional equilibria. These equilibria take into account the transitional dynamics that arise when the economy starts out in a state of balanced supply and demand, but then experiences an unexpected supply shock—e.g., because of a pandemic. In both stationary and transitional equilibria, welfare decreases in the degree of equilibrium stockpiling.

As mentioned previously, we assume (i) that the market price for the good is fixed and (ii) that households cannot resell the good to other households. We make these assumption for two reasons. First, in the context that motivates our analysis, constraints on price adjustments are often imposed by governments,[4] and the resale of consumer goods between households, while possible, is uncommon. Second, our assumptions eliminate any speculative motive for stockpiling. Flexible market prices that balance supply and demand will, by definition, clear the market and prevent rationing. However, in dynamic markets for storable goods, this does not imply that the incentive to accumulate inventories goes away—to the contrary, it is precisely the expectation of changing prices that can cause agents to stockpile for speculative reasons.[5] Moreover, because goods held in storage for speculative reasons are not consumed immediately, consumption is still reduced despite market-clearing prices. To steer clear of these complications and to isolate the role of stockpiling as a consumption smoothing mechanism—and to examine how stockpiling amplifies shortages—we assume that prices are fixed and inventories cannot be resold.

Stockpiling by consumers has been studied in the marketing and industrial organization literatures. Meyer and Assunção (1990), Mela *et al.* (1998), Hong *et al.* (2002), Hendel and Nevo (2006a), Hendel and Nevo (2006b), and Ching and Osborne (2020) examine consumer's propensity to stockpile in response to temporary promotional discounts, both theoretically and empirically. An implication from these studies is that, for certain storable consumer goods, price decreases can lead to large increases in units sold even though the underlying consumption demand is relatively price inelastic. That is, demand responses to price changes often merely reflect shifts in the timing of purchases. This, in turn, has implications for firms' pricing and promotion strategies; see, e.g., Bell *et al.* (2002), Guo and Villas-Boas (2007), Su (2010), and Gangwar *et al.* (2013). Our model of consumer stockpiling differs from this literature in that prices and aggregate supply quantities are exogenously fixed. This choice allows us to focus on consumer stockpiling as an optimal response to other consumers' stockpiling behavior (instead of an optimal response to expected price increases).

[4]For example, at least 38 U.S. states currently have price gauging statutes that limit retailers' ability to increase prices during natural disasters, emergencies, or major economic disruptions. See www.ncsl.org/financial-services/price-gouging-state-statutes (retrieved December 10, 2024).

[5]A classic literature in economics and finance examines price formation in competitive forward markets for storable commodities; see Telser (1958), Turnovsky (1983), Scheinkman and Schechtman (1983), Kawai (1983), Sarris (1984), and Hirshleifer (1989), among others. This line of inquiry was later extended to imperfectly competitive markets; see, e.g., Allaz (1991) and Thille (2003). The models in this literature are meant to characterize the strategies of professional traders in, e.g., agricultural markets. (Another literature examines producers' inventory behavior driven by the stockout-avoidance motive; see, e.g., Kahn 1987; Wen 2005, 2011.) These models are, therefore, much differently motivated than the one we examine, in which inventories are maintained at the consumer level and no sales from inventories can occur.

Two other recent theoretical papers are also motivated by consumer behavior observed in the early days of the COVID-19 pandemic. Awaya and Krishna (2024) study a two-period model with fixed but uncertain supply. Consumption takes place in the second period, but consumers can purchase in either period. Awaya and Krishna (2004) show that fixed prices tend to induce delayed buying while flexible prices tend to induce early buying (at a welfare loss). Moreover, with higher-order supply uncertainty, early "panic buying" becomes more likely. In our model, on the other hand, there is no uncertainty over the aggregate supply—households only face *individual* uncertainty in that they may be rationed with some probability, but this uncertainty is amplified through the stockpiling behavior of other households.

Noda and Teramoto (2024) examine a model with infinitely-lived consumers with the ability to accumulate inventories, as we do. Their model features continuous time, search costs, a fixed positive price, and both household-level and store-level inventories (whereas our model features discrete time, a storage cost, a zero price, and only household-level inventories). More importantly, Noda and Teramoto (2024) do not consider equilibrium multiplicity, which is the main focus of this paper. Their focus is on the transitional dynamics following a search cost shock, which they examine primarily using numerical methods.

Finally, our model shares similarities with some monetary search models. Berentsen (2000) shows that, in a simple extension of the Kiyotaki-Wright (1989) model, multiple stationary equilibria exist that have the same money stock and in which agents are willing to accumulate either one or two units of money. These equilibria resemble, in some ways, the 1-storage and 2-storage equilibria in our model. However, there are several important differences. First, money is a financial asset whereas the storable good in our model is a consumption good. Second, the supply of consumption goods in monetary search models is endogenously determined by agents' production decisions and the supply of money is (explicitly or implicitly) a policy variable. In our model, there is no money and the supply of the consumption good is exogenously fixed and cannot be increased by fiat. Third, while in both types of models inventories provide insurance against non-consumption events, the consequences of inventory accumulation are not the same. In our model, one household's inventory causes a negative externality, in that it increases the likelihood of non-consumption events faced by other households (thereby amplifying their stockpiling incentives). Money inventories, on the other hand, have a positive externality, in that they increase trading opportunities for other households.[6]

The remainder of the paper is organized as follows. In Section 2 we set up the theoretical model and define equilibrium. In Section 3 we define z-storage rules and

[6]For example, in Berentsen (2000), the high-inventory equilibrium exhibits a higher velocity of money, which results in higher welfare compared to the lower-inventory equilibrium.

derive properties of the dynamic system generated by these rules. Section 4 contains the main results, characterizing stationary equilibria in z-storage rules as well as transitional equilibria. Section 5 contains a welfare analysis of the various equilibria of the model. Section 6 discusses policy implications and some open questions. Most proofs are in the Appendix.

2 Model

2.1 The economy

The economy is populated by a continuum of households of measure 1. Time is divided into periods indexed by $t = 1, 2, 3, \ldots$. There is one consumption good, which can be bought and consumed in integer quantities only. In every period, a household requires one unit of the good. Each household can store up to $K \geq 2$ units of the good from one period to the next (where K is an integer and exogenous). A household receives a flow utility of 1 if it consumes the good in a given period, and a flow utility of 0 if it does not consume. In each period, a household pays a storage cost $\lambda > 0$ per unit stored.

The economy-wide supply of the good per period is a continuum of measure $m \leq 1$. If $m = 1$, the economy produces exactly as much of the good as is required to meet every household's consumption need. If $m < 1$, the economy experiences an *aggregate supply shortage*.[7] There is a single store in the economy at which households can obtain the good. The price of the good is fixed and normalized to zero.

Period t unfolds as follows:

1. A given household enters period t with some inventory $s^t \in \{0, 1, \ldots, K\}$ of the good in storage, and pays the resulting storage cost, λs^t. If $s^t > 0$, the household consumes one unit, reducing its inventory by one. If $s^t = 0$, the household cannot consume the good in this period.

2. Every household decides how many new units of the good it wants to obtain and makes a trip to the store. The store puts the entire economy-wide supply of the good, m, on its shelves.

3. Whether a household can obtain its desired quantity in the store or not depends on the aggregate demand by all households, denoted θ^t. If $\theta^t \leq m$, then every household is able to buy its desired quantity. If $\theta^t > m$, then some households will be rationed. We assume that the rationing is probabilistic and symmetric, i.e., all

[7] In Section 6, we discuss how the model can be adjusted in order to apply to the case of an aggregate demand increase, instead of a supply shortage.

households face the same probability of not being able to obtain the good in period t.[8] We say that these households experience an *in-store shortage*.

4. A households end-of-period inventory is determined by their beginning-of-period inventory (if any), minus consumption (if any), plus additional units obtained from the store (if any). This inventory is stored, and becomes the household's inventory at the beginning of the period $t+1$.

All households discount the future using a common discount factor $\beta \in (0,1)$. To prevent trivial outcomes where maintaining a zero inventory is optimal, we assume that $\lambda < \beta$.

We make the following additional assumptions. First, a household cannot resell or otherwise transfer the good to another household. The role of this assumption (along with the fixed-price assumption) is discussed in Section 1. Second, households cannot dispose of the good they have in storage, which implies that the only way for a household to reduce its inventory is through consumption.[9] Third, if the store has a positive amount of the good remaining after all households have shopped in a given period, the unsold amount is disposed (this does not happen in any of the equilibria we examine).

Finally, recall that we impose an exogenous upper bound K on the inventories held by each household. This simplifies certain steps in our analysis; however, because storage is costly it would not be optimal to accumulate unlimited inventories even if doing so was possible.[10]

2.2 State variables and decision rules

From the perspective of an individual household, it will be convenient to focus on the household's beginning-of-period-t inventory,

$$s^t \in \{0, \dots, K\},$$

[8] Such a rationing mechanism would arise, for instance, if arrival times at the store were random due to exogenous factors (such as traffic congestion) and such that, for every household, the measure of households arriving earlier was uniformly distributed.

[9] Without this assumption, the disposal quantity becomes an additional decision variable for each household. This would be an unnecessary complication: If the good had a price sufficiently above the cost of storage, reducing inventories through disposal would clearly not be optimal. Our no-disposal assumption achieves the same effect when the price is zero.

[10] Since a household wishes to consume only one unit per period, the present value of the cost of storing the k^{th} unit is $\lambda(1-\beta^k)/(1-\beta)$ and the present value of consuming this unit is β^k. It is straightforward to verify that storing the k^{th} unit has a non-negative net present value if and only if $k \leq \left[\ln\lambda - \ln(2-\beta)\right]/\ln\beta$, generating an endogenous upper bound on each household's inventory.

as the relevant state variable. A *decision rule* for the household in period t is a mapping that assigns to each value of s^t the household's desired beginning-of-next-period inventory:

$$\sigma^t(s^t) \in \{\max\{0, s^t - 1\}, \ldots, K\}.$$

Thus, in period t, the household obtains $\sigma^t(s^t) - \max\{0, s^t - 1\}$ units of the good if it does not experience an in-store shortage, and zero units otherwise.

From the perspective of the aggregate economy, the relevant state in each period is the distribution of inventories across households. This state is given by the vector

$$x^t = \left(x_0^t, x_1^t, \ldots, x_{K-1}^t, x_K^t\right),$$

where x_k^t is the fraction of households that enter period t with k units in storage. The space of all such states is the K-dimensional unit simplex,

$$\Delta_K \;\equiv\; \left\{x \in \mathbb{R}_+^{K+1} : \sum_{k=0}^{K} x_k = 1\right\}.$$

For $x, y \in \Delta_K$, we write $x \succsim y$ (or $y \precsim x$) if x weakly dominates y, in the sense of first-order stochastic dominance.[11]

Suppose all households use the same decision rule σ^t in period t. The aggregate measure of the good that would be purchased in period t if there was no supply constraint (i.e., if $m = \infty$) is given by

$$\theta^t \;=\; \sum_{k=0}^{K} x_k^t \left[\sigma^t(k) - \max\{0, k-1\}\right]. \tag{1}$$

Because a measure $m \leq 1$ is available, the probability that a household arrives at the store and is able to purchase the good is

$$p^t \;=\; \min\left\{\frac{m}{\theta^t}, 1\right\}. \tag{2}$$

The probability that a household experiences an in-store shortage is, therefore, equal to $1 - p^t$.

Households that find the good are able to execute their desired purchases and will enter period $t+1$ with $\sigma^t(s^t)$ units in storage. Households that experience an in-store shortage will enter period $t+1$ with $\max\{0, s^t - 1\}$ units in storage. Therefore, given aggregate state $x^t \in \Delta_L$ and common decision rule σ^t, we can compute next period's

[11] That is, $x \succsim y$ if $\sum_{k'=0}^{k} x_{k'} \leq \sum_{k'=0}^{k} y_{k'}$ $\forall k = 0, \ldots, K$.

state as follows:

$$x_k^{t+1} = \begin{cases} p^t \sum_{k':\sigma^t(k')=K} x_{k'}^t & \text{if } k = K, \\ p^t \sum_{k':\sigma^t(k')=k} x_{k'}^t + (1-p^t)x_{k+1}^t & \text{if } 0 < k < K, \\ p^t \sum_{k':\sigma^t(k')=0} x_{k'}^t + (1-p^t)\left[x_1^t + x_0^t\right] & \text{if } k = 0. \end{cases} \quad (3)$$

The economy is in a *stationary state* if the distribution of inventories across households is time-invariant, that is

$$x^{t+1} = x^t = x^* \quad \forall t.$$

A special case of a stationary state is a steady state, in which no individual household's inventory changes from period to period.

2.3 Equilibrium definition

Our notion of equilibrium is, essentially, that of Jovanovic and Rosenthal (1988) for anonymous sequential games: In every period, every agent chooses an action that maximizes the agent's discounted continuation payoff in that period, given state variables, and state variables in each period are determined from the previous period's state variables and the distribution of actions across individuals in the previous period. These conditions imply behavior that is the same as that in a subgame perfect equilibrium.[12]

We can, however, apply two simplifications to the Jovanovic/Rosenthal definition. First, while Jovanovic and Rosenthal (1988) allow for continuous individual state and action spaces, ours are finite, and the equilibrium definition below is stated accordingly. Second, Jovanovic and Rosenthal (1988) permit mixed strategy equilibria, or—which is equivalent with a continuum of identical agents—asymmetric pure strategy equilibria. We restrict attention to symmetric pure strategy equilibria, that is, equilibria in which all agents adopt the same decision rule in a given period.

We denote by $V^t(s^t)$ the continuation value of being in individual state $s^t = 0, \ldots, K$ in period t. This continuation value can be expressed recursively as follows:

$$V^t(s^t) = \begin{cases} \max\limits_{\sigma^t \in \{s^t-1,\ldots,K\}} \left\{ 1 - \lambda s^t + \beta\left[p^t V^{t+1}(\sigma^t) + (1-p^t)V^{t+1}(s^t-1)\right]\right\} & \text{if } s^t \geq 1, \\ \max\limits_{\sigma^t \in \{0,\ldots,K\}} \left\{ \beta\left[p^t V^{t+1}(\sigma^t) + (1-p^t)V^{t+1}(0)\right]\right\} & \text{if } s^t = 0. \end{cases} \quad (4)$$

[12]Technically, these conditions describe a Nash equilibrium, as they do not impose optimality at aggregate states that are not reached in equilibrium. However, no behavior that would be ruled out by the more stringent requirement of subgame perfection can emerge: With a continuum of agents, any aggregate state that is not reached in equilibrium could only be reached through a coordinated deviation by a positive measure of agents. Therefore, the Nash equilibrium requirement implies subgame perfect play in our model.

That is, in period t a household chooses a feasible next-period target inventory (denoted σ^t, as the decision is made in period t) that maximizes the net flow benefit received in period t plus the discounted expected continuation value of period $t+1$. For households that enter period t with inventory $s^t \geq 1$, the net flow benefit is the consumption utility minus the cost of storage $(1-\lambda s^t)$. For households that enter period t with inventory $s^t = 0$, the net flow benefit is zero. Both types of households can achieve their next-period target inventory if they do not experience an in-store shortage in period t; this event has probability p^t. With probability $1-p^t$, they experience an in-store shortage. In that case, households enter period $t+1$ with $s^t - 1$ units (if $s^t \geq 1$) or with zero units (if $s^t = 0$).

Since each household is quantitatively negligible, they choose actions that are optimal given economic aggregates and ignore the impact of their actions on these aggregates. With this in mind, we make the following definition:

Definition 1. Given $x^1 \in \Delta_K$, an *equilibrium* is a sequence of probabilities, states, continuation values, and decision rules

$$\Big((p^t, x^{t+1}, V^t(\cdot), \sigma^t(\cdot)\Big)_{t=0,1,2,\ldots}$$

such that for each $t \geq 1$ the following holds:

(i) p^t and x^{t+1} are determined from x^t and $\sigma^t(\cdot)$ via (1)–(3),

(ii) $V^t(s^t)$ and $\sigma^t(s^t)$ are the value and policy functions that solve the Bellman equations (4), for $s^t = 0, \ldots, K$.

The equilibrium is *stationary* if $x^t = x^*$ for all t.

3 z-Storage Rules

In this section, we introduce a specific class of decision rules, called z-storage rules. These rules are defined as follows:

Definition 2. For given $z \in \{1, \ldots, K\}$, the decision rule $\sigma^t(s^t) = \max\{s^t - 1, z\}$ is called the *z-storage rule.*

A household that uses a z-storage rule in period t tries to achieve a desired inventory of z units at the beginning of period $t+1$. If the household has more than z units in storage in period t, it uses one unit in period t and enters period $t+1$ with one less unit in storage. If the household has z or fewer units in period t, it tries to purchase enough so as to enter period $t+1$ with z units in storage. We call the z-storage rule the *maximum-storage rule* if $z = K$, and the *minimum-storage rule* if $z = 1$.[13] If a household uses a z-storage rule with $z > 1$, we say that the household *stockpiles*.

We now establish two intermediate results that we use later to characterize the stationary equilibria as well as the transitional dynamics of the economy. First, Lemma 1 below shows that, when all households use the same z-storage rule in every period, the distribution of household inventories in the economy converges to a unique stationary distribution associated with this rule.

Lemma 1. *Suppose every household uses the same z-storage rule in every period, for $z \in \{1, \dots, K\}$.*

(a) The unique stationary state associated with this rule is given by

$$x^* \;=\; \Big(\underbrace{(1-p^*)^z,\, p^*(1-p^*)^{z-1},\, p^*(1-p^*)^{z-2}, \dots,\, p^*(1-p^*)\,,\, p^*}_{x_0^*, \dots, x_z^*},\, \underbrace{0,\, \dots\,,\, 0}_{x_{z+1}^*, \dots, x_K^*}\Big),$$

where

$$p^* \;=\; 1 - (1-m)^{1/z}$$

is the probability that, in any given period, a household finds the good in the store.

(b) The economy converges to the stationary state x^* from any initial state x^1. Furthermore, if $x^1 \succsim (\precsim)\, x^*$ then p^t converges to p^* from above (below).

Since p^* is the probability that a household is able to purchase the good, the probability that a household experiences an in-store shortage when all households use the same z-storage rule is $1 - p^* = (1-m)^{1/z}$. If $m = 1$, this probability is zero for all z; yet, if $z > 1$ every household stockpiles. These observations may appear contradictory but they are, in fact, consistent: If $m = 1$, the stationary (in fact, steady) state associated with the z-storage rule is for every household to have z units in storage, consume one unit per period, and buy one unit per period to keep the household inventory constant at z units. What Lemma 1 (a) implies, then, is that stockpiling cannot cause permanent in-store shortages if the aggregate supply is sufficient to meet aggregate consumption needs.[14]

If $m < 1$, there must necessarily be in-store shortages: It is not possible to have an aggregate supply shortfall without some rationing on the individual level. The formula

[13] Technically, a household could also follow a 0-storage rule, that is, a policy of not buying the good even if it has nothing stored. Because we assume that $\lambda < \beta$, this rule is never optimal, and we can ignore it.

[14] At the same time, without in-store shortages there is clearly no need for costly storage. In Section 4, we use this observation to show that there is a unique equilibrium when $m = 1$, which is for all households to use the minimum-storage rule, i.e. $z = 1$.

for p^* in Lemma 1 (a) allows us to compute the frequency of these in-store shortages. Suppose there is a one percent aggregate shortfall in the toilet paper supply ($m = 0.99$) and households store two units ($z = 2$). In this case $p^* = 0.9$, meaning that one out of every ten trips to the store will be unsuccessful. At the same time, the fraction of households that do not consume in any given period is

$$x_0^* \; = \; (1 - p^*)^z \; = \; 1 - m,$$

which is independent of z. Thus, while the probability that households cannot make their desired purchase is 10%, the probability that their consumption needs are not satisfied is only 1%. This is so because the stockpiling behavior that generates in-store shortages also creates inventories that allows household to consume in the event of an in-store shortage. If households increase their storage to $z = 5$ units, we have $p^* = 0.6019$, meaning that the probability that households cannot make their desired purchases increases to 40%; at the same time, the probability of non-consumption in any period is still only 1%.

Next, we turn to the optimality of z-storage rules. We need the following notation. Let $(p^1, p^2, \ldots)$ be any sequence of probabilities. For $z \in \{2, \ldots, K\}$, define

$$\overline{\lambda}_z(p^1, p^2, \ldots) \;\equiv\; \frac{\beta^{z-1} \displaystyle\prod_{\tau=1}^{z-1} (1 - p^\tau)}{1 + \displaystyle\sum_{k=1}^{z-1} \beta^k \prod_{\tau=1}^{k} (1 - p^\tau)}. \tag{5}$$

Given $t \geq 1$, let $\mathbf{p}^t$ denote the subsequence $(p^t, p^{t+1}, \ldots)$. Thus, $\overline{\lambda}_z(\mathbf{p}^t) = \overline{\lambda}_z(p^t, p^{t+1}, \ldots)$.

The following Lemma 2 establishes conditions under which using a z-storage rule is optimal for an individual household.

Lemma 2. *Let p^t be the probability that a household finds the good in the store in period t (i.e., $1 - p^t$ is the probability of an in-store shortage).*

(a) For $z \in \{2, \ldots, K\}$, if

$$\overline{\lambda}_{z+1}(\mathbf{p}^t) \;\leq\; \lambda \;\leq\; \overline{\lambda}_z(\mathbf{p}^t) \quad \textit{for all } t \geq 1, \tag{6}$$

then the household's optimal decision rule is to use the z-storage rule in every period. The minimum-storage rule is optimal if the left inequality in (6) holds; and the maximum-storage rule is optimal if the right inequality in (6) holds.

(b) If $p^t < 1$ for all t, then $\overline{\lambda}_z(\mathbf{p}^t)$ is strictly increasing in β for all t. Moreover, if $p^t > \hat{p}$ for all t, then $\overline{\lambda}_z(\mathbf{p}^t) < \overline{\lambda}_z(\hat{\mathbf{p}}^t)$ for all t.

The values $\overline{\lambda}_z(\mathbf{p}^t)$ and $\overline{\lambda}_{z+1}(\mathbf{p}^t)$ represent lower and upper threshold values on the storage cost λ for which the z-storage rule is an optimal decision for the household. In particular, the inequality $\lambda \leq \overline{\lambda}_z(\mathbf{p}^t)$ in (6) means that the storage cost is low enough for the household to want to store z units in period t; and the inequality $\overline{\lambda}_{z+1}(\mathbf{p}^t) \leq \lambda$ in condition (6) means that the storage cost is high enough for the household to not want to store $z+1$ units.

Lemma 2 will be used several times to characterize equilibria of our model. For example, suppose all households use the same z-storage rule, and this strategy profile generates the probabilities p^t. If condition (6) in Lemma 2 (a) holds, then the z-storage rule is a best response to itself—in other words, all households using this z-storage rule constitutes an equilibrium of the model.

4 Equilibrium Analysis

We now examine the equilibria of the economy, as defined formally in Section 2. We consider three scenarios:

1. When there is no aggregate supply shortage, we show that there exists a unique equilibrium. In this equilibrium, all households use the minimal storage rule ($z = 1$) in every period. (See Section 4.1.)

2. When there is an aggregate supply shortage, we show that there exist one or more stationary equilibria in which all households use the same z-storage rule in every period. Equilibria with stockpiling ($z > 1$) exist for a large set of parameter values. (See Section 4.2.)

3. We also consider the case when the economy experiences an unexpected supply shock. In this case, the economy transitions from scenario 1 to scenario 2 above. However, this transition is no immediate, and the question is: What decision rules will arise along the convergence path from the previous stationary state to the new one? We characterize conditions under which adoption of a z-storage rule following the shock is an equilibrium. (See Section 4.3.)

4.1 Equilibrium when there is no aggregate shortage

When there is no aggregate shortage, it is straightforward to see that all households using the minimum-storage rule in every period must be an equilibrium. Since no household is rationed, every household consumes one unit of the good in every period. Moreover, every household achieves this consumption pattern in the least costly way. Therefore, no household could adopt a different decision rule and, by doing so, improve its utility.

Lemma 2 confirms this equilibrium: If $m = 1$ and every household uses the minimum-storage rule in every period, $p^t = 1$ for all t. In this case, the optimality condition for $z = 1$ stated in Lemma 2 (a) collapses to $\lambda \geq \overline{\lambda}_2(1, 1, \dots) = 0$, which is clearly satisfied.

The following result shows that the minimum-storage rule is, in fact, the *only* equilibrium when $m = 1$:

Proposition 3. *Suppose $m = 1$ and let $x^1 \in \Delta_K$ be any initial state. The unique equilibrium is for every household to use the minimum-storage rule in every period.*

The main implication of Proposition 3 is that stockpiling cannot merely arise because it is a "self-fulfilling prophecy." One might imagine a situation where there is no fundamental aggregate shortage, but where in-store shortages nonetheless arise, caused by consumer stockpiling, which itself is an optimal response to the in-store shortages. As Proposition 3 shows, this situation cannot occur in our model. The reason—formally shown in the proof of Proposition 3 Appendix—is the following: For households to stockpile in period t, they must anticipate in-store shortages in period $t + 1$. Without a fundamental supply shortage, however, in-store shortages in period $t + 1$ can only arise if some households stockpile in period $t + 1$. Stockpiling in period $t + 1$ can only be optimal if households anticipate in-store shortages in period $t + 2$, and those in-store shortages can only arise if some households stockpile in period $t + 2$—and so on. These dynamics would result in some households' inventories growing without bound, which is impossible.

In order to keep the notation and analysis manageable throughout the paper, our formal equilibrium definition (see Definition 1) involves only symmetric strategy profiles, i.e., all households use the same decision rule in a given period. In principle, Proposition 3 leaves open the possibility that asymmetric equilibria exist in which some households use a decision rule that is not the minimum-storage rule. However, as we show in the Appendix, no such additional equilibria can exist either if $m = 1$. Thus, when there is no aggregate supply shortage, we unambiguously conclude that all households use the minimum-storage rule in all periods.

4.2 Stationary equilibria under an aggregate shortage

We now assume that $m < 1$. We will be looking for equilibria in which all households use the same z-storage rule in every period. We call such profiles *z-storage equilibria.* We emphasize that we do not restrict households to use a z-storage rule, but rather identify conditions under which the use of a z-storage rule is optimal for a household if all other households use this rule.

For the time being, we also focus on stationary z-storage equilibria. These equilibria can be thought of as describing long-run outcomes of the economy, in which the economy is at stationary state x^*, characterized in Lemma 1, and remains at this state permanently absent any changes in fundamentals (i.e., absent any changes in m, K, β, or λ). This means that the probability of finding the good in the store is also constant and given by the value p^* in Lemma 1.

Stationary z-storage equilibria: Characterization

In a stationary z-storage equilibrium, the z-storage rule must be optimal for any individual household if (i) all other households use the same z-storage rule and (ii) the economy is in the stationary state associated with this z-storage rule.

Recall that Lemma 2 stated general conditions under which a given z-storage rule is optimal for a household, if the household faces an arbitrary sequence of probabilities $p^1, p^2, \ldots$ of finding the good in the store. To characterize stationary z-storage equilibria, therefore, we substitute p^* for each of these probability terms. Condition (6) in Lemma 2 can then be written as follows:

$$\frac{\gamma(z)^z\big(1-\gamma(z)\big)}{1-\gamma(z)^{z+1}} \;\le\; \lambda \;\le\; \frac{\gamma(z)^{z-1}\big(1-\gamma(z)\big)}{1-\gamma(z)^z}, \tag{7}$$

where $\gamma(z) \equiv \beta(1-m)^{1/z}$. For $z \in \{2, \ldots, K-1\}$, a stationary z-storage equilibrium exists if both inequalities in condition (7) are satisfied. In other words, the storage cost must be must be low enough for households to want to store at least z units, and also high enough for households to not want store more than z units, assuming all other households use the z-storage rule. If $z = 1$ or $z = K$, only one of the two inequalities in (7) remains relevant: A minimum-storage equilibrium exists if the left inequality in (7) holds for $z = 1$, and a maximum-storage equilibrium exists if the right inequality in (7) holds for $z = K$.

The following result shows that these equilibrium conditions are satisfied for at least one value of z:

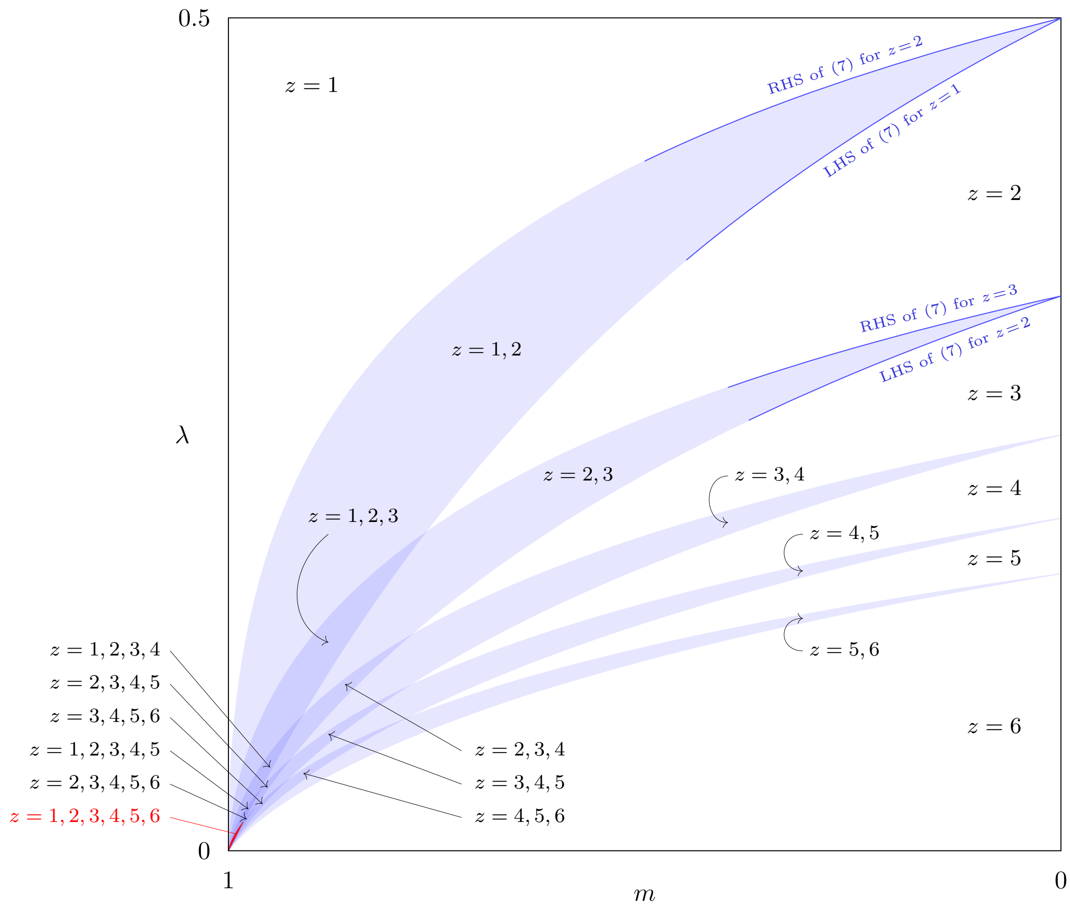
Proposition 4. *If $m < 1$, a stationary z-storage equilibrium exists for some $z \in \{1, \ldots, K\}$.*

Figure 1 shows the stationary z-storage equilibria for the case where $\beta = 0.999$ and $K = 6$ are fixed, but varying m and λ. The regions in Figure 1 are constructed by plotting the left-hand side and right-hand side of (7), for different integer values of z.[15] In the white parameter regions, a unique z-storage equilibrium exists; in the colored regions multiple equilibria exist. Consistent with the previous Proposition 3, if $m = 1$ (i.e., along the left edge of the box in Figure 1 the only equilibrium is for every household to employ

[15]This is visualized by the blue curves, drawn for select z-values, on the right side of the figure. For example, a stationary 2-storage equilibrium exists in the region between the curve labeled "LHS of (7) for $z = 2$" and the curve labeled "RHS of (7) for $z = 2$." (Note that these curves extend all the way to the bottom left corner of the figure, but are only partly drawn in order to avoid cluttering the figure.)

the minimum-storage rule. On the other hand, when $m < 1$, stationary equilibria with stockpiling can emerge. In particular, when the supply shortage is relatively small and storage is relatively inexpensive, a large number of different z-storage equilibria can exist simultaneously.

Figure 1: Stationary z-storage equilibria ($\beta = .999$, $K = 6$).



Note: The white regions in the figure indicate (m, λ)-combinations for which a unique stationary z-storage equilibrium exists. The shaded regions indicate (m, λ)-combinations for which two or more stationary z-storage equilibria exists. The equilibrium value(s) for z are noted in the figure. The small red region consists of (m, λ)-combinations for which stationary z-storage equilibria exists for all $z = 1, \ldots, K$.

Figure 1 further suggests that, when multiple stationary z-storage equilibria exist, the equilibrium set is "connected" in the sense that the set of equilibrium values for z consists of consecutive integers. The following result confirms this:

Proposition 5. *If stationary z-storage and z'-storage equilibria exist, with $z < z'$, then stationary z''-storage equilibria exist for all integers $z \leq z'' \leq z'$.*

In the small red-colored region in Figure 1, *every* z-storage rule constitutes a stationary equilibrium, including the minimum and maximum-storage rules. The following result shows that this is a generic possibility for arbitrarily small but positive aggregate supply shortages.

Proposition 6. *Fix $K > 1$ and $\beta < 1$. There exists $\overline{m} < 1$ such that the following is true. For every $\overline{m} < m < 1$, there exists an open interval of storage costs $\Lambda \subset (0, 1)$ such that, if $\lambda \in \Lambda$, a stationary z-storage equilibrium exists for all $z = 1, \ldots, K$.*

Finally, we examine how the equilibrium set changes when the model parameters change. As shown in Proposition 5 above, the set z-values for which the z-storage rule is a stationary equilibrium consists of consecutive integers z. The following result shows that this set of integers increases—in the sense that it shifts toward higher values—if either the storage cost decreases, or the aggregate supply decreases, or the discount factor increases.

Proposition 7. *Fix some values for the storage cost λ, the aggregate supply m, and the discount factor β, and let $Z(\lambda, m, \beta)$ be the set of consecutive integers such that a stationary z-storage equilibrium exists for all $z \in Z$. Suppose that $\lambda' < \lambda$, or $m' < m$, or $\beta' > \beta$. Then*

$$\min Z(\lambda', m', \beta') \;\geq\; \min Z(\lambda, m, \beta)$$

and

$$\max Z(\lambda', m', \beta') \;\geq\; \max Z(\lambda, m, \beta).$$

The effect of changes in λ or m on the equilibrium set described in Proposition 7 can be easily seen in Figure 1. With regard to β, Proposition 7 shows that, as households become more patient, equilibria with less stockpiling disappear and equilibrium with more stockpiling appear. The intuition for this result is the following. A household that follows a z-storage rule in period t can consume the good with certainty in the following z periods. Thus, the benefit of storing an additional $z + 1^{\text{st}}$ unit, if any, occurs $z + 1$ periods into the future; however, the additional cost incurred by storing that extra unit must be paid in the $z + 1$ periods after the unit is first put into storage. Therefore, the cost of storing a marginal unit is front-loaded relative to the benefit the unit provides. As households become more patient, the future benefit of storage weighs relatively more in their utility function, and the present cost of storage weighs relatively less, making storing an extra unit more attractive.

Excess stockpiling

In order to better understand what factors drive the decision to maintain inventories in equilibrium, we now decompose households' stockpiling incentives into two forces.

The first force is the fundamental, aggregate supply shortfall—only $m < 1$ new units of the good are available in each period despite households wanting to consume 1 unit. Without this shortfall, Proposition 3 implies that stockpiling would not arise in any equilibrium. However, if there is an aggregate supply shortage, it may be optimal for a household to maintain an inventory of more than one unit of the good in order to smooth consumption. We call this the *direct effect* of the shortage. The second force is an *indirect effect*: If every household decides to stockpile, in-store shortages become more frequent and, as a result, further stockpiling incentives are created. Thus, there exists a feedback mechanism from stockpiling to even more stockpiling. The equilibria of the model reflect the combined effect of both direct incentive and indirect feedback.

To measure the strength of the direct effect only, we can compute the rule that is optimal for an *individual household* if the aggregate supply is $m < 1$ but no other households stockpile. In this case, the likelihood that a household finds the good in the store in any given period is then $p^* = m$. Lemma 2 implies that the y-storage rule that is a best response in this situation satisfies the condition

$$\overline{\lambda}_{y+1}(m, m, \ldots) \ \leq \ \lambda \ \leq \ \overline{\lambda}_y(m, m, \ldots)$$

or, equivalently,

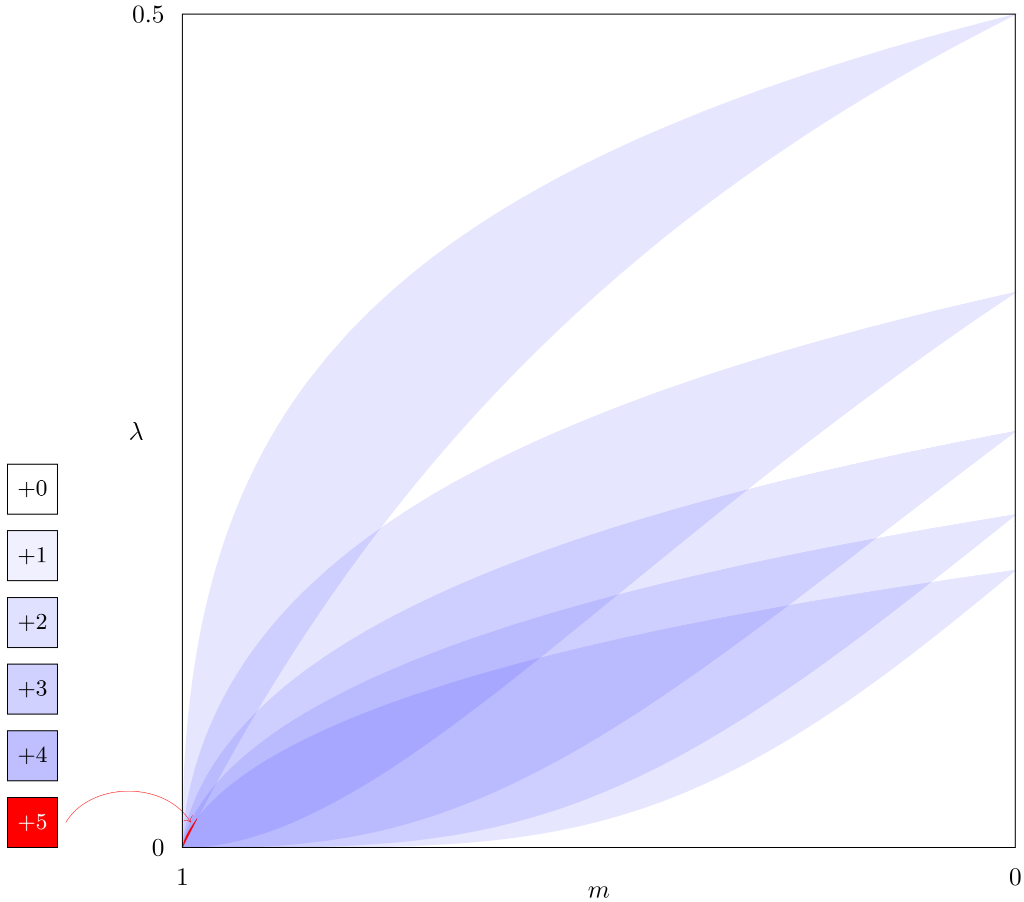
$$\frac{\gamma(1)^y\big(1-\gamma(1)\big)}{1-\gamma(1)^{y+1}} \ \leq \ \lambda \ \leq \ \frac{\gamma(1)^{y-1}\big(1-\gamma(1)\big)}{1-\gamma(1)^y}. \tag{8}$$

The value y that satisfies condition (8) represents a single household's unilateral best response to the shortage, assuming that none of the other households stockpiled. (Similar to condition (7), if $y = 1$ then only the left inequality in (8) applies; and if $y = K$ then only the right-inequality in (8) applies.)

Now suppose that a z-storage equilibrium exists for some value of z, and take the difference between the equilibrium value of z and the value y that satisfies condition (8). This difference can be interpreted as a measure of *excess stockpiling* arising from the aforementioned feedback loop; that is, it quantifies of the strength of the indirect stockpiling incentive.

Figure 2 depicts the amount of excess stockpiling for the same parameter values as those used to plot Figure 1. When multiple stationary z-storage equilibria exist, we selected the one with the largest z in order to measure the strength of the indirect incentive when it is the strongest. As Figure 2 shows, the feedback mechanism can generate a significant amount of excess stockpiling for certain parameter configurations. In particular, consider the set of parameter combinations for which both the minimum and maximum-storage equilibrium exist (indicated in red in Figure 2 as well). If all households followed the minimum-storage strategy, it would be a unilateral best response to use the minimum-storage strategy as well. Therefore, if the maximum-storage equilibrium is

Figure 2: Excess stockpiling ($\beta = .999$, $K = 6$).



Note: Colors represent the strongest degree of excess stockpiling that can arise, measured by units stockpiled in the highest z-storage equilibrium minus units stockpiled as a unilateral best response to the minimum storage rule. In the white regions in the figure, there is no excess stockpiling; in the shaded regions, there is excess stockpiling.

played in this case, the incentive to maintain inventories above a single unit is driven *entirely* by the fact that other households follow the same strategy. In other words, the feedback loop from stockpiling to more stockpiling accounts for $K - 1$ out of the K units in each household's target inventory.

4.3 Transitional dynamics

In Section 4.1 we characterized the unique outcome of the economy when there was no aggregate supply shortage; and in Section 4.2 we described the possible stationary, long-run outcomes of the economy when there was an aggregate supply shortage. We now examine what happens as the economy transitions between these two cases. Specifically, we consider the economy's response to an unanticipated supply shortage. In many applications—including the motivating examples discussed in Section 1—one may be particularly interested in this short-run response.

Consider a situation in which supply and demand are balanced (i.e., $m = 1$) in every period, until an unexpected shock reduces the supply to $m < 1$. Without loss of generality, we let period $t = 1$ be the period in which the shock occurs. In response to this shock, households may begin to accumulate inventories, thereby initiating the transition from the previous stationary state, $x^1 = (0, 1, 0, \ldots, 0)$, to a new stationary state. This transition will itself be governed by decision rules that are best responses to each other. For example, suppose that every household adopts the same z-storage rule immediately after the shock. In this case, a measure 1 of households will attempt to purchase z units of the good each, but only a measure m/z of households will be successful. Thus, next period's state is

$$x^2 \;=\; \Big(\underbrace{1 - \frac{m}{z}}_{x_0^2},\, 0\,,\, \ldots\,,\, 0\,,\, \underbrace{\frac{m}{z}}_{x_z^2}\,,\, 0\,,\, \ldots\,,\, 0\Big)\,.$$

If all households continue to follow the same z-storage rule, the state will further evolve and, as shown in Lemma 1, over time converge to the stationary state x^*. If the z-storage rule remains optimal in every period for a household that expects all other households to use the same z-storage rule in all periods, we call the resulting outcome a *transitional z-storage equilibrium.* Note that the equilibrium is non-stationary because the aggregate state x^t changes over time; however, the households' optimal strategies remain unchanged.

We now derive a condition on the storage cost λ under which a transitional z-storage equilibrium exists. It should be clear that this condition will be more stringent than the corresponding condition (7) for stationary equilibria, as the same z-storage rule must now be optimal in a larger set of circumstances. We begin with the following result:

Lemma 8. *Let $m < 0$ and fix initial state $x^1 = (0, 1, 0, \ldots, 0)$. Suppose all households use the same z-storage rule in every period $t = 1, 2, \ldots$. Let p^t denote the probability that a household finds the good in the store in period t in this scenario. Then $p^{t+1} \geq p^t$ for all t, and $p^t \to p^* = 1 - (1 - m)^{1/z}$.*

As before, we let $\mathbf{p}^t$ denote the subsequence $(p^t, p^{t+1}, \ldots)$. Note that Lemma 8 implies that $\mathbf{p}^1 \leq \mathbf{p}^2 \leq \ldots$, and $\mathbf{p}^t \to (p^*, p^*, \ldots)$ uniformly.

Now consider the question whether using the z-storage rule in every period is optimal for an individual household if all other households use the z-storage rule. By Lemma 2 (a), this is the case if two conditions are satisfied. The first condition applies when $z > 1$ and imposes an upper bound on the storage cost: $\lambda \leq \overline{\lambda}_z(\mathbf{p}^t)$. Lemma 2 (b) in conjunction with Lemma 8 implies that the upper-bound condition holds for all t if and only if it holds at $\lim_{t\to\infty} \mathbf{p}^t = (p^*, p^*, p^*, \ldots)$:

$$\lambda \;\leq\; \overline{\lambda}_z\big(p^*, p^*, p^*, \ldots\big) \;=\; \frac{\gamma(z)^{z-1}\big(1-\gamma(z)\big)}{1-\gamma(z)^z}. \tag{9}$$

The second condition applies when $z < K$ and imposes a lower bound on the storage cost: $\lambda \geq \overline{\lambda}_{z+1}\big(\mathbf{p}^t\big)$. Lemma 2 (b) in conjunction with Lemma 8 implies that the lower-bound condition holds for all t if and only if it holds at $\mathbf{p}^1 = (p^1, p^1, \ldots)$:

$$\lambda \;\geq\; \overline{\lambda}_{z+1}\big(p^1, p^2, p^2, \ldots\big). \tag{10}$$

Combining (9) and (10), we get:

$$\overline{\lambda}_{z+1}\big(p^1, p^2, p^3, \ldots\big) \;\leq\; \lambda \;\leq\; \frac{\gamma(z)^{z-1}\big(1-\gamma(z)\big)}{1-\gamma(z)^z}. \tag{11}$$

A transitional z-storage equilibrium exists if condition (11) is satisfied. (If $z = 1$ the right inequality becomes irrelevant; and if $z = L$ the left inequality becomes irrelevant.)

Note that, while the upper bound on the storage cost in (11) is the same as the previous upper bound for stationary z-storage equilibrium, the lower bound is now larger than it was in the stationary case. Therefore, a transitional z-storage equilibrium may fail to exist for parameter values under which a stationary z-storage equilibrium exists.[16]

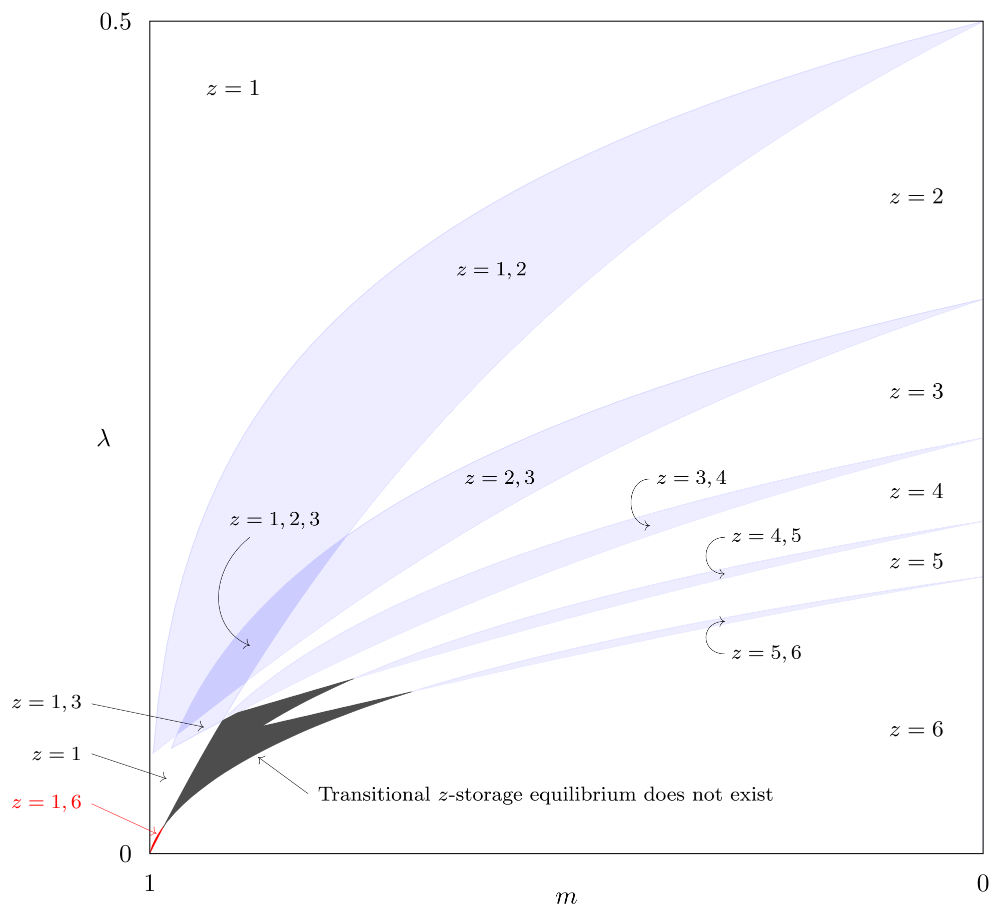
Figure 3 depicts the set of transitional z-storage equilibria for the same parameter values as were used in Figure 1 (i.e., $\beta = .999$ and $K = 6$), assuming initial state $x^1 = (0, 1, 0, 0, 0, 0, 0)$. As the new equilibrium condition (11) is more stringent than the previous condition (7), some of the z-storage rules that were stationary equilibria have disappeared. Moreover, Figure 3 demonstrates that the previous Propositions 4, 6, and 5 do not carry over to the transitional case: Transitional z-storage equilibria need not exist (as indicated by black region); when they exist the range of z-values for which the z-storage rule constitutes an equilibrium does not always consist of consecutive integers; and there is no longer region of parameter values for which every z-storage rule (i.e., $z = 1, \ldots, K$) is a transitional equilibrium.

However, a transitional minimum-storage equilibrium exists whenever a stationary minimum-storage equilibria exists, and the same is true for maximum-storage equilibria. For example, as shown in Figure 1 and Figure 3, in the red-colored parameter region for which the full range of stationary z-storage equilibrium exists, the minimum and

[16] This means the following: Starting at the new stationary state associated with $m < 1$ and a particular z-storage rule ($z > 1$), it would be mutually optimal for all households to use this z-storage rule. However, starting at the previous stationary state associated with $m = 1$, immediately switching to the new z-storage rule as part of the transition to the new stationary state would not be mutually optimal.

maximum-storage rules survive as transitional equilibria. To see why, note that if the initial state is $x^1 = (0, 1, 0, \ldots, 0)$ and households use the minimum-storage rule, the probability of finding the good in the store is m in every period. This is the same probability as in the stationary state $x^* = (1 - m, m, 0, \ldots, 0)$. Thus, the condition for a stationary minimum-storage equilibrium (i.e., the left inequality in (7), for $z = 1$) is identical to the condition for a transitional minimum-storage equilibrium (i.e., the left inequality in (11), for $z = 1$). Similarly, the condition for a stationary maximum-storage equilibrium (i.e., the right inequality in (7), for $z = K$) is identical to the condition for a

Figure 3: Transitional z-storage equilibria ($\beta = .999$, $K = 6$).



Note: The white regions in the figure indicate (m, λ)-combinations for which a unique transitional z-storage equilibrium exists. The shaded regions indicate (m, λ)-combinations for which two or more transitional z-storage equilibria exists. The equilibrium value(s) for z are noted in the figure. The small red region consists of (m, λ)-combinations in which both the minimum-storage rule and the maximum-storage rule are transitional equilibria. The black region consists of (m, λ)-combinations for which no transitional equilibrium exists in which every household uses the same z-storage rule in every period.

transitional maximum-storage equilibrium (i.e., the right inequality in (11), for $z = K$). We summarize this observation in the following result:

Proposition 9. *Let $x^1 = (0, 1, 0, \ldots, 0)$. A transitional minimum-storage equilibrium exists if and only if a stationary minimum-storage equilibrium exists; and a transitional maximum-storage equilibrium exists if and only if a stationary maximum-storage equilibrium exists*

5 Welfare Analysis

In any situation where multiple equilibria exist, it is natural to ask if these equilibria can be ranked by their welfare. We address address this question in Section 5.1 below, where we compare the welfare in the long-run, stationary z-storage equilibria of our model. In 5.2 we then examine more closely the welfare losses experienced during the transition from one such equilibrium to another.

5.1 Welfare in stationary equilibria

We measure welfare through a representative household's expected lifetime utility stream. Since a household consumes in period t if and only if enters the period with positive inventory s^t, we can express this welfare measure as follow:

$$W \; = \; (1-\beta)E\left[\sum_{t=1}^{\infty} \beta^{t-1} \mathbb{I}_{s^t>0}\left(1-\lambda s^t\right)\right]. \tag{12}$$

Note that we normalize the present value (i.e., we multiply it by $1-\beta$) in order to isolate the effect of changes in strategies from the "mechanical" effect of changing the preference parameter β on the utility function.

As shown in Section 3, in the stationary state x^* associated with any z-storage rule, the fraction of households that are able to consume the good in any period is m. Therefore, in a stationary z-storage equilibrium, welfare is

$$W \; = \; (1-\beta)\sum_{t=1}^{\infty} \beta^{t-1}\left(m - E\left[\lambda s^t\right]\right) \; = \; m - E\left[\lambda s^t\right].$$

This implies that welfare differs across the stationary states only in terms of the storage costs paid by households. The expected storage cost incurred by households in any given period is

$$E\big[\lambda s^t\big] \; = \; \lambda\big[x_1^* + 2x_2^* + \ldots + zx_z^*\big] \; = \; \lambda\left[z - \frac{(1-m)^{1/z}}{1-(1-m)^{1/z}} m\right],$$

where x^* denotes the stationary state associated with a given z-storage rule, characterized in Lemma 1. It is straightforward to verify that, if x^{**} is the stationary state associated with the z'-storage rule and $z' > z$, then x^{**} stochastically dominates x^*.[17] Thus, z-storage rules with lower values for z lead to lower storage cost payments. It follows that, if more than one z-storage equilibrium equilibrium exists, the equilibrium with the lowest z is the equilibrium with the highest welfare.

In Proposition 7, we described how the set of z-storage rules that are stationary equilibria responds to changes in important model parameters: the per-unit storage cost, the aggregate supply, and the discount factor. These comparative statics result have interesting welfare implications.

First, note that a decrease in the storage cost λ would unambiguously increase welfare *if households did not change their behavior*. However, as shown in Proposition 7, a decrease in λ has the countervailing effect of shifting the set of equilibria towards strategies with more stockpiling, which reduces welfare (everything else equal). It is possible for a small decrease in storage costs to have a negative net effect on welfare, if the decrease eliminates a low-storage equilibrium. Second, a decrease in the aggregate supply m decreases welfare if households do not change their behavior. As shown in Proposition 7, a decrease in m also has the effect of shifting the set of equilibria towards those with more stockpiling. If such a shift occurs, it further decreases welfare. Finally, because our welfare measure is normalized, an increase in the discount factor β has no effect on welfare unless it triggers a switch to a different equilibrium. Proposition 7 shows that an increase in β shifts the set of equilibria away from those with low stockpiling and towards outcomes with high stockpiling. Thus, if households become more patient and, as a result, low-storage equilibria disappear, welfare can be reduced.

5.2 Transitional welfare effects

In the applications we have in mind (i.e., the stockpiling of everyday household items), the storage cost is minor, as the per-unit cost λ is likely insignificant in comparison to the flow utility of consumption. Therefore, any welfare differences across different stationary equilibria are moderate at best.[18] These relatively small long-run welfare

[17]To see this, let $z' = z + 1$. From Lemma 1 (a), $x_0^* = 1 - m = x_0^{**}$. Moreover, for $0 < k \leq z$ and $m \in (0, 1)$,

$$x_k^* \;=\; \left(1 - (1-m)^{\frac{1}{z}}\right)(1-m)^{\frac{z-k}{z}} \;>\; \left(1 - (1-m)^{\frac{1}{z+1}}\right)(1-m)^{\frac{z-k+1}{z+1}} \;=\; x_k^{**}.$$

Thus, for all $k = 0, \ldots, K$ we have $\sum_{k'=0}^{k} x_{k'}^* \geq \sum_{k'=0}^{k} x_{k'}^{**}$, with strict inequality if $k = 1, \ldots, z$, and it follows that $x^{**} \succ x^*$.

[18]In principle, an additional welfare effect arises from the fact that household inventories bind resources whose value cannot be spent on consumption of other goods. Our model abstracts from this effect because the price of the stockpiled good is zero. However, in reality this price is positive. Thus, the total value of resources tied up in inventories is, therefore, lower in an equilibrium with less stockpiling. This welfare

losses associated with stockpiling can mask potentially larger welfare losses experienced during the transitional phase, i.e., when households begin to accumulate inventories in response to a supply shock.

To explicitly account for the short-run effect of stockpiling, we now return to the non-stationary equilibria introduced in Section 4.3. Recall that these equilibria capture the transitional dynamics from the initial state $x^1 = (0, 1, 0 \ldots, 0)$ to a new stationary state following a supply restriction, and it is along this transition path that the *accumulation* of inventories (and not merely the maintenance of inventories) occurs. Given a fixed per-period supply, the economy-wide inventory accumulation has a more significant impact on a household's contemporaneous utility than do storage costs alone, as the economy-wide inventory accumulation reduces consumption for some time.

For concreteness, we consider the following parameter configuration:

$$\beta = 0.999,\ K = 6,\ m = 0.9995,\ \lambda = 0.0005.$$

If the time period is one week, a discount factor of $\beta = 0.999$ implies an annual discount rate of approximately 5%. A per-unit storage cost of $\lambda = 0.0005$ means that the cost of storing a one week's worth of toilet paper is 0.05% of the utility the household obtains from the using toilet paper for a week (relative to consuming the next best substitute). And a supply of $m = 0.9995$ implies that—*absent any stockpiling by consumers*—a household would experience an in-store shortage of toilet paper during one week out of every 38 years on average. This parameterization is in the red-colored region in Figure 3, so that transitional minimum and maximum-storage equilibria both exist. Thus, despite the negligible supply shortage, it is an equilibrium for all households to accumulate and maintain inventories lasting $K = 6$ weeks.[19] Let period $t = 1$ denotes the start of the unanticipated permanent supply shortage. Prior to period 1, we assume that the supply was $m = 1$, and the economy was in the unique minimum-storage equilibrium.

Compared to the pre-shock economy, the supply shock will reduce welfare regardless of which of the transitional equilibria is played. Our goal is to compare the welfare reduction in the equilibrium with the least amount of stockpiling ($z = 1$) to the welfare reduction in the equilibrium with the highest amount of stockpiling ($z = 6$). To do so, we measure welfare in two different ways.

First, in any given period t we measure welfare as in (12), i.e., as the normalized expected continuation utility of a household. This captures both short-run and long-run welfare effects of stockpiling (if any). Our baseline welfare measurement is the welfare of a household that stores and consumes exactly one unit in every period, which is the

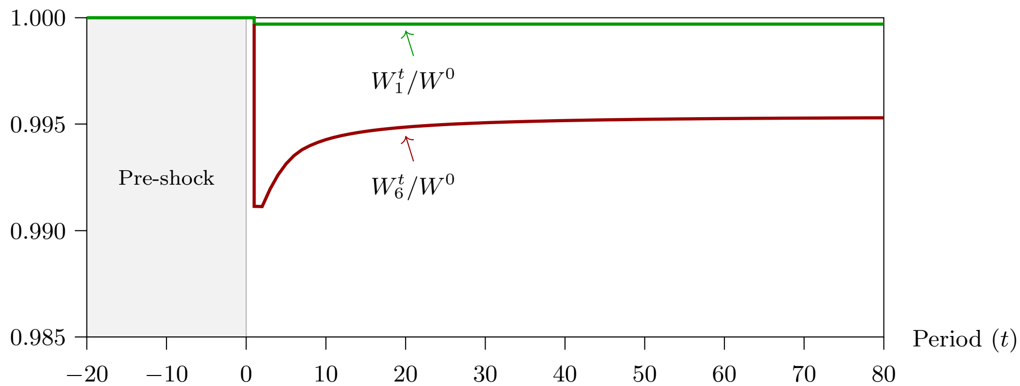
improvement, too, should be moderate, as goods such as toilet paper or dried pasta do not account for significant expenditure shares in most households.

[19]Similar examples can be constructed for larger values of K.

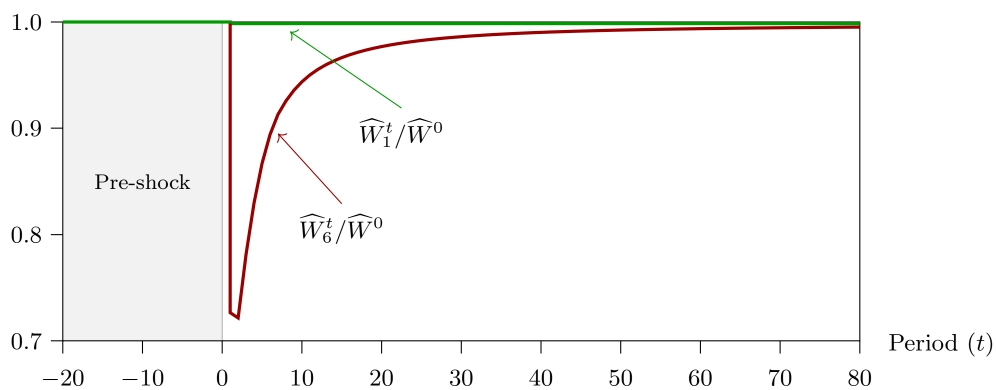
Figure 4: Welfare effects of a supply shock.

$\beta = .999$, $K = 6$, $m = 0.9995$, $\lambda = 0.0005$

(a) Relative welfare (untruncated)



(b) Relative welfare (truncated after 13 periods)



Note: Green curve represents transitional welfare loss in the minimum-storage equilibrium; red curve represents transitional welfare loss in the maximum-storage equilibrium.

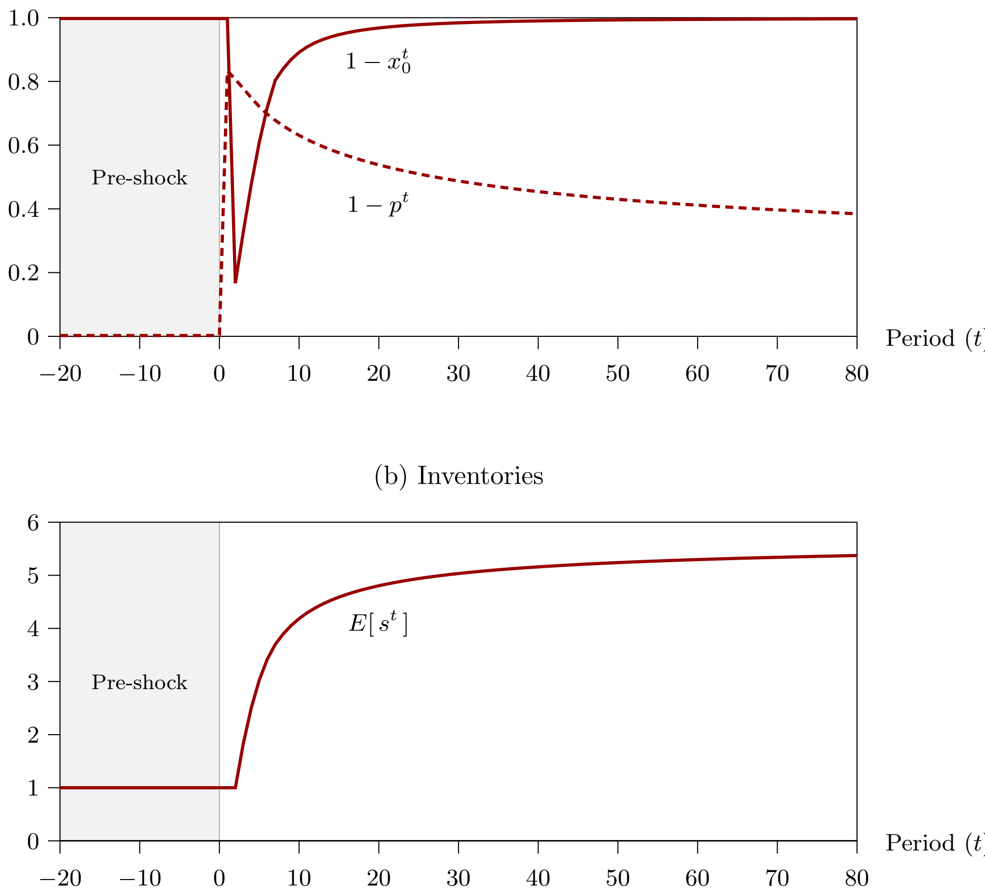
unique equilibrium behavior before the shock. We call this value $W^0 = 1 - \lambda$. Following the unanticipated shock, we numerically compute a household's welfare in every period $t = 1, 2, \ldots$, assuming that (i) all households adopt the minimum-storage rule ($z = 1$); and (ii) all households adopt the maximum-storage rule ($z = 6$). Both strategy profiles are transitional equilibria following the shock. We call these welfare values W_1^t and W_6^t, respectively.

Figure 4 (a) plots the ratios W_1^t/W^0 and W_6^t/W^0. The first ratio shows the welfare effect of the supply shock if the minimum-storage equilibrium $z = 1$ was maintained. The expected welfare loss would be less than 0.05% (i.e., consumption decreases by 0.05% and so does the corresponding storage cost). Moreover, the economy reaches the new

Figure 5: Consumption and inventory response to a supply shock.

$\beta = .999$, $K = 6$, $m = 0.9995$, $\lambda = 0.0005$

(a) Consumption and in-store shortage probability



stationary equilibrium immediately. The second ratio shows the welfare effect of the supply shock if the shock were to trigger a switch to the maximum-storage equilibrium. In this case, welfare drops by 0.89% initially and recovers about half of this loss over time, converging to a level 0.47% below the baseline. Thus, stockpiling significantly amplifies the welfare loss due to the supply shock, even though the overall loss remains modest.[20]

[20]This is in line with Noda and Teramoto (2024), who decompose transitional welfare losses into a direct effects arising from a parameter shock (an increase in search costs) and an indirect effect arising from the change in consumer behavior. In their calibration, the indirect effect is approximately five times larger than the direct effect (see Noda and Teramoto 2024, Table 2). However, in Noda and Teramoto (2024) the equilibrium is unique. In contrast, we compare the welfare loss in a "good equilibrium" ($z = 1$) to that in a "bad equilibrium" ($z = 6$). In the good equilibrium the loss is due to the direct effect only, and in the bad equilibrium the loss is due to both the direct and the indirect effect.

Second, in order to focus on the short-run welfare effects we use a 13-period rolling window and compute a household's expected discounted utility over this truncated timespan only. If the model period is one week, then a 13-period window corresponds to a welfare assessment for the leading quarter-year. In practice, decision makers often evaluate outcomes using such rolling-windows; for example, the results of a given policy or strategic plan may be assessed using monthly, quarterly, or annual time horizons. (We also considered 4-week (monthly) and 26-week (semi-annual) rolling windows, with similar results similar.) Corresponding to the previous notation, we denote the corresponding truncated welfare measures by $\widehat{W}^0$, $\widehat{W}_1^t$, and $\widehat{W}_6^t$, respectively.

Figure 4 (b) plots the ratios $\widehat{W}_1^t/\widehat{W}^0$ and $\widehat{W}_6^t/\widehat{W}^0$. Stockpiling now has a very large effect on short-run welfare. Compared to the baseline, if the minimum-storage equilibrium was maintained after the supply shock, the expected welfare loss in the short-run is still only 0.05%. However, if a switch to the maximum-storage equilibrium occurred, short-run welfare would drop by over 27% initially (and recover nearly all of this loss over time). Unlike in panel (a), the large loss in short-run welfare is no longer masked by the relatively small loss in long-run welfare. As expected, shorter rolling windows would make the initial drop more pronounced, while longer rolling windows would attenuate the initial drop.

For the same example, Figure 5 shows how households' consumption patterns and inventories respond to the supply shock, assuming the shock triggers a switch to the maximum-storage equilibrium. Panel (a) plots a household's per-period consumption probability $(1 - x_0^t)$ and the probability that a household experiences an in-store shortage $(1 - p^t)$; panel (b) plots average household inventories $(E[s^t])$. As households start accumulating inventories following the shock, they experience an in-store shortage with probability 83.3%. Accordingly, their consumption probability next period drops to 16.7%. Over time, as inventories get built up and the speed of inventory accumulation slows, the consumption probability rises and converges to the stationary state probability, which is $m = 0.9995$. In-store shortages, however, remain stubbornly common: The probability of experiencing an in-store shortage exceeds 42% even 52 weeks after the initial shock. This does not affect long-run consumption, however, as households maintain their large inventories precisely to tide them over these shortages.

6 Discussion

6.1 Shortages due to demand increases

In our model, the supply-demand imbalances that generated the stockpiling incentives were driven by exogenous supply reductions, holding demand constant. In reality, the same imbalances can also be caused by demand increases, holding supply constant.

For example, during COVID-19 many consumers stockpiled common household goods to last through a potential quarantine period. In addition, shortages of flour, yeast, and other baking supplies were primarily due to demand spikes, as many consumers took on baking as a new hobby during the lockdown. Furthermore, consumption of certain goods shifted from workplaces to homes, increasing the demand for home-use varieties of these goods. For example, the toilet paper used in many workplaces comes in large rolls that fit high-capacity dispensers, and the coffee consumed in offices is often sold in packs suitable for commercial coffee makers. Offices generally procure these items from specialized vendors that do not sell to consumers. Thus, separate markets exist for away-from-home and at-home varieties of certain goods, and the supply of away-from-home varieties cannot be redirected (in the short run) to meet increased demand for at-home varieties.

Demand-driven shortages can be accommodated in our model, by assuming that supply is always fixed $m < 1$ units per period and that each household requires one unit of the good with probability m per period. If these household-level demands are independent, the aggregate demand will be exactly equal to aggregate supply. Thus, the scenario where households require one unit with probability m is the balanced scenario. Relative to this scenario, a demand increase happens when the consumption probability changes from m to 1, holding supply fixed at m. This setting is mathematically equivalent to our model with a supply shortage. Therefore, our main results apply regardless of whether the shortages are generated on the supply side or the demand side of the economy.

6.2 Policy implications

Given that stockpiling creates inefficiencies both in the short and long run, welfare can be improved through policy interventions.

Governments may attempt to impose limits on inventory accumulation. Such restrictions are not uncommon. Many countries have anti-hoarding laws in place to prevent stockpiling by businesses or households during emergencies.[21] In addition, in times of shortages retail stores often limit quantities sold per customer. These store-imposed policies are, of course, not perfectly enforceable, as customers could visit more than one store, or make more than one trip to the same store. Nevertheless, to the extent that such policies can be enforced, in situations where multiple z-storage equilibria exist a limit on sales to relatively few units per household per period would result in a Pareto improvement. Interestingly, individual households would not feel constrained by such a

[21] In the United States, for example, 50 U.S. Code § 4512 states that "[i]n order to prevent hoarding, no person shall accumulate (1) in excess of the reasonable demands of business, personal, or home consumption, or (2) for the purpose of resale at prices in excess of prevailing market prices, materials which have been designated by the President as scarce materials or materials the supply of which would be threatened by such accumulation." 50 U.S. Code § 4512 specifies a maximum fine of $10,000 and a maximum prison sentence of one year for violations.

purchasing limit, if that limit is itself an equilibrium storage quantity. In other words, in cases where both a z-storage equilibrium and a z'-storage equilibrium exist, with $z' < z$, a purchasing cap of z' units is merely an equilibrium selection device.[22] On the other hand, if the z'-storage rule is not an equilibrium (and no z''-storage equilibrium exists for $z'' < z'$), the purchasing cap would be experienced as a binding constrained, but social welfare would still be higher under that constraint than in the original z-storage equilibrium.

Governments can also communicate with the public about supply conditions. When the true supply of a good is not observable to households, *mistaken* perceptions of even a slight shortage can trigger a switch to stockpiling strategies. In this case, the role of government communications is to correct these mistaken beliefs. In particular, governments can communicate to the public that any underlying supply-demand imbalances are negligible. In the early days of the COVID-19 pandemic, for instance, Dutch prime minister Mark Rutte famously told shoppers at a grocery store that the Netherlands had sufficient toilet paper for its citizens to be able to "poop for 10 years."[23] Similarly, on March 20, 2020, Germany's federal minister for consumer protection, Julia Klöckner, posted pictures on social media that depicted her in a wholesale warehouse stocked with toilet paper and other necessities. Ms. Klöckner wrote: "There is enough for all of us!"[24] These public announcements can be interpreted as messages aimed at breaking a stockpiling equilibrium, by addressing incorrect perceptions of a severe fundamental supply shortage.

6.3 Open questions

As we have shown in this paper, in-store shortages may be *one out of many* possible equilibrium outcomes that can arise in response to small underlying supply-demand imbalances. Therefore, the underlying conditions that create in-store shortages could occur frequently, but in only a small percentage of these instances do consumers coordinate on equilibria that involve in-store shortages. It remains an important question to understand the factors that cause consumers to coordinate on one equilibrium instead of another.

Our analysis of transitional equilibria in Section 4.3 also leaves open the question how the transition from x^0 to some new long-run state looks like when a z-storage equilibrium does not exist. It is conceivable that the equilibrium is asymmetric, with a fraction of households using one z-storage rule and another fraction using a different z-storage rule.

[22] Eckert *et al.* (2017) examine the use of quantity limits as equilibrium selection devices in an antitrust context. Their model, too, has the feature that the constraints will be non-binding in the equilibrium they select.

[23] See https://www.reuters.com/article/us-health-coronavirus-netherlands-toilet-idUSKBN21627A (retrieved December 10, 2024).

[24] See https://www.facebook.com/298621020195483/posts/3098208980236659 (retrieved December 10, 2024; authors' translation from German).

It is also conceivable that all households use the same z-storage rule but change the value of z over time. Yet another possibility is that the equilibrium does not involve simple decision rules like the z-storage rule, or that it does not involve convergence to a stationary state at all. A full examination of these possibilities is beyond the scope of this paper and a topic for future research.

Appendix

Proof of Lemma 1 (a): Characterization of the stationary state

Consider a single household that uses a z-storage rule and enters period t with k units in storage. If $1 \leq k \leq z$, the household first consumes one unit and then attempts to purchase $z-(k-1)$ units of the good, in order to enter the next period with an inventory of z units. If $k=0$, the household does not consume in period t and will attempt to purchase z units. If $k>z$, the household consumes one unit and will then purchase zero units.

Now recall that x_k^t is the fraction of households who enter period t with k units in storage. If all households use the same z-storage rule in period t, the aggregate quantity of the good households attempt to purchase—which was defined generally in (1)—becomes

$$\theta(x^t|z) \;=\; zx_0^t + zx_1^t + (z-1)x_2^t + (z-2)x_3^t + \ldots + 3x_{z-2}^t + 2x_{z-1}^t + x_z^t, \tag{13}$$

and the probability that a household finds the good in period t—which was defined generally in (2)—becomes

$$p(x^t|z) \;=\; \min\left\{\frac{m}{\theta(x^t|z)}, 1\right\}. \tag{14}$$

The transition rule (3) can now be written as follows:

$$x_k^{t+1} \;=\; T_k(x^t|z) \;\equiv\; \begin{cases} 0 & \text{if } k=K>z, \\ p(x^t|z) & \text{if } k=z=K, \\ x_{k+1}^t & \text{if } z<k<K, \\ x_{z+1}^t + p(x^t|z)(x_0^t+\ldots+x_z^t) & \text{if } k=z<K, \\ (1-p(x^t|z))x_{k+1}^t & \text{if } 0<k<z, \\ (1-p(x^t|z))(x_1^t+x_0^t) & \text{if } k=0. \end{cases} \tag{15}$$

A stationary state of the economy is a fixed point of $T(\cdot|z) : \Delta_K \to \Delta_K$. Since T is continuous, and Δ_K is compact and convex, a fixed point x^* exists by Brouwer's Fixed Point Theorem. To characterize x^*, fix z and let $p^* \equiv p(x^*|z)$ and $\theta^* \equiv \theta(x^*|z)$. Then

(15) implies that $x^*_k = 0$ for all $k > z$ and

$$\begin{aligned}
x^*_z &= p^*, \\
x^*_{z-1} &= p^*(1-p^*), \\
x^*_{z-2} &= p^*(1-p^*)^2, \\
&\vdots \\
x^*_1 &= p^*(1-p^*)^{z-1}, \\
x^*_0 &= 1 - \sum_{k=1}^{z} x^*_k \;=\; 1 - p^* \sum_{k=0}^{z} (1-p^*)^k \;=\; (1-p^*)^z.
\end{aligned}$$

Thus, in the stationary state, the attempted purchase quantity, (13), becomes

$$\theta^* \;=\; \sum_{k=1}^{z} (z-k+1) p^* (1-p^*)^{z-k} \;+\; z(1-p^*)^z \;=\; \frac{1-(1-p^*)^z}{p^*} \;\geq\; 1, \tag{16}$$

and the probability that a household finds the good in the store, (14), becomes

$$p^* \;=\; \frac{m}{\theta^*} \;=\; \frac{mp^*}{1-(1-p^*)^z}. \tag{17}$$

(17) can be solved uniquely for $p^* = 1-(1-m)^{1/z}$. □

Proof of Lemma 1 (b): Convergence

Note that the mapping T defined in (15) fails to be a contraction. For example, suppose $m = 1$ and $K = 5$. Consider the states

$$x = \big(0,0,0,0,0,1\big) \;\text{ and }\; y = \left(\tfrac{1}{2},0,0,0,0,\tfrac{1}{2}\right).$$

Under the standard Euclidean norm (i.e., ℓ_2), the distance between x and y is $d(x-y) = \sqrt{1/2}$. Consider now how these states change if consumers use the maximum-storage rule, i.e., $z = 5$. Applying the laws of motion (13)–(15), we obtain

$$x' = T(x|5) = \big(0,0,0,0,0,1\big) \;\text{ and }\; y' = T(y|5) = \left(\tfrac{1}{3},0,0,0,\tfrac{1}{3},\tfrac{1}{3}\right).$$

This means that $d(x'-y') = \sqrt{2/3}$; therefore, the mapping T expands the distance between certain points in Δ_K (and the same is true under alternative metrics d). For this reason, we cannot simply appeal to the Contraction Mapping Theorem to establish convergence. Instead, in the following we establish convergence "from the ground up."

Outline of the proof

It is sufficient to prove the result for the maximum-storage rule, i.e., $z = K$. Observe that the law of motion (15) implies $z_k^t = 0$ for all $k > z$ and all $t \geq K - z + 1$. Therefore, after at most $K - z$ iterations of $T(\cdot|z)$, we have $x^t \in \Delta_z \times \{0\}^{K-z}$, and the projection of $T(\cdot|z)$ onto Δ_z becomes

$$x_k^{t+1} = T_k(x^t|z) \equiv \begin{cases} p^t & \text{if } k = z, \\ (1-p^t)x_{k+1}^t & \text{if } 0 < k < z, \\ (1-p^t)(x_1^t + x_0^t) & \text{if } k = 0, \end{cases} \tag{18}$$

which is the same as (15) when $z = K$. Without loss of generality, therefore, we can restrict attention to the maximum-storage rule.[25] To save on notation, for the remainder of this proof we write $T(\cdot)$ instead of $T(\cdot|K)$ for the law of motion associated with the maximum-storage rule.

Take an initial state $x^1 \in \Delta_K$. For $t = 1, 2, \ldots$ define

$$p^t \equiv p(x^t) \quad \text{and} \quad x^{t+1} \equiv T^t(x^1),$$

where $p : \Delta_K \to [0,1]$ is defined via (13)–(15) in the proof of part (a) of Lemma 1. We will construct a sequence $\underline{q}^t \to p^*$ such that $p^t \geq \underline{q}^t$ $\forall t$. We will construct a second sequence $\overline{q}^t \to p^*$ such that $p^t \leq \overline{q}^t$ $\forall t$. This implies that $p^t \to p^*$. By definition of the mapping T in (18), $p^t \to p^*$ implies $x^t \to x^*$.[26]

The proof that $p^t \to p^*$ is divided into a series of steps. In Step 1 we establish several preliminary results that will be applied repeatedly in the subsequent steps. In Step 2 we construct the sequence $\underline{q}^t$, in Step 3 we show that $\underline{q}^t \to p^*$, and in Step 4 we show that $p^t \geq \underline{q}^t$ for all t. Step 5 repeats Steps 2–4 to establish analogous results for $\overline{q}^t$. Finally, Step 6 establishes that $x^1 \succsim (\precsim)\ x^*$ implies convergence of p^t to p^* from above (below).

Step 1: Preliminaries

Define a function $f : \Delta_K \times [0,1] \to \Delta_K$ as follows:

$$f_k(x, q) = \begin{cases} q & \text{if } k = K, \\ (1-q)x_{k+1} & \text{if } 0 < k < K, \\ (1-q)(x_1 + x_0) & \text{if } k = 0. \end{cases}$$

[25] Put differently: If $z < K$, we can redefine $K := z$ and proceed with proving the result for $z = K$, thus redefined.

[26] To see this, note that $p^t \to p^*$ implies $x_K^t \to p^* = x_K^*$. Therefore, $x_{K-1}^t \to (1-p^*) \lim_t x_K^{t-1} = (1-p^*)p^* = x_{K-1}^*$. Therefore, $x_{K-2}^t \to (1-p^*) \lim_t x_{K-1}^{t-1} = (1-p^*)^2 p^* = x_{K-2}^*$; and so on.

Note that $T(x) = f(x, p(x))$. In Step 4 and Step 6, we will apply the following result:

Lemma 10.

(a) If $x \succsim \hat{x}$ then $p(x) \geq p(\hat{x})$.

(b) If $x \succsim \hat{x}$ then $f(x,q) \succsim f(\hat{x},q)$ for all $q \in [0,1]$.

(c) If $q \geq \hat{q}$ then $f(x,q) \succsim f(x,\hat{q})$ for all $x \in \Delta_K$.

Proof. Part (a) is readily apparent from (13)–(14). To show part (b), suppose $x \succsim \hat{x}$, that is,

$$\sum_{k'=0}^{k} x_{k'} \ \leq \ \sum_{k'=0}^{k} \hat{x}_{k'} \quad \forall k = 0, \ldots, K.$$

Fix $q \in [0,1]$ and let $y = f(x,q)$ and $\hat{y} = f(\hat{x},q)$. Then we have

$$\sum_{k'=0}^{k} y_{k'} \ = \ (1-q) \sum_{k'=0}^{k+1} x_{k'} \ \leq \ (1-q) \sum_{k'=0}^{k+1} \hat{x}_{k'} \ = \ \sum_{k'=0}^{k} \hat{y}_{k'}$$

for all $k = 0, \ldots, K-1$, and $\sum_{k'=0}^{K} y_{k'} = 1 = \sum_{k'=0}^{K} \hat{y}_{k'}$. It follows that $y \succsim \hat{y}$. Finally, to show part (c), suppose $q \geq \hat{q}$. Fix $x \in \Delta_K$ and let $y = f(x,q)$ and $\hat{y} = f(x,\hat{q})$. Then we have

$$\sum_{k'=0}^{k} y_{k'} \ = \ (1-q) \sum_{k'=0}^{k+1} x_{k'} \ \leq \ (1-\hat{q}) \sum_{k'=0}^{k+1} x_{k'} \ = \ \sum_{k'=0}^{k} \hat{y}_{k'}$$

for all $k = 0, \ldots, K-1$, and $\sum_{k'=0}^{K} y_{k'} = 1 = \sum_{k'=0}^{K} \hat{y}_{k'}$. It follows that $y \succsim \hat{y}$. □

Step 2: Construction of the sequence $\underline{q}^t$

Associated with the sequence $\underline{q}^t$ will be a sequence of states, $\underline{x}^t \in \Delta_K$, defined through $\underline{x}^1 = (1, 0, \ldots, 0)$ and $\underline{x}^{t+1} = f(\underline{x}^t, \underline{q}^t)$. For each t, define $\underline{p}^t = p(\underline{x}^t)$. Note that $p^1 = m/K$.

We build the sequence $\underline{q}^t$ in pieces of K elements at a time. We begin by setting the first K values of $\underline{q}^t$ to

$$\underline{q}^1, \ldots, \underline{q}^K \ = \ p^1.$$

Given the definition of f, in period $K+1$ we have

$$\underline{x}^{K+1} \ = \ \left((1-\underline{p}^1)^K, (1-\underline{p}^1)^{K-1}\underline{p}^1, (1-\underline{p}^1)^{K-2}\underline{p}^1, \ldots, (1-\underline{p}^1)\underline{p}^1, \underline{p}^1 \right).$$

Using the same formulas as in (16)–(17), we can write

$$\underline{p}^{K+1} \ = \ \frac{m\underline{p}^1}{1-(1-\underline{p}^1)^K}.$$

We then set the next K values of $\underline{q}^n$ to

$$\underline{q}^{K+1}, \ldots, \underline{q}^{2K} \;=\; \underline{p}^{K+1}.$$

Therefore, in period $2K+1$ we have

$$\begin{aligned}\underline{x}^{2K+1} \;=\; \big((1-\underline{p}^{K+1})^K, (1-\underline{p}^{K+1})^{K-1}\underline{p}^{K+1}, (1-\underline{p}^{K+1})^{K-2}\underline{p}^{K+1},& \\ \ldots, (1-\underline{p}^{K+1})\underline{p}^{K+1}, \underline{p}^{K+1}\big)&\end{aligned}$$

and

$$\underline{p}^{2K+1} \;=\; \frac{m\underline{p}^{K+1}}{1-(1-\underline{p}^{K+1})^K}.$$

We then set the next K values of $\underline{q}^t$ to

$$\underline{q}^{2K+1}, \ldots, \underline{q}^{3K} \;=\; \underline{p}^{2K+1}.$$

Proceeding in the same fashion for $\ell = 3, 4, \ldots$, we obtain

$$\underline{q}^{\ell K+1}, \ldots, \underline{q}^{(\ell+1)K} \;=\; \underline{p}^{\ell K+1} \;=\; \frac{m\underline{p}^{(\ell-1)K+1}}{1-(1-\underline{p}^{(\ell-1)K+1})^K}. \tag{19}$$

Step 3: $\underline{q}^t \to p^*$ as $t \to \infty$

Write the second equality on the right side of (19) as $\underline{p}^{\ell K+1} = A\big(\underline{p}^{(\ell-1)K+1}\big)$, where

$$A(p) \;=\; \frac{mp}{1-(1-p)^K}. \tag{20}$$

The unique fixed point of $A : [0,1] \to [0,1]$ is $p^* = 1-(1-m)^{1/K}$. We will show that $0 < A'(p) < 1$ for all $p \in (0,1)$. This implies that $\underline{p}^{\ell K+1} \to p^*$ as $\ell \to \infty$, and thus $\underline{q}^t \to p^*$ as $t \to \infty$.

Note that

$$\begin{aligned} A'(p) \quad &= \quad m\frac{1-(1-p)^K - pK(1-p)^{K-1}}{\big(1-(1-p)^K\big)^2} \;<\; 1 \\ &\Leftarrow \quad \frac{1-(1-p)^K - pK(1-p)^{K-1}}{\big(1-(1-p)^K\big)^2} \;<\; 1 \\ &\Leftrightarrow \quad K \;>\; \frac{(1-p)-(1-p)^{K+1}}{p} \;=\; \sum_{k=1}^{K}(1-p)^k, \end{aligned}$$

which is true if $p \in (0,1)$. Likewise, note that

$$\begin{aligned} A'(p) &= m\frac{1-(1-p)^K - pK(1-p)^{K-1}}{\left(1-(1-p)^K\right)^2} \;>\; 0 \\ &\Leftarrow \frac{1-(1-p)^K - pK(1-p)^{K-1}}{\left(1-(1-p)^K\right)^2} \;>\; 0 \\ &\Leftrightarrow K \;<\; \frac{1-(1-p)^K}{p(1-p)^{K-1}} \;=\; \sum_{k=0}^{K-1} \frac{(1-p)^k}{(1-p)^{K-1}}, \end{aligned}$$

which, too, is true if $p \in (0,1)$. Therefore $0 < A'(p) < 1 \ \forall p \in (0,1)$.

Step 4: $p^t \geq \underline{q}^t$ for all t

It will be convenient to construct a sequence $\underline{\underline{x}}^t \in \Delta_L$ as follows: $\underline{\underline{x}}^1 = (1,0,\ldots,0)$ and $\underline{\underline{x}}^{t+1} = T(\underline{\underline{x}}^t)$. Also define $\underline{\underline{p}}^t = p(\underline{\underline{x}}^t)$. Note that $\underline{\underline{x}}^2 \succsim \underline{\underline{x}}^1$ necessarily. Using Lemma 10 (a)–(c), we can write

$$\underline{\underline{x}}^3 = T(\underline{\underline{x}}^2) = f(\underline{\underline{x}}^2, \underline{\underline{p}}^2) \succsim f(\underline{\underline{x}}^2, \underline{\underline{p}}^1) \succsim f(\underline{\underline{x}}^1, \underline{\underline{p}}^1) = T(\underline{\underline{x}}^1) = \underline{\underline{x}}^2,$$

$$\underline{\underline{x}}^4 = T(\underline{\underline{x}}^3) = f(\underline{\underline{x}}^3, \underline{\underline{p}}^3) \succsim f(\underline{\underline{x}}^3, \underline{\underline{p}}^2) \succsim f(\underline{\underline{x}}^2, \underline{\underline{p}}^2) = T(\underline{\underline{x}}^2) = \underline{\underline{x}}^3,$$

and so on. Therefore $\underline{\underline{x}}^t$ is increasing in the sense that $\underline{\underline{x}}^{t+1} \succsim \underline{\underline{x}}^t \ \forall t$. This implies that $\underline{\underline{p}}^{t+1} \geq \underline{\underline{p}}^t \ \forall t$.

We will show that $p^t \geq \underline{\underline{p}}^t \geq \underline{p}^t \geq \underline{q}^t$ for all t. All four sequences are illustrated in Figure 6 below. (The same figure also shows the corresponding sequence that will bound p^t from above; see Step 5.)

First, to show that $p^t \geq \underline{\underline{p}}^t \ \forall t$, note that $x^1 \succsim \underline{\underline{x}}^1$ necessarily. Thus, using Lemma 10 (a)–(c), we have

$$x^2 = T(x^1) = f(x^1, p(x^1)) \succsim f(x^1, \underline{\underline{p}}^1) \succsim f(\underline{\underline{x}}^1, \underline{\underline{p}}^1) = T(\underline{\underline{x}}^1) = \underline{\underline{x}}^2,$$

$$x^3 = T(x^2) = f(x^2, p(x^2)) \succsim f(x^2, \underline{\underline{p}}^2) \succsim f(\underline{\underline{x}}^2, \underline{\underline{p}}^2) = T(\underline{\underline{x}}^2) = \underline{\underline{x}}^3,$$

and so on. It follows that $x^t \succsim \underline{\underline{x}}^t \ \forall t$. By Lemma 10 (a), this implies $p^t \geq \underline{\underline{p}}^t \ \forall t$.

Second, to show that $\underline{\underline{p}}^t \geq \underline{p}^t \ \forall t$, observe that $\underline{\underline{x}}^1 = \underline{x}^1$ implies $\underline{\underline{p}}^1 = \underline{p}^1$ $(= \underline{q}^1, \ldots, \underline{q}^K)$. Using Lemma 10 (a)–(c) and the fact that $\underline{\underline{x}}^t$ is increasing, it follows that

$$\underline{\underline{x}}^2 = T(\underline{\underline{x}}^1) = f(\underline{\underline{x}}^1, \underline{\underline{p}}^1) = f(\underline{\underline{x}}^1, \underline{p}^1) = f(\underline{x}^2, \underline{p}^1) = \underline{x}^2,$$

$$\underline{\underline{x}}^3 = T(\underline{\underline{x}}^2) = f(\underline{\underline{x}}^2, \underline{\underline{p}}^2) \succsim f(\underline{\underline{x}}^2, \underline{\underline{p}}^1) = f(\underline{\underline{x}}^2, \underline{p}^1) \succsim f(\underline{x}^2, \underline{p}^1) = \underline{x}^3,$$

$$\vdots$$

$$\underline{\underline{x}}^{K+1} = T(\underline{\underline{x}}^K) = f(\underline{\underline{x}}^K, \underline{\underline{p}}^K)) \succsim f(\underline{\underline{x}}^K, \underline{\underline{p}}^1) = f(\underline{\underline{x}}^K, \underline{p}^1) \succsim f(\underline{x}^K, \underline{p}^1) = \underline{x}^{K+1}.$$

This implies that $\underline{\underline{p}}^{K+1} \geq \underline{p}^{K+1}$ $(= \underline{q}^{K+1}, \ldots, \underline{q}^{2K})$. Once again using Lemma 10 (a)–(c) and the fact that $\underline{\underline{x}}^t$ is increasing, it follows that

$$\begin{aligned}
\underline{\underline{x}}^{K+2} &= T(\underline{\underline{x}}^{K+1}) = f(\underline{\underline{x}}^{K+1}, \underline{\underline{p}}^{K+1}) \\
&\qquad \succsim f(\underline{\underline{x}}^{K+1}, \underline{p}^{K+1}) \succsim f(\underline{x}^{K+1}, \underline{p}^{K+1}) = \underline{x}^{K+2}, \\
\underline{\underline{x}}^{K+3} &= T(\underline{\underline{x}}^{K+2}) = f(\underline{\underline{x}}^{K+2}, \underline{\underline{p}}^{K+2}) \succsim f(\underline{\underline{x}}^{K+2}, \underline{\underline{p}}^{K+1}) \\
&\qquad \succsim f(\underline{\underline{x}}^{K+2}, \underline{p}^{K+1}) \succsim f(\underline{x}^{K+2}, \underline{p}^{K+1}) = \underline{x}^{K+3}, \\
&\vdots \\
\underline{\underline{x}}^{2K+1} &= T(\underline{\underline{x}}^{2K}) = f(\underline{\underline{x}}^{2K}, \underline{\underline{p}}^{2K})) \succsim f(\underline{\underline{x}}^{2K}, \underline{\underline{p}}^{K+1})) \\
&\qquad \succsim f(\underline{\underline{x}}^{2K}, \underline{p}^{K+1}) \succsim f(\underline{x}^{2K}, \underline{p}^{K+1}) = \underline{x}^{2K+1}.
\end{aligned}$$

Proceeding in the same fashion (i.e., in blocks of K elements at a time) we can establish that $\underline{\underline{x}}^t \succsim \underline{x}^t$ $\forall t$. By Lemma 10 (a), this implies $\underline{\underline{p}}^t \geq \underline{p}^t$ $\forall t$.

Finally, we show $\underline{p}^t \geq \underline{q}^t$ $\forall t$. As before, we proceed in blocks of K elements at a time. Note that $\underline{q}^1, \ldots, \underline{q}^K = \underline{p}^1$. Furthermore $\underline{x}^2 \succsim \underline{x}^1$ necessarily. Using Lemma 10 (b)–(c), we have

$$\begin{aligned}
\underline{x}^3 &= f(\underline{x}^2, \underline{q}^2) \succsim f(\underline{x}^1, \underline{q}^2) = f(\underline{x}^1, \underline{p}^1) = \underline{x}^2, \\
&\vdots \\
\underline{x}^{K+1} &= f(\underline{x}^K, \underline{q}^K) \succsim f(\underline{x}^{K-1}, \underline{q}^K)) = f(\underline{x}^{K-1}, \underline{p}^1) = \underline{x}^K.
\end{aligned}$$

This implies that $\underline{x}^K \succsim \ldots \succsim \underline{x}^1$, and hence $\underline{p}^K \geq \ldots \geq \underline{p}^1 = \underline{q}^1, \ldots, \underline{q}^K$. Next, note that $\underline{q}^{K+1}, \ldots, \underline{q}^{2K+1} = \underline{p}^{K+1}$. Furthermore, by (19) and Step 3 we have $\underline{p}^{K+1} = A(\underline{p}^1) > \underline{p}^1$, and Lemma 10 (b)–(c) implies

$$\underline{x}^{K+2} = f(\underline{x}^{K+1}, \underline{q}^{K+1}) \succsim f(\underline{x}^K, \underline{q}^{K+1}) \succsim f(\underline{x}^K, \underline{p}^1) = \underline{x}^{K+1}.$$

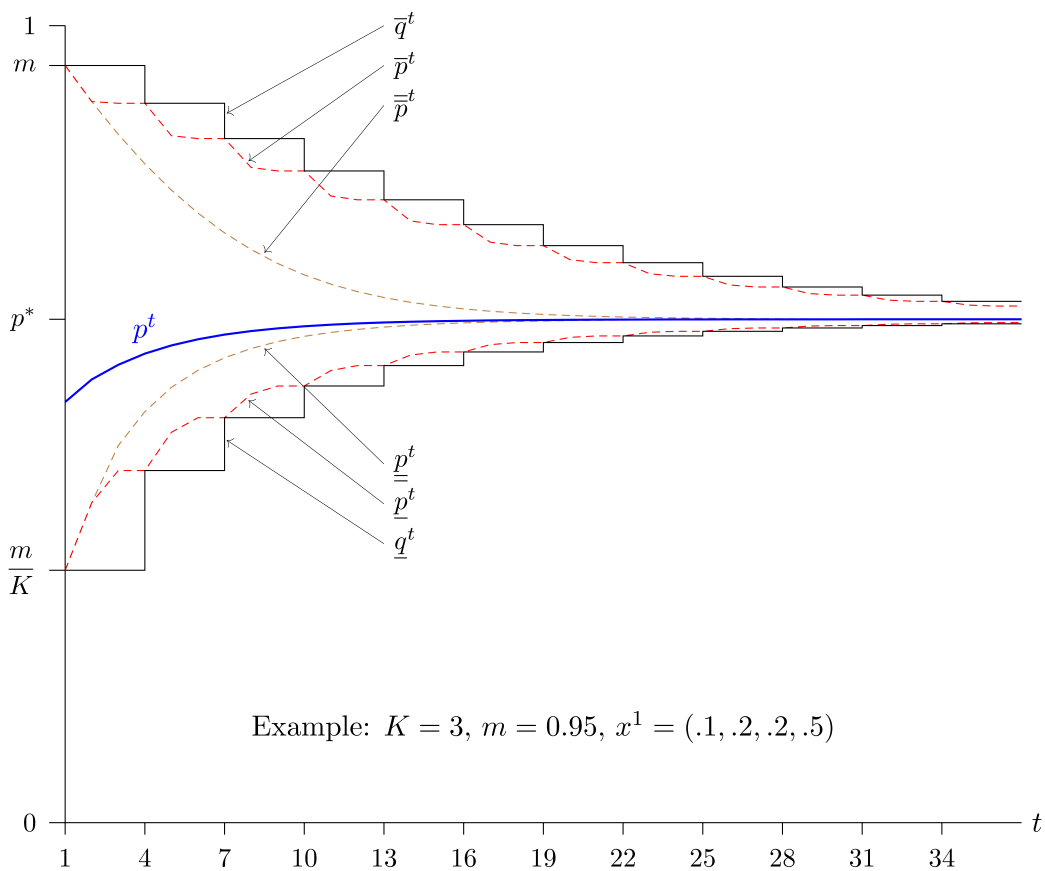
Using Lemma 10 (b)–(c) again, we have

$$\begin{aligned}
\underline{x}^{K+3} &= f(\underline{x}^{K+2}, \underline{q}^{K+2}) \succsim f(\underline{x}^{K+1}, \underline{q}^{K+2}) = f(\underline{x}^{K+1}, \underline{p}^{K+1}) = \underline{x}^{K+2}, \\
&\vdots
\end{aligned}$$

$$\underline{x}^{2K+1} = f(\underline{x}^{2K}, \underline{q}^{2K}) \succsim f(\underline{x}^{2K-1}, \underline{q}^{2K})) = f(\underline{x}^{K-1}, \underline{p}^{K+1}) = \underline{x}^{2K}.$$

This implies that $\underline{x}^{2K} \succsim \ldots \succsim \underline{x}^{K+1}$, and hence $\underline{p}^{2K+1} \geq \ldots \geq \underline{p}^{K+1} = \underline{q}^{K+1}, \ldots, \underline{q}^{2K}$. Proceeding in the same fashion, we can establish that $\underline{p}^t \succsim \underline{q}^t$ $\forall t$.

Figure 6: Illustration of the sequences constructed in the proof of Lemma 1 (b).



Step 5: Construction of $\overline{q}^t \to p^*$ such that $p^t \leq \overline{q}^t$ for all t

This step is analogous to Steps 2–4. In Step 2, the sequences $\underline{x}^t$, $\underline{p}^t$, $\underline{q}^t$ are replaced with $\overline{x}^t$, $\overline{p}^t$, $\overline{q}^t$, where $\overline{x}^1 = (0, \ldots, 0, 1)$, $\overline{x}^{t+1} = f(\overline{x}^t, \overline{q}^t)$, $\overline{p}^t = p(\overline{x}^t)$,

$$\begin{aligned}
\overline{q}^1, \ldots, \overline{q}^K &= \overline{p}^1, \\
\overline{q}^{K+1}, \ldots, \overline{q}^{2K} &= \overline{p}^{K+1}, \\
\overline{q}^{2K+1}, \ldots, \overline{q}^{3K} &= \overline{p}^{2K+1},
\end{aligned}$$

and so on. For $\ell = 1, 2, 3, \ldots$, we can write $\overline{p}^{\ell K+1} = A(p^{(\ell-1)K+1})$, where A is defined in (20) as before. Thus, we can apply the previous Step 3 to establish that $\overline{q}^t \to p^*$. In Step 4, the sequences $\underline{\underline{x}}^t$ and $\underline{\underline{p}}^t$ are replaced with $\overline{\overline{x}}^t$ and $\overline{\overline{p}}^t$, where $\overline{\overline{x}}^1 = (0, \ldots, 0, 1)$, $\overline{\overline{x}}^{t+1} = T(\overline{\overline{x}}^t)$ and $\overline{\overline{p}}^t = p(\overline{\overline{x}}^t)$. One can then show that $p^t \le \overline{\overline{p}}^t \le \overline{p}^t \le \overline{q}^t$ for all t.

Step 6: Convergence from above/below

The previous steps together establish that $x^t \to x^*$ and $p^t \to p^*$. We now show that $p^t \to p^*$ from above if $x^1 \succsim z^*$, and $p^t \to p^*$ from below if $x^1 \precsim z^*$. Suppose $x^1 \succsim x^*$. Using Lemma 10 (a)–(c), we have

$$
\begin{aligned}
x^2 &= T(x^1) = f(x^1, p(x^1)) \succsim f(x^1, p(x^*)) = f(x^1, p^*) \succsim f(x^*, p^*) = T(x^*) = x^*, \\
x^3 &= T(x^2) = f(x^2, p(x^2)) \succsim f(x^2, p(x^*)) = f(x^1, p^*) \succsim f(x^*, p^*) = T(x^*) = x^*,
\end{aligned}
$$

and so on. It follows that $x^t \succsim x^* \ \forall t$. By Lemma 10 (a) this implies $p^t \ge p^* \ \forall t$. The argument when $x^1 \precsim x^*$ is analogous. □

Proof of Lemma 2

Consider the value function in (4) and note that $V^t(0) - V^t(1) = \lambda - 1 < 0 \, \forall t$. Furthermore, for $k \ge 2$, we have

$$
\begin{aligned}
V^t(k-1) - V^t(k) \;=\; & \lambda + \beta\Big(p^t \overbrace{\Big[\max_{\sigma \ge k-2} V^{t+1}(\sigma) - \max_{\sigma \ge k-1} V^{t+1}(\sigma)\Big]}^{\ge 0} \\
& + (1 - p^t)\Big[V^{t+1}(k-2) - V^{t+1}(k-1)\Big]\Big).
\end{aligned}
$$

Proceeding recursively, we have

$$
\begin{aligned}
V^t(k-1) - V^t(k) \;\ge\; & \lambda + \beta(1-p^t)\Big[V^{t+1}(k-2) - V^{t+1}(k-1)\Big] \\
\ge\; & \lambda + \beta(1-p^t)\Big[\lambda + \beta(1-p^{t+1})\Big[V^{t+2}(k-3) - V^{t+2}(k-2)\Big]\Big] \\
& \vdots \\
\ge\; & \lambda\left[1 + \sum_{k'=1}^{k-2} \beta^{k'} \prod_{\tau=0}^{k'-1}(1-p^{t+\tau})\right] \\
& + \beta^{k-1} \prod_{\tau=0}^{k-2}(1-p^{t+\tau}) \underbrace{\Big[V^{t+k-1}(0) - V^{t+k-1}(1)\Big]}_{=\lambda - 1} \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda \left[1 + \sum_{k'=1}^{k-1} \beta^{k'} \prod_{\tau=0}^{k'-1} (1 - p^{t+\tau}) \right] - \beta^{k-1} \prod_{\tau=0}^{k-2} (1 - p^{t+\tau}) \\
&\equiv D_k(\mathbf{p}^t),
\end{aligned}
$$

where $\mathbf{p}^t$ denotes the sequence $(p^t, p^{t+1}, p^{t+2}, \ldots)$. From (21) it is apparent that

$$
D_K(\mathbf{p}^t) \;\geq\; D_{K-1}(\mathbf{p}^t) \;\geq\; \ldots \;\geq D_2(\mathbf{p}^t) \;\geq\; D_1(\mathbf{p}^t) \;=\; \lambda - 1. \tag{22}
$$

Now suppose the z-storage rule is optimal for a household from period t onward. Then, for all $k \leq z + 1$, we have

$$
\max_{\sigma \geq k-2} V^{t+1}(\sigma) \;=\; \max_{\sigma \geq k-1} V^{t+1}(\sigma).
$$

This implies that the inequality (21) becomes an equality for all $k \leq z + 1$:

$$
V^t(k-1) - V^t(k) \;=\; D_k(\mathbf{p}^t). \tag{23}
$$

Moreover, if $z \in \{2, \ldots, K-1\}$ is such that

$$
D_z(\mathbf{p}^t) \;\leq\; 0 \quad \text{and} \quad D_{z+1}(\mathbf{p}^t) \;\geq\; 0, \tag{24}
$$

then (22), (23), and (24) together imply that

$$
V^t(1) \;\leq\; \ldots \;\leq\; V^t(z) \quad \text{and} \quad V^t(z) \;\geq\; V^t(z+1) \;\geq\; \ldots \;\geq\; V^t(K). \tag{25}
$$

This means that the z-storage rule is also optimal in period $t - 1$. It follows that, if (24) holds for all periods $t \geq 1$, then using the z-storage rule in every period must be optimal for the household when facing the sequence probabilities $\mathbf{p}^1 = (p^1, p^2, \ldots)$ of finding the good in the store. If we now define $\overline{\lambda}_z(\mathbf{p}^t)$ as in (5), condition (24) is equivalent to

$$
\overline{\lambda}_{z+1}(\mathbf{p}^t) \;\leq\; \lambda \;\leq\; \overline{\lambda}_z(\mathbf{p}^t),
$$

which is condition (6) in the statement in Lemma 2 (a).

Because $\overline{\lambda}_z(\cdot)$ is undefined for $z = 1$ and $z = K$, two special cases must be considered. First, if $z = 1$ and the left inequality in (6) holds, then (25) becomes $V^t(1) \geq V^t(2) \geq \ldots \geq V^t(K)$, which means the minimum-storage rule is optimal. Second, if $z = K$ and the right inequality in (6) holds, then (25) becomes $V^t(1) \leq V^t(2) \leq \ldots \leq V^t(K)$, which means the maximum-storage rule is optimal. This completes the proof of Lemma 2 (a).

To show the comparative results stated in Lemma 2 (b), without loss of generality consider $t = 1$. For $z \in \{2, \ldots, K\}$, write

$$\frac{1}{\overline{\lambda}_z(\mathbf{p}^1)} = \frac{1 + \sum_{k=1}^{z-1} \beta^k \prod_{\tau=1}^{k} (1 - p^\tau)}{\beta^{z-1} \prod_{\tau=1}^{z-1} (1 - p^\tau)} = \sum_{k=1}^{z} \frac{1}{\beta^{z-k} \prod_{\tau=k}^{z-1} (1 - p^\tau)}. \tag{26}$$

Suppose $p^\tau < 1$ for all τ. Then an increase in β strictly decreases (26) and, therefore, strictly increases $\overline{\lambda}_z(\mathbf{p}^1)$. Similarly, if p^τ decreases for all τ, then (26) decreases strictly and, therefore, $\overline{\lambda}_z(\mathbf{p}^1)$ increases strictly. □

Proof of Proposition 3

When $m = 1$, existence of a minimum-storage equilibrium was shown in the main text. We now show that no other equilibria exist.

Recall that the assumption $\lambda < \beta$ ensures that each household wants to store at least one unit in every period. Furthermore, if $\lambda > \beta^2/(1 + \beta)$ then no household wants to store more than one unit in every period.[27] In this case, the 1-storage rule is strictly dominant in every period, and the proof is complete. Thus, assume $\lambda \leq \beta^2/(1 + \beta)$.

Fix some period $t \geq 1$. Suppose that

$$p^t \; > \; \delta \equiv 1 - \frac{\lambda}{\beta(1 - \lambda)} \in (0, 1)$$

(where $\delta \in (0, 1)$ follows from $0 < \lambda \leq \beta^2/(1 + \beta) < \beta/(1 + \beta)$). Using the notation introduced in the Proof of Lemma 2, this implies $D_2(\mathbf{p}^t) > 0$ and, by (22),

$$V^t(1) \; > \; V^t(2) \; > \; \ldots \; > \; V^t(K).$$

At the same time, $V^t(0) - V^t(1) = \lambda - 1 < 0$. Thus, if $p^t > \delta$ then every household strictly prefers to enter period t with one unit in storage, and every additional unit a household has in storage decreases the expected continuation utility for that household. In this case, it is optimal for every household to use the minimum-storage rule in period $t - 1$. If every household uses the minimum-storage rule in period $t - 1$, the aggregate demand for the good in period $t - 1$ is $\theta^{t-1} \leq 1$; and since $m = 1$, this implies $p^{t-1} = 1 > \delta$. Repeating the same steps, we can show that every household follows the minimum-storage rule in period $t - 2$, $t - 3$, and so on.

[27]The marginal cost of storing the second unit is $\lambda + \beta\lambda$ (as the unit would be stored for two periods) and the marginal benefit is β^2 (as the unit would be consumed after it was stored for two periods); thus, if $\lambda > \beta^2/(1 + \beta)$ it is not optimal to store more than one unit.

From the above argument, we conclude the following: If $m = 1$, then in any equilibrium in which some household does *not* follow the minimum-storage rule in every period, there must exist t^* such that $p^t \leq \delta \; \forall t \geq t^*$. Suppose that this is the case. Then, in any period $t \geq t^* + K$, the fraction of households who have experienced K or more in-store shortages in a row, and hence enter period t with a zero quantity in storage, is

$$(1 - p^{t-1})(1 - p^{t-2}) \cdots (1 - p^{t-K}) \;\geq\; (1-\delta)^K \;>\; 0.$$

Since households with a zero inventory cannot consume; starting in period $t^* + K$ at most a measure $1 - (1-\delta)^K$ of the good is being consumed in each period. Note that, because $p^t \leq \delta < 1 \; \forall t \geq t^*$, the store sells the entire supply in period t. But since households cannot resell or dispose of the good, the amount of the good that is not consumed must end up in storage. Since each household can store at most K units, at the latest in period $t^{**} = t^* + K + \lfloor K/(1-\delta)^K \rfloor$ every household must have K units in storage. But this means that $\theta^{t^{**}} \leq 1$, which implies $p^{t^{**}} = 1$, a contradiction.

It follows that, when $m = 1$, there does exist an equilibrium in which some household does not follow the minimum-storage rule in every period. $\square$

Remark: Recall that we defined equilibrium in a way that assumes all households use a common decision rule σ^t in a given period. However, the argument presented above is based solely on the sequence of probabilities p^t with which a household is able to buy the good in period t, but does not depend on the assumption that these probabilities are derived from a common decision rule applied by all households in period t. Thus, even if we extended the equilibrium definition to include the possibility that different households apply different decision rules, the current Proposition 3 would remain true.

Proof of Proposition 4

Define

$$L(z) \;=\; \begin{cases} 0 & \text{if } z = K, \\ \dfrac{\gamma(z)^z\big(1-\gamma(z)\big)}{1-\gamma(z)^{z+1}} & \text{if } z < K, \end{cases} \tag{27}$$

and

$$U(z) \;=\; \begin{cases} \dfrac{\gamma(z)^{z-1}\big(1-\gamma(z)\big)}{1-\gamma(z)^{z}} & \text{if } z > 1, \\ 1 & \text{if } z = 1, \end{cases} \tag{28}$$

where $\gamma(z) = \beta(1-m)^{1/z}$. Note that $m \in (0,1)$ implies $\gamma(1) < \gamma(2) < \ldots < \gamma(K) < 1$.

A stationary z-storage equilibrium exists if $\lambda \in [L(z), U(z)]$ for some $z \in \{1, \ldots, K\}$. To show that such a z exists, we establish three properties:

(i) $0 \leq L(z) < U(z) \leq 1 \; \forall z = 1, \ldots, K$.

(ii) $L(z) > L(z+1)$ and $U(z) > U(z+1) \; \forall z = 1, \ldots, K-1$.

(iii) $L(z) < U(z+1) \; \forall z = 1, \ldots, K-1$.

Together, these three properties imply that $\cup_{z=1,\ldots,K}[L(z), U(z)] = [0, 1]$, which implies $\lambda \in [L(z), U(z)]$ for at least one $z \in \{1, \ldots, K\}$.

Property (i): First, $0 \leq L(z)$ is obvious from (27). Next, we show $L(z) < U(z)$. For $z = 1$ and $z = K$ this is obvious; for $1 < z < K$ we have

$$L(z) < U(z) \;\Leftrightarrow\; \frac{\gamma(z)}{1-\gamma(z)^{z+1}} < \frac{1}{1-\gamma(z)^{z}} \;\Leftrightarrow\; \gamma(z)^{z} - \gamma(z)^{z+1} < 1 - \gamma(z)^{z+1}$$
$$\Leftrightarrow\; \gamma(z)^{z} < 1 \;\Leftrightarrow\; \gamma(z) < 1,$$

which is satisfied. Lastly, we show $U(z) \leq 1$. For $z = 1$ this is obvious from (28); for $z > 1$ we have

$$U(z) \leq 1 \;\Leftrightarrow\; \gamma(z)^{z-1} - \gamma(z)^{z} \leq 1 - \gamma(z)^{z} \;\Leftrightarrow\; \gamma(z)^{z-1} \leq 1 \;\Leftrightarrow\; \gamma(z) \leq 1,$$

which is satisfied.

Property (ii): We show that $L(z) > L(z+1)$. The proof that $U(z) > U(z+1)$ is analogous and omitted. For $z = K-1$, the inequality $L(z) > L(z+1)$ is obvious from (27). For $z < K-1$, substitute $\gamma(z) = \beta(1-m)^{1/z}$ into the definition of $L(z)$ to write $L(z) < L(z+1)$ as follows:

$$\frac{\beta^{z}(1-m)\left(1-\beta(1-m)^{1/z}\right)}{1-\beta^{z+1}(1-m)^{(z+1)/z}} > \frac{\beta^{z+1}(1-m)\left(1-\beta(1-m)^{1/(z+1)}\right)}{1-\beta^{z+2}(1-m)^{(z+2)/(z+1)}}$$

$$\Leftrightarrow\; \frac{1-\beta(1-m)^{1/z}}{1-\beta^{z+1}(1-m)^{(z+1)/z}} > \beta\frac{1-\beta(1-m)^{1/(z+1)}}{1-\beta^{z+2}(1-m)^{(z+2)/(z+1)}},$$

$$\Leftrightarrow\; 1-\beta(1-m)^{\frac{1}{z}} - \beta^{z+2}(1-m)^{\frac{z+2}{z+1}} + \beta^{z+3}(1-m)^{\frac{z^2+3z+1}{z(z+1)}}$$
$$> \beta\Big[1-\beta(1-m)^{\frac{1}{z+1}}\Big] - \beta^{z+2}(1-m)^{\frac{z+1}{z}} + \beta^{z+3}(1-m)^{\frac{z^2+3z+1}{z(z+1)}}$$

$$\Leftrightarrow\; 1-\beta(1-m)^{\frac{1}{z}} - \beta^{z+2}(1-m)^{\frac{z+2}{z+1}} > \beta\Big[1-\beta(1-m)^{\frac{1}{z+1}}\Big] - \beta^{z+2}(1-m)^{\frac{z+1}{z}}. \tag{29}$$

To show (29), it sufficient to show that

$$1-\beta(1-m)^{\frac{1}{z}} - \beta^{z+2}(1-m)^{\frac{z+2}{z+1}} > \Big[1-\beta(1-m)^{\frac{1}{z+1}}\Big] - \beta^{z+2}(1-m)^{\frac{z+1}{z}}$$

$$\Leftrightarrow \ \beta(1-m)^{\frac{1}{z+1}} - \beta^{z+2}(1-m)^{\frac{z+2}{z+1}} \ > \ \beta(1-m)^{\frac{1}{z}} - \beta^{z+2}(1-m)^{\frac{z+1}{z}}$$
$$\Leftrightarrow \ \beta(1-m)^{\frac{1}{z+1}}\left[1-\beta^{z+1}(1-m)\right] \ > \ \beta(1-m)^{\frac{1}{z}}\left[1-\beta^{z+1}(1-m)\right]$$
$$\Leftrightarrow \ \beta(1-m)^{\frac{1}{z+1}} \ > \ \beta(1-m)^{\frac{1}{z}}$$
$$\Leftrightarrow \ \gamma(z+1) \ > \ \gamma(z),$$

which is satisfied.

Property (iii): For $z = 1, \ldots, K-1$, we have

$$\begin{aligned} L(z) \ < \ U(z+1) \quad &\Leftrightarrow \quad \frac{\gamma(z)^z\big(1-\gamma(z)\big)}{1-\gamma(z)^{z+1}} \ < \ \frac{\gamma(z+1)^z\big(1-\gamma(z+1)\big)}{1-\gamma(z+1)^{z+1}} \\ &\Leftrightarrow \quad \frac{1}{\gamma(z+1)^z}\frac{1-\gamma(z+1)^{z+1}}{1-\gamma(z+1)} \ < \ \frac{1}{\gamma(z)^z}\frac{1-\gamma(z)^{z+1}}{1-\gamma(z)} \\ &\Leftrightarrow \quad \frac{1}{\gamma(z+1)^z}\sum_{k=0}^{z}\gamma(z+1)^k \ < \ \frac{1}{\gamma(z)^z}\sum_{k=0}^{z}\gamma(z)^k \\ &\Leftrightarrow \quad \sum_{k=0}^{z}\gamma(z+1)^{-k} \ < \ \sum_{k=0}^{z}\gamma(z)^{-k}, \end{aligned}$$

which is implied by $\gamma(z) < \gamma(z+1) < 1$. □

Proof of Proposition 5

The result follows from property (ii) shown in the proof of Proposition 4. Suppose that a stationary z-storage equilibrium exists and that a z' stationary storage equilibrium exists, with $z < z'$. Let z'' be an integer such that $z < z'' < z'$. Because a stationary z-storage equilibrium exists, $L(z) \le \lambda \le U(z)$; similarly, because a stationary z'-storage equilibrium exists, $L(z') \le \lambda \le U(z')$. Since $z'' > z$, by property (ii) we have $L(z'') \le L(z) \le \lambda$. Similarly, since $z'' < z'$, by property (ii) we have $U(z'') \ge U(z') \le \lambda$. Therefore, $L(z'') \le \lambda \le U(z'')$, which means that a stationary z''-storage equilibrium exists. □

Proof of Proposition 6

By Proposition 5, we only need to verify the conditions for which the minimum and maximum-storage equilibria exist at the same time. Using (7), the minimum-storage equilibrium exists if

$$\lambda \ \ge \ \frac{\gamma(1)\big(1-\gamma(1)\big)}{1-\gamma(1)^2} \ = \ \frac{\beta(1-m)\big(1-\beta(1-m)\big)}{1-\beta^2(1-m)^2} \ > \ 0,$$

and the maximum-storage equilibrium exists if

$$\lambda \;\le\; \frac{\gamma(K)^{K-1}\bigl(1-\gamma(K)\bigr)}{1-\gamma(K)^K} \;=\; \frac{\beta^{K-1}(1-m)^{(K-1)/K}\bigl(1-\beta(1-m)^{1/K}\bigr)}{1-\beta^K(1-m)} \;<\; 1.$$

Thus, both equilibria coexist as long as

$$\frac{\beta(1-m)\bigl(1-\beta(1-m)\bigr)}{1-\beta^2(1-m)^2} \;\le\; \lambda \;\le\; \frac{\beta^{K-1}(1-m)^{(K-1)/K}\bigl(1-\beta(1-m)^{1/K}\bigr)}{1-\beta^K(1-m)}. \tag{30}$$

We need to show that, if m is sufficiently large (but smaller than one), the left-hand side of (30) is strictly smaller than the right-hand side. After rearranging, this condition becomes

$$\frac{1-\beta^K(1-m)}{1-\beta^2(1-m)^2} \;<\; \frac{\beta^{K-2}\bigl(1-\beta(1-m)^{1/K}\bigr)}{(1-m)^{1/K}\bigl(1-\beta(1-m)\bigr)}. \tag{31}$$

As $m \to 1$ from below, the left-hand side in (31) converges to 1 and the right-hand side converges to $+\infty$. Thus, for fixed K and b, there exists $\overline{m} < 1$ such that (31) holds for all $\overline{m} < m < 1$. (The open interval Λ is then defined by replacing the weak inequalities in (30) with strict inequalities.) □

Proof of Proposition 7

As in the proof of Proposition 4, we denote the left-hand side of (7) by $L(z)$ and the right-hand side of (7) by $U(z)$. Recall that a stationary z-storage equilibrium exists if and only if $L(z) \le \lambda \le U(z)$. The value $\min Z(\lambda, m, \beta)$ in Proposition 7 is hence the smallest integer z such that $L(z) \le \lambda \le U(z)$, and the value $\min Z(\lambda, m, \beta)$ is the largest integer z such that $L(z) \le \lambda \le U(z)$.

The comparative result with respect to the storage cost λ follows immediately from part (ii) in the proof of Proposition 4, establishing that $L(z{+}1) \le L(z)$ and $U(z{+}1) \le U(z)$. To establish the comparative result with respect to the aggregate supply m and the discount factor β, we show that $L(z)$ and $U(z)$ are strictly decreasing in m and strictly increasing in β for $z \in \{2, \ldots, K\}$. Note that $L(z) = \overline{\lambda}_{z+1}(p^*, p^*, \ldots)$ and $U(z) = \overline{\lambda}_z(p^*, p^*, \ldots)$, where $\overline{\lambda}_z(\cdot)$ is defined in (5) and $p^* = 1-(1-m)^{1/z}$. If m increases then p^* increases. Applying Lemma 2 (b), we see that $\overline{\lambda}_z(p^*, p^*, \ldots)$ and $\overline{\lambda}_{z+1}(p^*, p^*, \ldots)$ decrease for all $z \in \{2, \ldots, K\}$. Similarly, if β increases, Lemma 2 (b) implies that $\overline{\lambda}_z(p^*, p^*, \ldots)$ and $\overline{\lambda}_{z+1}(p^*, p^*, \ldots)$ increase. □

Proof of Lemma 8

Fix $z \in \{1, \ldots, K\}$ and $x^1 = (0, 1, 0, \ldots, 0)$, and let p_z^t be defined as in the statement of Lemma 8. From (13)–(15),

$$p_z^1 \;=\; \frac{m}{z} \;\leq\; \frac{m}{z - (m/z)(z-1)} \;=\; \frac{m}{(1-m/z)\,z + m/z} \;=\; p_z^2.$$

Below, we show that $x^* \succsim x^2$. Lemma 1 then implies that $p_z^1 \leq p_z^2 \leq p_z^3 \ldots$ and $p_z^t \to p^* = 1 - (1-m)^{1/z}$, and the result follows.

Note that $x_k^* = x_k^2 = 0$ for all $k > z$. For $k < z$, we have

$$\sum_{k'=0}^{k} x_{k'}^* \;\leq\; \sum_{k'=0}^{z-1} x_{k'}^* \;=\; (1-m)^{1/z} \quad \text{and} \quad \sum_{k'=0}^{k} x_{k'}^2 \;=\; 1 - \frac{m}{z}.$$

Thus, we need to show that

$$(1-m)^{1/z} \;\leq\; 1 - \frac{m}{z}. \tag{32}$$

If $m = 0$, then (32) holds as an equality. Therefore, it is sufficient to show that

$$\frac{\partial}{\partial m}\left[(1-m)^{1/z}\right] \;=\; -(1-m)^{1/z-1} \;\leq\; -\frac{1}{z} \;=\; \frac{\partial}{\partial m}\left[1 - \frac{m}{z}\right]. \tag{33}$$

The left-hand side of (33) decreases in m, and the right-hand side is independent of m. Therefore, it is sufficient that (33) holds at $m = 0$. This is satisfied, as $-1 \leq -1/z$. $\square$

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