Two-Sided Matching with Spatially Differentiated Agents*

Tilman Klumpp†
Department of Economics
Emory University

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Abstract

We consider the problem of assigning sellers and buyers into stable matches. The agents are located along a line and the match surplus function is decreasing in the distance between partners. We investigate the structure of stable assignments under both non-transferable utility (NTU) and transferable utility (TU). If the surplus function is sufficiently convex, the TU-stable assignments are a subset of the NTU-stable assignments. Furthermore, if trade is restricted to uni-directional flows the unique TU-stable assignment coincides with the unique NTU-stable assignment for every convex surplus function. We also examine the graph-theoretic representation of stable assignments and show that the graph structure can be exploited to compute surplus shares in TU-stable assignments.

Keywords: Spatial heterogeneity, bilateral exchange, two-sided matching, assignment game, stable marriage problem.

JEL classification: C65, D83, D85, L14.

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†Rich Building 316, 1602 Fishburne Dr., Atlanta, GA 30322. E-mail: `tklumpp@emory.edu`.

1 Introduction

In many economic environments, agents or goods are spatially differentiated: Preferences over trading partners or traded goods depend on their distance in some attribute space. In models of spatial competition, for example, this attribute is geographical location—a buyer's value for the good of a seller depends on the physical proximity to the seller's store. In this paper, we examine the role of spatial heterogeneity in two-sided matching markets. That is, each agent is interested in a "close match." In job markets, for example, a worker is more productive in jobs he has specialized training for and less so in others. In marriage markets, men and women are interested in finding mates who share their interests.

We assume that sellers and buyers of a good are located at various integer points along the line. We use the seller-buyer terminology by default, and one may equally well think of job seekers and employers, or of women and men. Sellers supply one unit each of an indivisible good, and buyers demand one unit of the good. If a pair of agents exchanges the good, match surplus is generated which decreases in the distance between the seller and the buyer. Our aim to characterize the structure of stable matchings which arise under both non-transferable utility (NTU) and transferable utility (TU). We assume that with non-transferable utility surpluses are split by allocating a fixed fraction of it to the seller, and the residual to the buyer. Under transferable utility, buyers and sellers can agree on any feasible surplus share. The usual stability definitions apply to each case.

We first show that NTU-stable assignments can be found by applying a simple recursive algorithm.[1] We then go on to characterize the NTU-stable assignment as collections of *nested clusters*. Roughly speaking, this has the interpretation that the population is segregated into spatially separated communities within (but not across) which matching occurs; this property is called *clustering*. Furthermore, within a community there will be no "overlapping matches;" this property is called *nesting*.[2] We then show that TU-stable assignments in general will possess the first property, but not the second one, unless the match surplus function is sufficiently convex in the distance between the agents. In this case, the TU-stable assignments are a subset of the NTU-stable assignments. We also examine the model when trade is uni-directional, that is, the seller in a given match cannot be located to the right of the buyer (or equivalently, to the left). For example, such a friction can arise in an intertemporal setting where goods can be consumed only after they have been produced. We show that in this case a unique NTU-stable assignment exists

[1] This algorithm is similar to the one Alcalde (1995) uses in a one-sided setting, as well as to the ones developed in Eeckhout (2000) and Clark (2003, 2006). Unlike these algorithms, however, ours may yield multiple NTU-stable assignments because preferences need not be strict. We discuss this issue in Section 3.1 and Section 4.4.

[2] This means that it cannot be the case that exactly one party of a match is located between the two parties of another match. A precise definition of nested clusters is given in Section 3.2.

and coincides with the unique TU-stable assignment for every convex surplus function. This equivalence result provides a condition under which one can compute the TU-stable assignment without having to compute surplus shares at the same time. The task of computing surplus shares can then be addressed separately, knowing what the stable matches are. We demonstrate this approach by developing an algorithm which computes TU-stable surplus shares based on the stable matching that arises in the NTU case with uni-directional trade. This is done by showing that the nested structure of the assignment has a graph-theoretic representation as a forest; the algorithm then iterates over the nodes of each tree in the forest.

Our model is a version of the stable marriage problem examined first by Gale and Shapley (1962) for the NTU case, and by Shapley and Shubik (1972) for the TU case. In the general setup of the stable marriage problem, no specific assumptions are made with regard to preferences or the match surplus function. Despite this generality, we know that stable matchings exist, how to find them, and that they possess a lattice structure (a survey of these and other results can be found in Roth and Sotomayor (1990)). In many applications, however, some additional structure can be put on either preferences or surpluses in order to obtain a finer characterization of equilibrium outcomes. Below we review some of these assumptions and relate them to the ones made in this paper.

A widely used assumption is *vertical heterogeneity*. In the case of non-transferable utility, this means that there exists a common ordering among both sets of agents, such that each agent prefers partners which are ranked higher in this ordering to those ranked lower. With strict preferences, there exists a unique NTU-stable assignment, which is positive assortative (e.g., richer men marry richer women). Positive sorting continues to hold if utility is transferable and the match surplus is supermodular in the two agents' vertical characteristics (Becker (1973); see Legros and Newman (2002) for a weaker condition), or when search frictions are introduced (Shimer and Smith (2000)). With *horizontal heterogeneity*, on the other hand, each agent has a subjective ranking over potential match partners. However, there is no unambiguous definition of the term in the literature. Eeckhout (2000) defines it through a "mutually ideal matches" condition which says that each individual is the ideal partner of her own ideal partner. He shows this to be part of a larger class of preferences (that includes vertical heterogeneity) for which a unique NTU-stable assignment exist provided all preferences are strict (see Clark (2006) for a weaker condition). Note that horizontal heterogeneity, thus defined, puts no restrictions on the ranking of partners who are not an agent's top choice. In this paper, we use the term *spatial heterogeneity* for an alternative preference structure in which each agent can be characterized by her location on a line, and utility or match surplus decreases in the distance between match partners. Thus, we make an additional assumption on preferences over alternatives that are not top ranked (we assume a single-peaked profile), but we discard the assumption of mutually ideal matches.[3] Clark (2003), on the other hand, uses the term horizontal heterogeneity for precisely this class of preferences. He shows that with non-transferable utility positive assortative matching need not occur, in contrast to models with vertical heterogeneity. Instead, the nature of sorting depends on the relative location of the agents. Some negative sorting will occur if some individuals have to settle for distant partners because closer partners are unavailable, as they themselves are in even closer matches. The same effect occurs in our model and generates the nested structure of NTU-stable assignments. As we show in this paper, it is then sufficiently strong convexity of the match surplus in the distance between match partners—instead of supermodularity in the locations of the agents—which forces this structure to carry over to the case of transferable utility.[4]

The rest of the paper is organized as follows. In Section 2 we present the basic matching framework and our stability definitions. In Section 3 we examine stable assignments in the NTU case, and in Section 4 we treat the TU case. An application to the computation of surplus shares is presented in Section 5, and concluding remarks are in Section 6. All but very short proofs are contained in the Appendix.

2 The Matching Model

We formulate a model of bilateral exchange between sellers on the one side and buyers on the other. This terminology may suggest that a physical good is transferred between the agents. However the model applies equally to bilateral exchange between workers and firms (the good is a labor service) or business consultants and clients (the good is advice). More generally, the model applies to any matching environment in which the input of two distinguishable sides of a market are needed in order for a successful match to form. The obvious example here is the marriage market.

2.1 The market

Let the set of integers be $\mathbb{Z}$, and let $\mathcal{S} \subseteq \mathbb{Z}$ and $\mathcal{B} \subseteq \mathbb{Z}$. Each $i \in \mathcal{S}$ represents the location of one seller and each $j \in \mathcal{B}$ represents the location of one buyer. These sets will remain fixed throughout the paper.[5]

If seller i's good is transferred to buyer j, we say that i and j have *matched*. Note that

[3] Horizontal heterogeneity is therefore neither sufficient nor necessary for spatial heterogeneity. However, for spaces with more than one dimension this can change; see our discussion of this issue in Section 6.

[4] Note that supermodularity in locations would be equivalent to concavity in distance. However, convexity in distance is not the same as submodularity in locations.

[5] For a discussion of the assumption that a location must be an integer see Section 6.

$(i,j) \neq (j,i)$ unless $i = j$. Given any match (i,j) we use the notation

$$\lfloor i,j \rfloor \equiv \min\{i,j\}, \quad \lceil i,j \rceil \equiv \max\{i,j\}, \quad |i,j| \equiv \lceil i,j \rceil - \lfloor i,j \rfloor.$$

The latter value is called the *match distance*.[6]

An agent who is not matched with another agent receives a zero payoff. If seller i and buyer j match, the surplus generated from the match is denoted $v(i,j)$. We assume that $v(i,j)$ depends on the match distance $|i,j|$, that is

$$v(i,j) \;=\; u(|i,j|)$$

for some strictly decreasing $u : \{0,1,2,\ldots\} \to \mathbb{R}$. We further assume that there exists $0 \leq \hat{d} < \infty$ such that $u(d) > 0$ if and only if $d \leq \hat{d}$.[7] Define

$$D \equiv \{(i,j) \in \mathcal{S} \times \mathcal{B} : |i,j| \leq \hat{d}\}.$$

If $(i,j) \in D$, then i and j can generate a positive surplus if they match. For $d = 0, \ldots, \hat{d}$ we also define the sets

$$D(d) \equiv \{(i,j) \in D : |i,j| = d\}.$$

A subset $M \subseteq D$ is called an *assignment* if $(i,j) \in M$ implies $(i,k) \notin M$ for all $k \neq j$ and $(k',j) \notin M$ for all $k' \neq i$ (i.e., no agent can be in more than one match).

Given an assignment M, denote by $A_S(M) = \{i \in \mathcal{S} : \exists j \in \mathcal{B} \text{ s.t. } (i,j) \in M\}$ the set of matched sellers, by $A_B(M) = \{j \in \mathcal{B} : \exists i \in \mathcal{S} \text{ s.t. } (i,j) \in M\}$ the set of matched buyers, by $U_S(M) = \mathcal{S} \backslash A_S(M)$ the set of unmatched sellers, and by $U_B(M) = \mathcal{B} \backslash A_B(M)$ the set of unmatched buyers.

2.2 Non-transferable utility

A fixed sharing rule is an instance of non-transferable utility (NTU), where match partners obtain fixed shares of $v(i,j)$ and neither of them can transfer any fraction of that share to the other party. Under fixed sharing rule $\alpha \in (0,1)$, the seller i obtains $\alpha v(i,j)$ and buyer j obtains $(1-\alpha)v(i,j)$, where α is an exogenously given parameter. The value that seller $i \in \mathcal{S}$ obtains in M is then given by

$$V_S(i|M) \;=\; \begin{cases} \alpha v(i,j) & \text{if } i \in A_S(M), \\ 0 & \text{if } i \in U_S(M). \end{cases}$$

[6] $|i,j|$ is obviously the same as $|i-j|$, where $|\cdot|$ denotes absolute value. We use notation $|i,j|$ only because of its more compact appearance.

[7] An interpretation of this assumption is that the gross surplus of a match goes to zero as the match distance increases, and there is a transaction cost from matching which is independent of the match distance. Thus a match generates positive net surplus if and only if the match distance is not too large.

The value of buyer j, $V_B(j|M)$, is defined similarly. If there exists $(i, j) \in D$ such that

$$v(i, j) \ > \ \max\{V_S(i|M)/\alpha, V_B(j|M)/(1-\alpha)\},$$

we say that (i, j) *blocks* M. That is, if i and j were to match with each other instead of their assigned partners in M, and allocate a fraction α of the surplus to i and the rest to j, then both would be made better off. We can now define stability of an assignment under non-transferable utility:

Definition 1. An assignment M is *NTU-stable* if M is not blocked by any $(i, j) \in D$.

2.3 Transferable utility

Suppose now that match surpluses can be split between trading partners in any way they agree on. This case is called transferable utility (TU). With TU, if an agent can leave a match and trade with someone else, she can use this possibility as a "threat" to obtain a larger fraction of the match surplus. Conversely, an agent may offer her trading partner a larger fraction of the match surplus to induce the partner to stay in the match and not trade with another agent. Given an assignment M, let $V : M \to \mathbb{R}^2$ be a non-negative function that assigns to each match $(i, j) \in M$ a pair of surplus shares $(V_S(i), V_B(j))$ such that $V_S(i) + V_B(j) = v(i, j)$. V is called a *value assignment* for M, and the pair (M, V) is called a *match-value assignment*. The value that seller $i \in \mathcal{S}$ obtains in (M, V) can then be written as

$$V_S(i|M, V) \ = \ \begin{cases} V_S(i) & \text{if } i \in A_S(M), \\ 0 & \text{if } i \in U_S(M). \end{cases}$$

Buyer j's value, $V_B(j|M, V)$, is defined similarly. We will often drop the qualifiers M, V from the value notations, but this will not cause confusion. If there exists $(i, j) \in D$ such that

$$v(i, j) \ > \ V_S(i|M, V) + V_B(j|M, V),$$

we say that (i, j) *blocks* (M, V). Stability under transferable utility is now defined as follows:

Definition 2. A match-value assignment (M, V) is *TU-stable* if (M, V) is not blocked by any $(i, j) \in D$.

We also say that the assignment M is TU-stable if there exists V such that the match-value assignment (M, V) is TU-stable. In this case V *supports* M. Note that for a given TU-stable M, the supporting surplus shares are typically not unique. That is, (M, V) and (M, V') can be two TU-stable match-value assignments with $V \neq V'$.

3 Stable Assignments under Non-Transferable Utility

3.1 The inside-out algorithm

In this section we derive an algorithm to find the NTU-stable assignments. Consider the set of potential matches D. If $(i,j) \in D$, then i and j have coincidence of wants and could trade with each other. Recall that all agents prefer matches of shorter distance to matches of longer distance. Thus, both i and j may decline the transaction if a closer match partner is available who agrees to enter into a match. For this to be the case, the third agent must not have an even closer potential match partner available herself. For instance, buyer j would decline to obtain the good from seller i if there exists another seller k such that $|j,k| < |i,k|$, *and* if there does not exist a buyer l such that $|k,l| < |k,j|$ *and* l does not herself decline the transaction with k. From these arguments one can see that NTU-stable assignments can be computed recursively. To do so, consider a family φ of $\hat{d}+1$ functions

$$\begin{aligned}\varphi_0 &: D(0) \to \{0,1\}, \\ &\vdots \\ \varphi_{\hat{d}} &: D(\hat{d}) \to \{0,1\},\end{aligned}$$

where $\varphi_d(i,j) = 1$ means that match $(i,j) \in D(d)$ "clears the market" and will be included in the assignment. To construct φ we will work "from the inside out" and begin with φ_0; then we build φ_1, and so on until $\varphi_{\hat{d}}$. The set of all clearing matches will then constitute an NTU-stable assignment.

Consider the closest possible matches first. Observe that if $(i,j) \in D(0)$ (i.e. $i = j$), then these agents must trade since $v(i,j) = u(0) > 0$ is the maximal possible match value. That is, for any $\alpha \in (0,1)$ and $d > 0$, $\alpha u(0) > \alpha u(d)$ and $(1-\alpha)u(0) > (1-\alpha)u(d)$. Therefore define $\varphi_0 : D(0) \to \{0,1\}$ as follows:

$$\varphi_0(i,j) = 1. \tag{1}$$

Whether a pair $(i,j) \in D(d)$ clears for $d > 0$ is then determined recursively. Fix $d > 0$ and suppose $\varphi_{d'}(i,j)$ has been defined for all $d' < d$. Now let $\varphi_d : D(d) \to \{0,1\}$ satisfy the following:

$$\begin{aligned}\varphi_d(i,j) = 1 \quad \Leftrightarrow \quad \Big[&\varphi_{d'}(i,k) = \varphi_{d'}(k',j) = 0 \;\; \forall (i,k),(k',j) \in D(d') \;\; \forall d' < d \\ &\text{and} \\ &\varphi_d(i,k) = \varphi_d(k',j) = 0 \;\; \forall (i,k),(k',j) \in D(d) \;\; \text{s.t.} \;\; k \neq j, \; k' \neq i\Big].\end{aligned} \tag{2}$$

That is, the pair $(i,j) \in D(d)$ clears if and only if it is not blocked by a match of distance $d-1$ or less and neither i nor j are in another match of distance d. Note that there can be several functions φ_d which satisfy (2).

The set of all clearing matches, thus defined, constitutes an NTU-stable assignment. That is, M is an NTU-stable assignment if and only if there exists a family of functions $\varphi = (\varphi_d : D(d) \to \{0,1\})_{d=0,\ldots,\hat{d}}$, satisfying (1)–(2), such that

$$M \;=\; \{(i,j) \in D : \varphi_{|i,j|}(i,j) = 1\}.$$

We denote by $\mathcal{M}^{NTU}$ be the set of all NTU-stable assignments.

Observe that M does not depend on the particular value of α that is used to split the surplus between a seller and a buyer, and also not on the shape of u (except for monotonicity). Nonetheless, TU-stable assignments are typically not unique, as the following example shows:

Example 1. Suppose $\mathcal{S} = \{2\}$ and $\mathcal{B} = \{1,3\}$. Then $M = \{(2,1)\}$ and $M' = \{(2,3)\}$ are both NTU-stable assignments.

The multiplicity of NTU-stable assignments owes to the fact that an agent is indifferent between matching with a partner d positions to her left or d positions to her right. Technically, the possibility of indifference is what necessitates the inclusion of the second requirement on the right-hand side of (2). Had we ruled out indifference by assumption, then this requirement would be unnecessary; in fact, the inside-out algorithm would then essentially be the same as the constructions in Eeckhout (2000) and Clark (2006). In these papers, a unique NTU-stable assignment is shown to exist, but this requires strict preferences.[8]

3.2 The structure of NTU-stable assignments

Let $(i,j),(i',j') \in D$ be such that $i \neq i'$ and $j \neq j'$. We say that the matches (i,j) and (i',j') are *nested* if

$$\lfloor i,j \rfloor < \lfloor i',j' \rfloor \leq \lceil i',j' \rceil < \lceil i,j \rceil \quad \text{or} \quad \lfloor i',j' \rfloor < \lfloor i,j \rfloor \leq \lceil i,j \rceil < \lceil i',j' \rceil$$

(in the first case (i',j') is nested in (i,j), and vice versa in the second case). We say that (i,j) and (i',j') are *side-by-side* if

$$\lceil i,j \rceil < \lfloor i',j' \rfloor \quad \text{or} \quad \lfloor i,j \rfloor > \lceil i',j' \rceil$$

(in the first case (i',j') is to the right of (i,j), and vice versa in the second case). Finally, we say that (i,j) and (i',j') are *overlapping* if

$$\lfloor i,j \rfloor \leq \lfloor i',j' \rfloor \leq \lceil i,j \rceil \leq \lceil i',j' \rceil \quad \text{or} \quad \lfloor i',j' \rfloor \leq \lfloor i,j \rfloor \leq \lceil i',j' \rceil \leq \lceil i,j \rceil.$$

[8] Clark's (2006) no-crossing condition can be shown to hold in our model; however, the no-crossing condition alone is sufficient for a unique NTU-stable assignment only in conjunction with the outer assumption that preferences are always strict. Further, had we ruled out indifference, our setup would also satisfy Eeckhout's condition (1).

For example, consider the assignment depicted in Figure 1. The match $(4, 2)$ is nested in $(1, 6)$, the match $(1, 6)$ is to the left of $(9, 8)$, and the matches $(12, 15)$ and $(14, 16)$ overlap. Note that any pair of matches in an assignment satisfies exactly one of these properties.

A *cluster* $C \subseteq M$ is a maximal set of matches such that each pair of elements of C is either nested or overlapping, or nested or overlapping with a common third match in C. A cluster C is *nested* if it does not contain overlapping matches. The *length* of a cluster C is defined as

$$\delta(C) = \max_{(i,j),(i',j') \in C} \left\{ \lceil i, j \rceil - \lfloor i', j' \rfloor \right\}.$$

In Figure 1 there are three clusters, indicated by the shaded areas. The lengths of the clusters are 5, 1, and 4 respectively. The left and middle clusters are nested, but the right cluster is not.

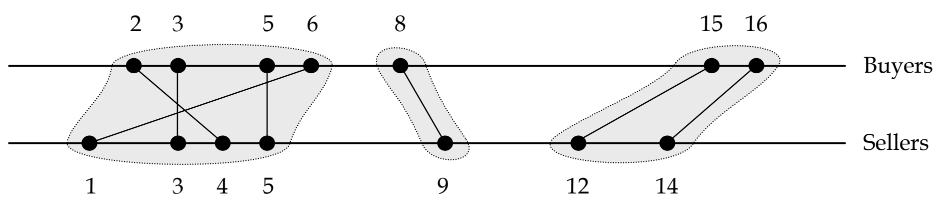

Figure 1: An assignment

One may think of a cluster as a community of agents, located along a street the length of the cluster. These communities are segregated from each other, and no streets are necessary between clusters as no trade takes place between them. The following result characterizes an NTU-stable assignment M in terms of its clusters.

Theorem 1. *Let $M \in \mathcal{M}^{NTU}$ be an NTU-stable assignment, and suppose $\hat{d} < \infty$. M can be partitioned into a family $\mathcal{C}$ of nested clusters of length $\hat{d}$ or less.*

Proof. Let $M \in \mathcal{M}^{NTU}$ and recall that $(i, j) \in M \subset D$ only if $|i, j| \leq \hat{d}$. To prove the result, then, we only need to show that M does not contain overlapping matches. Suppose $(i, j) \in M$ and $(i', j') \in M$ are overlapping; without loss of generality let $i < i' \leq j < j'$ (the other cases are similar). Then $v(i', j) = u(|i', j|) > u(|i, j|) = v(i, j)$ and $v(i', j) = u(|i', j|) > u(|i', j'|) = v(i', j')$; thus $(i', j) \in D$ blocks M, a contradiction. $\square$

4 Stable Assignments under Transferable Utility

4.1 The structure of TU-stable assignments

In this section we characterize the stable assignments under transferable utility. We denote by $\mathcal{M}^{TU}$ the set of all assignments that are TU-stable (that is, for each $M \in \mathcal{M}^{TU}$ there exists a value assignment V that supports M).

In general, TU-stable assignments differ from NTU-stable assignments. We begin with an example to demonstrate this:

Example 2. Let $u(1) = 1$, $u(2) = .9$, $u(3) = .7$. Suppose $\mathcal{S} = \{1, 2\}$ and $\mathcal{B} = \{3, 4\}$. In this case $M = \{(2, 3), (1, 4)\}$ is the unique NTU-stable assignment. However, the following is easily verified to be a TU-stable match-value assignment:

$$M' = \{(1, 3), (2, 4)\},\ V'(1, 3) = (.36, .54),\ V'(2, 4) = (.54, .36).$$

Furthermore, it is impossible to find a TU-stable (M'', V'') such that $M'' = M$: It would need to be the case that $V''_S(1) + V''_B(4) \geq u(2) = 0.9$ and $V''_S(2) + V''_B(3) \geq u(2) = 0.9$, but $[V''_S(1) + V''_B(4)] + [V''_S(2) + V''_B(3)] = u(1) + u(3) = 1.7$.

We see from Example 2 that under transferable utility the nested structure that arises with non-transferable utility can be lost. That is, TU-stable assignments may contain overlapping matches. We can nevertheless prove the following:

Theorem 2. *Let $M \in \mathcal{M}^{TU}$ be a TU-stable assignment. There exists $\eta < \infty$, independent on $\mathcal{S}$ and $\mathcal{B}$, such that M can be partitioned into a family $\mathcal{C}$ of (not necessarily nested) clusters of length η or less.*

Theorem 2 is not as straightforward to prove as the corresponding result for the NTU case (Theorem 1). To see why, recall that TU-stable assignments can contain overlapping matches, as demonstrated in Example 2. While each individual match must be of distance $\hat{d}$ or less, in principle there can be chains of overlapping matches. Theorem 2 shows that such chains must be of bounded length.

4.2 Efficiency

We say that a match-value assignment (M, V) is *efficient* if there does not exist another match-value assignment (M', V') such that $V_S(i|M', V') \geq V_S(i|M, V)\ \forall i \in \mathcal{S}$ and $V_B(j|M', V') \geq V_B(j|M, V)\ \forall j \in \mathcal{B}$, with at least one inequality strict. Note that if (M, V) is efficient, so is (M, V') for any strictly positive V'. Thus, efficiency is essentially a property of the assignment M alone. For this reason we say that an assignment M is efficient if (M, V) is efficient for some V.

Example 2 also shows that the matching outcomes under fixed sharing rules may not be efficient under transferable utility. If $\alpha = 1/2$, for example, then agents 1 and 4 obtain values $V_S(1|M) = V_B(4|M) = .35$, while agents 2 and 3 obtain $V_S(2|M) = V_B(3|M) = .5$. These values are clearly dominated by the (feasible) values V'. On the other hand, V' cannot be improved upon. This will be the case for any TU-stable (M, V):

Theorem 3. *Every TU-stable match-value assignment is efficient.*

Proof. Suppose that (M, V) is TU-stable but not efficient. Then there exists another assignment (M', V') s.t. $V_S(i|M', V') \geq V_S(i|M, V)$ $\forall i \in \mathcal{S}$ and $V_B(j|M', V') \geq V_B(j|M, V)$ $\forall j \in \mathcal{B}$, with at least one inequality strict. Take any match $(i, j) \in M'$ for which at least one agent obtains a strictly higher value in (M', V') than in (M, V). Then $v(i, j) = V_S(i|M', V') + V_B(j|M', V') > V_S(i|M, V) + V_B(j|M, V)$, so (i, j) blocks (M, v), a contradiction. □

The converse of Theorem 3 is not true, however. There are efficient assignments which are not TU-stable:

Example 3. Suppose $\mathcal{S} = \{1, 2\}$ and $\mathcal{B} = \{2\}$. Then $M = \{(1, 2)\}$ is efficient, as (M, V) is efficient for every V such that $V_S(1) > 0$. However, (M, V) is not TU-stable for any V (the blocking match is $(2, 2)$).

4.3 Relationship to the NTU case

In this section and the next, we derive sufficient conditions under which the TU-stable assignments have the same nested structure as NTU-stable assignments. We make the following definitions. The surplus function u is *convex* if

$$\forall d \geq 0 : \quad u(d) - u(d + 1) \; > \; u(d + 1) - u(d + 2).$$

That is, not only does the match surplus increase when match partners move closer together, but the incremental surplus from reducing the distance between match partners increases as well. We say that u is *strongly convex* if

$$\forall d \geq 0 : \quad u(d) - u(d + 1) \; > \; u(d + 1).$$

In this case, the match surplus more than doubles every time the distance between the partners is reduced by one unit.

Our next result applies to the case of strongly convex surpluses:

Theorem 4. *If the match surplus function u is strongly convex, then $\mathcal{M}^{TU} \subseteq \mathcal{M}^{NTU}$.*

The inclusion in Theorem 4 is generally strict: There are NTU-stable assignments which are not TU-stable. However, for finite $\mathcal{B}$ and $\mathcal{S}$ the result of Shapley and Shubik (1972) implies that the TU-stable assignments are those in which the sum of surpluses are maximized. Thus, if $\mathcal{M}^{NTU}$ is not overly large the set $\mathcal{M}^{TU}$ can be easily found by comparing aggregate surpluses over the assignments in $\mathcal{M}^{NTU}$. The following example demonstrate this observation.

Example 4. Suppose $\mathcal{S} = \{1, 3\}$ and $\mathcal{B} = \{2, 4\}$. For every strongly convex u there are two NTU-stable assignments, $M = \{(1, 2), (3, 4)\}$ and $M' = \{(3, 2), (1, 4)\}$. M is the only TU-stable assignment, and it is easy to verify that M' has a lower aggregate surplus than M.

Furthermore, if u is convex but not strongly convex, it is possible that $\mathcal{M}^{TU} \not\subseteq \mathcal{M}^{NTU}$:

Example 5. Suppose $\mathcal{S} = \{1, 4\}$ and $\mathcal{B} = \{3, 6\}$. For every decreasing u the unique NTU-stable assignment is $M = \{(4, 3), (1, 6)\}$. Now suppose $u(d) = (2/3)^d - \varepsilon$ (for some small $\varepsilon > 0$), which is convex but not strongly convex. Then the unique TU-stable assignment is $M' = \{(1, 3), (4, 6)\}$.

4.4 Uni-directional trade

Suppose now that we restrict the direction of trade flows that are allowed to take place. Assume that (i, j) can match only if $j \geq i$ (only left-to-right trade is permitted), or if $j \leq i$ (only right-to-left trade is permitted). To introduce uni-directional trade in the model, all we must do is change the distance function $|\cdot|$ to one the following:

$$|\overrightarrow{i,j}| \equiv \begin{cases} j - i & \text{if } j \geq i, \\ \infty & \text{otherwise,} \end{cases} \qquad |\overleftarrow{i,j}| \equiv \begin{cases} i - j & \text{if } i \geq j, \\ \infty & \text{otherwise.} \end{cases}$$

Uni-directional trade is a strong friction which can arise in certain contexts. Consider, for example, the intertemporal trade of a storable but depreciable good. Suppose sellers produce goods only on certain dates, and buyers require these goods only on certain dates. The location of a seller (resp. buyer) indicates the time at which she produces (resp. consumes) the good. Assume also that at each point in time, future supply and demand dates are known the agents.[9] At the time a buyer requires the good, she prefers the freshest unit available—her ideal good would be one that is produced "just-in-time." On the other hand, the right to a yet to be produced unit has no value, giving rise to a uni-directional trade setup.[10]

The following result shows that with uni-directional trade, convexity of the surplus function is sufficient for equivalence of TU-stability and NTU-stability:

Theorem 5. *Suppose trade is uni-directional. Then $\mathcal{M}^{NTU}$ is a singleton, and if u is convex then $\mathcal{M}^{TU} = \mathcal{M}^{NTU}$.*

Observe that uni-directional trade forces strict preferences. The matching market now satisfies the aforementioned uniqueness criteria of Eeckhout (2000) and Clark (2006), so that $\mathcal{M}^{NTU}$ is a singleton accordingly.[11] We further remark that if trade is uni-directional but u fails to be convex, the TU-stable assignment may not coincide with the NTU-stable assignment (see Example 2, which can be applied to this case).

[9] If this was not the case, then negotiations over the terms of trade in the TU-case would have to take place in ignorance of future trading possibilities, and we would need a different definition of TU-stability.

[10] Models of money as a medium for exchange use similar one-way street assumptions; e.g. Townsend (1980), Kiyotaki and Wright (1989).

[11] In addition, preferences now satisfy Alcalde's (1995) α-reducibility criterion for uniqueness. In our two-sided context, this means that for every subset of sellers $\mathcal{S}' \subseteq \mathcal{S}$ and buyers $\mathcal{B}' \subseteq \mathcal{B}$, there is a pair $(i, j) \in \mathcal{S}' \times \mathcal{B}'$ such that $v(i, j) > v(i, j')$ for all $j' \in \mathcal{B}' \backslash \{j\}$ and $v(i, j) > v(i', j)$ for all $i' \in \mathcal{S}' \backslash \{i\}$.

5 Application: Computation of Stable Surplus Shares

We now demonstrate a potentially useful application of our results. Consider the task of finding a TU-stable match-value assignment. In principle, the assignment M and the surplus shares V must be found simultaneously. This can be done, for example, by using auctioning methods based on the Hungarian algorithm, (Demange et al., 1986) or through dynamic market processes (Crawford and Knoer, 1981). For the case of uni-directional trade and convex match surpluses (i.e., the case treated in Theorem 5), we show that it is also possible to use the clustering structure of stable assignments to achieve the same. That is, we first use the inside-out algorithm to find M. Then we use the cluster structure of M to compute V, by means of an "outside-in algorithm" described further below.

5.1 Tree structure of nested clusters

Without loss of generality, we shall assume that trade is left-to-right. Let M be a TU-stable assignment. If the assumptions of Theorem 5 are satisfied then M can be partitioned into a family of nested clusters $\mathcal{C}$. The family $\mathcal{C}$ can be found easily by the inside-out algorithm described in Section 3.1. Each $C \in \mathcal{C}$ has a graph-theoretic representation as a (directed) tree: Every match $(i, j) \in C$ is a node, and there is an edge from (i, j) to $(i', j') \in C$ if and only if the following holds: (i) $i < i' \leq j' < j$, and (ii) there does not exist $(i'', j'') \in C$ such that $i < i'' < i' \leq j' < j''$ (see Figure 2). For each non-terminal node (i, j), we let $S(i, j)$ be the set of successor nodes of (i, j). For all nodes (i, j) other than the initial node, we let $p(i, j)$ be the parent node of (i, j). Because each $C \in \mathcal{C}$ is a tree, M is a forest.[12] Figure 2 shows the representation of a nested cluster as a tree.

5.2 The outside-in algorithm

The outside-in algorithm is based on the following idea. In any given neighborhood of a seller-buyer pair $(i, j) \in M$ there may be more alternative buyers close to seller i than there are alternative sellers close to buyer j (or vice versa). Loosely speaking, there may exist either local excess demand or local excess supply of the traded good. The existence of such local excess supply or demand shifts surplus from the "disadvantaged side" to the "advantaged side." The algorithm assigns values to i and j based on the relative bargaining power of these agents. As we will show, when computing the relative bargaining power of i versus j it is sufficient to examine only one alternative seller for j and one alternative buyer for i, and these will be the agents in the parent node $p(i, j)$.

Consider now a single nested cluster $C \in \mathcal{C}$. We will assign values V^C to the matches in C. This will be done recursively, starting with the longest match in C (i.e., the initial

[12] Note that this is true not only for uni-directional trade, but in general provided M does not contain overlapping matches.

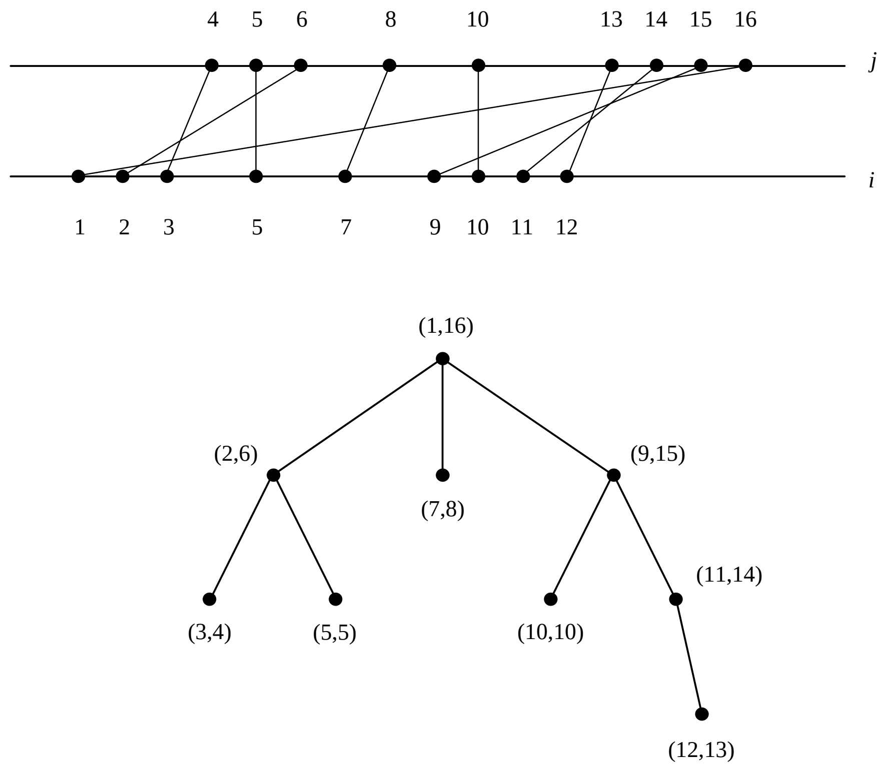

Figure 2: Tree structure of a nested cluster

node in the tree representing C) and working inwards (i.e., along the paths of successor nodes). The resulting pair (C, V^C) is called a *local* match-value assignment. The global match-value assignment (M, V) is then defined as the union of the local assignments over all nested clusters $C \in \mathcal{C}$:

$$(M, V) = \bigcup_{C \in \mathcal{C}} (C, V^C)$$

Given $C \in \mathcal{C}$, let (i_0, j_0) be the initial node in the tree representing C. For $(i, j) \in C$ such that $(i, j) \neq (i_0, j_0)$, suppose that values have already been assigned to the match $(i', j') = p(i, j)$. Then we assign values to (i, j) as follows. First, define

$$w_S(i) = u(j' - i) - V_B^C(j'), \quad w_B(i) = u(j - i') - V_S^C(i'), \tag{3}$$

and set $\Delta(i, j) = w_S(i) - w_B(j)$. The difference $\Delta(i, j)$ is a measure of the "relative bargaining power" in the match (i, j). Then the values $V^C(i, j)$ are given by

$$V_S^C(i) = \frac{1}{2}\left(u(j - i) + \Delta(i, j)\right), \quad V_B^C(j) = \frac{1}{2}\left(u(j - i) - \Delta(i, j)\right). \tag{4}$$

With the exception of (i_0, j_0), this procedure can be used iteratively to assign values to all matches in C by working "from the outside in" (i.e., from the successor nodes of (i_0, j_0) toward the terminal nodes).

What is left to do is to assign starting values to (i_0, j_0). This will be done in almost the same fashion as described above. Define the *pseudo-parent* of (i_0, j_0), (i_0^*, j_0^*), as follows. Let $i_0^* = \max\{i \in U_S(M) : i < i_0\}$ if such an agent exists; otherwise set $i_0^* = -\infty$. Similarly let $j_0^* = \min\{j \in U_B(M) : j > j_0\}$ if this agent exists; otherwise set $j_0^* = \infty$. Define

$$w_S(i_0) = \max\{u(j_0^* - i_0), 0\}, \quad w_B(j_0) = \max\{u(j_0 - i_0^*), 0\} \tag{5}$$

and set $\Delta(i_0, j_0) = w_S(i_0) - w_B(j_0)$. Now $V^C(i_0, j_0)$ is given by

$$V_S^C(i_0) = \frac{1}{2}\left(u(j_0 - i_0) + \Delta(i_0, j_0)\right), \quad V_B^C(j_0) = \frac{1}{2}\left(u(j_0 - i_0) - \Delta(i_0, j_0)\right). \tag{6}$$

It is easily seen that for all $(i, j) \in C$, $V^C(i, j)$ is a well-defined set of values satisfying $V_S^C(i) > 0$, $V_B^C(j) > 0$, and $V_S^C(i) + V_B^C(j) = v(i, j)$.

5.3 Stability

Note again that the outside-in algorithm is simple in that the *only* alternative matches considered when assigning values to (i, j) are those that involve an agent belonging to the parent node (or, if (i, j) is the initial node of a cluster, the pseudo-parent). Clearly there are more alternative matches that could potentially block (M, V), and to be stable the match-value assignment must not be blocked by any of these additional matches either. As we will show, this is indeed satisfied:

Theorem 6. *If u is convex and trade is uni-directional, then the match-value assignment (M, V) generated by the outside-in algorithm is TU-stable.*

To prove Theorem 6, we start with the following definition of pairwise stability:

Definition 3. Given a match-value assignment (M, V), $(i, j) \in M$ and $(i', j') \in M$ are *pairwise stable* if $V_S(i) + V_B(j') \geq v(i, j')$ and $V_S(i') + V_B(j) \geq v(i', j)$.

It is easily seen that for a match-value assignment (M, V) to be TU-stable per Definition 2, it is necessary that all $(i, j) \in M$ and $(i', j') \in M$ be pairwise stable. Furthermore, TU-stability is equivalent to the following: (i) pairwise stability of all matches in M, (ii) the requirement that (M, V) not be blocked by any (i, j) where at least one of i and j is unmatched in M, and (iii) the requirement that all matched agents obtain a non-negative value. Note that (iii) is clearly satisfied by the outside-in algorithm, so we need to show that the values generated by the outside-in algorithm satisfy (i) and (ii). This is done in the following two steps.

First, we show that each local match-value assignment (C, V^C) is *internally stable*, in the sense that any $(i, j) \in C$ and $(i', j') \in C$ are pairwise stable. To this end, we use the following result (the proof is in the Appendix):

Lemma 1. *Suppose trade is uni-directional and* u *is convex, and let* (M, V) *be a TU-stable match-value assignment. Then the following holds:*

(a) If $(i, j) \in M$ *and* $(i', j') \in M$ *are pairwise stable side-by-side matches,* $(i'', j'') \in M$ *is nested in* (i', j')*, and* (i', j') *and* (i'', j'') *are pairwise stable, then* (i, j) *and* (i'', j'') *are pairwise stable.*

Furthermore, let $C \in \mathcal{C}$ *be a nested cluster. For the local match-value assignment* (C, V^C) *generated by the outside-in algorithm, the following properties hold:*

(b) Suppose $(i', j') \in C$ *is nested in* $(i, j) \in C$*. Then* (i, j) *and* (i', j') *are pairwise stable.*

(c) Suppose $(i, j) \in C$ *and* $(i', j') \in C$ *are such that* $p(i, j) = p(i', j')$*. Then* (i, j) *and* (i', j') *are pairwise stable.*

Lemma 1 implies internal stability of (C, V^C), as any two matches in C can be related to each other by applying a combination of properties (a), (b), and (c) of Lemma 1.[13] Second, we prove the *external stability* of the match-value assignment generated by the outside-in algorithm, by establishing the following result (proven in the Appendix):

Lemma 2. *Suppose trade is uni-directional and* u *is convex, and let* (M, V) *be the TU-stable match-assignment generated by the outside-in algorithm. Then the following properties hold:*

(a) (M, V) *is not blocked by a potential match involving at least on agent who is unmatched in* M*.*

(b) If $(i, j) \in M$ *and* $(i', j') \in M$ *are matches belonging to different nested clusters in* $\mathcal{C}$*, then* (i, j) *and* (i', j') *are pairwise stable.*

Note that property (a) of Lemma 2 corresponds to requirement (ii) for TU-stability above, and property (b) together with internal stability of each $C \in \mathcal{C}$ corresponds to requirement (i). Hence we have TU-stability globally, proving Theorem 6.

6 Conclusion

We conclude with a brief discussion of some technical aspects of this paper. In our model we assume that agents can only reside in integer addresses on the real line. This assumption drives the multiplicity of NTU-stable assignments, as indifference is now a generic

[13] For example, the matches $(3, 4)$ and $(12, 13)$ in Figure 2 are shown to be pairwise stable by the following chain of steps: 1. Apply (b) to show that $(3, 4)$ and $(2, 6)$ are pairwise stable; 2. Apply (b) to show that $(12, 13)$ and $(9, 15)$ are pairwise stable; 3. Apply (c) to show that $(2, 6)$ and $(9, 5)$ are pairwise stable; 4. Apply (a) to show that $(2, 6)$ and $(12, 13)$ are pairwise stable; 5. Apply (a) to show that $(3, 4)$ and $(12, 13)$ are pairwise stable.

possibility that cannot easily be assumed away. More importantly, this assumption is crucial for Theorem 4 which relies on our definition of strongly convex surpluses. Imagine that we refine the integer grid and include, say, the halfway points between the integers. Then to still have $\mathcal{M}^{TU} \subseteq \mathcal{M}^{NTU}$ the surplus u must double everytime we decrease the distance between match partners by 1/2. Of course, this would simply be a relabeling of the old model with integer addresses, and would not affect Theorem 4 in a substantial way. However, when passing to a continuous space, there will be a difference: The only surplus function that would still satisfy strong convexity is discountinuous and positive only at zero. That is, surplus is created only if two identical agents match. Theorem 5 for the uni-direcational case would still hold, however, as it requires u to be convex only.

As argued in the introduction, the spatial preferences we consider are different, in important aspects, from other notions of horizontal differentiation in the literature. In particular, our spatial structure is neither sufficient nor necessary for horizontal heterogeneity as defined in Eeckhout (2000). It is interesting to note that if we allow the space in which agents reside to have more than one dimension, these differences become less pronounced. In particular, more horizontal models will fall into the spatial category because the single-peakedness condition becomes less stringent. Consider three sellers $\{a, b, c\}$ and three buyers $\{A, B, C\}$. The matrix on the left side of Figure 3 contains the possible match surpluses, to be split evenly among the agents in a match:

	A	B	C
a	2	1	0
b	0	2	1
c	1	0	2

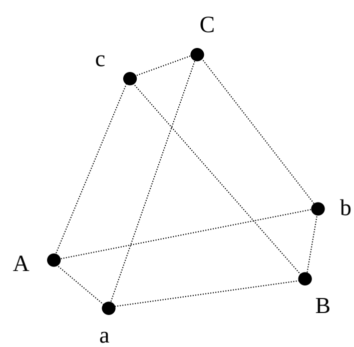

Figure 3: Spatial representation of cyclic preferences

Clearly these preferences satisfy the "mutual ideal match" condition. But the cyclicity of the surpluses makes it impossible to arrange the six agents along a line so that the surpluses are ordered in the same way as the distance between the matching agents. However, it is easy to arrange them in such a way on a plane, as the graph on the right shows.

Finally, this paper can be extended in a number of directions. In particular, it appears promising to examine spatial heterogeneity in more than one dimension, as well as in other matching environments, for example in one-sided markets and roommate assignment problems.

Appendix: Proofs

Preliminaries

The following intermediate result will be used later in the Appendix.

Lemma 3. *Let* $u : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$ *be decreasing and convex. Then*

(a) $m, n > r$ *implies* $u(m + n - r) > u(m) + u(n) - u(r)$*, and*

(b) $m < n < r$ *implies* $u(m - n + r) < u(m) - u(n) + u(r)$*.*

Proof. To show (a), suppose $u(m + n - r) > u(m) + u(n) - u(r)$ holds for some $m, n > r$. If we increase m, both sides of the inequality decrease. However, since $n - r > 0$, convexity implies that the left-hand side decreases by less than the right-hand side; thus the inequality still holds. The same argument applies when we increase n. Therefore, it is sufficient to show the inequality for $m = n = r + 1$. That is, $u(r + 2) > 2u(r + 1) - u(r)$, which is true for convex u. To show (b), an argument similar to the one used in part (a) implies that it is sufficient to show the desired inequality for $n = m+1$ and $r = m+2$. That is, $u(m-(m+1)+(m+2)) < u(m)-u(m+1)+u(m+2)$, or $2u(m+1) < u(m)+u(m+2)$, which is true for convx u. □

Proof of Theorem 2

Let (M, V) be TU-stable. Clearly $|i, j| \leq \hat{d}$ for all $(i, j) \in M$. However, since M may contain overlapping matches, to prove the result we must show that there exists a constant $\hat{m} < \infty$ (independent of $\mathcal{S}$ and $\mathcal{B}$) such that the following is true: $\nexists$ a sequence of matches $((i_t, j_t))_{t=0,\ldots,\hat{m}} \subset M$ s.t. (i_t, j_t) and (i_{t+1}, j_{t+1}) are overlapping $\forall 0 \leq t < \hat{m}$. Letting $\eta = (\hat{m} + 1)\hat{d}$, the result follows.

First, we show that if $(i, j) \in M$ and $(i', j') \in M$ are overlapping matches, they must be oriented in the same way: Either $i < j$ and $i' < j'$, or $i > j$ and $i' > j'$. To see this, suppose without loss of generality that $i < j$ and $i' > j'$ (note that if the matches overlap then $i \neq j$ and $i' \neq j'$). Because (i, j) and (i', j') are not nested, we have $i \leq j' < j \leq i'$. This in turn implies $v(i, j') > v(i, j) = V_S(i) + V_B(j)$ and $v(i', j) > v(i', j') = V_S(i') + V_B(j')$. Thus, by deleting (i, j) and (i', j') from M and replacing these matches with (i, j') and (i', j), it is possible to increase the payoffs to all four agents i, j, i', j' without affecting the payoff to any other agent. But by Theorem 3 (M, V) is efficient, a contradiction.[14]

Now let $((i_t, j_t))_{t=0,1,\ldots,\hat{m}} \subset M$ be a sequence of pairwise overlapping matches. By the previous argument, it is without loss of generality to assume $i_t < i_{t+1} \leq j_t < j_{t+1}$ $\forall t$. Let

$$\epsilon \;\equiv\; \max_{d=1,\ldots,\hat{d}} \; \max_{d'=0,\ldots,d-1} \big(u(d) - u(d')\big) \;<\; 0.$$

[14]Note that even though Theorem 3 is stated and proven after Theorem 2 which we are currently proving, its proof does not rely on Theorem 2.

(Because $\hat{d} < \infty$, ϵ is well-defined.) It follows that $v(i_{t+1}, j_{t+1}) - v(i_{t+1}, j_t) \leq \epsilon \ \forall t$. Because (M, V) is TU-stable,

$$V_S(i_1) \ \geq \ v(i_1, j_0) - V_B(j_0)$$

and thus

$$V_B(j_1) \ = \ v(i_1, j_1) - V_S(i_1) \ \leq \ v(i_1, j_1) - v(i_1, j_0) + V_B(j_0). \tag{7}$$

Also by TU-stability

$$V_S(i_2) \ \geq \ v(i_2, j_1) - V_B(j_1) \ \geq \ v(i_2, j_1) - v(i_1, j_1) + v(i_1, j_0) - V_B(j_0)$$

where the second inequality is due to (7). Thus

$$V_B(j_2) \ = \ v(i_2, j_2) - V_S(i_2) \ \leq \ v(i_2, j_2) - v(i_2, j_1) + v(i_1, j_1) - v(i_1, j_0) + V_B(j_0).$$

Continuing in this fashion we can write

$$V_B(j_m) \ \leq \ \sum_{t=1}^{m} \big[v(i_t, j_t) - v(i_t, j_{t-1}) \big] + V_B(j_0) \ \leq \ m\epsilon + V_B(j_0) \ \leq \ m\epsilon + u(0).$$

Since $\epsilon < 0$, $\exists \hat{m} \geq 1$ s.t. $V_B(j_m) < 0 \ \forall m > \hat{m}$, and thus $(i_m, j_m) \notin M$. Since ϵ is independent of $\mathcal{S}$ and $\mathcal{B}$, so is $\hat{m}$. This completes the proof. □

Proof of Theorem 4

The following intermediate result will be used in the proof.

Lemma 4. *Suppose u is strongly convex. Let $M \in \mathcal{M}^{TU}$ and $(i, j) \in D(d)$. Suppose that there does not exist integers k, k' such that $k \neq j$, $k' \neq i$, $|i, k| \leq d$, $|k', j| \leq d$, and such that either $(i, k) \in M$ or $(k', j) \in M$ or both. Then $(i, j) \in M$.*

Proof. Let (M, V) be TU-stable and let $(i, j) \in D(d)$. Suppose that the condition in the Lemma holds. That is: $(i, k) \notin M \ \forall k$ s.t. $k \neq j$ and $|i, k| \leq d$, and $(k', j) \notin M \ \forall k'$ s.t. $k' \neq i$ and $|k', j| \leq d$. Suppose to the contrary of the claim that $(i, j) \notin M$. Then $i \in U_S(M)$, or $(i, k) \in M$ for some k with $|i, k| > d$. In either case, $V_S(i) \leq u(d + 1) < u(d)/2$. Similarly, $j \in U_B(M)$ or $(k', j) \in M$ for some k' with $|k', j| > d$. In either case, $V_B(j) \leq u(d + 1) < u(d)/2$. Thus we have $V_S(i) + V_B(j) < u(d) = v(i, j)$, so (i, j) blocks (M, V), a contradiction. □

Let u be strongly convex. Recall that $M \in \mathcal{M}^{NTU}$ if and only if there exists a family of functions $\varphi = (\varphi_d : D(d) \rightarrow \{0, 1\})_{d=0,\ldots,\hat{d}}$, satisfying (1)–(2), such that

$$M \ = \ \{(i, j) \in D : \varphi_{|i,j|}(i, j) = 1\}.$$

Now fix $M' \in \mathcal{M}^{TU}$ and define the family $\varphi' = (\varphi'_d : D(d) \rightarrow \{0, 1\})_{d=0,\ldots,\hat{d}}$ from M' as follows:

$$\varphi'_{|i,j|}(i, j) = 1 \ \Leftrightarrow \ (i, j) \in M'.$$

We will show that φ'_0 satisfies (1) and φ'_d $(d > 1)$ satisfies (2); this will imply $M' \in \mathcal{M}^{NTU}$. The proof is by induction on d.

Consider $(i, j) \in D(0)$ first. Lemma 4 implies $(i, j) \in M'$, which implies $\varphi'_0(i, j) = 1$ $\forall (i, j) \in D(0)$. Thus, φ'_0 satisfies (1). If $\hat{d} = 0$ we are done, so suppose $1 \leq d \leq \hat{d}$ and assume $\varphi'_{d'}$ satisfies (1)–(2) for all $d' < d$. We show that φ'_d satisfies (2). Let $(i, j) \in D(d)$ and consider the following three possibilities:

(A) Suppose $\varphi'_{d'}(i, k) = 1$ or $\varphi'_{d'}(k', j) = 1$ for some $d' < d$. By construction of φ' this implies $(i, k) \in M'$ for some $k \neq j$ or $(k', j) \in M'$ for some $k' \neq i$ (or both). Because an agent cannot be in two matches, $(i, j) \notin M'$ and thus $\varphi'_d(i, j) = 0$.

(B) Suppose $\varphi'_d(i, k) = 1$ for some $k \neq j$, or $\varphi'_d(k', j) = 1$ for $k' \neq i$. Then again $(i, j) \notin M'$ because an agent cannot be in two matches, so $\varphi'_d(i, j) = 0$.

(C) In all other cases, $\varphi'_{d'}(i, k) = \varphi'_{d'}(k', j) = 0$ $\forall d' < d$, and $\varphi'_d(i, k) = \varphi'_d(k', j) = 0$ $\forall k, k'$ s.t. $|i, k| = |k', j| = d$, $k \neq i$, $k' \neq j$. Lemma 4 then implies that $(i, j) \in M'$; therefore $\varphi'_d(i, j) = 1$.

Close inspection of (A)–(C) and (2) then reveals that φ'_d satisfies (2). □

Proof of Theorem 5

Suppose trade is uni-directional. Without loss of generality, assume left-to-right trade. The set of potential matches of distance d now becomes $\overrightarrow{D}(d) \equiv \{(i, j) \in D : |\overrightarrow{i, j}| = d\}$. Replace $D(d)$ with $\overrightarrow{D}(d)$ in (1)–(2). Applying (1)–(2) iteratively on $\overrightarrow{D}(0), \ldots, \overrightarrow{D}(\hat{d})$ yields a family of functions $\overrightarrow{\varphi} = (\overrightarrow{\varphi}_d : \overrightarrow{D}(d) \to \{0, 1\})_{d=0,\ldots,\hat{d}}$ such that

$$\overrightarrow{\varphi}_{|\overrightarrow{i,j}|} = 1 \;\Leftrightarrow\; (i, j) \in M,$$

where M is an NTU-stable assigmnet. Observe that for each $d \geq 0$ there is at most one potential match partner located within distance d from each agent, and since u strictly decreases in d no agent can be indifferent between potential partners. Therefore $\overrightarrow{\varphi}$ is unique and generates a unique M.

Now let u be convex and let (M', V') be TU-stable. We need to show that $(i, j) \in M \Leftrightarrow (i, j) \in M'$. The proof is by induction on the match distance.

The induction step for $d > 0$. Let $d \in \{0, \ldots \hat{d} - 1\}$ and assume that $(i, j) \in M \Leftrightarrow (i, j) \in M'$ $\forall (i, j) \in \cup_{d'=0,\ldots,d} \overrightarrow{D}(d')$. We show that $(i, j) \in M \Leftrightarrow (i, j) \in M'$ $\forall (i, j) \in \overrightarrow{D}(d)$. To show necessity, suppose $(i, i + d + 1) \notin M$. Then either $i \notin \mathcal{S}$ or $(i + d + 1) \notin \mathcal{B}$ and thus $(i, i + d + 1) \notin M'$, or (by construction of $\overrightarrow{\varphi}$) at least one of the following holds: $(i, k) \in M$ for some $i < k < i + d + 1$, or $(k', i + d + 1) \in M$ for some $i < k' < i + d + 1$. Then by the induction hypothesis $(i, k) \in M'$ or $(k', i + d + 1) \in M'$ (or both), and since each agent can be in at most one match, $(i, i + d + 1) \notin M'$.

Next we show sufficiency. Take any $(i, i+d+1) \in M$ and suppose $(i, i+d+1) \notin M'$. Then at least one of the following must be true, for otherwise $(i, i+d+1)$ would block (M', V'): (i) $(i, k) \in M'$ for some $k > i$ s.t. $k \neq i+d+1$, or (ii) $(k', i+d+1) \in M'$ for some $k' < i+d+1$ s.t. $k' \neq i$. We show that both (i) and (ii) hold. Suppose (i). By the induction hypothesis we can rule out $k < i+d+1$: If $(i, k) \in M'$ for $i < k < i+d+1$ then $(i, k) \in M$ and therefore $(i, i+d+1) \notin M$ since each agent can be in at most one match. Thus $k > i+d+1$ and therefore $V'_S(i) \leq u(k-i) < u(d+1)$. This implies that $V'_B(i+d+1) \geq u(d+1) - V'_S(i) > 0$, for otherwise $(i, i+1)$ would block (M', V'). But $V'_B(i+d+1) > 0$ implies $(k', i+d+1) \in M'$ for some $k' < i+d+1$, so (ii) must be true. Invoking the induction hypothesis then shows that $k' < i$. A parallel argument establishes that (ii) $\Rightarrow$ (i). Therefore, $(i, k) \in M'$ for some $k > i+d+1$, and $(k', i+d+1) \in M'$ for some $k' < i$. Because $(i, k) \in M'$ and $(k', i+d+1) \in M'$, it must be true that $k' \in \mathcal{S}$ and $k \in \mathcal{B}$. The sum of utilities obtained by k and k' can be written as

$$V'_S(k') + V'_B(k) \;=\; u(i+d+1-k') + u(k-i) - V'_B(i+d+1) - V'_S(i). \tag{8}$$

For $(i, i+d+1)$ not to block (M', V'), it is necessary that

$$u(d+1) \;\leq\; V'_S(i) + V'_B(i+d+1). \tag{9}$$

Substituting (9) into (8) yields

$$V'_S(k') + V'_B(k) \;\leq\; u(i+d+1-k') + u(k-i) - u(d+1). \tag{10}$$

Letting $r = d+1$, $m = i+d+1-k' > r$, and $n = k-i > r$, we can apply Lemma 3 (a) to show $u(k-k') > u(i+d+1-k') + u(k-i) - u(d+1)$. Together with (10), this implies that (k', k) blocks (M', V'), a contradiction. Therefore $(i, i+d+1) \in M \Rightarrow (i, i+d+1) \in M'$.

The initial step for $d = 0$. We now prove that $(i, i) \in M \Leftrightarrow (i, i) \in M'$ for all $(i, i) \in D(0)$. Necessity is obvious given $\overrightarrow{\varphi}$: If $(i, i) \notin M$ then $i \notin \mathcal{S}$ or $(i+1) \notin \mathcal{B}$ (or both); this implies $(i, i) \notin M'$.

For sufficiency, take any $(i, i) \in M$ and suppose that $(i, i) \notin M'$. Then at least one of the following must hold in order for (i, i) not to block (M', V'): (i) $(i, k) \in M'$ for some $k > i$, or (ii) $(k', i) \in M'$ for some $k' < i$. We show that (i) and (ii) both hold. If (i), then $V'_S(i) \leq u(k-i) < u(0)$; this implies that $V'_B(i) > 0$ as otherwise (i, i) would block (M', V'). But $V'_B(i) > 0$ implies $(k', i) \in M'$ for some $k' < i$, so (ii) must be true. A parallel argument shows that (ii) $\Rightarrow$ (i). Therefore, $(i, k) \in M'$ for some $k > i$, and $(k', i) \in M'$ for some $k' < i$. Because $(i, k) \in M'$ and $(k', i) \in M'$, it must be true that $k' \in \mathcal{S}$ and $k \in \mathcal{B}$. The sum of utilities obtained by k and k' can be written as

$$V'_S(k') + V'_B(k) \;=\; u(i-k') + u(k-i) - V'_B(i) - V'_S(i). \tag{11}$$

For (i, i) not to block (M', V'), it is necessary that

$$u(0) \;\leq\; V'_S(i) + V'_B(i). \tag{12}$$

Substituting (12) into (11) yields

$$V_S'(k') + V_B'(k) \ \leq \ u(i-k') + u(k-i) - u(0). \tag{13}$$

Letting $r = 0$, $m = i - k' > r$, and $n = k - i > r$, we can apply Lemma 3 (a) to show $u(k-k') > u(i-k') + u(k-i) - u(0)$. Together with (13), this implies that (k', k) blocks (M', V'), a contradiction. Therefore $(i,i) \in M \Rightarrow (i,i) \in M'$. □

Proof of Lemma 1

Proof of property (a). Suppose (i'', j'') is nested in (i', j'), and without loss of generality that (i,j) is to the left of (i', j'). Thus $i \leq j < i' < i'' \leq j'' < j'$. Since trade is assumed left-to-right, to show that (i,j) and (i'', j'') are pairwise stable we only need to show $V_S(i) + V_B(j'') > u(j'' - i)$. Because (i', j') and (i,j) are pairwise stable,

$$V_S(i) + V_B(j') \geq u(j' - i). \tag{14}$$

Because (i'', j'') and (i', j') are pairwise stable,

$$V_S(i') + V_B(j'') \geq u(j'' - i'). \tag{15}$$

Adding (14)–(15) and using the fact that $V_S(i') + V_B(j') = u(j' - i')$, we get

$$V_S(i) + V_B(j'') \geq u(j'-i) + u(j''-i') - u(j'-i') > u(j''-i),$$

where the last inequality follows from Lemma 3 (b).

Proof of property (b). We proceed in two steps. First we show that if $(i', j') \in S(i,j)$ then (i', j') and (i,j) are pairwise stable. Next, we show that if (i,j) and (i', j') are pairwise stable and (i', j') is nested in (i,j), then for every $(i'', j'') \in S(i', j')$, (i'', j'') and (i,j) are pairwise stable. The two steps together imply property (a).

For the first step, take $(i', j') \in S(i,j)$ and note that $i < i' \leq j' < j$. Using (3)–(4), write the value $V_S(i')$ as follows:

$$V_S(i') = \frac{1}{2}\left(u(j'-i') + [u(j-i') - V_B(j)] - [u(j'-i) - V_S(i)]\right).$$

Thus

$$\begin{aligned} V_S(i') + V_B(j) &= \frac{1}{2}\left(u(j'-i') + [u(j-i') + V_B(j)] - [u(j'-i) - V_S(i)]\right) \\ &= \frac{1}{2}\left(u(j'-i') - u(j'-i) + u(j-i)\right) + \frac{1}{2}u(j-i') \\ &> \frac{1}{2}u(j-i') + \frac{1}{2}u(j-i') \ = \ u(j-i'), \end{aligned}$$

where the second line uses the fact that $V_S(i) + V_B(j) = u(j-i)$, and the inequality is by Lemma 3 (b). In similar fashion we can show $V_S(i) + V_B(j') > u(j'-i)$. Thus (i,j) and (i',j') are pairwise stable.

For the second step, suppose (i',j') is nested in (i,j), and $(i,j),(i',j')$ are pairwise stable. Suppose further $(i'',j'') \in S(i',j')$, so that $i < i' < i'' \leq j'' < j' < j$. By the previous step, we know that (i'',j'') and (i',j') are pairwise stable, which implies

$$V_S(i'') + V_B(j') \geq u(j'-i''). \tag{16}$$

Furthermore, pairwise stability of (i,j) and (i',j') implies

$$V_S(i') + V_B(j) \geq u(j-i'). \tag{17}$$

Adding (16)–(17) and using the fact that $V_S(i') + V_B(j') = u(j'-i')$, we get

$$V_S(i'') + V_B(j) \geq u(j'-i'') + u(j-i') - u(j'-i') > u(j-i''),$$

where the last inequality is by Lemma 3 (b). In similar fashion we can show $V_S(i) + V_B(j'') > u(j''-i)$. Thus (i,j) and (i'',j'') are pairwise stable.

Proof of property (c). Suppose $p(i,j) = p(i',j') = (i'',j'')$. Without loss of generality assume (i,j) is to the left of (i',j'), so that $i'' < i \leq j < i' \leq j' < j''$. Since trade is assumed left-to-right, to show that (i,j) and (i',j') are pairwise stable we only need to show $V_S(i)+V_B(j') > u(j'-i)$. Using (3)–(4), write the values $V_S(i)$ and $V_B(j')$ as follows:

$$V_S(i) = \frac{1}{2}\left(u(j-i) + [u(j''-i) - V_B(j'')] - [u(j-i'') - V_S(i'')]\right), \tag{18}$$

$$V_B(j') = \frac{1}{2}\left(u(j'-i') + [u(j'-i') - V_S(i'')] - [u(j''-i') - V_B(j'')]\right). \tag{19}$$

Adding (18)–(19) and rearranging terms, we have

$$\begin{aligned} V_S(i) + V_B(j') &= \frac{1}{2}\left(u(j-i) - u(j-i'') + u(j'-i'')\right) \\ &\quad + \frac{1}{2}\left(u(j'-i') - u(j''-i') + u(j''-i')\right) \\ &> \frac{1}{2}u(j'-i) + \frac{1}{2}u(j'-i) = u(j'-i), \end{aligned}$$

where the inequality is by Lemma 3 (b). □

Proof of Lemma 2

Proof of property (a). Let i' be any unmatched seller with $x_{i'} = 1$, and let j' be any unmatched buyer with $y_{j'} = 1$. Since M is the set of matches generated by the inside-out algorithm (Theorem 5), it follows that $u(j'-i') < 0$, so (i',j') does not block (M,V).

We now show that (M, V) is also not blocked by (i', j) or (i, j') for some $(i, j) \in M$. Let $C \in \mathcal{C}$ be the nested cluster containing (i, j), and let (i_0, j_0) be the initial node in the tree representing C. Let (i_0^*, j_0^*) be the "pseudo-predecessor" of (i_0, j_0), as defined in Section 5.2. Note that $i' \leq i_0^* < i_0 \leq j_0 < j_0^* \leq j'$. We proceed in two steps.

We first show $V_S(i_0) > u(j_0^* - i_0)$. Using (5)–(6) write

$$V_S(i_0) = \frac{1}{2}\big(u(j_0 - i_0) + \max\{u(j_0^* - i_0), 0\} - \max\{u(j_0 - i_0^*), 0\}\big). \tag{20}$$

Since $V_S(i_0) > 0$, the inequality automatically holds if $u(j_0^* - i_0) \leq 0$. Thus suppose $u(j_0^* - i_0) > 0$. If $u(j_0 - i_0^*) \leq 0$ then (20) becomes

$$V_S(i_0) = \frac{1}{2}\left(u(j_0 - i_0) + u(j_0^* - i_0)\right) > \frac{1}{2}\left(u(j_0^* - i_0) + u(j_0^* - i_0)\right) = u(j_0^* - i_0)$$

and the inequality holds. If $u(j_0 - i_0^*) > 0$ then (20) becomes

$$\begin{aligned} V_S(i_0) &= \frac{1}{2}\big(u(j_0 - i_0) + u(j_0^* - i_0) - u(j_0 - i_0^*)\big) \\ &= \frac{1}{2}\big(u(j_0 - i_0) - u(j_0 - i_0^*)\big) + \frac{1}{2}u(j_0^* - i_0). \end{aligned} \tag{21}$$

Note that

$$u(j_0 - i_0) - u(j_0 - i_0^*) > u(j_0 - i_0) - u(j_0 - i_0^*) + u(j_0^* - i_0^*) > u(j_0^* - i_0), \tag{22}$$

where the first inequality follows from $u(j_0^* - i_0^*) < 0$ and the second follows from Lemma 3 (b). Plugging (22) back into (21), we have $V_S(i_0) > u(j_0^* - i_0)$, as desired.

Next, we show that $V_S(i) + V_B(j') > u(j' - i)$. It follows from definition j_0^* that $V_S(i_0) > u(j' - i_0)$. Furthermore, by Lemma 1 (b) we know that (i, j) and (i_0, j_0) are pairwise stable. Thus

$$[V_S(i) + V_B(j_0)] + V_S(i_0) \; > \; u(j_0 - i) + u(j' - i_0).$$

Using the fact that $V_B(j_0) + V_S(i_0) = u(j_0 - i_0)$ and applying Lemma 3 (b), this becomes

$$V_S(i) > u(j_0 - i) + u(j' - i_0) - u(j_0 - i_0) > u(j' - i).$$

A similar argument shows $V_B(j) > u(j - i')$, establishing property (a).

Proof of property (b). Let $C \in \mathcal{C}$ be the nested cluster containing (i, j) and $C' \in \mathcal{C}$ be the nested cluster containing (i', j') (with $C \neq C'$). Let (i_0, j_0) and (i_0', j_0') be the initial nodes in the trees representing C and C', respectively. We will show that (i_0, j_0) and (i_0', j_0') are pairwise stable; pairwise stability of (i, j) and (i', j') is then implied by Lemma 1 (a).

Without loss of generality assume $i_0 \leq j_0 < i'_0 \leq j'_0$. Because trade is left-to-right, we only need to show that $V_S(i_0) + V_B(j'_0) \geq u(j'_0 - i_0)$. We denote by (i_0^*, j_0^*) the pseudo-predecessor of (i_0, j_0), and by $(i_0^{'*}, j_0^{'*})$ the pseudo-predecessor of (i'_0, j'_0). Note that $j_0^* \neq i_0^{'*}$ because j_0^* and $i_0^{'*}$ are unmatched. There are then five cases to consider:

$$\begin{array}{ll} \text{Case 1:} & i_0^* < i_0 \leq j_0 < j_0^* < i_0^{'*} < i'_0 \leq j'_0 < j_0^{'*}, \\ \text{Case 2:} & i_0^* < i_0 \leq j_0 < i_0^{'*} < j_0^* < i'_0 \leq j'_0 < j_0^{'*}, \\ \text{Case 3:} & i_0^* < i_0 \leq j_0 < i_0^{'*} < i'_0 \leq j'_0 < j_0^* = j_0^{'*}, \\ \text{Case 4:} & i_0^* = i_0^{'*} < i_0 \leq j_0 < j_0^* < i'_0 \leq j'_0 < j_0^{'*}, \\ \text{Case 5:} & i_0^* = i_0^{'*} < i_0 \leq j_0 < i'_0 \leq j'_0 < j_0^* = j_0^{'*}. \end{array}$$

Observe that these cases exhaust all possibilities regarding the arrangement of i_0^*, i_0, j_0, j_0^*, $i_0^{'*}$, i'_0, j'_0, and $j_0^{'*}$.

In cases 1–3, we have

$$V_S(i_0) + V_B(j'_0) \geq V_B(j'_0) \geq u(j'_0 - i_0^{'*}) > u(j'_0 - i_0),$$

where the second inequality is by property (a) and the third inequality is because $i_0 < i_0^{'*}$. Similarly, in case 4 we have

$$V_S(i_0) + V_B(j'_0) \geq V_S(i_0) \geq u(j_0^* - i_0) > u(j'_0 - i_0),$$

where the second inequality is by property (a) and the third inequality is because $j_0^* < j'_0$. Finally, consider case 5. Using (5)–(6), write the values $V_S(i_0)$ and $V_B(j'_0)$ as follows:

$$\begin{aligned} V_S(i_0) &= \frac{1}{2}\big(u(j_0 - i_0) + \max\{u(j_0^* - i_0), 0\} - \max\{u(j_0 - i_0^*), 0\}\big), \\ V_B(j'_0) &= \frac{1}{2}\big(u(j'_0 - i'_0) + \max\{u(j'_0 - i_0^{'*}), 0\} - \max\{u(j_0^{'*} - i'_0), 0\}\big). \end{aligned}$$

Using the fact that $i_0^* = i_0^{'*}$ and $j_0^* = j_0^{'*}$ and rearranging terms, we can write $V_S(i_0) + V_B(j'_0) = (E_1 + E_2)/2$, where

$$\begin{aligned} E_1 &= u(j_0 - i_0) - \max\{u(j_0 - i_0^*), 0\} + \max\{u(j'_0 - i_0^*), 0\}, \\ E_2 &= u(j'_0 - i'_0) - \max\{u(j_0^* - i'_0), 0\} + \max\{u(j_0^* - i_0), 0\}. \end{aligned}$$

Consider the term E_1. If $u(j_0 - i_0^*) \leq 0$ then

$$E_1 = u(j_0 - i_0) - 0 + \max\{u(j'_0 - i_0^*), 0\} \geq u(j_0 - i_0) > u(j'_0 - i_0),$$

where the last inequality is because $j'_0 > j_0$. If $u(j_0 - i_0^*) > 0$ then

$$\begin{aligned} E_1 &= u(j_0 - i_0) - u(j_0 - i_0^*) + \max\{u(j'_0 - i_0^*), 0\} \\ &\geq u(j_0 - i_0) - u(j_0 - i_0^*) + u(j'_0 - i_0^*) \ > \ u(j'_0 - i_0), \end{aligned}$$

where the last inequality is by Lemma 3 (b). We can similarly show that $E_2 > u(j_0' - i_0)$, and thus $V_S(i_0) + V_B(j_0') > u(j_0' - i_0)$. $\square$

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