Fresnel Equations: Amplitude Coefficients

• We want to compute the fraction of light (power) that is reflected and transmitted by a flat interface between two dielectric media with different indices of refraction. We are also interested in possible phase shifts.

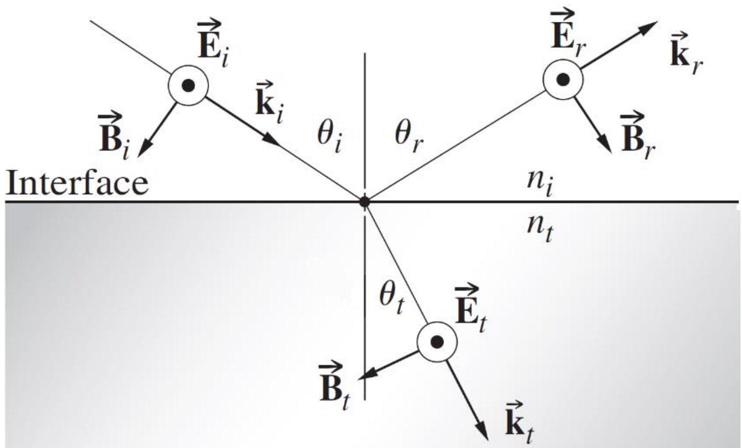
• As a first step in achieving this, we investigate the relationship between the incident, reflected and refracted *electric* fields at the interface. The relationship is summarized by the **Fresnel Equations**.

• Suppose that a monochromatic plane wave is incident on the planar surface separating two isotropic media. Whatever the polarization of the incident wave, we shall resolve its $\vec{\mathbf{E}}$ (and $\vec{\mathbf{B}}$) field into components ∥ and ⊥ to the ***plane-of-incidence***. The Fresnel Equations treat these components separately.

Note: Some Optics literature use s and p to refer to $\vec{\mathbf{E}}_{\perp}$ and $\vec{\mathbf{E}}_{\parallel}$, respectively. (In this notation, s derives from the German word "senkrecht" which means "perpendicular" and p stands for parallel.) In other literature, $\vec{\mathbf{E}}_{\perp}$ is labelled "transverse electric" or TE, indicating that the electric field is transverse (perpendicular) to the plane-of-incidence. In this notation, $\vec{\mathbf{E}}_{\parallel}$ is labelled TM for "transverse magnetic".

• Consider the following scenario for the situation where the incident $\vec{\mathbf{E}}$ is *perpendicular* to the plane-of-incidence (i.e., the page):

["s-polarized" or "TE polarized"]



($\theta_r = \theta_i$)

(θ_t related to θ_i via Snell's Law)

Fig.1

- As usual, the symbol "dot within circle" (as shown in Fig. 1) indicates the vector is coming out of the page. [The symbol "cross within circle" would be used to indicate the vector is going into the page.]

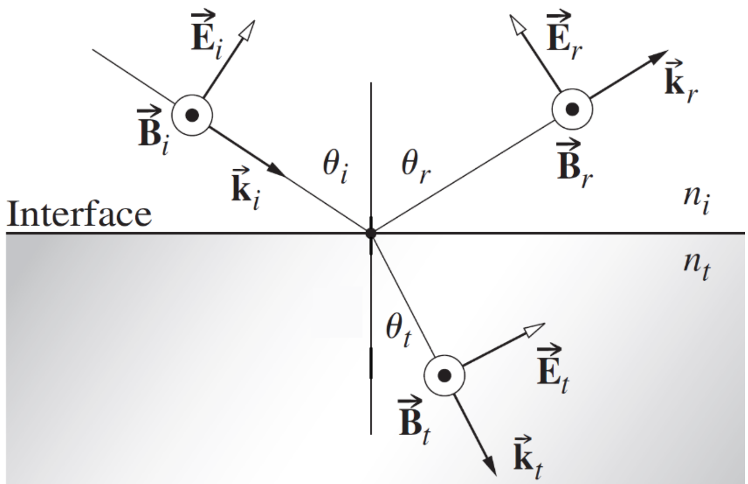
- $\vec{\mathbf{E}}_r$ (the reflected $\vec{\mathbf{E}}$) and $\vec{\mathbf{E}}_t$ (the transmitted $\vec{\mathbf{E}}$) are perpendicular to the plane-of-incidence, but in principle could be phase-shifted compared to the incident $\vec{\mathbf{E}}_i$ (more details later).

• In this scenario where $\vec{\mathbf{E}}_i$ is $\perp$ to the plane-of-incidence, i.e., *perpendicular* polarization, the **amplitude reflection coefficient** $r_\perp$ and the **amplitude transmission coefficient** $t_\perp$ are relevant:

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp \quad \text{and} \quad t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp$$

• Consider the following scenario for the situation where the incident $\vec{\mathbf{E}}$ is *parallel* to the plane-of-incidence (i.e., the page):

[“p-polarized” or “TM polarized”]



$(\theta_r = \theta_i)$

(θ_t related to θ_i via Snell’s Law)

Fig.2

- $\vec{\mathbf{E}}_r$ (the reflected $\vec{\mathbf{E}}$) and $\vec{\mathbf{E}}_t$ (the transmitted $\vec{\mathbf{E}}$) are entirely parallel to the plane-of-incidence but in principle could be phase-shifted compared to the incident $\vec{\mathbf{E}}_i$ (more details later).

• In this scenario where $\vec{\mathbf{E}}_i$ is ∥ to the plane-of-incidence, i.e., *parallel* polarization, the **amplitude reflection coefficient** $r_\parallel$ and the **amplitude transmission coefficient** $t_\parallel$ are relevant:

$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel \quad \text{and} \quad t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel$$

Fresnel Equations

for homogeneous, non-magnetic, dielectric media

(see the derivation handout)

Augustin-Jean Fresnel (1788-1827)

Amplitude coefficients

$$r_\perp \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\perp \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_\parallel \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}$$

$$t_\parallel \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}$$

Eqn (1)

Alternatively, by using Snell's law:

$$r_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_\perp = +\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)}$$

$$r_\parallel = +\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$t_\parallel = +\frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

Eqn (2)

Note: In the Optics literature, different sign variations have been labeled as the Fresnel Equations (in particular, for $r_\parallel$). To avoid confusion, the equations should be related to the specific field directions from which they were derived. (The above equations are referred to the directions indicated in Fig. 1 and Fig. 2.)

$\Rightarrow$ It can be shown that:

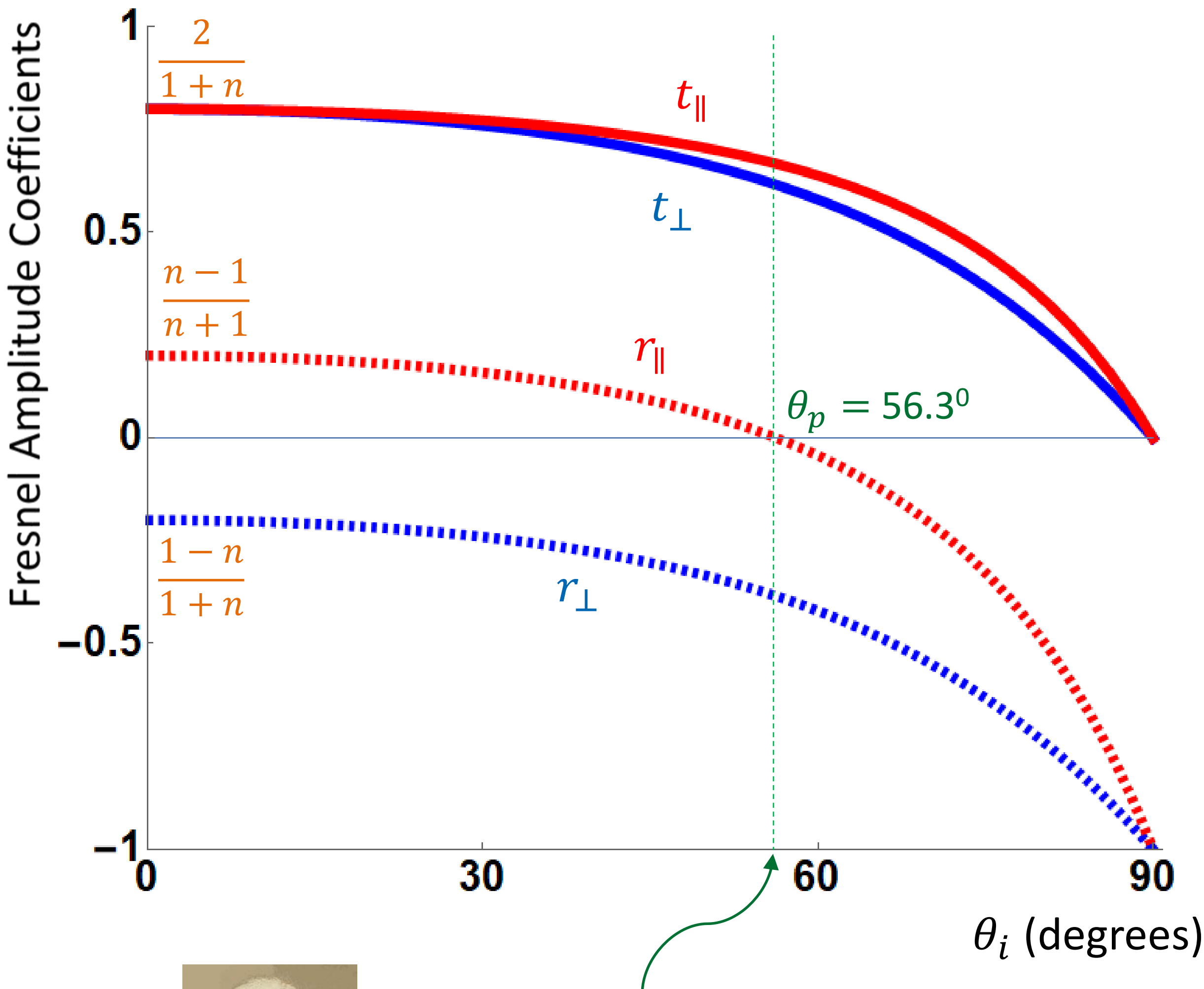
$t_\perp + (-r_\perp) = 1$ always

$t_\parallel + r_\parallel = 1$ only at normal incidence

(I) **External Reflection** $[n_i < n_t]$,
transmitted medium is optically denser

Example:
Light going from air ($n_i = 1.0$) to glass ($n_t = n = 1.5$).

Fig.3



"**Polarization Angle**" or
"**Brewster's Angle**" $\boldsymbol{\theta_P}$
($r_\parallel = 0$)

Sir David Brewster
(1781-1868)

Fresnel Equations: Amplitude Coefficients (con't)

Notes:

$\Rightarrow$ Although the material is considered to be "transparent", in general, light is actually both transmitted and reflected from the material.

$\Rightarrow$ At *normal incidence*, i.e., $\theta_i = 0^0$, and hence $\theta_t = 0^0$ from Snell's Law Eqn (1) then gives [Eqn (2) might be harder to use]:

$$r_{\parallel} = -r_{\perp} = \frac{n_t - n_i}{n_t + n_i} \qquad (3)$$

$$t_{\parallel} = t_{\perp} = \frac{2n_i}{n_t + n_i} \qquad (4)$$

(The expressions displayed in the graph can then be obtained by substituting $n_i = 1$ and $n_t = n$).

$\Rightarrow$ If the incident ray is at *glancing incidence* to the interface, i.e., $\theta_i = 90^0$, Eqn (2) and some trigonometric identities yield $r_{\parallel} = r_{\perp} = -1$ and $t_{\parallel} = t_{\perp} = 0$. This implies that *all the light is reflected*.

For example:

$$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} = -\frac{\sin(90^0 - \theta_t)}{\sin(90^0 + \theta_t)} = -\frac{\cos(\theta_t)}{\cos(\theta_t)} = -1$$

$$r_{\parallel} = +\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = \frac{\tan(90^0 - \theta_t)}{\tan(90^0 + \theta_t)} = \frac{\cot(\theta_t)}{-\cot(\theta_t)} = -1$$

etc.

$\Rightarrow$ Notice that the Fresnel amplitude coefficients can take on both positive and negative values. This implies the reflected light can undergo phase shifts (more later...).

Polarization Angle (Brewster's Angle)

⇒ At the so-called **polarization angle** or **Brewster's angle** $\boldsymbol{\theta_p}$ the value $r_{\parallel} = 0$. This means that light that is polarized parallel to the plane of incidence (p-polarized light) is *not* reflected. Note that if *unpolarized* light is incident on the interface, the reflected light will be fully polarized ⊥ to the plane-of-incidence.

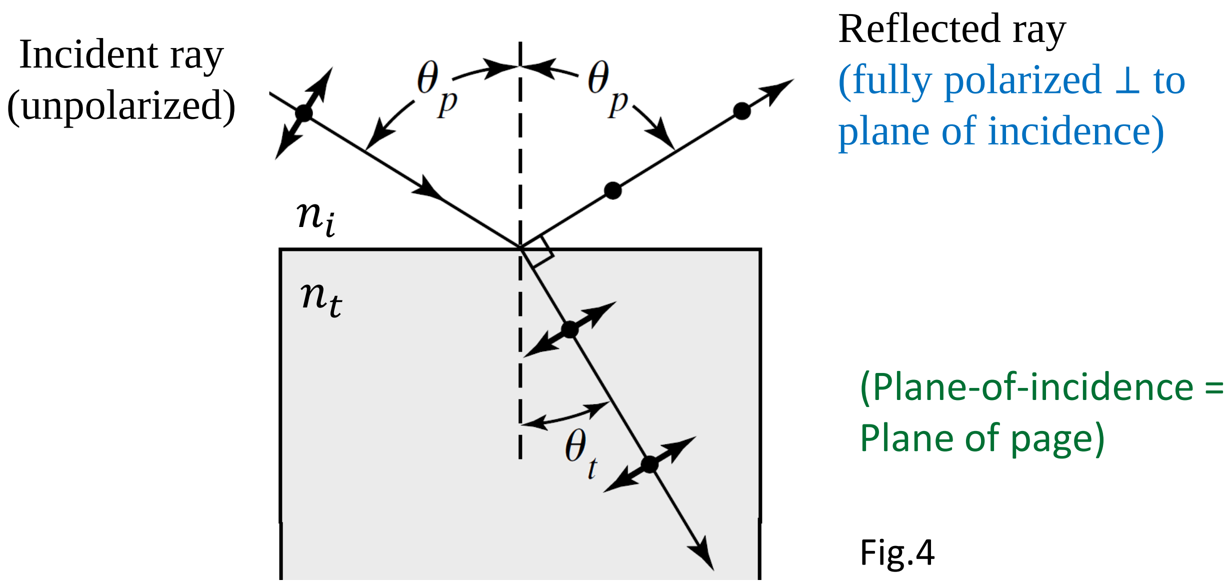


Fig.4

Note that in Eqn (2), $r_{\parallel} = 0$ occurs when the denominator is $\tan 90^0 = \infty$. Hence,

$$\boxed{\theta_p + \theta_t = 90^0} \qquad (5)$$

i.e., the reflected and transmitted rays are perpendicular.

Physically, the ∥-polarized (p-polarized) light cannot reflect because the reflected $\vec{\mathbf{k}}_{\mathbf{r}}$ and the transmitted $\vec{\mathbf{k}}_t$ are perpendicular. A reflection would require the microscopic dipoles at the surface of the second material (with n_t) to radiate along their axes, which they cannot do.

Polarization Angle (Brewster's Angle) con't

From Snell's Law:

$$n_i \sin\theta_p = n_t \sin\theta_t$$
$$= n_t \sin\left(90^0 - \theta_p\right)$$
$$= n_t \cos\theta_p$$

$$\boxed{\tan\theta_p = \frac{n_t}{n_i}} \qquad (6)$$

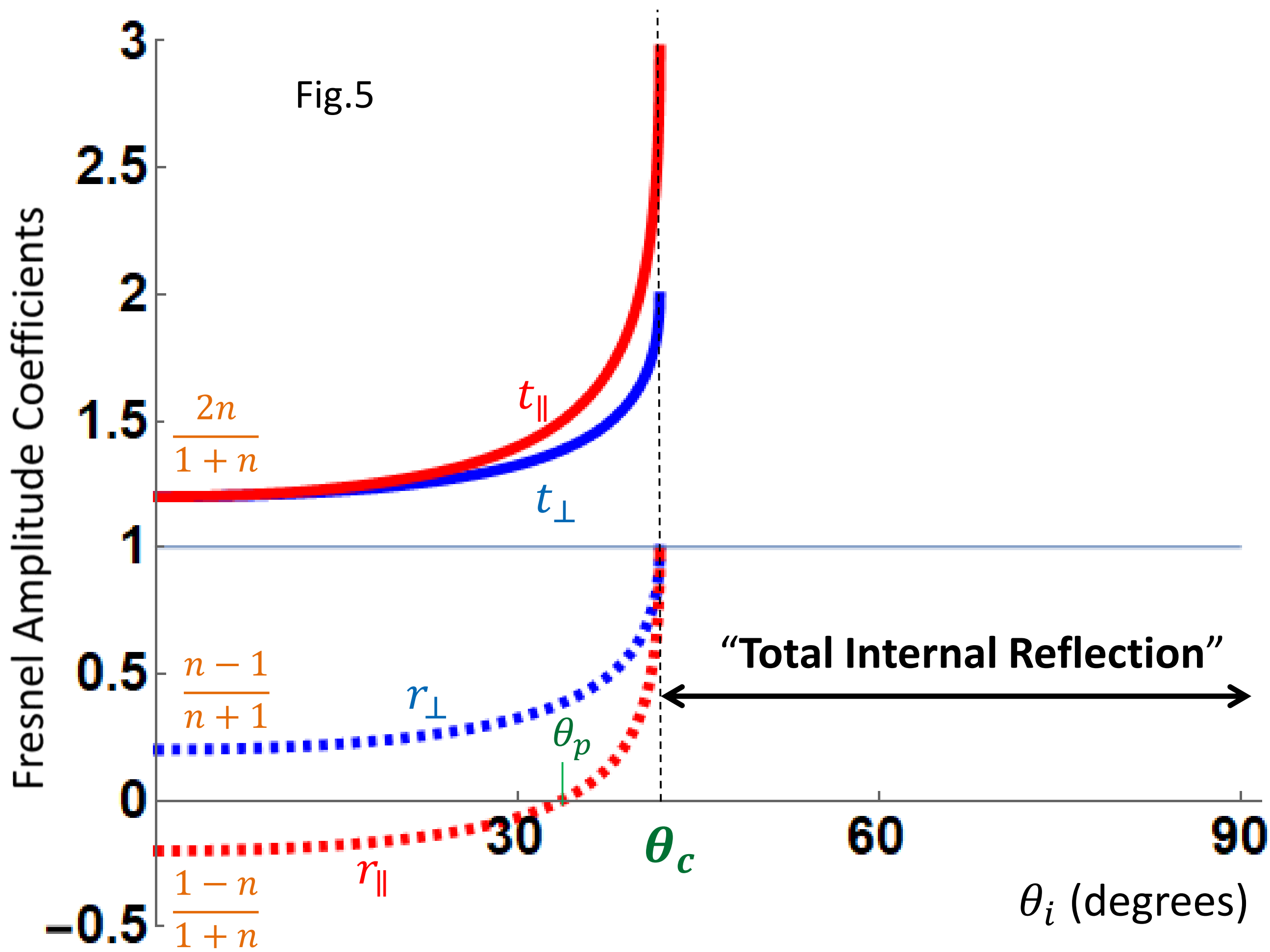
e.g. $n_i = 1.00;\ n_t = 1.50$

$$\tan\theta_p = \frac{n_t}{n_i} = \frac{1.50}{1.00} \quad \Rightarrow \quad \theta_p = 56.3^0$$

(II) **Internal Reflection** $[n_i > n_t]$, incident medium is optically denser

Example:

Light going from glass ($n_i = n = 1.5$) to air ($n_t = 1.0$).

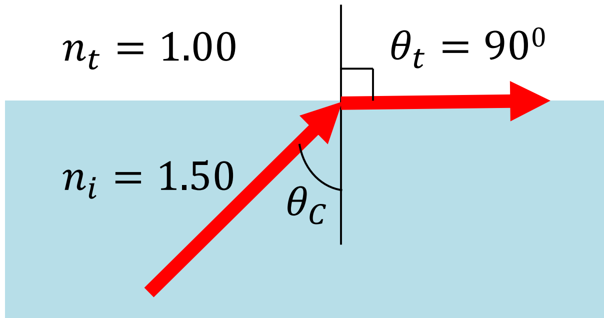


Additional Notes:

$\Rightarrow$ *New behavior*: Both $r_\parallel$ and $r_\perp$ reach +1 at the **critical angle θ_C**. As we shall discuss later, if the incident angle is larger than θ_C, the light ray is fully reflected, i.e., the regime of **total internal reflection (TIR)**. (In this regime, the Fresnel coefficients are complex.) The numerical value of θ_C can be found using Snell's Law with the refracted angle set to 90^0 (it is equal to 41.8^0 in the figure above).

Internal Reflection (con't)

re: *Critical angle* θ_C (con't)



$$n_i \sin\theta_C = n_t \sin 90^0$$

$$1.50 \sin\theta_C = 1.00 \sin 90^0$$

$$\theta_C = 41.8^0$$

$\Rightarrow$ Here, the polarization angle (Brewster's angle) is $\theta_p = 33.7^0$.

i.e., from $\tan\theta_p = \frac{n_t}{n_i} = \frac{1.00}{1.50}$

Note: Recall that the polarization (Brewster's) angle for external reflection, i.e., from air to glass, was $\theta_p' = 56.3^0$. Hence, $\theta_p + \theta_p' = 90^0$. This is a general result: i.e., polarization (Brewster's) angles for internal and external reflection at a given interface are complementary.

$\Rightarrow$ Note that the transmission coefficients $t_\parallel > 1$ and $t_\perp > 1$, implying that the amplitudes of the transmitted electric fields are larger than that of the incident electric field...!

(At θ_C, the values of $t_\parallel$ and $t_\perp$ are finite; here $t_\parallel = 3$ and $t_\perp = 2$.)