Derivation of Fresnel Equations

• The Fresnel Equations can be obtained by making use of the *boundary conditions* of EM fields at the interface of two dielectrics (for derivations of these boundary conditions, see most textbooks, etc. on Electromagnetism). These boundary conditions state that the following are ***continuous*** across the interface:

- *tangential* component of $\vec{\mathbf{E}}$.
- *tangential* component of $\vec{\mathbf{B}}/\mu$.

- *normal* component of $\epsilon\vec{\mathbf{E}}$.
- *normal* component of $\vec{\mathbf{B}}$.

• Assume linearly polarized waves. Resolve a wave's $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields into components parallel and perpendicular to the plane-of incidence and treat them separately.

Situation 1: Incident $\vec{\mathbf{E}}$ is ⊥ to the plane-of-incidence

("s-polarized" or "Transverse Electric (TE) polarized")

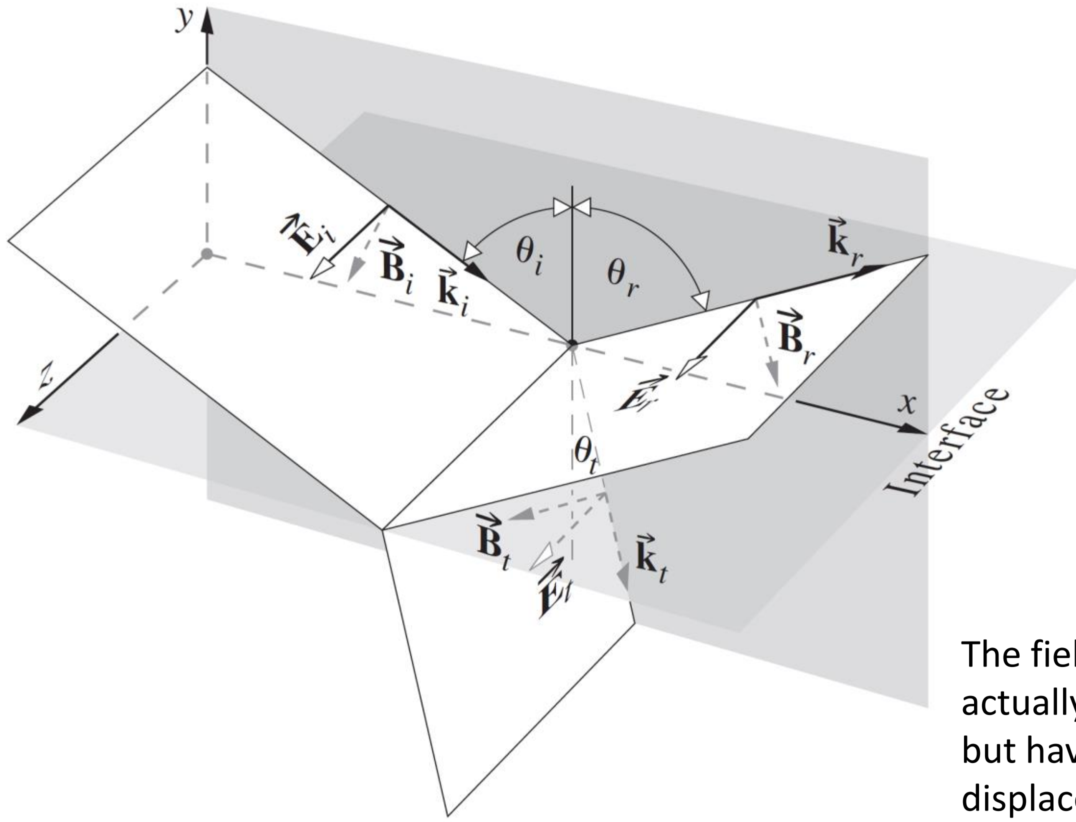

Fig. 1a

The fields shown are actually at the interface but have been drawn displaced so the vectors can be seen more clearly.

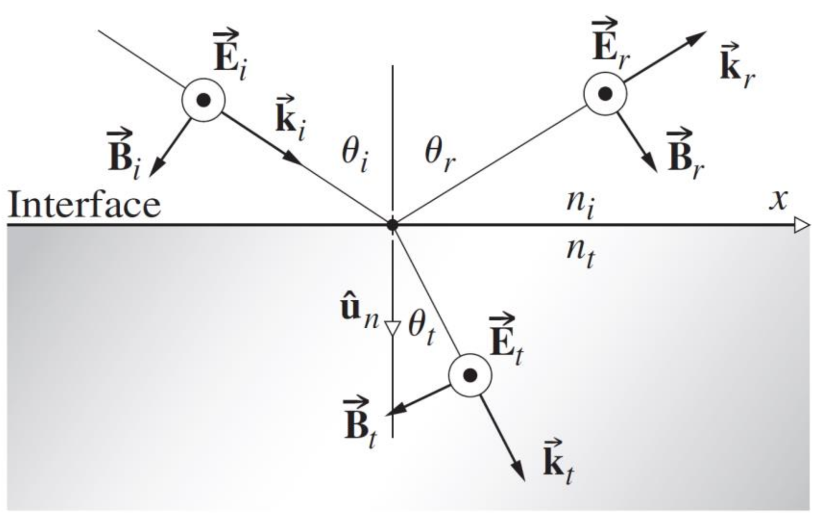

Fig. 1b

• The continuity of the tangential component of $\vec{\mathbf{E}}$ gives

$$\vec{\mathbf{E}}_i + \vec{\mathbf{E}}_r = \vec{\mathbf{E}}_t \qquad \text{(A-1)}$$

Each of the terms can be written in the complex form that looks like "$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 \exp\left[i\left(\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t + \varepsilon\right)\right]$". Since eqn (A-1) is valid at any location on the interface and for any time, it must be independent of time t and position $\vec{\mathbf{r}}$. This implies that $\vec{\mathbf{E}}_i$, $\vec{\mathbf{E}}_r$ and $\vec{\mathbf{E}}_t$ have the same functional dependence on the variables t and $\vec{\mathbf{r}}$, and hence the phases (which look like "$\vec{\mathbf{k}} \cdot \vec{\mathbf{r}} - \omega t + \varepsilon$") are always the same. Hence, the exponential phase terms cancel out in eqn (A-1) and the continuity condition can be applied to the amplitudes.

$$\vec{\mathbf{E}}_{0i} + \vec{\mathbf{E}}_{0r} = \vec{\mathbf{E}}_{0t} \qquad \text{(A-2a)}$$

or $$E_{0i} + E_{0r} = E_{0t} \qquad \text{(A-2b)}$$

• The continuity of the tangential component (along the x direction in Fig. 1) of $\vec{\mathbf{B}}/\mu$ gives:

$$-\frac{B_i}{\mu_i}\cos\theta_i + \frac{B_r}{\mu_i}\cos\theta_r = -\frac{B_t}{\mu_t}\cos\theta_t \qquad \text{(A-3)}$$

As an example, Fig. 2 shows how to obtain the tangential component of the magnetic field of the incident light (see Fig. 1b), i.e., the 1st term in eqn (A-3).

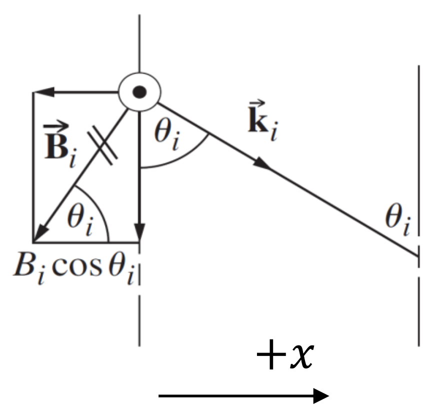

Fig. 2

[When the tangential component of the $\vec{\mathbf{B}}$-field points in the negative x-direction it is entered with a minus sign.]

• By using "$B = E/v$" and since $v_i = v_r$ ("i" and "r" are in the same medium) and $\theta_i = \theta_r$ (Law of Reflection), eqn (A-3) yields

$$\frac{1}{\mu_i v_i}(E_i - E_r)\cos\theta_i = \frac{1}{\mu_t v_t}E_t\cos\theta_t \qquad \text{(A-4)}$$

Recall from our above discussion that the phases of the electric fields are the same; hence the exponential phase terms that are present in E_i, E_r, E_t of eqn (A-4) cancel. We can also substitute "$v = nc$" into eqn (A-4). Hence,

$$\frac{n_i}{\mu_i c}(E_{0i} - E_{0r})\cos\theta_i = \frac{n_t}{\mu_t c}E_{0t}\cos\theta_t \qquad \text{(A-5a)}$$

$$\frac{n_i}{\mu_i}(E_{0i} - E_{0r})\cos\theta_i = \frac{n_t}{\mu_t}E_{0t}\cos\theta_t \qquad \text{(A-5b)}$$

By substituting (A-2b) into the RHS of (A-5b) and re-arranging:

$$\left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{\frac{n_i}{\mu_i}\cos\theta_i - \frac{n_t}{\mu_t}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \qquad \text{(A-6)}$$

The subscript $\perp$ in (A-6) serves as a reminder that we are dealing with $\vec{\boldsymbol{E}}$ that is $\perp$ to the plane-of-incidence.

• Furthermore, the LHS of (A-6) can be written as, using (A-2b),

$$\frac{E_{0r}}{E_{0i}} = \frac{E_{0t} - E_{0i}}{E_{0i}} = \frac{E_{0t}}{E_{0i}} - 1 \qquad \text{(A-7)}$$

Hence,

$$\left(\frac{E_{0t}}{E_{0i}}\right)_\perp = 1 + \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_i + \frac{n_t}{\mu_t}\cos\theta_t} \qquad \text{(A-8)}$$

using eqn (A-6)

• Eqns (A-6) and (A-8) apply to any linear, isotropic, homogeneous media, and are two of the Fresnel Equations.

• If we are primarily interested in non-magnetic materials, then $\mu_i \approx \mu_r \approx \mu_0$, and (A-6) and (A-8) can be simplified to yield the commonly used forms of the Fresnel equations:

$$r_\perp \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \qquad \text{(A-9)}$$

$$t_\perp \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} \qquad \text{(A-10)}$$

Situation 2: Incident $\vec{\mathbf{E}}$ is ∥ to the plane-of-incidence

("p-polarized" or "Transverse Magnetic (TM) polarized")

- The calculations for this situation proceeds in an analogous way to Situation 1.

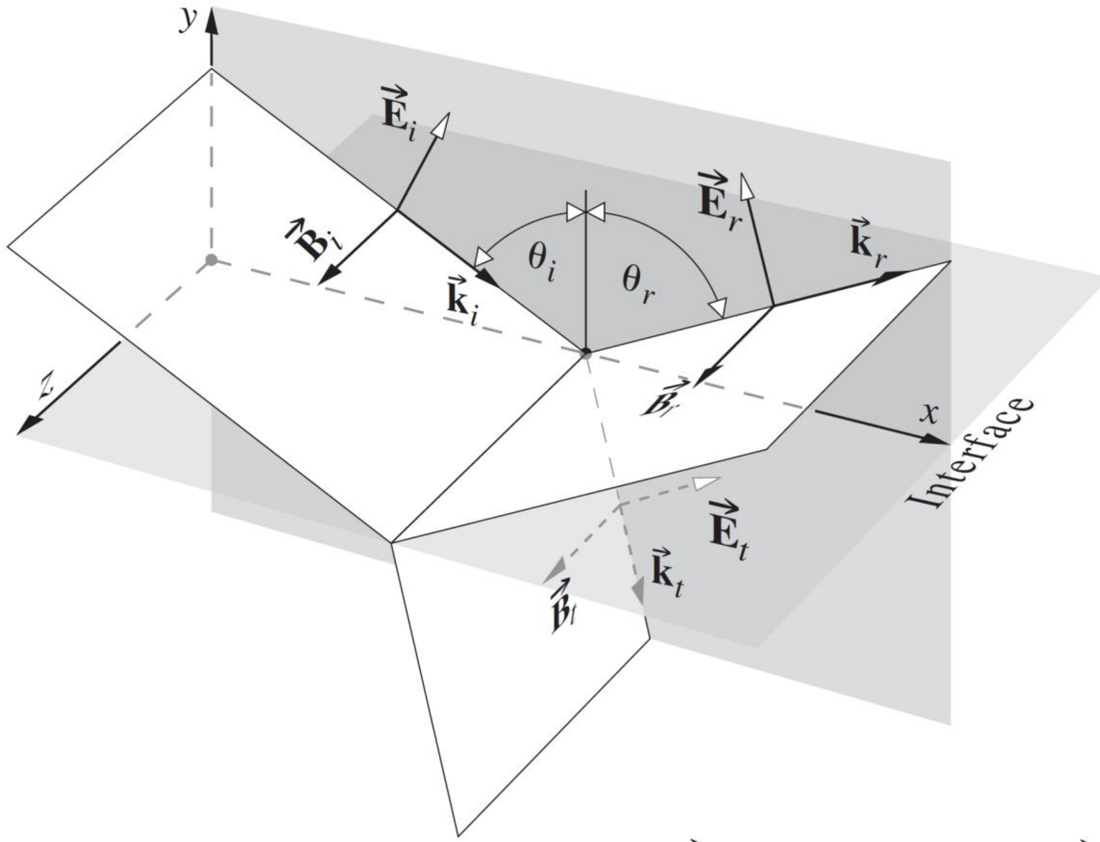

Fig. 3a

The fields shown are actually at the interface but have been drawn displaced so the vectors can be seen more clearly.

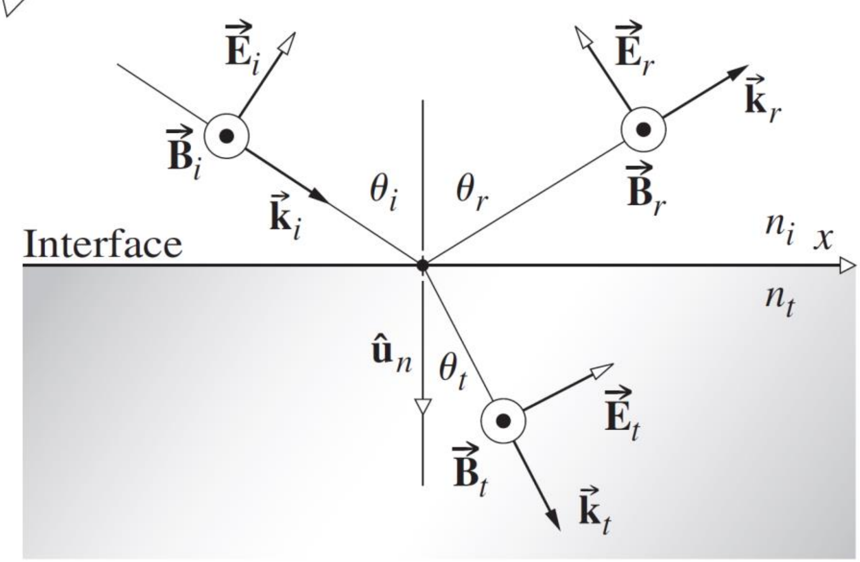

Fig. 3b

• The continuity of the tangential component of $\vec{\mathbf{E}}$ gives

$$E_i\cos\theta_i - E_r\cos\theta_r = E_t\cos\theta_t \qquad \text{(A-11)}$$

Again, the phases of electric fields are the same; hence the exponential phase terms that are present in E_i, E_r, E_t of eqn (A-11) cancel. This yields:

$$E_{0i}\cos\theta_i - E_{0r}\cos\theta_r = E_{0t}\cos\theta_t \qquad \text{(A-12)}$$

• The continuity of the tangential component of $\vec{\mathbf{B}}/\mu$ gives:

$$\frac{B_i}{\mu_i} + \frac{B_r}{\mu_i} = \frac{B_t}{\mu_t} \qquad \text{or} \qquad \frac{B_{0i}}{\mu_i} + \frac{B_{0r}}{\mu_i} = \frac{B_{0t}}{\mu_t} \qquad \text{(A-13)}$$

$\Rightarrow$ By using "$B = E/v$“, $v_i = v_r$ and "$v = nc$" , (A-13) yields:

$$\frac{n_i}{\mu_i}E_{0i} + \frac{n_i}{\mu_i}E_{0r} = \frac{n_t}{\mu_t}E_{0t} \qquad \text{(A-14)}$$

• Dividing both (A-12) and (A-14) by E_{0i}, use Law of Reflection $\theta_i = \theta_r$, (and carrying out some slight rearrangement of terms) we obtain the two simultaneous linear equations:

$$\left(\frac{E_{0t}}{E_{0i}}\right)\cos\theta_t + \left(\frac{E_{0r}}{E_{0i}}\right)\cos\theta_i = \cos\theta_i \qquad \text{(A-15)}$$

$$\left(\frac{E_{0t}}{E_{0i}}\right)\frac{n_t}{\mu_t} - \left(\frac{E_{0r}}{E_{0i}}\right)\frac{n_i}{\mu_i} = \frac{n_i}{\mu_i} \qquad \text{(A-16)}$$

• Of course, the two equations (A-15) and (A-16) can be solved by various methods for E_{0t}/E_{0i} and E_{0r}/E_{0i}. For example, in matrix notation:

$$\begin{bmatrix} \cos\theta_t & \cos\theta_i \\ \frac{n_t}{\mu_t} & -\frac{n_i}{\mu_i} \end{bmatrix} \begin{pmatrix} E_{0t}/E_{0i} \\ E_{0r}/E_{0i} \end{pmatrix} = \begin{pmatrix} \cos\theta_i \\ \frac{n_i}{\mu_i} \end{pmatrix} \qquad \text{(A-17)}$$

⇒ One method is to use Cramer's rule (forming the appropriate ratio of the determinants):

$$\frac{E_{0r}}{E_{0i}} = \frac{\begin{vmatrix} \cos\theta_t & \cos\theta_i \\ \frac{n_t}{\mu_t} & \frac{n_i}{\mu_i} \end{vmatrix}}{\begin{vmatrix} \cos\theta_t & \cos\theta_i \\ \frac{n_t}{\mu_t} & -\frac{n_i}{\mu_i} \end{vmatrix}} = \frac{\frac{n_t}{\mu_t}\cos\theta_i - \frac{n_i}{\mu_i}\cos\theta_t}{\frac{n_i}{\mu_i}\cos\theta_t + \frac{n_t}{\mu_t}\cos\theta_i} \qquad \text{(A-18)}$$

$$\frac{E_{0t}}{E_{0i}} = \frac{\begin{vmatrix} \cos\theta_i & \cos\theta_i \\ \frac{n_i}{\mu_i} & -\frac{n_i}{\mu_i} \end{vmatrix}}{\begin{vmatrix} \cos\theta_t & \cos\theta_i \\ \frac{n_t}{\mu_t} & -\frac{n_i}{\mu_i} \end{vmatrix}} = \frac{2\frac{n_i}{\mu_i}\cos\theta_i}{\frac{n_i}{\mu_i}\cos\theta_t + \frac{n_t}{\mu_t}\cos\theta_i} \qquad \text{(A-19)}$$

- Eqns (A-18) and (A-19) apply to any linear, isotropic, homogeneous media, and are the remaining two of the Fresnel Equations.

- If we are primarily interested in non-magnetic materials, then $\mu_i \approx \mu_r \approx \mu_0$, and (A-18) and (A-19) can be simplified to yield the two remaining commonly used forms of the Fresnel equations:

$$r_\parallel \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t + n_t\cos\theta_i}$$

$$t_\parallel \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i} \qquad \text{(A-20)}$$

The subscript ∥ in (A-20) serves as a reminder that we are dealing with $\vec{\boldsymbol{E}}$ that is ∥ to the plane-of-incidence.