Fermat's Principle; Optical Path Length

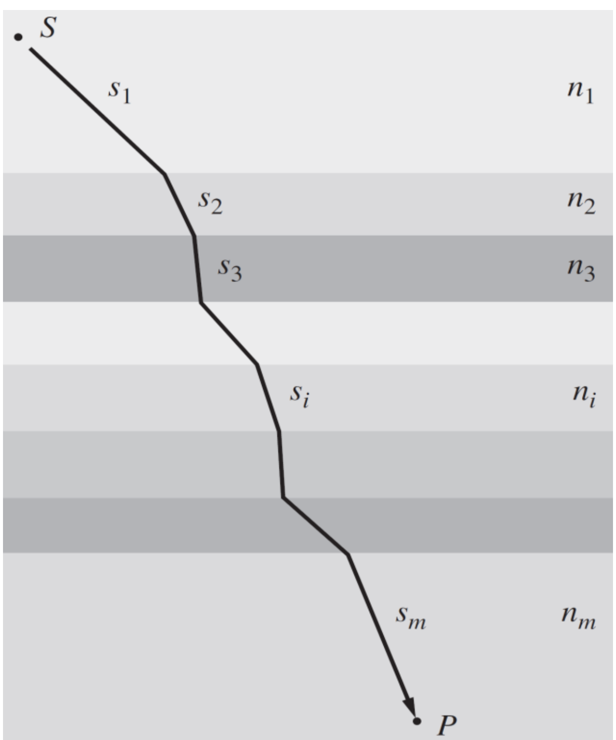
Optical Path Length (*OPL*):

- In a medium of constant index of refraction n, the *optical path length* (*OPL*) for a light ray that has traveled a geometric distance s is:

$$OPL = ns \qquad (1)$$

- More generally, if n varies along the path C, the *OPL* is given by a line integral:

$$OPL = \int_C n(s)ds \qquad (2)$$



Example:

Ray goes from S to P through the various layers shown in the figure.

$$OPL = \int_S^P n(s)ds = \sum_{i=1}^{m} n_i s_i$$

$$= n_1 s_1 + n_2 s_2 + \cdots + n_m s_m$$

Comments about *OPL*:

• The *OPL* corresponds to the **distance in vacuum that is equivalent to the distance traversed (s) in the medium of refractive index n.** Both quantities will (see example next page)

- correspond to the same number of wavelengths.

  e.g. Suppose EM wave travels a distance s in a medium of refractive index n, and has wavelength λ.

$$\text{Number of wavelengths} = \frac{s}{\lambda} = \frac{OPL}{\lambda_0}$$

since $\lambda = \lambda_0/n$ and $OPL = ns$

- correspond to the same phase change as the light advances.

  i.e., An EM wave propagating along a path Q has the phase change over Q as if it was propagating a path in a vacuum whose length is equal to the OPL of Q.

• The time taken to traverse the *OPL* at speed c is the same as the time taken to traverse s at speed $v = c/n$ (in a medium with constant n).

i.e.,
$$t = \frac{s}{v} = \frac{ns}{c} = \frac{OPL}{c}$$

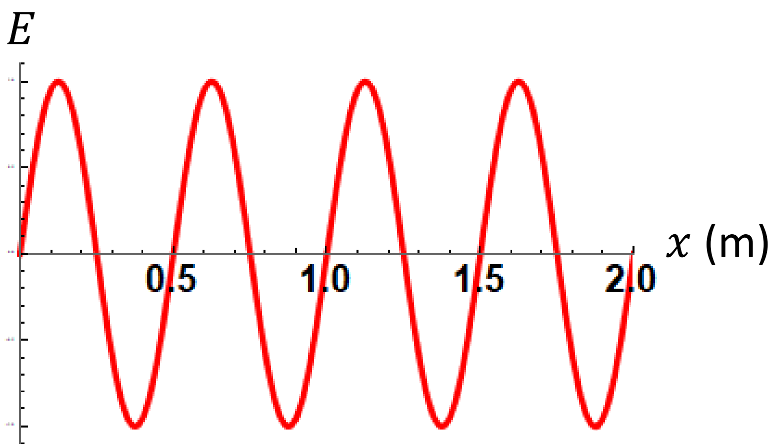
time taken to traverse s at speed v

time taken to traverse *OPL* at speed c

(We will repeatedly encounter the *OPL* in this course.)

Example (regarding *OPL*):

- Assume the wavelength of the light in vacuum is $\lambda_0 = 1.00$ m
- Now, suppose the light traverses a distance $s = 2.00$ m (in a straight line) in a medium with $n = 2.00$. The profile might look like this:



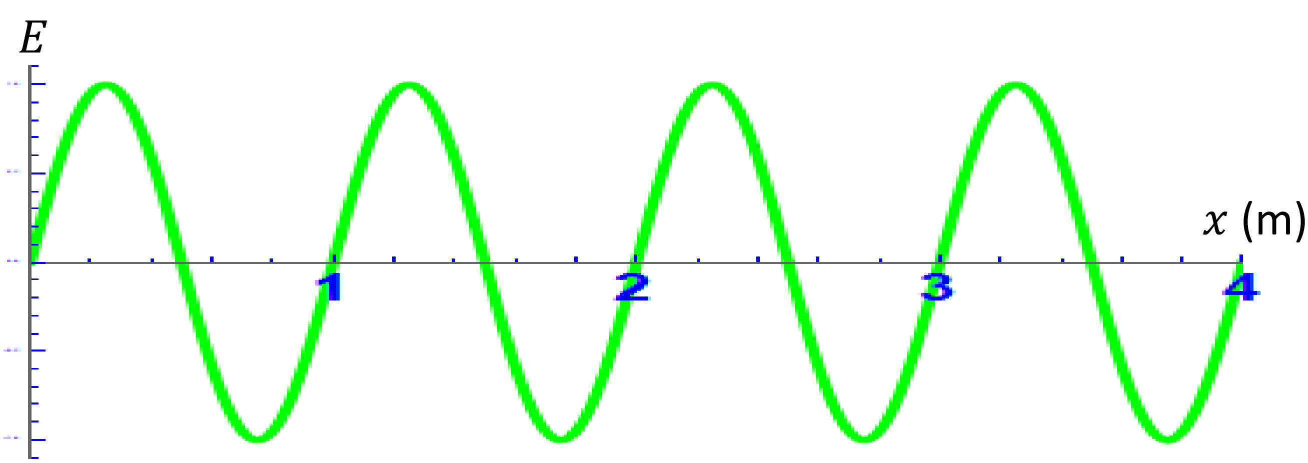
- Note that the wavelength **in the medium** is $\lambda = \lambda_0/n = 0.500$ m.
- In the distance s (= 2 m), there are 4 wavelengths, or 8π radians.

- Now, consider the following situation in vacuum:

$$OPL = ns = (2.00)(2.00\text{ m}) = 4.00\text{ m}$$

Recall that the wavelength **in vacuum** is $\lambda_0 = 1.00$ m

- The profile in the vacuum would look like this:



- Note that in the distance OPL (= 4 m) , there are also 4 wavelengths, or 8π radians.

Fermat's Principle

Pierre de Fermat (1580-1626)

- Fermat's principle states:

A ray of light in going from one point to another follows, regardless of the media involved, a route which corresponds to a stationary value of the optical path length.

Hence, the actual path is one for which the derivative of the *OPL* is zero.

(The extremum may be a minimum, maximum or a point of inflection.)

Note: (Valid if the extremum is a minimum.)

- Although not as generally valid, Fermat's principle is also commonly stated as:

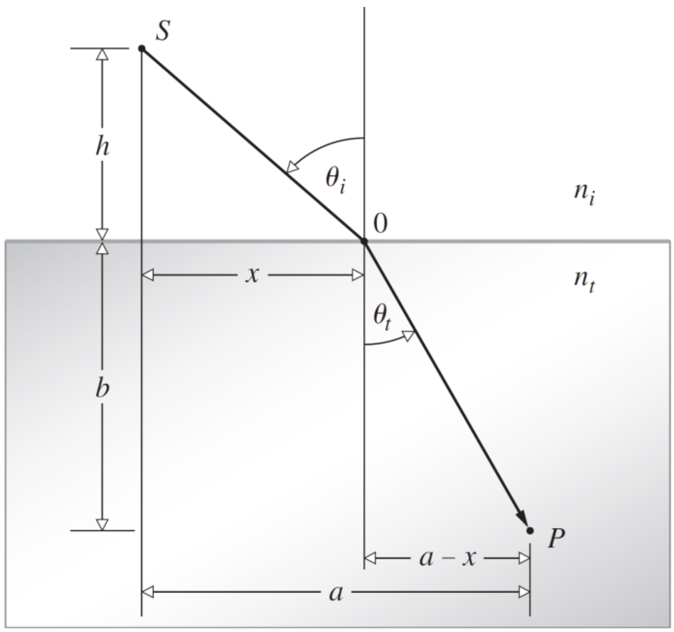
A ray traverses a route which corresponds to the shortest optical path length

or the "original" statement:

The actual path between two points taken by a beam of light is the one that is traversed in the least time.

Principle of reversibility: if a ray travels from *S* to *P* along some path, it will also travel back from *P* to *S* along the same path. i.e., The light follows the same path if the direction of travel is reversed.

Example: Use Fermat's Principle to derive Snell's Law.



$$OPL = n_i\overline{SO} + n_t\overline{OP}$$

$$= n_i(x^2 + h^2)^{1/2} + n_t[(a-x)^2 + b^2]^{1/2}$$

The variable is x so from Fermat's Principle:

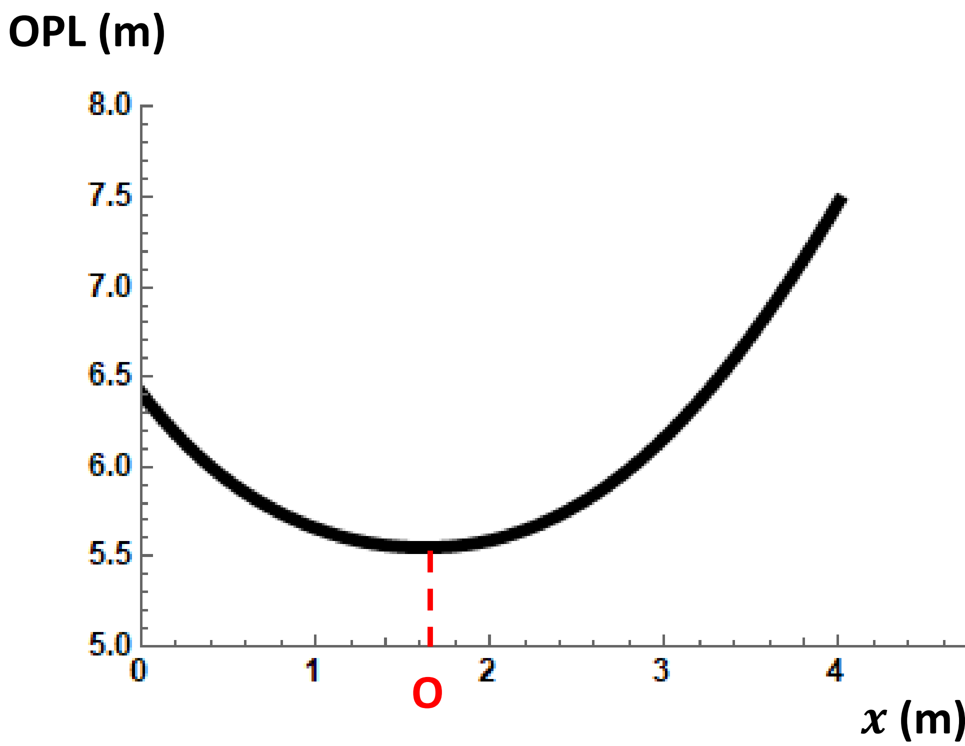
$$\frac{d(OPL)}{dx} = 0 = n_i x(x^2 + h^2)^{-1/2} - n_t(a-x)[(a-x)^2 + b^2]^{-1/2}$$

$$0 = n_i\sin\theta_i - n_t\sin\theta_t$$

Hence,

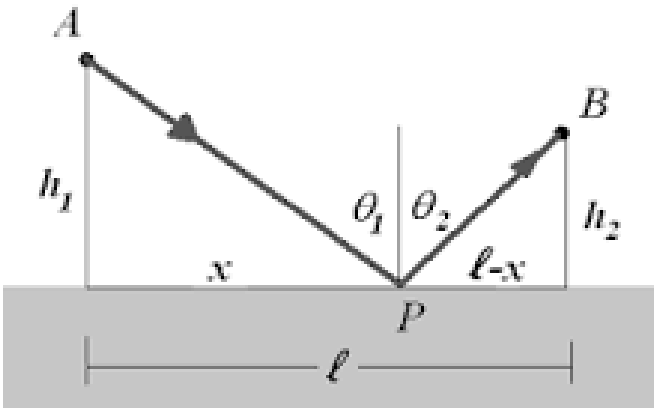
$$\boxed{n_i\sin\theta_i = n_t\sin\theta_t}$$

ps. In this problem, the location of point O corresponds to a path of minimum *OPL.*



The above plot was made using the following values:
$n_i = 1.00$, $n_t = 1.50$, $h = 1.00$ m, $b = 2.00$ m, $a = 3.00$ m.
The point O corresponds to $x = 1.62169$ m, or $\theta_i = 58.34^0$ and $\theta_t = 34.57^0$.

Example: Use Fermat's Principle to derive the Law of Reflection.



$$OPL = n_i\overline{AP} + n_i\overline{PB}$$

$$= n_i(x^2 + h_1^2)^{1/2} + n_i[(l-x)^2 + h_2^2]^{1/2}$$

The variable is x so from Fermat's Principle:

$$\frac{d(OPL)}{dx} = 0 = n_i\left(\frac{1}{2}\right)\frac{2x}{(x^2 + h_1^2)^{1/2}} + n_i\left(\frac{1}{2}\right)\frac{2(l-x)(-1)}{[(l-x)^2 + h_2^2]^{1/2}}$$

(value of n_i does not matter)

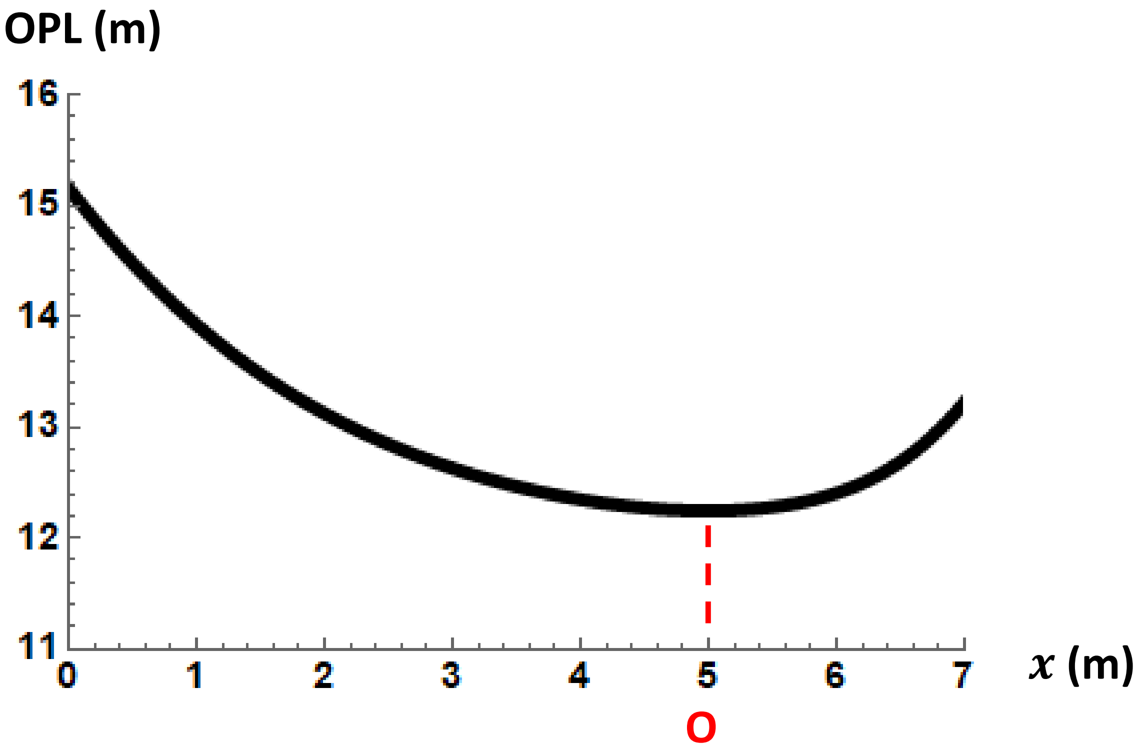
This reduces to:

$$\frac{x}{(x^2 + h_1^2)^{1/2}} = \frac{(l-x)}{[(l-x)^2 + h_2^2]^{1/2}}$$

which is $\sin\theta_1 = \sin\theta_2$

Hence, $\boxed{\theta_i = \theta_t}$

ps. In this problem, the location of point O corresponds to a path of minimum *OPL.*



The above plot was made using the following values:
$n_i = 1.50$, $h_1 = 3.00$ m, $l = 7.00$ m, $h_2 = 1.20$ m.
The point O corresponds to $x = 5.000$ m, or $\theta_1 = \theta_2 = 30.96^0$.