Transmit Radiation Pattern Invariance in MIMO Radar With Application to DOA Estimation

Aboulnasr Hassanien, Sergiy A. Vorobyov, and Arash Khabbazibasmenj

Abstract—The desired property of having the same beampattern for different columns of a beamspace transformation matrix (beamforming vectors) often plays a key importance in practical applications. At most $2^{M-1} - 1$ beamforming vectors with the same beampattern can be generated from any given beamforming vector, where M is the size of the beamforming vector. Thus, one can start with a single (mother) beamforming vector, which gives a desired beampattern, but may not satisfy some other desired properties, and generate all other beamforming vectors, which give the same beampattern, in a computationally efficient way. Then the beamforming vectors, which in addition satisfy other desired properties that the mother beamforming vector may not satisfy, can be selected. Such procedure is developed in this letter in the application to the transmit beamspace design that ensures practically important properties for multiple-input multiple-output radar. A computationally efficient sub-optimal method for selecting best beamforming vectors from a population of vectors that give the same beampattern is also developed.

Index Terms—Array processing, beamforming, beampattern design, dimensionality reduction, multiple-input multiple-output (MIMO) radar.

I. INTRODUCTION

Beamspace transformation [1], [2] and beamforming [3] techniques are the key approaches, among others, in array signal processing [4]-[6], radar [7], multiple-input multiple-output (MIMO) radar [8]-[21], wireless communications [22]-[24], data compression and dimensionality reduction [25], [26], biomedical engineering [27], etc.

In the traditional applications in array processing and dimensionality reduction, it is often desirable to reduce the high dimensional space into a lower one by means of the beamspace transformations. In more recent applications to MIMO radar, it has been required not only to design a lower dimensional transmit beamspace but also to transmit a number of orthogonal waveforms from a larger number of transmit antenna elements while achieving transmit coherent processing gain. While designing such a transmit beamspace, certain properties have to be satisfied such as a uniform power distribution for different transmit waveforms in the desired sector where the targets are likely to be located. The latter enables, for example, to enforce at the transmitter the very useful rotational invariance property [28], [29] which can

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significantly simplify and improve, for example, the direction-of-arrival (DOA) estimation at the receive antenna array.

Practically, generating a transmit beamspace that satisfies some additional to transmit radiation pattern invariance properties is desirable [29]. Thus, it is of interest to design simple technique that starts with a single beamforming vector, which we call the mother beamforming vector by analogy with mother wavelet [30], and generate a number of other beamforming vectors that all have the same beampattern as the mother beamforming vector. Then, beamforming vectors satisfing some additional practical properties, e.g., a uniform power distribution across the transmit array elements can then be selected from a population of so generated vectors. In wavelets, self-similarity is an important property where basis functions are all obtained from a single prototype mother wavelet using scaling and translation. A similar property can be used also in our beamspace design problem. To the best of the authors knowledge, such approach has not been used before to the transmit beamspace design in MIMO radar.

II. TRANSMIT RADIATION PATTERN INVARIANCE

Consider a uniform linear array (ULA) of size M. The steering vector of the array towards direction θ is denoted as $\mathbf{a}(\theta)$. The transmit array beampattern can be expressed as

$$p(\theta) = \|\mathbf{w}^H \mathbf{a}^*(\theta)\|^2 \qquad (1)$$

where $\mathbf{w} \triangleq [w_1, w_2, \cdots, w_M]^T$ is the $M \times 1$ beamforming vector and $\| \cdot \|$, $(\cdot)^T$, $(\cdot)^H$ and $(\cdot)^*$ stand for the Euclidean norm, transpose, Hermitian transpose of a vector and conjugation, respectively. Let the beampattern corresponding to a given beamforming vector $\mathbf{w}$, referred to as the mother beamforming vector, satisfy certain shape design requirements, but it does not satisfy other practically important requirements. Such a requirement is, for example, a uniform power distribution across the antenna elements. The question then arises about existence of other distinct beamforming vectors which generate the same exact beampattern as the mother beamforming vector $\mathbf{w}$ and which in addition satisfy other possible design requirements.

The total number of other distinct beamforming vectors with the same exact beampattern as the mother beamforming vector $\mathbf{w}$, which corresponds to a ULA of size M, is at most $2^{M-1} - 1$. This can be justified by considering the fact that the beampattern in equation (1) is a non-linear mapping of the frequency response of a finite impulse response (FIR) filter with coefficients $w_i, i = 1, \cdots, M$. Such an FIR filter has at most $M - 1$ distinct roots and it is quite well-known that

reflecting each root against the unit circle, i.e., inverse conjugating, does not change the frequency response magnitude [31]. More details can be found in [32]. From the viewpoint of constructing a population of all possible beamforming vectors that have the same beampattern as the mother beamforming vector $\mathbf{w}$, let us consider (1) as an invariant for the following function of a single variable x

$$f(x) \triangleq \overbrace{(w_1 + w_2 x + w_3 x^2 + \cdots + w_M x^{M-1})}^{\text{First Multiplicative Term}} \qquad (2)$$
$$\times \underbrace{(w_1^* + w_2^* x^{-1} + w_3^* x^{-2} + \cdots + w_M^* x^{-M+1})}_{\text{Second Multiplicative Term}}.$$

From equation (1), it can be immediately concluded that $p(\theta) = f(e^{j\pi \sin(\theta)})$. Let x_0 be a non-zero root of the first multiplicative term in (2), i.e., $w_1 + w_2 x + w_3 x^2 + \cdots + w_M x^{M-1}$. Then, it is simple to verify that $1/x_o^*$ is also a root of the second multiplicative term $w_1^* + w_2^* x^{-1} + w_3^* x^{-2} + \cdots + w_M^* x^{-M+1}$ of $f(x)$. One implication of this observation is that the inverse conjugate of every root of the function (2) is also a root of $f(x)$ and, therefore, the roots of $f(x)$ can be denoted as x_i and $1/x_i^*$, $i = 1, \cdots, M-1$ and $f(x)$ can be decomposed as

$$f(x) = |w_M|^2 \left(\frac{w_1}{w_M} + \frac{w_2}{w_M} x + \frac{w_3}{w_M} x^2 + \cdots + x^{M-1} \right)$$
$$\times \left(\frac{w_1^*}{w_M^*} + \frac{w_2^*}{w_M^*} x^{-1} + \frac{w_3^*}{w_M^*} + \cdots + x^{-M+1} \right)$$
$$= |w_M|^2 \prod_{i=1}^{M-1} (x - x_i) \prod_{i=1}^{M-1} (x^{-1} - x_i^*). \qquad (3)$$

Furthermore, it is easy to verify that the product $(x - x_i)(x^{-1} - x_i^*)$ can be equivalently expressed as

$$(x - x_i)(x^{-1} - x_i^*) = |x_i|^2 \left(x - \frac{1}{x_i^*} \right) \left(x^{-1} - \frac{1}{x_i} \right). \qquad (4)$$

Note that the product terms $\prod_{i=1}^{M-1}(x - x_i)$ and $\prod_{i=1}^{M-1}(x^{-1} - x_i^*)$ that appear in (3) will preserve the structure of the first and second multiplicative terms in (2) for any arbitrary $x_i, i = 1, \cdots, M-1$. Based on these observations, the function (2) can be decomposed as the multiplication of two terms in the form of $v_1 + v_2 x + v_3 x^2 + \cdots + v_M x^{M-1}$ and $v_1^* + v_2^* x^{-1} + v_3^* x^{-2} + \cdots + v_M^* x^{-M+1}$ in 2^{M-1} different ways depending on whether x_i (or $1/x_i^*$,) $i = 1, \cdots, M-1$ is the root of the first polynomial.

III. APPLICATION TO TRANSMIT BEAMSPACE DESIGN IN MIMO RADAR

Consider a MIMO radar with transmit ULA of M antenna elements spaced half a wavelength apart. The total transmit power is normalized to $P_{\text{t}} = M$. Waveform diversity in tandem with transmit beamforming have been employed in the literature via transmitting multiple orthogonal waveforms over multiple transmit beams. The $M \times 1$ vector of baseband representation of the signals at the input of the transmit antennas is given as

$$\mathbf{x}(t) = \sum_{k=1}^{K} \psi_k(t) \mathbf{w}_k^* \qquad (5)$$

where t is the fast time index, $\psi_k(t)$, $k = 1, \ldots, K$ are K orthogonal waveforms, and $\mathbf{w}_k$, $k = 1, \ldots, K$ are the associated transmit weight vectors. Existing methods for designing transmit beamforming in MIMO radar adopt joint design approach which can be computationally expensive especially if the involved optimization is non-convex. Moreover, the resulting solution always yields transmit beamforming weight vectors which have different individual transmit power radiation patterns. As a result, the received data at the output of the matched-filters at the receive array becomes contaminated with non-uniform noise, interference, and/or clutter components which may deteriorate the DOA estimation performance.

Here we develop a simpler approach via designing a single mother transmit beamforming weight vector. Then, the required K transmit beamforming weight vectors, which are guaranteed to have the exact same transmit radiation pattern, can be selected from the population of $2^{M-1} - 1$ associated weight vectors. Note that existing sophisticated and computationally efficient FIR filter design techniques can be used to obtain the mother beamforming weight vector [31], [33], [34].

One key practical requirement in MIMO radar is to have uniform power distribution across the transmit array elements. The selection of K weight vectors from the population of $2^{M-1} - 1$ vectors while satisfying the uniform power constraint can be cast as the following optimization problem

$$\min_{\mathbf{w}_1, \ldots, \mathbf{w}_K} \eta$$
$$\text{s.t.} \sum_{k=1}^{K} |\mathbf{w}_{[k,m]}|^2 \leq \eta, \quad m = 1, \ldots, M \qquad (6)$$
$$\{\mathbf{w}_1, \ldots, \mathbf{w}_K\} \in \mathbf{W}_{\text{pop}}$$

where $\mathbf{W}_{\text{pop}}$ is the population of $2^{M-1} - 1$ associated weight vectors obtained from the mother weight vector and $\mathbf{w}_{[k,m]}$ refers to the mth entry of $\mathbf{w}_k$. The optimization problem (6) is computationally demanding when exhaustive search is done over all vectors on $\mathbf{W}_{\text{pop}}$.

It is worth noting, however, that while building the population $\mathbf{W}_{\text{pop}}$, the larger the magnitude of a certain root x_i ($x_i \geq 1$), the larger the deviation between the two weight vectors associated with x_i and $1/x_i^*$, respectively. Using this observation, a computationally efficient way of solving the optimization problem (6), at least sub-optimally, is to obtain a reduced size population $\tilde{\mathbf{W}}_{\text{pop}} \subset \mathbf{W}_{\text{pop}}$ and then search within this reduced size population. This can be done by dividing the roots (3) into two groups. The first group consists of the Q roots with the largest magnitude and the second group contains the remaining $M - Q - 1$ roots. Therefore, (3) can be rewritten as

$$f(x) = |w_M|^2 \prod_{i=1}^{Q} (x - x_i) \prod_{i=1}^{M-Q-1} (x - x_i)$$
$$\times \prod_{i=1}^{Q} (x^{-1} - x_i^*) \prod_{i=1}^{M-Q-1} (x^{-1} - x_i^*)$$
$$= |w_M|^2 h(x) \prod_{i=1}^{Q} (x - x_i) \prod_{i=1}^{Q} (x^{-1} - x_i^*) \qquad (7)$$

where $h(x) \triangleq \prod_{i=1}^{M-Q-1}(x-x_i)\prod_{i=1}^{M-Q-1}(x^{-1}-x_i^*)$. Using (7), a reduced population of size $2^Q - 1$ can be obtained. Thus, the use of small to moderate values of Q will result in a computationally achievable sub-optimal solution to the problem (6) even if exhaustive search is used.

IV. SIMULATION RESULTS

We assume a ULA of $M = 10$ transmit antenna elements spaced half a wavelength apart from each other. Two mother transmit beamforming weight vectors are designed in two different ways to focus the transmit energy within the sector $\boldsymbol{\Theta} = [-10°, \ 10°]$. The first mother beamforming weight vector is designed using spheroidal sequences technique [2] (see also [19] in application to transmit beamforming design in MIMO radar). Specifically, it is computed as $\mathbf{w}_{\mathrm{SPH}} = \sqrt{M/2}(\mathbf{u}_1 + \mathbf{u}_2)$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the two principle eigenvectors of the matrix $\mathbf{A} = \int_{\boldsymbol{\Theta}} \mathbf{a}(\theta)\mathbf{a}^H(\theta)d\theta$. The second mother beamforming weight vector is designed using convex optimization to control the sidelobe levels. In particular, it is obtained by solving the following convex optimization problem [19]

$$\min_{\mathbf{w}} \max_i \|\mathbf{w}^H\mathbf{a}(\theta_i) - e^{-j\phi_i}\|, \quad \theta_i \in \boldsymbol{\Theta}, \ i = 1, \ldots, I$$
$$\text{subject to } \|\mathbf{w}^H\mathbf{a}(\theta_k)\| \le \delta, \quad \theta_k \in \bar{\boldsymbol{\Theta}}, \ k = 1, \ldots, K$$

combined with the ripple and transition band control design capability [33], [34]. Here $\bar{\boldsymbol{\Theta}}$ combines a continuum of all out-of-sector directions, i.e., directions lying outside the sector-of-interest $\boldsymbol{\Theta}$; $\phi_i, i = 1, \ldots, I$ is the desired transmit phase profile of user choice; and $\delta > 0$ is the parameter of the user choice that characterizes the worst acceptable level of transmit power radiation in the out-of-sector region $\bar{\boldsymbol{\Theta}}$. The phase $\phi_i = 2\pi\sin(\theta_i)$ and the parameter $\delta = 0.1$ are chosen, i.e, the sidelobe levels are kept below $20\log\delta = -20$ dB. The resulting mother beamforming weight vector is referred to hereafter as $\mathbf{w}_{\mathrm{CVX}}$. The transmit beampatterns associated with $\mathbf{w}_{\mathrm{SPH}}$ and $\mathbf{w}_{\mathrm{CVX}}$ are shown as the dotted and solid curves, respectively, in Fig. 1.

The mother beamforming weight vector $\mathbf{w}_{\mathrm{SPH}}$ is used to generate a population of $2^{10-1} - 1 = 511$ other vectors. To implement a MIMO radar system with four orthogonal transmit waveforms, four beamforming weight vectors among the population that achieve the best transmit power distribution across the transmit array elements are chosen by solving (6) using exhaustive search for $Q = 4$ and $Q = M$. It is found that the final solution for both selections of Q are identical. Therefore, the choice of $Q << M$ is sufficient in practice. The four chosen weight vectors are denoted as $\mathbf{w}_{\mathrm{SPH}}^{(j)}$, $j = 1, \ldots, 4$, and are scaled such that $\sum_j \|\mathbf{w}_{\mathrm{SPH}}^{(j)}\|^2 = M$. Each of the vectors $\mathbf{w}_{\mathrm{SPH}}^{(j)}$, $j = 1, \ldots, 4$ has the same transmit radiation pattern as the mother vector except for a magnitude scaling factor of $1/4$. Note that the beampattern magnitude in the mainlobe as well as in the sidelobe regions is scaled by the same scaling factor, i.e., the relative attenuation of the sidelobes with respect to the mainlobe remains unchanged. The transmit power distribution across the transmit array elements for the mother transmit beamforming vector $\mathbf{w}_{\mathrm{SPH}}$ operated
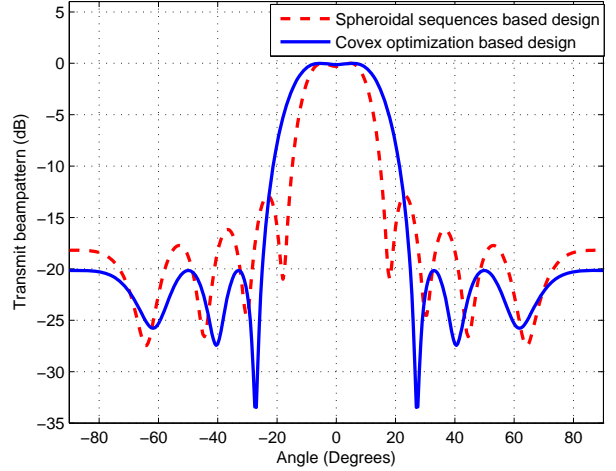

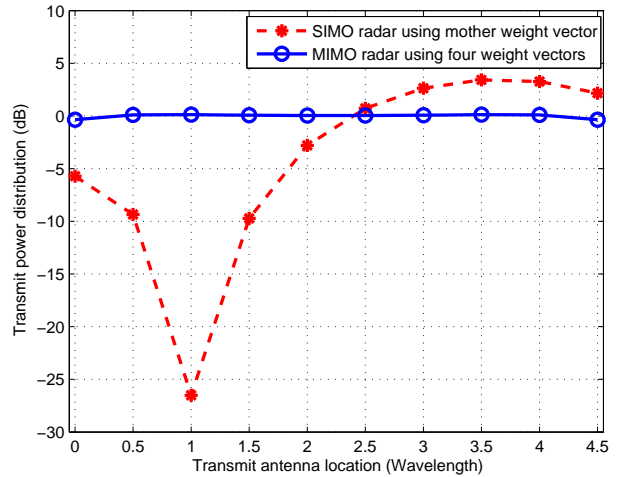Fig. 1. Transmit beampattern using spheroidal sequences and convex optimization based designs.

Fig. 2. Transmit power distribution across the transmit array elements for spheroidal-sequences based transmit beamforming design.

in a single-input-multiple-output (SIMO) radar mode and for the chosen vectors $\mathbf{w}_{\mathrm{SPH}}^{(j)}$, $j = 1, \ldots, 4$ operated in a MIMO radar mode are shown in Fig. 2. It can be seen from the figure that the mother beamforming vector has very poor transmit power distribution across antenna elements. For example, the power radiated from the third transmit array element is over 25 dB less than the average transmit power per transmit array element. On the other hand, the four chosen beamforming weight vectors exhibit transmit power distribution that is almost uniform, which is desirable in practice.

Similarly, the mother beamforming weight vector $\mathbf{w}_{\mathrm{CVX}}$ is used to generate a population of 511 beamforming vectors which have the exact same beampattern. The four beamforming weight vectors among the population that achieve the best transmit power distribution across the transmit array elements are then chosen by solving (6) using exhaustive search for $Q = 4$. The four chosen weight vectors are denoted as $\mathbf{w}_{\mathrm{CVX}}^{(j)}$, $j = 1, \ldots, 4$. The transmit power distributions
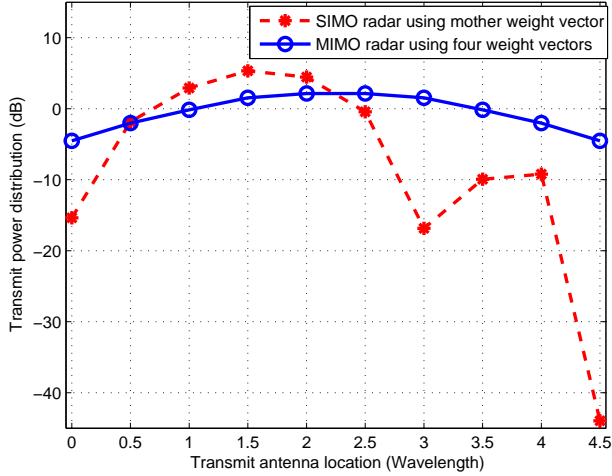
Fig. 3. Transmit power distribution across the transmit array elements for convex optimization based transmit beamforming design.

Fig. 4. RMSE versus SNR.

Fig. 5. Probability of source resolution versus SNR.

across the transmit antenna array elements for the mother weight vector and for the vectors $\mathbf{w}_{\text{CVX}}^{(j)}$, $j = 1, \ldots, 4$ are shown in Fig. 3. It can be seen from the figure that the four chosen vectors yield much better transmit power distribution as compared to the mother beamforming weight vector.

In the last example, we test our design via estimating the DOAs of two targets that are assumed to be located at directions $3°$ and $5°$, respectively. The total transmit power is fixed to $P_\text{t} = M = 10$. We compare the performance of the SIMO radar with the spheroidal-sequences based mother weight vector $\mathbf{w}_{\text{SPH}}$ to the performance of MIMO radar with $\mathbf{w}_{\text{SPH}}^{(j)}$, $j = 1, \ldots, 4$ from the previous example. For both scenarios tested, we perform the simulations for the (practically unattractive) case when no transmit power clipping is used as well as for the (practically attractive) case of restricting each antenna to transmit maximum power which is set to be unity. In the latter case, if the power of the signals fed to each antenna is less than unity it is left unchanged while if the power is more that unity then the weights associated with this antenna are scaled such that the power of signal fed to the corresponding antenna is unity, i.e., power clipping is enforced. Figs. 4 and 5 show the DOA estimation root-mean-square-error (RMSE) and the probability of source resolutions, respectively. It can be seen form the figures that the MIMO radar case has better performance than the SIMO radar case even without power clipping. When power clipping is enforced, the performance of SIMO radar deteriorates even more. On the other hand, the performance of the MIMO radar with power clipping is almost the same as that of the MIMO radar without power clipping which can be attributed to the careful selection of the best four weight vector among the population that achieve near uniform transmit power distribution across the array elements.

V. CONCLUSION

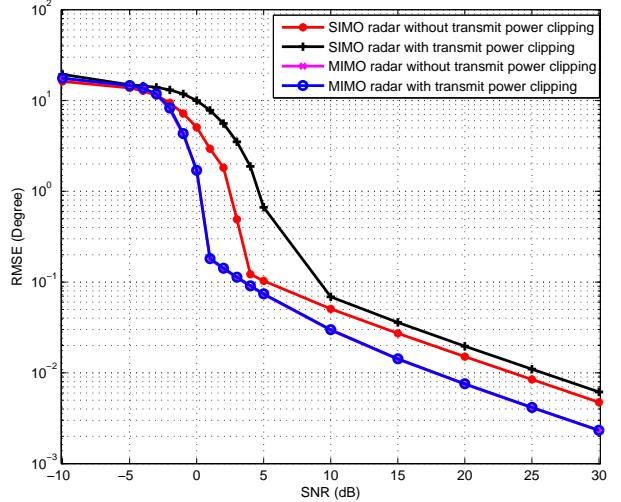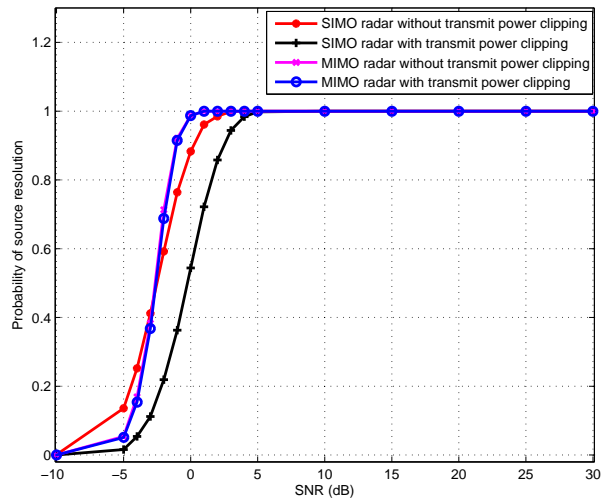Efficient approach for designing a transmit beamspace transformation in MIMO radar that satisfies transmit radiation pattern invariance property and other practically significant properties has been developed. It starts with designing a mother beamforming vector, which gives a desired radiation pattern, and is based on generating other at most $2^{M-1} - 1$ vectors with the same radiation pattern, and selecting the vectors, which also satisfy additional properties. It has been shown how this design can be utilized in the field of transmit beamspace design for MIMO radar, where it is desirable that different transmit waveforms are radiated with the same transmit beampattern and the distribution of transmit power across antenna elements is uniform. The computationally efficient sub-optimal approach for selecting best beamforming vectors from a population of vectors that give the same beampattern has been developed. The proposed approach has been also tested by simulations in application to DOA estimation.

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