2011 4th IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing

A Computationally Efficient Robust Adaptive Beamforming for General-Rank Signal Model with Positive Semi-Definite Constraint

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***Abstract*—The robust adaptive beamforming problem for general-rank signal model with positive semi-definite (PSD) constraint is considered. The existing approaches for solving the corresponding non-convex optimization problem are iterative methods for which the convergence is not guaranteed. Moreover, these methods solve the problem only suboptimally. We revisit this problem and develop a new beamforming method based on a new solution for the corresponding optimization problem. The new proposed method is iterative and is based on a reformulation and then linearization of a single non-convex difference-of-two-convex functions (DC) constraint. Our simulation results confirm that the new proposed method finds the global optimum of the problem in few iterations and outperforms the state-of-the-art robust adaptive beamforming methods for general-rank signal model with PSD constraint.**

I. Introduction

Most of the robust adaptive beamforming methods have been developed for the case of point source for which the rank of the desired source covariance matrix is equal to one [1]- [4]. However, in some practical applications such as the incoherently scattered signal source or source with fluctuating (randomly distorted) wavefronts, the rank of the source covariance matrix is higher than one. The robust adaptive beamformer for the general-rank signal model that is based on explicit modeling of the error mismatches has been developed in [5]. Despite its simplicity, the robust adaptive beamformer of [5] is known to be overly conservative [6], [7]. Thus, less conservative approaches have been developed in [6] and [7] by adding an additional positive semi-definite (PSD) constraint to the beamformer of [5] which eventually leads to a non-convex optimization problem. The proposed iterative approaches for this non-convex problem do not have guaranteed convergence and are able to find only suboptimal solutions. It motivates us to look for a new efficient and exact solution for the aforementioned non-convex problem. Thus, we develop an iterative method based on a reformulation of the aforementioned problem and a linearization of a single non-convex difference-of-two-convex functions (DC) constraint. Simulation results show that the proposed method finds the global optimum in few iterations and outperforms the other state-of-the-art methods.

II. System Model and Problem Formulation

The narrowband signal received by a linear antenna array with M omni-directional antenna elements at the time instant k can be expressed as

$$\mathbf{x}(k) = \mathbf{s}(k) + \mathbf{i}(k) + \mathbf{n}(k) \tag{1}$$

where $\mathbf{s}(k)$, $\mathbf{i}(k)$, and $\mathbf{n}(k)$ are the statistically independent $M \times 1$ vectors of the desired signal, interference, and noise, respectively. The beamformer output at the time instant k is given as

$$y(k) = \mathbf{w}^H \mathbf{x}(k) \tag{2}$$

where $\mathbf{w}$ is the $M \times 1$ complex beamforming vector of the antenna array and $(\cdot)^H$ stands for the Hermitian transpose. The beamforming problem for general-rank signal model is formulated as finding the vector $\mathbf{w}$ which maximizes the beamformer output signal-to-noise-plus-interference ratio (SINR) given as

$$\text{SINR} = \frac{\mathbf{w}^H \mathbf{R}_s \mathbf{w}}{\mathbf{w}^H \mathbf{R}_{i+n} \mathbf{w}} \tag{3}$$

where $\mathbf{R}_s \triangleq E\{\mathbf{s}(k)\mathbf{s}(k)^H\}$ and $\mathbf{R}_{i+n} \triangleq E\{(\mathbf{i}(k) + \mathbf{n}(k))(\mathbf{i}(k) + \mathbf{n}(k))^H\}$ denote the desired signal and interference-plus-noise covariances matrices, respectively, and $E\{\cdot\}$ stands for the statistical expectation. The sample matrix inversion (SMI)-based solution to the problem of maximizing the beamformer output SINR can be found as [1]

$$\mathbf{w}_{\text{SMI}} = \rho\{\hat{\mathbf{R}}^{-1}\mathbf{R}_s\} \tag{4}$$

where $\rho(\cdot)$ stands for the principal eigenvector operator and $\hat{\mathbf{R}}$ denotes the data sample covariance matrix calculated for K training data samples as

$$\hat{\mathbf{R}} = \frac{1}{K}\sum_{i=1}^{K} \mathbf{x}(i)\mathbf{x}^H(i). \tag{5}$$

This matrix replaces the matrix $\mathbf{R}_{i+n}$ which is typically unavailable.

Robust adaptive beamforming techniques for general-rank signal model address the situation when the desired signal covariance matrix $\mathbf{R}_s$ is not known precisely as well as the sample estimate of the data covariance matrix (5) is inaccurate (for example, because of small sample size).

In order to provide robustness against the norm-bounded mismatches $\|\mathbf{\Delta}_1\| \leq \epsilon$ and $\|\mathbf{\Delta}_2\| \leq \gamma$ (here $\|\cdot\|$ denotes the Frobenius norm of a matrix) in the desired signal and data sample covariance matrices, respectively, the following worst-case-based robust adaptive beamformer has been derived in [5]

$$\mathbf{w} = \rho\{(\hat{\mathbf{R}} + \gamma\mathbf{I})^{-1}(\mathbf{R}_s - \epsilon\mathbf{I})\}. \quad (6)$$

Although it is a simple closed-form solution, it is overly conservative due to the fact that the negatively diagonally loaded signal covariance matrix can be indefinite [5]– [7]. A less conservative robust adaptive beamforming problem formulation, which enforces the matrix $\mathbf{R}_s + \mathbf{\Delta}_1$ to be PSD has been considered in [6] and [7]. Defining $\mathbf{R}_s = \mathbf{Q}^H\mathbf{Q}$, the corresponding robust adaptive beamforming problem for a norm bounded-mismatch $\|\mathbf{\Delta}\| \leq \eta$ to the matrix $\mathbf{Q}$ is given as [6]

$$\begin{aligned} &\min_{\mathbf{w}} \max_{\|\mathbf{\Delta}_2\|\leq\gamma} \mathbf{w}^H(\hat{\mathbf{R}} + \mathbf{\Delta}_2)\mathbf{w} \\ &s.t. \min_{\|\mathbf{\Delta}\|\leq\eta} \mathbf{w}^H(\mathbf{Q}+\mathbf{\Delta})^H(\mathbf{Q}+\mathbf{\Delta})\mathbf{w} \geq 1. \end{aligned} \quad (7)$$

If the mismatch of the signal covariance matrix is small enough, the optimization problem (7) can be equivalently recast as [6]

$$\min_{\mathbf{w}} \mathbf{w}^H(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{w} \quad \text{s.t.} \quad \|\mathbf{Q}\mathbf{w}\| - \eta\|\mathbf{w}\| \geq 1 . \quad (8)$$

Due to the non-convex DC constraint, the problem (8) is non-convex. DC problems are believed to be NP-hard in general. This problem is solved using an iterative semidefinite relaxation (SDR)-based algorithm in [6] for which, however, the convergence is not guaranteed and the iterative method does not result in the optimal solution. Another iterative algorithm has been recently proposed in [7]. Using the facts that the constraint in (8) is active at optimality and that any scaled version of the optimal beamforming vector does not change the output SINR, the following problem has been considered instead of (8)

$$\begin{aligned} &\min_{\mathbf{w}} \quad \mathbf{w}^H(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{w} \\ &\text{s.t.} \quad 1 - \eta\|\mathbf{w}\| = c, \quad \|\mathbf{Q}\mathbf{w}\| = 1 \end{aligned} \quad (9)$$

where $0 < c < 1$ and its optimal value is defined to be the one for which the objective of (9) takes its minimum value. An iterative bisection method for finding the optimal value of c has been proposed in [7] which is only valid for the special case of well separated interference and desired signals. Since c is also dependent on $\mathbf{w}$, the problem (9) is not equivalent to (8). This fact has not been considered in [7]. Therefore, the method proposed in [7] also leads to a suboptimal solution, and furthermore, it is not valid if interference is not well-separated from the desired signal.

III. New Proposed Method

By introducing the auxiliary optimization variable $\alpha \geq 1$ and setting $\|\mathbf{Q}\mathbf{w}\| = \sqrt{\alpha}$ (here $\|\cdot\|$ is the Euclidian norm of a vector), the problem (8) can be equivalently rewritten as

$$\begin{aligned} &\min_{\mathbf{w},\alpha} \quad \mathbf{w}^H(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{w} && (10a) \\ &\text{s.t.} \quad \mathbf{w}^H\mathbf{Q}^H\mathbf{Q}\mathbf{w} = \alpha && (10b) \\ &\qquad \mathbf{w}^H\mathbf{w} \leq \frac{(\sqrt{\alpha}-1)^2}{\eta^2} && (10c) \\ &\qquad \alpha \geq 1. && (10d) \end{aligned}$$

First, let us find the interval of α for which the optimization problem (10) is feasible. Using (10b) and (10c), it can be found that the following inequality must be satisfied

$$\frac{\alpha\eta^2}{(\sqrt{\alpha}-1)^2} \leq \frac{\mathbf{w}^H\mathbf{Q}^H\mathbf{Q}\mathbf{w}}{\mathbf{w}^H\mathbf{w}}. \quad (11)$$

Since the maximum value of the expression in the right hand-side of (11) corresponds to the largest eigenvalue of the matrix $\mathbf{Q}^H\mathbf{Q}$ denoted as γ_{max}, the optimization problem (10) (or equivalently (8)) is feasible if and only if

$$\frac{\alpha\eta^2}{(\sqrt{\alpha}-1)^2} \leq \gamma_{max} \quad (12)$$

From (12), it can be immediately found that the problem (10) is infeasible if $\gamma_{max} \leq \eta^2$. Thus, hereafter, it is assumed that $\gamma_{max} > \eta^2$. Moreover, using (12), it can be found that the feasible set of the problem (10) corresponds to $\alpha \geq 1/(1-\eta/\sqrt{\gamma_{max}})^2 \geq 1$. An upper-bound for the optimal value of α in (10) can be obtained by using the following lemma.

Lemma 1: The optimal value of the optimization variable α in the problem (10), denoted as α_{opt}, is upper-bounded by $\Delta_{max}\mathbf{w}_0^H(\hat{\mathbf{R}}+\gamma\mathbf{I})\mathbf{w}_0$, where $\mathbf{w}_0$ is any arbitrary feasible point of the problem (10) and Δ_{max} is the largest eigenvalue of the matrix $(\hat{\mathbf{R}} + \gamma\mathbf{I})^{-1}\mathbf{Q}^H\mathbf{Q}$.

Proof: Let us consider the following auxiliary optimization problem

$$\begin{aligned} g \triangleq \; &\min_{\mathbf{w}} \quad \mathbf{w}^H(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{w} && (13a) \\ &\text{s.t.} \quad \mathbf{w}^H\mathbf{Q}^H\mathbf{Q}\mathbf{w} = \alpha_{opt} && (13b) \end{aligned}$$

which is obtained from the problem (10) by dropping the constraints (10c) and (10d) and fixing α to be equal to α_{opt}. It is easy to verify that g equals to $\alpha_{opt}/\Delta_{max}$. Since the optimal value of the problem (10) does not change when α is set to be equal to α_{opt} and also its feasible set for $\alpha = \alpha_{opt}$ is a subset of the feasible set of the problem (13), it can be concluded that g is upper-bounded by the optimal value of the objective of (10) and, as a result, it is upper-bounded by $\mathbf{w}_0^H(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{w}_0$, where $\mathbf{w}_0$ is any arbitrary feasible point of (10). The latter implies that $\alpha_{opt} \leq \Delta_{max}\mathbf{w}_0^H(\hat{\mathbf{R}}+\gamma\mathbf{I})\mathbf{w}_0$. ■

Using Lemma 1, the problem (10) can be equivalently stated as

$$\begin{aligned} &\min_{\mathbf{w},\,\alpha} \quad \mathbf{w}^H(\hat{\mathbf{R}}+\gamma\mathbf{I})\mathbf{w} && (14a) \\ &\text{s.t.} \quad \mathbf{w}^H\mathbf{Q}^H\mathbf{Q}\mathbf{w} = \alpha, \quad \mathbf{w}^H\mathbf{w} \leq \frac{(\sqrt{\alpha}-1)^2}{\eta^2} && (14b) \\ &\qquad \theta_1 \leq \alpha \leq \theta_2 && (14c) \end{aligned}$$

where $\theta_1 = 1/(1-\eta/\sqrt{\gamma_{max}})^2$ and $\theta_2 = \Delta_{max}\mathbf{w}_0^H(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{w}_0$, $\theta_2 \geq \theta_1$.

By introducing $\mathbf{W} \triangleq \mathbf{w}\mathbf{w}^H$ and using the fact that for any arbitrary matrix $\mathbf{A}$, $\mathbf{w}^H\mathbf{A}\mathbf{w} = Tr\{\mathbf{A}\mathbf{w}\mathbf{w}^H\}$ (here $Tr\{\cdot\}$ stands for the trace of a matrix), the problem (14) can be equivalently rewritten as

$$\min_{\mathbf{W},\alpha} Tr\{(\hat{\mathbf{R}}+\gamma\mathbf{I})\mathbf{W}\} \tag{15a}$$

$$\text{s.t. } Tr\{\mathbf{Q}^H\mathbf{Q}\mathbf{W}\}=\alpha,\ Tr\{\mathbf{W}\}\leq\frac{(\sqrt{\alpha}-1)^2}{\eta^2} \tag{15b}$$

$$\mathbf{W} \succeq 0,\quad rank(\mathbf{W}) = 1,\ \theta_1 \leq \alpha \leq \theta_2. \tag{15c}$$

Note that if $\mathbf{W}_\alpha$ denotes the optimal solution of the problem (15) for a fixed value of $\alpha \in [\theta_1, \theta_2]$ without considering the rank-one constraint, it is always possible to construct another optimal rank-one solution based on $\mathbf{W}_\alpha$ (see [3] for the detailed algorithm). The later fact implies that for fixed α, the optimal value of objective of (15) with respect to $\mathbf{W}$ remains the same independent on whether the rank-one constraint is considered or not. Based on this fact, we can drop the rank-one constraint, solve the resulting problem, and then construct another optimal rank-one solution once the optimal $\mathbf{W}_{opt}$ and α_{opt} are obtained.

The problem (15) without rank-one constraint can be solved using exhaustive search over α. Specifically, for a fixed α, the problem (15) without rank-one constraint is a semi-definite programming (SDP) problem which can be solved efficiently, while α can be found using exhaustive search in the interval $[\theta_1, \theta_2]$. Although the exhaustive search is inefficient, it can be used as a benchmark to assess the performance of the algorithm which we propose next.

After dropping the rank-one constraint, the only other non-convex constraint is $Tr\{\mathbf{W}\} \leq (\sqrt{\alpha}-1)^2/\eta^2$. Indeed, this constraint can be rewritten as $\eta^2 Tr\{\mathbf{W}\} - (\alpha+1) + 2\sqrt{\alpha} \leq 0$ where the term $\sqrt{\alpha}$ is concave. This constraint, however, can be handled by linearizing the non-convex term $\sqrt{\alpha}$ at a suitably selected point. Then the optimization problem can be solved iteratively through the sequence of linearizations. At the first iteration, an arbitrary α_c is chosen from the interval $[\theta_1, \theta_2]$ and $\sqrt{\alpha}$ is replaced by its linear approximation at α_c, i.e., $\sqrt{\alpha} \approx \sqrt{\alpha_c} + (\alpha - \alpha_c)/(2\sqrt{\alpha_c})$. It leads to the following SDP problem

$$\min_{\mathbf{W},\alpha} Tr\{(\hat{\mathbf{R}}+\gamma\mathbf{I})\mathbf{W}\} \tag{16a}$$

$$\text{s.t. } Tr\{\mathbf{Q}^H\mathbf{Q}\mathbf{W}\} = \alpha \tag{16b}$$

$$\eta^2 Tr\{\mathbf{W}\}+(\sqrt{\alpha_c}-1)+\alpha\left(\frac{1}{\sqrt{\alpha_c}}-1\right)\leq 0 \tag{16c}$$

$$\mathbf{W} \succeq 0,\quad \theta_1 \leq \alpha \leq \theta_2 \tag{16d}$$

which can be efficiently solved using, for example, the interior-point methods. In the second iteration, $\sqrt{\alpha}$ is linearized again at the value α_c, which equals to the optimal α found at the first iteration, and the corresponding optimization problem (16) is solved again, and so on. The corresponding iterative algorithm is summarized in the table at the top of this column, where $\mathbf{W}_{opt,i}$ and $\alpha_{opt,i}$ denote the optimal values of $\mathbf{W}$ and α, respectively, at iteration i and ζ is the desired accuracy of solving the problem. As it is shown in the next lemma, the optimal value of the optimization problem under consideration is non-increasing over the iterations.

Lemma 2: The optimal value of the objective $Tr\{(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{W}_{opt,i}\}$ is non-increasing over the iterations of the algorithm summarized in the Algorithm 1.

Proof: As it has been explained above, the corresponding optimization problem at iteration i, $i \geq 2$ is obtained by linearizing $\sqrt{\alpha}$ at $\alpha_{opt,i-1}$. Since $\mathbf{W}_{opt,i-1}$ and $\alpha_{opt,i-1}$ are feasible for the corresponding optimization problem at iteration i, it can be straightforwardly concluded that the optimal value of the objective at iteration i is less than or equal to the optimal value at the previous iteration, i.e., $Tr\{(\hat{\mathbf{R}} + \gamma\mathbf{I}))\mathbf{W}_{opt,i}\} \leq Tr\{(\hat{\mathbf{R}} + \gamma\mathbf{I})\mathbf{W}_{opt,i-1}\}$ which completes the proof. ■

The iterations of the algorithm summarized in the table terminate and the optimal solution is found when no more improvement for the objective function with respect to both $\mathbf{W}$ and α can not be achieved, i.e., the difference between the values of the objective at two consecutive iterations is less than or equal to a certain desired threshold ζ. The global convergence of Algorithm 1 is observed throughout extensive simulations.

Algorithm 1 The iterative method

Require: An arbitrary $\alpha_c \in [\theta_1, \theta_2]$,
the termination threshold ζ,
set i equal to 1.

repeat

Solve the following optimization problem using α_c to obtain $\mathbf{W}_{opt}$ and α_{opt}

$$\min_{\mathbf{W},\alpha} Tr\{(\hat{\mathbf{R}}+\gamma\mathbf{I})\mathbf{W}\}$$
$$\text{s.t. } Tr\{\mathbf{Q}^H\mathbf{Q}\mathbf{W}\} = \alpha$$
$$\eta^2 Tr\{\mathbf{W}\}+(\sqrt{\alpha_c}-1)+\alpha\left(\frac{1}{\sqrt{\alpha_c}}-1\right)\leq 0$$
$$\mathbf{W} \succeq 0,\quad \theta_1 \leq \alpha \leq \theta_2$$

and set

$$\mathbf{W}_{opt,i} \leftarrow \mathbf{W}_{opt},\quad \alpha_{opt,i} \leftarrow \alpha_{opt}$$
$$\alpha_c \leftarrow \alpha_{opt},\quad i \leftarrow i+1$$

until

$$Tr(\mathbf{R}\mathbf{W}_{opt,i-1}) - Tr(\mathbf{R}\mathbf{W}_{opt,i}) \leq \zeta \text{ for } i \geq 2$$

IV. Simulation Results

A uniform linear array of 10 omni-directional antenna elements with the inter-element spacing of half wavelength is considered. Additive noise in antenna elements is modeled as spatially and temporally independent complex Gaussian noise with zero mean and unit variance. In addition to the desired source, an interference source with the interference-to-noise ratio (INR) of 40 dB is assumed. The desired and interference sources are assumed to be locally incoherently scattered (see

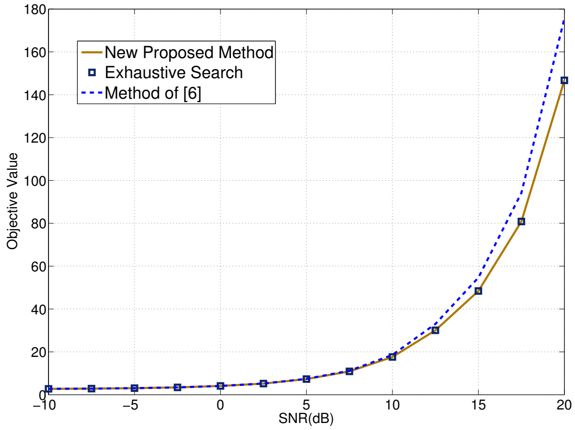

Fig. 1. Objective function of (9) versus SNR.

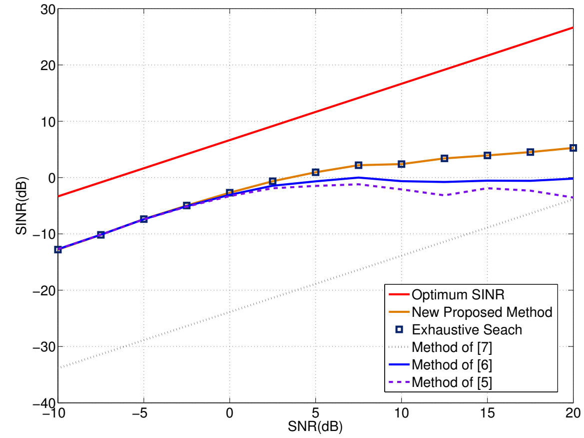

Fig. 2. Output SINR versus SNR.

[5]) with, correspondingly, Gaussian and uniform angular power densities and with central angles of $30°$ and $10°$, respectively. Both sources have the same angular spread of $4°$. The presumed knowledge of the desired source is different from the actual one and is characterized by an incoherently scattered source with Gaussian angular power density whose central angle and angular spread are $32°$ and $1°$, respectively. It is worth stressing that for a fair comparison of different methods, the central angel of the interference source has been assumed to be well separated from the desired source. For obtaining each point, 200 independent runs have been used and the sample data covariance matrix has been estimated using $K = 20$ snapshots.

The new proposed method is compared, in terms of the output SINR, to the method based on the exhaustive search over α (to provide a benchmark) and the robust adaptive beamforming methods of [5], [6] and [7]. The proposed method and the best previous one, that is the method of [6], are also compared in terms of the achieved values for the objective of (8). The diagonal loading parameters of $\gamma = 10$ and $\eta = 0.3\sqrt{Tr\{\mathbf{R}_s\}}$ are chosen for all the aforementioned methods. The initial α_c in the first iteration of the proposed method equals to $(\theta_1 + \theta_2)/2$. The termination threshold ζ for the proposed algorithm is chosen to be equal to 10^{-6}.

In Figs. 1 and 2, the values of the objective function of the problem (8) and the output SINRs corresponding to the beamforming vectors obtained by using the aforementioned beamforming methods are plotted versus SNR. It can be observed from the figures that the proposed new method has superior performance over the robust adaptive beamforming methods of [5], [6] and [7]. Our simulation results show that the method of [7] is sensitive to the simulation setup and does not always results in a near optimal solution even for the well separated desired and interference sources. Although the method of [6] does not have a guaranteed convergence, it results in a better average performance as compared to the method of [7]. Moreover, the figures confirm that the new proposed method archives the global minimum of the optimization problem (8) since the corresponding performance coincides completely with the benchmark performance. The proposed algorithm converges to a global optimum in 3-5 iterations.

V. Conclusion

The robust adaptive beamforming problem for general-rank signal model with PSD constraint has been considered. An iterative method which uses the reformulation and then linearization of the non-convex DC constraint in the corresponding optimization problem has been developed. Simulation results demonstrate the superiority of the proposed method over the other state-of-the-art methods and its global optimality.

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