

# Power Allocation in Decode-and-Forward Cooperative Networks via SEP Minimization

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**Abstract**—In this paper, the problem of optimal power allocation among the relays in two-hop decode-and-forward cooperative networks is considered, and a new power allocation scheme which minimizes the average symbol error probability under limited total power of all relays and limited maximum powers of individual relays is developed. It is shown that the corresponding power allocation optimization problem is convex and, therefore, can be efficiently solved using interior point methods. Moreover, an approximate closed-form solution in the form of water-filling is also developed. Simulation results confirm the improved performance of the proposed power allocation scheme over other schemes.

**Index Terms**—Power allocation, decode-and-forward relay networks, cooperative communications, convex optimization.

## I. INTRODUCTION

It has been recently shown that cooperative relay networks enjoy most of the advantages of multiple-input multiple-output (MIMO) systems such as high data rate and low probability of outage [1]– [3]. In cooperative networks, the source message is received by multiple relays, and then retransmitted to the destination. Multiple relays can process their received signals using different cooperation protocols such as decode-and-forward (DF), amplify-and-forward (AF), coded cooperation, and compress-and-forward techniques [3]. If some sort of channel information is available at the relays and/or destination, the relay transmission can be optimized by appropriately allocating the available resources at the relays [3]– [5].

In [4], a power allocation scheme which aims at minimizing an upper bound on the outage probability<sup>1</sup> under limited total power has been derived for DF relaying by assuming that the average channel gains are known at the relays. The resulting strategy divides the total power equally only among the relays with 'better' channel conditions. However, the outage probability is meaningful as a relay network performance metric only when the rate of change of channel fading is slow. The latter requirement limits the applications of the method of [4] to slow fading case. Alternatively, if the rate of channel fading is high, a well known figure of merit for measuring the system performance is the average symbol error probability (SEP) [6].

<sup>1</sup>The outage occurs when the received signal-to-noise ratio (SNR) required for error free decoding is below a given threshold.

In this paper, we also assume that the average channel gains are available at the relays and derive new power allocation strategies for cooperative DF relay networks by minimizing the average SEP. It enables us to apply power allocation also in the case when the rate of change of channel fading is high. In addition to the total power constraint considered in [4], we also adopt the practically important constraints on the maximum individual relay powers. We find an exact numerical solution to the power allocation problem for cooperative DF relay networks via average SEP minimization by using the result that the problem considered is convex. Moreover, we derive a simple approximate closed-form power allocation strategy, and show by simulations that the exact numerical and closed-form approximate solutions provide close average SEP performance. We also show that the proposed power allocation schemes are superior to the one of [4].

The paper is organized as follows. In Section II, the relay network system model is introduced. New closed-form approximate and numerical exact power allocation strategies are presented in Section III. In Section IV, numerical results are reported followed by conclusion drawn in Section V.

## II. SYSTEM MODEL

Consider a wireless relay network consisting of a source-destination pair and  $N$  relay nodes. There is no direct link between the source and destination. All relay nodes are half-duplex and, thus, every data transmission from the source to destination occurs in two phases. In the first phase, the source node transmits its message to all relay nodes, and in the second phase, the relay nodes retransmit the source message through orthogonal channels to the destination by using the time division multiple access scheme (TDMA). The DF cooperation protocol is used by the relay nodes. Moreover, using advanced automatic repeat request (ARQ) scheme between the source and relays, we can ensure that error free decoding is archived at the relay nodes. Therefore, we assume that the relay nodes decode source message correctly.<sup>2</sup> The destination receives

<sup>2</sup>Note that an additional signaling can be required for the ARQ to inform each relay that the source signal is received correctly by other relays and it is time to retransmit it.

signals from the relays. Then, the received signal from the  $i$ th relay node can be written as

$$y_{i,d} = \sqrt{E_i} h_{i,d} x + n_{i,d} \quad i = 1, 2, \dots, N \quad (1)$$

where  $x$  is the source message that has unit energy,  $h_{i,d}$  is a complex Gaussian distributed channel coefficient from the  $i$ th relay to the destination,  $E_i$  is the average transmitted energy of the  $i$ th relay, and  $n_{i,d}$  is the complex additive white Gaussian noise (AWGN) with variance  $N_0$ .

If the destination has a perfect knowledge of the instantaneous channel state information (CSI), the maximal ratio combining can be used. Then, the received SNR at the destination is given by

$$\gamma_D = \sum_{i=1}^N \gamma_i \quad (2)$$

where  $\gamma_i \triangleq E_i |h_{i,d}|^2 / N_0$  is the received SNR from the  $i$ th relay to the destination. The random variable  $\gamma_i$  is an exponential random variable with mean  $E_i m_i / N_0$ , where  $m_i \triangleq E[|h_{i,d}|^2]$  and  $E[\cdot]$  denotes the expectation operation. Note that  $\gamma_i$ ,  $i = 1, \dots, N$  are statically independent.

### III. AVERAGE SEP MINIMIZATION FOR POWER CONSTRAINED NETWORKS

In the case of  $M$  phase shift keying (M-PSK) modulation<sup>3</sup>, the conditional SEP can be written as [7]

$$P_{e|\gamma_D} = \frac{1}{\pi} \int_0^{\frac{M-1}{M}\pi} e^{-\frac{g_{PSK} \gamma_D}{\sin^2(\theta)}} d\theta \quad (3)$$

where  $g_{PSK} \triangleq \sin^2(\pi/M)$ . Averaging (3) over  $\gamma_i$ ,  $i = 1, \dots, N$ , the average SEP can be found as

$$P_e = \frac{1}{\pi} \int_0^{\frac{M-1}{M}\pi} \prod_{i=1}^N \frac{\sin^2(\theta)}{\sin^2(\theta) + B_i E_i} d\theta \quad (4)$$

where  $B_i \triangleq g_{PSK} m_i / N_0$ .

Using partial fraction technique, the average SEP can be derived in closed-form. For example, if  $E_i m_i \neq E_j m_j$ ,  $i \neq j$ , the closed-form expression for the average SEP is given by

$$P_e = \sum_{i=1}^N \left( \frac{\arctan\left(\cot\left(\frac{M-1}{M}\pi\right) \sqrt{\frac{B_i E_i}{B_i E_i + 1}}\right) - \frac{1}{2}}{\pi} \right) \sqrt{\frac{B_i E_i}{B_i E_i + 1}} \\ \times \prod_{j=1, j \neq i}^N \frac{1}{1 - \frac{B_j E_j}{B_i E_i}} + \left( \frac{1}{2} - \frac{\arctan\left(\cot\left(\frac{M-1}{M}\pi\right)\right)}{\pi} \right). \quad (5)$$

In the following, we develop a power allocation strategy by minimizing the average SEP (4), while satisfying the constraints on the total power  $E_T$  and the maximum power per each relay  $E_i^{max}$ . It is assumed that each relay has the knowledge of the corresponding average channel gains, i.e.,

<sup>3</sup>Note that our method can be straightforwardly extended to any other modulation scheme, but because of space limitations only M-PSK is considered.

$m_i = E[|h_{i,d}|^2]$ ,  $E_T$ , and  $E_i^{max}$ . Then, the power allocation optimization problem can be formulated as

$$\min_{\mathbf{E}} P_e(\mathbf{E}) \quad (6)$$

$$s.t. \quad \sum_{i=1}^N E_i = E_T \quad (7)$$

$$0 \leq E_i \leq E_i^{max}, i = 1, 2, \dots, N \quad (8)$$

where  $P_e(\mathbf{E})$  is given by (4),  $\mathbf{E} \triangleq (E_1, E_2, \dots, E_N)^T$  is the  $N \times 1$  vector of relay transmit powers, and  $(\cdot)^T$  stands for the transposition. The following result regarding the problem (6)-(8) is in order.

**Result 1:** *The problem (6)-(8) is a convex optimization problem.*

Due to space limitations, the proof of Result 1 is left for the subsequent journal paper.

It follows from Result 1 that any local minimum of (6)-(8) is also a global minimum [8]. However, the problem (6)-(8) does not have a simple exact closed-form solution. Therefore, in the following two subsections, we solve this problem through an approximation and also develop a numerical procedure for finding the exact solution using the well-established interior point methods.

#### A. Sub-Optimal Solution

Since the problem (6)-(8) is a convex optimization problem which satisfies Slater's conditions, the corresponding Karush-Kuhn-Tucker (KKT) equations provide necessary and sufficient optimality conditions for the problem [8]. Therefore, the optimum solution of (6)-(8) can be obtained by solving the corresponding KKT system of equations.

The Lagrangian function of the problem (6)-(8) can be written as

$$L(\mathbf{E}, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \nu) = \frac{1}{\pi} \int_0^{\frac{M-1}{M}\pi} \prod_{i=1}^N \frac{\sin^2(\theta)}{\sin^2(\theta) + B_i E_i} d\theta - \sum_{i=1}^N \lambda_i E_i \\ + \sum_{i=1}^N \gamma_i (E_i - E_i^{max}) + \nu (\sum_{i=1}^N E_i - E_T) \quad (9)$$

where  $\boldsymbol{\lambda} \triangleq (\lambda_1, \lambda_2, \dots, \lambda_N)^T$  and  $\boldsymbol{\gamma} \triangleq (\gamma_1, \gamma_2, \dots, \gamma_N)^T$  are the  $N \times 1$  vectors,  $\lambda_i$ ,  $\gamma_i$ ,  $i = 1, \dots, N$ , and  $\nu$  are the Lagrange multipliers associated with the inequality constraints  $E_i \geq 0$ ,  $i = 1, 2, \dots, N$  and  $E_i \leq E_i^{max}$ ,  $i = 1, 2, \dots, N$  and the equality constraint  $\sum_{i=1}^N E_i = E_T$ , respectively.

Using the Lagrangian (9), the KKT conditions for the problem (6)-(8) can be written as

$$\lambda_i \geq 0, \quad \gamma_i \geq 0, \quad 0 \leq E_i \leq E_i^{max}, \quad i = 1, \dots, N \quad (10)$$

$$\lambda_i E_i = 0, \quad \gamma_i (E_i - E_i^{max}) = 0, \quad i = 1, \dots, N \quad (11)$$

$$\frac{1}{\pi} \int_0^{\frac{M-1}{M}\pi} \frac{-B_i}{\sin^2(\theta) + B_i E_i} \prod_{i=1}^N \frac{\sin^2(\theta)}{\sin^2(\theta) + B_i E_i} d\theta \\ + \nu - \lambda_i + \gamma_i = 0, \quad i = 1, \dots, N \quad (12)$$

$$\sum_{i=1}^N E_i = E_T. \quad (13)$$

In general, the KKT conditions (10)-(13) cannot be solved in closed-form. However, an approximate closed-form solution can be found by using a Chebyshev-type approximation of the condition (12). Specifically, we observe that the function  $g(\theta) \triangleq \prod_{i=1}^N \sin^2(\theta)/(\sin^2(\theta) + B_i E_i)$  is strictly increasing and decreasing in the intervals  $(0, \pi/2)$  and  $(\pi/2, (M-1)\pi/M)$ , correspondingly. Then, for a large number of relay nodes, the slope of the increment and decrement in the aforementioned intervals is so high that it allows to approximate the KKT condition (12) by the following condition

$$-\frac{B_i}{\pi(1+B_i E_i)} \int_0^{\frac{M-1}{M}\pi} g(\theta) d\theta + \nu - \lambda_i + \gamma_i = 0, \quad i = 1, \dots, N \quad (14)$$

where the fact that  $\sin^2(\pi/2) = 1$  (due to the aforementioned approximation) is used in the denominator of the fraction  $B_i/(\sin^2(\theta) + B_i E_i)$ .

Dividing the Lagrange multipliers by the positive quantity  $\frac{1}{\pi} \int_0^{\frac{M-1}{M}\pi} g(\theta) d\theta$  and rearranging the equations, we can find the following approximate KKT conditions

$$\gamma_i^* \geq 0, \quad i = 1, \dots, N \quad (15)$$

$$\gamma_i^* (E_i - E_i^{max}) = 0, \quad i = 1, \dots, N \quad (16)$$

$$0 \leq E_i \leq E_i^{max}, \quad i = 1, \dots, N \quad (17)$$

$$\frac{1}{\frac{1}{B_i} + E_i} \leq \nu^* + \gamma_i^*, \quad i = 1, \dots, N \quad (18)$$

$$E_i \left( \nu^* + \gamma_i^* - \frac{1}{\frac{1}{B_i} + E_i} \right) = 0, \quad i = 1, \dots, N \quad (19)$$

$$\sum_{i=1}^N E_i = E_T \quad (20)$$

where  $\nu^* \triangleq \nu\pi / \int_0^{\frac{M-1}{M}\pi} g(\theta) d\theta$ ,  $\gamma_i^* \triangleq \gamma_i\pi / \int_0^{\frac{M-1}{M}\pi} g(\theta) d\theta$ ,  $i = 1, \dots, N$ , and the condition (18) is obtained from the approximate condition (14) in the following way. First, we rewrite the condition (14) as

$$-\frac{B_i}{1+B_i E_i} + \nu^* - \lambda_i^* + \gamma_i^* = 0, \quad i = 1, \dots, N \quad (21)$$

where  $\lambda_i^* \triangleq \lambda_i\pi / \int_0^{\frac{M-1}{M}\pi} g(\theta) d\theta$ . Then, by noticing that  $\lambda_i \geq 0$ , it can be shown that the condition (21) is equivalent to (18). The conditions  $\lambda_i E_i = 0$ ,  $i = 1, \dots, N$  are also rewritten accordingly as it is shown in (19).

Note that if  $B_i \leq \nu^*$ , then the condition (19) holds true only if  $E_i = 0$ . Therefore,  $E_i$  is zero when  $B_i \leq \nu^*$ . Furthermore, if  $B_i > \nu^*$ , then  $E_i$  can not be equal to 0. It is because the condition (16) implies that  $\gamma_i^* = 0$ . Then, substituting  $E_i = 0$  and  $\gamma_i^* = 0$  in the condition (18) yields that  $B_i \leq \nu^*$ , which contradicts the condition  $B_i > \nu^*$ . Therefore, using the condition (19) and the fact that  $E_i$  must be positive if  $B_i > \nu^*$ , we can infer that  $\nu^* + \gamma_i^* - 1/(1/B_i + E_i) = 0$  or, equivalently, that  $E_i = 1/(\nu^* + \gamma_i^*) - 1/B_i$ . Substituting

the latter expression for  $E_i$  to (16) and (17), we obtain

$$\gamma_i^* \left( \frac{1}{\nu^* + \gamma_i^*} - \frac{1}{B_i} - E_i^{max} \right) = 0 \quad (22)$$

$$0 \leq \frac{1}{\nu^* + \gamma_i^*} - \frac{1}{B_i} \leq E_i^{max}. \quad (23)$$

If  $1/\nu^* - 1/B_i > E_i^{max}$ , then the condition (23) holds true only if  $\gamma_i^* > 0$ . Moreover, the condition (22) implies that  $E_i = E_i^{max}$ . Furthermore, since  $\nu^*$  is not negative if  $1/\nu^* - 1/B_i \leq E_i^{max}$ , the conditions (22) and (23) hold true only if  $\gamma_i^* = 0$ . Therefore, we can conclude that  $E_i = 1/\nu^* - 1/B_i$ . Summarizing all the cases analyzed above and using the fact that  $B_i \triangleq g_{PSK} m_i / N_0$ , the following result can be formulated.

**Result 2:** For a set of DF relays, the approximate power allocation  $\{E_1, \dots, E_N\}$ , i.e., the approximate solution of the optimization problem (6)-(8), is

$$E_i = \begin{cases} 0, & \frac{1}{\nu^*} - \frac{N_0}{g_{PSK} m_i} \leq 0 \\ \frac{1}{\nu^*} - \frac{N_0}{g_{PSK} m_i}, & 0 < \frac{1}{\nu^*} - \frac{N_0}{g_{PSK} m_i} \leq E_i^{max} \\ E_i^{max}, & \frac{1}{\nu^*} - \frac{N_0}{g_{PSK} m_i} > E_i^{max}. \end{cases} \quad (24)$$

Result 2 shows that the total power should be distributed only among relays from a selected set. Moreover, the relays with 'better' channel conditions use bigger portion of the total power. Note that this solution can be viewed as a form of water-filling solution and has similar complexity.

## B. Optimal Solution Via Interior-Point Methods

Interior-point methods can also be used to solve the problem (6)-(8). Despite the higher complexity of  $O(M^3)$  as compared to the approximate closed-form solution summarized in Result 2, interior-point methods provide an exact solution, which can be used as a benchmark to evaluate the accuracy of the aforementioned approximate solution. For solving (6)-(8), we adopt one of the widely used interior-point methods, that is, the *barrier function method*. Then, the summary of the barrier function method is as follows.

Given strictly feasible  $\mathbf{E} = (E_1, E_2, \dots, E_N)$ ,  $t = t^{(0)}$ ,  $\mu > 1$ , and  $\epsilon > 0$ , where  $t > 0$  is the step parameter of the barrier function method,  $\mu$  is the step size and  $\epsilon$  is the allowed duality gap (accuracy parameter) (see [8]), do the following.

1. Compute  $\mathbf{E}(t) = \arg \min \left\{ \frac{t}{\pi} \int_0^{\frac{M-1}{M}\pi} g(\theta) d\theta - \sum_{i=1}^N \log E_i - \sum_{i=1}^N \log(E_i^{max} - E_i) \right\}$  subject to  $\sum_{i=1}^N E_i = E_T$  starting at current  $\mathbf{E}$  using Newton's method.
2. Update  $\mathbf{E} := \mathbf{E}(t)$ .
3. Stopping criterion: quit if  $N/t < \epsilon$
4. Increase  $t = \mu t$ , and go to step 1.

In this algorithm,  $\mathbf{E}(t)$  is called the central point and the first step of the algorithm is called the centering step. Then, at each iteration the central point  $\mathbf{E}(t)$  is recomputed using Newton's method until  $N/t < \epsilon$ , which guarantees that the solution is found with accuracy  $\epsilon$ , i.e.,  $P_e(\mathbf{E}) - P_e(\mathbf{E}_{opt}) < \epsilon$ .

#### IV. SIMULATION RESULTS

A cooperative network consisting of a source-destination pair and  $N = 10$  DF relays is considered. Relays are capable of decoding the source message correctly and retransmitting it to the destination using QPSK modulation. The noise power is assumed to be equal to 1. Four different power allocation schemes are compared to each other: the proposed optimal scheme that minimizes the average SEP using the Barrier function method, the proposed approximate power allocation scheme summarized in Result 2, the equal power allocation with channel selection scheme of [4], and the equal power allocation without channel selection scheme. In the proposed optimal scheme based on the barrier function method,  $t = 1$  and  $\mu = 2$  are selected. Moreover, at each SNR, the duality gap of the barrier function method is set initially so that the ratio of the average SEP to the duality gap is greater than  $10^4$ . For the equal power allocation with channel selection scheme of [4], the unit outage threshold is used.

Fig. 1 shows the average SEP of the four aforementioned power allocation schemes for all relays with average channel gains  $m_i = \{0, -3, -6, -8, -10, -12, -14, -16, -18, -20\}$  dB, that is, the same example as considered in [4]. The closed-form expression (5) is used to compute the average SEP. It can be seen from the figure that for two cases of  $E_{max} = E_T$  and  $E_{max} = E_T/8$ , the proposed optimal scheme based on the barrier function method and the approximate power allocation scheme summarized in Result 2 show approximately the same average SEP performance. Moreover, for  $E_{max} = E_T/8$  and SNR= 6 dB, the power distribution suggested by the barrier function-based method is  $[0.4976, 0.4976, 0.4976, 0.4976, 0.4976, 0.4976, 0.4975, 0.4973, 0.0004, 0.0002]$ , while the power distribution suggested by the method summarized in Result 2 is  $[0.4976, 0.4976, 0.4976, 0.4976, 0.4976, 0.4976, 0.4976, 0.4976, 0, 0]$ . It also confirms that the approximate power allocation is very close to the optimal one.

Since there is no limitation on the maximum allowable power at the relays for the equal power allocation with channel selection and equal power allocation without channel selection schemes, for fair comparison, we also show the improvements achieved by the proposed schemes when  $E_{max} = E_T$ . It can be seen in the figure that for low SNRs, the average SEP of the equal power allocation with channel selection scheme of [4] is approximately the same as the one for the proposed schemes. However, for high SNRs, the average SEP of the equal power allocation with channel selection of [4] converges to the equal power allocation without channel selection, while the proposed schemes enjoy up to 2 dB improvement. Limiting the maximum allowable transmission power for the relays to  $E_T/8$  causes more than 1 dB loss for the proposed schemes in comparison to the best achievable average SEP.

#### V. CONCLUSION

A new power allocation strategy for DF cooperative relay networks which is based on the average SEP minimization, has been developed. The exact and approximate power allocation

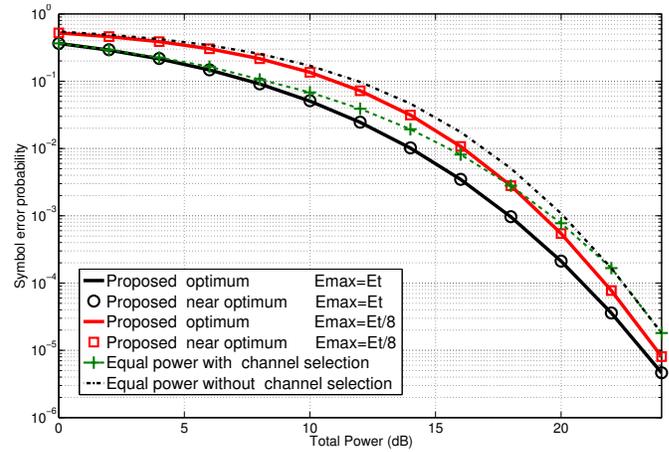


Fig. 1. Average SEP for different power allocation schemes.

solutions have been obtained and their close performance match has been demonstrated via simulations. The approximate solution can be viewed as a form of water-filling strategy, while the exact solution is based on the interior-point methods. According to our approximate solution, no power should be allocated to relays that have a average channel gain to noise ratio less than a certain threshold value, while the excess powers that are allocated to admitted relays should be proportional to the corresponding average channel gain to noise ratios for that relays. Our simulation results demonstrate improved performance of the proposed methods as compared to earlier developed power allocation scheme for DF cooperative relay networks.

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