

FIELDS WITH A UNIQUE ANISOTROPIC QUADRATIC FORM OF MAXIMAL DIMENSION

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ABSTRACT. The u -invariant of a field is the highest dimension of a non-degenerate anisotropic quadratic form over this field. We enhance author's recent result on existence of fields of the u -invariant 11, by constructing such a field over which all 11-dimensional non-degenerate anisotropic quadratic forms are similar to each other. For the u -invariant 9, the analogous result has been proved in 2000 by Oleg Izhboldin; unlike the original one, the new proof, also provided here, avoids unramified cohomology. Finally, for higher odd values of the u -invariant, we prove a similar result with the similarity of forms replaced by a weaker relation.

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1. ESSENTIAL FORMS

We work with finite-dimensional non-degenerate quadratic forms over arbitrary fields and use [1] as the main source of references.

Let F be a field. Let us say that non-degenerate quadratic form φ over F of an odd dimension $2n + 1 \geq 1$ is *essential*, if its even Clifford algebra $C_0(\varphi)$ is a division algebra (in particular, φ is anisotropic).

The following result is recently proved in [4]:

Theorem 1.1. *Let φ be an essential quadratic form of dimension $2n + 1$ over a field F . Assume that $2n + 1$ is neither a 2-power minus 1 nor a 2-power minus 3. Then there exists an overfield $L \supset F$ such that φ_L is anisotropic and every non-degenerate quadratic form over L of dimension at least $2n + 2$ is isotropic (in particular, $u(L) = 2n + 1$).*

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In this paper we prove the following enhancement for the two initial values of the dimension (taking Remark 1.3 into account):

Theorem 1.2. *Let φ be an essential quadratic form of dimension $2n + 1$ over a field F . Assume that $2n + 1$ is 9 or 11. Then there exists an overfield $L \supset F$ such that φ_L is anisotropic and all non-degenerate anisotropic quadratic forms over L of dimension at least $2n + 1$ are similar to φ (in particular, $u(L) = 2n + 1$).*

In §4, we prove Theorem 4.7 which, together with Theorem 5.3, provides a stronger version of Theorem 1.2.

Remark 1.3. A quadratic form of dimension $2n + 1 = 9$ which is essential in the above sense is also essential in the sense of [3, Definition 0.2]. Therefore Theorem 1.2 in dimension 9 is a consequence of [3, Theorem 0.3].

2. EVEN DIMENSION

The analogue of Theorem 1.2 for even dimension $2n$ is easy to get. We discuss it for completeness restricting to $n \geq 3$ in the discussion.

Let us say that a non-degenerate quadratic form φ over a field F of even dimension $2n$ with $n \geq 3$ is essential, if it has trivial discriminant and the index of its total Clifford algebra $C(\varphi)$ is 2^{n-1} (which is the maximal possible value). Any essential quadratic form is anisotropic. By the index reduction formula for quadrics [1, Theorem 30.5], if φ over the function field $F(\psi)$ (i.e., the function field of the corresponding projective quadric) of some non-degenerate quadratic form ψ over F of dimension at least $2n$ is not essential anymore, then the dimension of ψ is also $2n$, the discriminant of ψ is also trivial, and the Clifford algebra $C(\psi)$ is isomorphic to $C(\varphi)$. Therefore, modifying slightly [1, Proof of Theorem 38.4], one can construct an overfield $L \supset F$ such that $I_q^3(L) = 0$, φ_L is still essential, and every non-degenerate anisotropic quadratic form ψ over L of dimension at least $2n$ has dimension $2n$ and trivial discriminant, whereas its Clifford algebra $C(\psi)$ is isomorphic to $C(\varphi_L)$. Since $I_q^3(L) = 0$, the conditions on ψ imply that ψ is similar (and, moreover, isomorphic) to φ_L .

Remark 2.1. The following property, which is a consequence of the index reduction formula for quadrics, is important for the above construction to work (cf. Lemma 4.8): a non-trivial central simple algebra remains non-trivial over the function field of any non-degenerate quadratic form of dimension at least 5. In particular, if the Clifford algebras of two $2n$ -dimensional forms of trivial discriminant are not isomorphic, they remain so over the function fields of other quadratic forms of dimension at least $2n$ (due to the above assumption that $n \geq 3$).

3. A COMPUTATION

For a smooth projective quadric Q over a field F of odd dimension $2n - 1$, recall (from, e.g., [1]) that the Chow group $\text{CH}(\bar{Q})$ of the variety $\bar{Q}$ (which is Q with the scalars extended to an algebraic closure of F) is free with a basis

$$h^0, \dots, h^{n-1}, l_{n-1}, \dots, l_0.$$

The element $h \in \mathrm{CH}^1(\bar{Q})$ is the hyperplane section class, $h^i \in \mathrm{CH}^i(\bar{Q})$ is its i th power, and $l_i \in \mathrm{CH}_i(\bar{Q})$ is the class of an i -dimensional projective space lying on $\bar{Q}$.

We use the following notation. By $\bar{X}$ for a smooth quasi-projective F -variety X , we mean X with the base field extended to an algebraic closure of F . We write $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\mathrm{CH}(X)$ for the modulo 2 Chow group of X ; $\overline{\mathrm{Ch}}(X)$ stands for the image of the change of field homomorphism $\mathrm{Ch}(X) \rightarrow \mathrm{Ch}(\bar{X})$; similarly, $\overline{\mathrm{CH}}(X)$ is the image of $\mathrm{CH}(X) \rightarrow \mathrm{CH}(\bar{X})$.

For $i \geq 0$, we work with the cohomological Steenrod operations

$$\mathrm{St}^i: \mathrm{Ch}^*(X) \rightarrow \mathrm{Ch}^{*+i}(X)$$

on the modulo 2 Chow groups. Their history, construction, and basic properties are explained in [1, Chapter XI] for characteristic not 2 and in [7] for characteristic 2.

Proposition 3.1. *For an integer $n \geq 4$, let Q be a smooth projective quadric of dimension $2n - 1$ over a field F . Let V be a projective homogeneous variety over F . For an element $\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$, we consider the decomposition*

$$(3.2) \quad \bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^{n-1} \times \alpha_{n-1} + l_{n-1} \times \beta + al_{n-2} \times [\bar{V}]$$

of its image $\bar{\alpha} \in \mathrm{Ch}^{n+1}(\bar{Q} \times \bar{V})$ with $\alpha_i \in \mathrm{Ch}^{n+1-i}(\bar{V})$, $\beta \in \mathrm{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$.

(i) *If neither $n + 1$ nor $n + 2$ is a 2-power, then*

$$\alpha_0 + \sum_{i=1}^{[(n+1)/2]} a_i \mathrm{St}^i(\alpha_i) \in \overline{\mathrm{Ch}}^{n+1}(V)$$

for some $a_i \in \mathbb{Z}/2\mathbb{Z}$, where $[(n+1)/2]$ is the integral part of $(n+1)/2$.

(ii) *If $a = 0$ in decomposition (3.2) whereas n is even and not a 2-power, then*

$$\alpha_0 + \mathrm{St}^1(\alpha_1) + \alpha_1 \beta \in \overline{\mathrm{Ch}}^{n+1}(V).$$

Proof of Proposition 3.1(i). The statement of Proposition 3.1(i) and its proof are similar to [4, Proposition 5.1(i)], but $\dim Q$ here is $2n - 1$ instead of $2n$ there.

For every $i \geq 0$, let s^i be the image in $\mathrm{CH}^{n+1+i}(\bar{Q} \times \bar{V})$ of an element of the group $\mathrm{CH}^{n+1+i}(Q \times V)$ representing the value $\mathrm{St}^i(\alpha) \in \mathrm{Ch}^{n+1+i}(Q \times V)$ of the Steenrod operation. We will refer to these elements s^i later in the proof.

For $d := 2^r - 1$, where 2^r is the highest 2-power not exceeding n , we choose a d -dimensional smooth subquadric P of Q . Writing in for the imbedding $P \times V \hookrightarrow Q \times V$ and applying [1, Theorem 61.9 and Proposition 61.10] (along with their characteristic 2 counterparts from [7]), we have

$$(3.3) \quad pr_*^P \sum_{i=0}^d c_i(-T_P) in^* \mathrm{St}^{d-i} \alpha = \mathrm{St}^d pr_*^P in^* \alpha \in \mathrm{Ch}^{n+1}(V),$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \rightarrow V$. The summation on the left side in (3.3) is from 0 to d only because $c_i(-T_P) = 0$ for $i < 0$ whereas $\mathrm{St}^{d-i} \alpha = 0$ for $i > d$. If $d > n + 1 - d$, the right side in (3.3) vanishes because

$$\mathrm{St}^d pr_*^P in^* \alpha \in \mathrm{St}^d \mathrm{Ch}^{n+1-d}(V)$$

and $\mathrm{St}^d \mathrm{Ch}^{n+1-d}(V) = 0$ (see [1, Theorem 61.13] and [7, Corollary 7.5]). However the relation $d > n + 1 - d$ fails if (and only if) n has the highest possible value with respect

to $d = 2^r - 1$. Since n is not a 2-power minus 1 or minus 2, the highest possible value of n is $n = 2^{r+1} - 3$, in which case $d = n + 1 - d$ and the operation St^d on $\text{Ch}^{n+1-d}(V)$ is the squaring (by [1, Theorem 61.13] and [7, Corollary 7.3]).

Rewriting the left side of (3.3) with a help of the projection formula [1, Proposition 56.9], we obtain

$$pr_* \sum_{i=0}^d in_*(c_i(-T_P)) \text{St}^{d-i} \alpha,$$

where pr is the projection $Q \times V \rightarrow V$. By the computation [1, Lemma 78.1] of the Chern classes of the tangent bundle, it follows from (3.3) that

$$(3.4) \quad \sum_{i=0}^d \binom{-d-2}{i} \cdot pr_*(h^{2n-1-d+i} \cdot s^{d-i}) \in 2\overline{\text{CH}}^{n+1}(V) \subset \text{CH}^{n+1}(\bar{V})$$

provided that $n \neq 2^{r+1} - 3$. For $n = 2^{r+1} - 3$, we obtain relation (3.4) with $2\overline{\text{CH}}^{n+1}(V)$ on the right replaced by $4\text{CH}^{n+1}(\bar{V})$. So, in both cases the following combined relation holds:

$$(3.5) \quad \sum_{i=0}^d \binom{-d-2}{i} \cdot pr_*(h^{2n-1-d+i} \cdot s^{d-i}) \in 2\overline{\text{CH}}^{n+1}(V) + 4\text{CH}^{n+1}(\bar{V}) \subset \text{CH}^{n+1}(\bar{V}).$$

The sum showing up in (3.5) is a linear combination of the elements

$$pr_*(h^{2n-1-d}s^d), pr_*(h^{2n-d}s^{d-1}), \dots, pr_*(h^{2n-1}s^0) \in \overline{\text{CH}}^{n+1}(V) \subset \text{CH}^{n+1}(\bar{V}).$$

Let us compute the i th element $pr_*(h^{2n-1-d+i} \cdot s^{d-i}) \in \text{CH}^{n+1}(\bar{V})$ modulo $4\text{CH}^{n+1}(\bar{V})$. Since $2n - 1 - d + i \geq n$, the factor $h^{2n-1-d+i} \in \text{CH}(\bar{Q})$ is divisible by 2. The other factor modulo 2 is $\text{St}^{d-i}(\bar{\alpha})$ and it follows that

$$pr_*(h^{2n-d+i} \cdot s^{d-i}) \equiv 2 \sum_{k \geq 0} \binom{k}{d-i-k} \varepsilon_k \pmod{4\text{CH}^{n+1}(\bar{V})},$$

where $\varepsilon_k \in \text{CH}^{n+1}(\bar{V})$ is an integral representative of $\text{St}^k(\alpha_k) \in \text{Ch}^{n+1}(\bar{V})$ which in the case of $k > n + 1 - k$ we choose to be 0 taking into account that $\text{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \text{Ch}^{n+1-k}(\bar{Q} \times \bar{V})$. So, the sum over k runs up to the integral part of $(n+1)/2$ only.

Since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i \in \{0, 1, \dots, d\}$ (the binomial coefficient $\binom{-d-2}{i} = \binom{d+1+i}{i}$ is easy to compute modulo 2 using [1, Lemma 78.6]), we deduce from (3.5) that

$$(3.6) \quad 2 \sum_{i=0}^d \sum_{k=0}^{[(n+1)/2]} \binom{k}{d-i-k} \varepsilon_k \equiv 2\gamma \pmod{4\text{CH}^{n+1}(\bar{V})}$$

for some $\gamma \in \overline{\text{CH}}^{n+1}(V)$. The coefficient at ε_0 is 2. Therefore

$$2\varepsilon_0 + 2 \sum_{k=1}^{[(n+1)/2]} a_k \varepsilon_k \equiv 2\gamma \pmod{4\text{CH}^{n+1}(\bar{V})}$$

for some integers a_k . Dividing by 2 (we have the right to divide by 2 because the group $\mathrm{CH}^{n+1}(\bar{V})$ for a projective homogeneous variety V is free of torsion), we obtain the congruence

$$\varepsilon_0 + \sum_{k=1}^{[(n+1)/2]} a_k \varepsilon_k \equiv \gamma \pmod{2 \mathrm{CH}^{n+1}(\bar{V})}$$

meaning that

$$\alpha_0 + \sum_{k=1}^{[(n+1)/2]} a_k \mathrm{St}^k \alpha_k \in \overline{\mathrm{Ch}}^{n+1}(V). \quad \square$$

Proof of Proposition 3.1(ii). The statement of Proposition 3.1(ii) and its proof are similar to [4, Proposition 5.1(ii)], but $\dim Q$ here is $2n - 1$ instead of $2n$ there. This difference causes the need to additionally assume that n is not a 2-power here.

For every $i = 0, 1, \dots, n$, let s^i be the image in the group $\mathrm{CH}^{n+1+i}(\bar{Q} \times \bar{V})$ of an element of $\mathrm{CH}^{n+1+i}(Q \times V)$ representing the value $\mathrm{St}^i(\alpha) \in \mathrm{Ch}^{n+1+i}(Q \times V)$ of the Steenrod operation. We also set $s^i := 0$ for $i > n + 1$ as well as for $i < 0$. Finally, we set $s^{n+1} := (s^0)^2$.

Note that we have

$$s^0 := h^0 \times \alpha_0 + h^1 \times \alpha_1 + \dots + h^{n-1} \times \alpha_{n-1} + l_{n-1} \times \beta + bl_{n-2} \times [\bar{V}] \in \mathrm{CH}^{n+1}(\bar{Q} \times \bar{V})$$

with some $\alpha_i \in \mathrm{CH}^{n+1-i}(\bar{V})$, $\beta \in \mathrm{CH}^1(\bar{V})$, and an integer b . Since

$$s^0 \pmod{2 \mathrm{CH}^{n+1}(\bar{Q} \times \bar{V})} = \bar{\alpha},$$

the integer b is even. Since the element $2l_{n-2} = h^{n+1}$ is in $\overline{\mathrm{CH}}^{n+1}(Q) \subset \mathrm{CH}^{n+1}(\bar{Q})$, we remove the last summand from the above decomposition of s^0 .

Let d be any integer with $n \leq d \leq 2n - 1$. For a d -dimensional smooth subquadric P of Q , writing in for the imbedding $P \times V \hookrightarrow Q \times V$ and applying [1, Theorem 61.9 and Proposition 61.10]), we have

$$pr_*^P \sum_{i=d-n-1}^d c_i(-T_P) in^* \mathrm{St}^{d-i} \alpha = \mathrm{St}^d pr_*^P in^* \alpha \in \mathrm{Ch}^{n+1}(V),$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \rightarrow V$. The summation on the left side is from $d - n - 1$ to d only because $\mathrm{St}^{d-i} \alpha = 0$ outside of this interval, see [1, Theorem 61.13]. The right side is actually zero since

$$\mathrm{St}^d pr_*^P in^* \alpha \in \mathrm{St}^d \mathrm{Ch}^{n+1-d}(V)$$

and $\mathrm{St}^d \mathrm{Ch}^{n+1-d}(V) = 0$ because $d > 1 \geq n + 1 - d$. Rewriting the left side with a help of the projection formula [1, Proposition 56.9], we obtain the relation

$$pr_* \sum_{i=d-n-1}^d in_*(c_i(-T_P)) \mathrm{St}^{d-i} \alpha = 0 \in \mathrm{Ch}^{n+1}(V),$$

where pr is the projection $Q \times V \rightarrow V$. By the computation [1, Lemma 78.1] of the Chern classes of the tangent bundle, it follows that

$$(3.7) \quad \sum_{i=d-n-1}^d \binom{-d-2}{i} \cdot pr_*(h^{2n-1-d+i} \cdot s^{d-i}) \in 2 \overline{\mathrm{CH}}^{n+1}(V) \subset \mathrm{CH}^{n+1}(\bar{V}).$$

We are going to use (3.7) for various values of d . Note that when d varies, the sum showing up in (3.7) is a linear combination of always the same elements

$$pr_*(h^{n-2}s^{n+1}), pr_*(h^{n-1}s^n), \dots, pr_*(h^{2n-1}s^0) \in \overline{\text{CH}}^{n+1}(V).$$

However, the coefficients of this linear combination vary with d .

Let us compute the i th element $pr_*(h^{2n-1-d+i} \cdot s^{d-i}) \in \text{CH}^{n+1}(\bar{V})$ modulo $4\text{CH}^{n+1}(\bar{V})$. For $i \geq d - n + 1$, we have $2n - 1 - d + i \geq n$ so that the factor $h^{2n-1-d+i} \in \text{CH}(\bar{Q})$ is divisible by 2. The other factor modulo 2 is $\text{St}^{d-i}(\bar{\alpha})$ and it follows that

$$pr_*(h^{2n-1-d+i} \cdot s^{d-i}) \equiv 2 \sum_{k \geq 0} \binom{k}{d-i-k} \varepsilon_k \pmod{4} \quad \text{for } i \geq d - n + 1,$$

where $\varepsilon_k \in \text{CH}^{n+1}(\bar{V})$ is an integral representative of $\text{St}^k(\alpha_k) \in \text{Ch}^{n+1}(\bar{V})$ which in the case of $k > n + 1 - k$ we choose to be 0 taking into account that $\text{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \text{Ch}^{n+1-k}(\bar{Q} \times \bar{V})$. So, the sum over k runs up to $n/2$ only (recall that n is even). Besides, we choose $\varepsilon_0 = \alpha_0$.

For $i = d - n - 1$, the i th summand is

$$\begin{aligned} pr_*(h^{n-2}s^{n+1}) &= pr_* \left(h^{n-2} \cdot (h^0 \times \alpha_0 + h^1 \times \alpha_1 + \dots + h^{n-1} \times \alpha_{n-1} + l_{n-1} \times \beta)^2 \right) \\ &\equiv 2 pr_* \left(h^{n-2} \cdot (h^1 \times \alpha_1) \cdot (l_{n-1} \times \beta) \right) = 2\alpha_1\beta, \end{aligned}$$

where the congruence is modulo $4\text{CH}^{n+1}(\bar{V})$. Here again we use that n is even: for odd n the answer would be $2\alpha_1\beta + 2\alpha_{(n+1)/2}^2$.

There is no need to compute the last remaining summand $pr_*(h^{n-1}s^n)$ which occurs for $i = d - n$.

With the above computations in hand, we are going to use (3.7) for the following two values of d : for $d = 2^r - 1$ and for $d = 2^r$, where $2^r \leq 2n$ is the highest 2-power not exceeding $2n$. Note that $d \leq 2n - 1 = \dim Q$ because of the assumption that n is not a 2-power.

For the first choice of d , since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i = 0, 1, \dots, d$, we get that

$$2\alpha_1\beta + pr_*(h^{n-1}s^n) + 2 \sum_{i=d-n+1}^d \sum_{k=0}^{n/2} \binom{k}{d-i-k} \varepsilon_k \equiv 2a \pmod{4\text{CH}^{n+1}(\bar{V})}$$

for some $a \in \overline{\text{CH}}^{n+1}(\bar{V})$. For $k < n/2$, the coefficient at ε_k is twice the sum of all binomial coefficients $\binom{k}{\cdot}$ and therefore is divisible by 4 for $0 < k < n/2$. The coefficient at ε_0 is 2. The coefficient at $\varepsilon_{n/2}$ is twice the sum of all binomial coefficients $\binom{n/2}{\cdot}$ except $\binom{n/2}{n/2} = 1$ and so is congruent to 2 modulo 4. Therefore the congruence we get with the first choice of d is

$$(3.8) \quad 2\alpha_1\beta + pr_*(h^{n-1}s^n) + 2\varepsilon_0 + 2\varepsilon_{n/2} \equiv 2a \pmod{4\text{CH}^{n+1}(\bar{V})}.$$

For the second choice of d (namely, $d = 2^r$), the binomial coefficient $\binom{-d-2}{i}$ with $i \in \{0, 1, \dots, d\}$ is odd for even $i < d$ and is even for the remaining i . Since the integer

$d - n - 1$ is odd, we get that

$$pr_*(h^n s^n) + 2 \sum_{\substack{i=d-n+2 \\ i \text{ is even}}}^{d-2} \sum_{k=0}^{n/2} \binom{k}{d-i-k} \varepsilon_k \equiv 2b \pmod{4 \text{ CH}^{n+1}(\bar{V})}$$

for some $b \in \bar{\text{CH}}^{n+1}(\bar{V})$.

The coefficient at ε_0 is 0 here. Since for any $k \geq 1$, we have

$$\sum_{\text{even } l} \binom{k}{l} = 2^{k-1} = \sum_{\text{odd } l} \binom{k}{l},$$

only the coefficients at ε_1 and $\varepsilon_{n/2}$ survive modulo 4 (the coefficient at $\varepsilon_{n/2}$ survives because the binomial coefficient $\binom{n/2}{n/2}$ is missing).

Therefore the congruence we get with our second choice of d is

$$(3.9) \quad pr_*(h^{n-1} s^n) + 2\varepsilon_1 + 2\varepsilon_{n/2} \equiv 2b \pmod{4 \text{ CH}^{n+1}(\bar{V})}.$$

Adding together (3.8) and (3.9), we get

$$2\alpha_1\beta + 2pr_*(h^{n-1} s^n) + 2\varepsilon_0 + 2\varepsilon_1 \equiv 2(a + b) \pmod{4 \text{ CH}^{n+1}(\bar{V})}.$$

Dividing by 2 (we have the right to do so because the group $\text{CH}^{n+1}(\bar{V})$ is free of torsion), we obtain the congruence

$$\alpha_1\beta + \varepsilon_0 + \varepsilon_1 \equiv a + b - pr_*(h^{n-1} s^n) \pmod{2 \text{ CH}^{n+1}(\bar{V})}$$

implying that

$$e_{n+1} + \alpha_1\beta + \text{St}^1(\alpha_1) \in \bar{\text{Ch}}^{n+1}(V)$$

because $a + b - pr_*(h^{n-1} s^n) \in \bar{\text{CH}}^{n+1}(V)$. □

4. ALMOST ESSENTIAL FORMS

We start with several definitions.

Let us say that a non-degenerate quadratic form φ over a field F of odd dimension $2n + 1$ is *almost essential*, if the Schur index of its even Clifford algebra is at least 2^{n-1} .

Assume that φ is almost essential of dimension at least 7 or essential of dimension at least 5. A quadratic form ψ is called a *relative* of φ , if it is non-degenerate, has the same dimension as φ , and the anisotropic parts of the quadratic forms φ_L and ψ_L are similar, where L is the function field of the product of the *third last* (i.e., the $(n - 2)$ nd) grassmannians of φ and of ψ . Note that for a relative ψ of φ , its even Clifford algebra $C_0(\psi)$ is isomorphic to $C_0(\varphi)$ (see Remark 2.1). Besides, the quadratic forms ψ_L and φ_L are 5-dimensional and essential; they are similar if and only if at least one of them becomes isotropic over the function field of the other.

We say that a non-degenerate quadratic form φ over a field F of odd dimension $2n + 1$ is *strongly anisotropic*, if

$$e_{n+1} \notin \bar{\text{Ch}}^{n+1}(X_{n-1})$$

in notation of [4]. The variety X_{n-1} here is the *second last* (i.e., $(n - 1)$ st) grassmannian of φ , called *almost last* as well as *almost highest* in [4].

By [4, Proposition 3.1], any essential quadratic form is strongly anisotropic. By definition of e_{n+1} , given in [4, §3], any strongly anisotropic quadratic form satisfies $l_0 \notin \overline{\text{Ch}}_0(X_1)$, where X_1 is the corresponding quadrics, and therefore is anisotropic.

Here is the general core result of this paper. It can be viewed as an enhanced version of [4, Theorem 7.2]. It implies the general main result – Theorem 4.7, which follows further below.

Theorem 4.1. *Let $n \geq 4$ be an integer such that neither $n+1$ nor $n+2$ is a 2-power. Let φ be an almost essential, strongly anisotropic quadratic form over F of dimension $2n+1$. Let ψ be a non-degenerate quadratic form over F of dimension at least $2n+1$ which is not a relative of φ . Then the quadratic form $\varphi_{F(\psi)}$ over the function field $F(\psi)$ if ψ is still almost essential and strongly anisotropic.*

Proof. The form $\varphi_{F(\psi)}$ is almost essential according to the index reduction formula for quadrics. We just need to check that it is strongly anisotropic.

It suffices to consider the case where ψ is anisotropic. By [4, Theorem 7.2], we only need to deal with the case where $\dim \psi = \dim \varphi$.

We induct on $\dim \varphi \geq 9$. The proof below is an adaptation of [4, Proof of Theorem 7.2].

Let Q be the projective quadric of ψ and let V be the second last grassmannian of φ . Assume that $\varphi_{F(\psi)} = \varphi_{F(Q)}$ is not strongly anisotropic and consider a lift

$$\alpha \in \text{Ch}^{n+1}(Q \times V)$$

of an element in $\text{Ch}^{n+1}(V_{F(Q)})$ representing $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V_{F(Q)})$. Note that $\alpha_0 = e_{n+1}$ in the decomposition (3.2).

By Proposition 3.1(i), we have

$$(4.2) \quad \alpha_0 + \sum_{i=1}^{[(n+1)/2]} \text{St}^i(\alpha_i) \in \overline{\text{Ch}}^{n+1}(V).$$

For $i = 1, \dots, n$, let X_i be the i th grassmannian of φ and let Y_i be the i th grassmannian of ψ . By the index reduction formula for quadrics (the function field extension $F(Y_i)/F$ in the next formula can be replaced by a chain of function fields of quadrics), we have

$$(4.3) \quad \text{ind } C_0((\varphi)_{F(Y_i)}) \geq 2^{n-i} \quad \text{for } i \in \{1, \dots, n\}.$$

Viewing α as a correspondence $Q \rightsquigarrow V$ and applying the homomorphism α_* to $l_i \in \overline{\text{Ch}}(Q_{F(Y_{i+1})})$, we get

$$\alpha_i \in \overline{\text{Ch}}^{n+1-i}(V_{F(Y_{i+1})})$$

for $i < n$. Therefore by [4, Proposition 3.1] we may assume that for every $i \in \{1, \dots, n-1\}$, the element α_i is e_{n+1-i} or 0.

Let us choose some $i \in \{1, \dots, n-1\}$ and consider the anisotropic part φ^i of the quadratic form $\varphi_{F(X_i)}$. We have

$$\dim \varphi^i = \dim \varphi - 2i = 2(n-i) + 1$$

and $C_0(\varphi^i)$ is a division algebra. Let X_{n-i-1}^i be the second last grassmannian of φ^i , and let $e_{n-i+1}^i \in \text{Ch}^{n-i+1}(\overline{X}_{n-i-1}^i)$ be the last of the special elements from [4, §4]. Since the form

$\varphi_{F(X_i)(Y_i)}^i$ is essential and the anisotropic part of $\psi_{F(X_i)(Y_i)}$ is not its relative, it follows by the induction hypothesis of our proof that

$$(4.4) \quad e_{n-i+1}^i \notin \overline{\text{Ch}}^{n-i+1}((X_{n-i-1}^i)_{F(X_i)(Y_{i+1})})$$

provided that $n-i$ neither $n-i+1$ nor $n-i+2$ is a 2-power. If $n-i+2$ is a 2-power and $n-i \geq 2$, we still have (4.4): this time by Proposition 4.5 below.

By [4, Proposition 4.2], (4.4) implies that

$$e_{n-i+1} \notin \text{Ch}^{n-i+1}((X_{n-1})_{F(X_i)(Y_{i+1})}) = \text{Ch}^{n-i+1}(V_{F(X_i)(Y_{i+1})})$$

and, in particular,

$$e_{n-i+1} \notin \text{Ch}^{n-i+1}(V_{F(Y_{i+1})}).$$

Consequently, $\alpha_i = 0$ in (4.2) provided that $n-i$ is at least 2 and not a 2-power minus 1.

Since n is at least 4 in our setting and the sum over i in (4.2) runs up to $[n+1]/2$ only, the difference $n-i$ is at least 2 for all i appearing in (4.2). But if $n-i$ is a 2-power minus 1, we cannot conclude that $\alpha_i = 0$. So, let us assume that $\alpha_i = e_{n-i+1}$. By [9, Proposition 2.9], providing a formula for $\text{St}^i(e_{n-i+1})$, the coefficient at e_{n+1} is the binomial coefficient $\binom{n-i+1}{i}$. Note that $i < n-i+1$: otherwise $i = n-i+1$, i.e., $i = (n+1)/2$ and $n-i+1 = (n+1)/2$ is a 2-power – a contradiction with the assumption that n is not a 2-power minus 1. It follows that $\binom{n-i+1}{i} \equiv 0 \pmod{2}$. Thus (4.2) yields

$$e_{n+1} = \alpha_0 \in \overline{\text{Ch}}^{n+1}(V)$$

meaning that φ does not satisfy (4.2). $\square$

Proposition 4.5. *Let φ be an essential quadratic form over a field F of dimension a 2-power minus 3, at least 5. Let ψ be a non-degenerate quadratic form of the same dimension, not a relative of φ . Then $\varphi_{F(\psi)}$ is almost essential and strongly anisotropic.*

Proof. The form $\varphi_{F(\psi)}$ is almost essential by the index reduction formula for quadrics. We only need to check the strong anisotropy.

It suffices to consider the case where ψ is anisotropic.

Let Q be the projective quadric of ψ and let V be the second last grassmannian of φ . Assume that $\varphi_{F(\psi)}$ is not strongly anisotropic and consider a lift $\alpha \in \text{Ch}^{n+1}(Q \times V)$ of an element in $\text{Ch}^{n+1}(V_{F(Q)})$ representing $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V_{F(Q)})$. Note that $\alpha_0 = e_{n+1}$ in decomposition (3.2) of the image $\bar{\alpha} \in \text{Ch}^{n+1}(\bar{Q} \times \bar{V})$ of α . Also note that by [4, Proposition 3.1] the form $\varphi_{F(Q)}$ is not essential so that by the index reduction formula for quadrics the even Clifford algebra $C_0(\psi)$ is isomorphic to $C_0(\varphi)$. Since ψ is not a relative of φ , we have $l_{n-2} \notin \overline{\text{Ch}}(Q_{F(V)})$ implying that $a = 0$ in (3.2). Thus, we can apply Proposition 3.1(ii) and conclude that

$$(4.6) \quad \alpha_0 + \text{St}^1(\alpha_1) + \alpha_1\beta \in \overline{\text{Ch}}^{n+1}(V).$$

By [4, Proposition 3.1], $be_{n+1} \in \overline{\text{Ch}}^{n+1}(V)$, where $b \in \mathbb{Z}/2\mathbb{Z}$ is the coefficient at e_{n+1} in (4.6). The third summand of (4.6) does not provide any contribution to b . (Note that β is a multiple of c_1 because c_1 generates the group $\text{Ch}^1(\bar{V}) \ni \beta$.) By [9, Proposition 2.9], the contribution of $\text{St}^1(e_n)$ is also trivial: the coefficient $\binom{n}{1} = n$ at e_{n+1} in this formula is even by the assumption on n . Therefore $b = 1$ and $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V)$ as desired. $\square$

Theorem 4.7. *For n and φ as in Theorem 4.1, there exists an overfield $L \supset F$ such that φ_L is anisotropic and all non-degenerate anisotropic quadratic forms over L of dimension at least $2n + 1$ are relatives of φ (in particular, $u(L) = 2n + 1$).*

Proof. As usually, the field L is the union of all fields F_i from the infinite chain

$$F = F_0 \subset F_1 \subset \dots$$

constructed as follows. If F_i is already constructed for some i , the next field F_{i+1} is defined as the direct limit of function fields of finite products of quadrics given by non-degenerate anisotropic quadratic forms over F_i of dimension at least $2n + 1$ each of which is not a relative of φ_{F_i} . Then by Theorem 4.1 above and Lemma 4.8 below, the field L has all desired properties. $\square$

Lemma 4.8. *Let ψ be a quadratic form over a field F which is not a relative of an almost essential form φ over F of dimension at least 7. Let ρ be a non-degenerate quadratic form over F of dimension at least 9. Then $\psi_{F(\rho)}$ is still not a relative of $\varphi_{F(\rho)}$.*

Proof. We may assume that ψ is non-degenerate and of the same dimension as φ . Let L be the function field of the product of the third last grassmannians of φ and ψ . The anisotropic parts φ^0 and ψ^0 of φ_L and ψ_L are 5-dimensional. Since ψ is not a relative of φ , the form $\psi_{L(\varphi^0)}^0$ is anisotropic. By Separation Theorem [1, Theorem 26.5], it remains anisotropic over the function field of $\rho_{L(\varphi^0)}$ implying that $\psi_{F(\rho)}$ is not a relative of $\varphi_{F(\rho)}$. $\square$

5. SIMILARITY OF RELATIVES

We are using the Chow motives [1, §64] with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.1. *The motive of the quadric given by an essential quadratic form is indecomposable.*

Proof. We induct on the dimension of the quadratic form. In dimension 1 the statement is trivial. For higher dimension, writing X for the initial quadric and writing X' for the quadric given by the anisotropic part of the quadratic form over the function field of X , we know by the induction hypothesis that the motive of X' is indecomposable. Therefore, if the motive of X is decomposable, then one of the summands of a non-trivial decomposition is the motive of X' shifted by 1, whereas the other summand is given by a Rost projector in the sense of [5, Definition 1.3]. However, by [5, Corollary 6.6], since the quadratic form defining X is essential, there is no Rost projector on X .

Note that the base field is assumed to be of characteristic different from 2 in [5]. However, [5, Definition 1.3] applies in arbitrary characteristic. Moreover, the proof of [5, Corollary 6.6] works in characteristic 2 as well.

Another possibility is to refer to [2, Theorem 1.1], relating existence of the Rost projector with rationality of certain element in $\text{Ch}(\bar{X}_2)$, where X_2 is the 2nd grassmannian of the quadratic form. For an essential quadratic form, this element is not rational by [4, Proposition 3.1]. $\square$

Lemma 5.2. *The motive of the quadric given by an almost essential quadratic form of dimension different from a 2-power plus 1 is indecomposable.*

Proof. Let φ be an almost essential quadratic form over a field F of dimension different from a 2-power plus 1. Let X be the quadric of φ and let X' be the quadric given by the anisotropic part $\varphi'_{F(X)}$. Since the form φ' is essential, the motive of X' is indecomposable by Lemma 5.1. Therefore, if the motive of X is decomposable, then it contains the motive of X' , shifted by 1, as a direct summand. The complimentary summand is then given by a Rost projector. However, by [1, Corollary 80.11], since $\dim \varphi$ is not a 2-power plus 1, there is no Rost projector on X .

Note that this as well as another result from [1, Chapter XV], cited below, are stated there for characteristic $\neq 2$ by the only reason of non-availability of the Steenrod operations in characteristic 2 at the time; since nowadays, this obstacle is surpassed in [7], we apply them in arbitrary characteristic. $\square$

Theorem 5.3. *Let ψ be a relative of a quadratic form φ over a field F which is almost essential of dimension $2n + 1 \geq 7$ or essential of dimension $2n + 1 \geq 5$.*

- (i) *If $2n + 1 < 8$, then ψ is similar to φ .*
- (ii) *If $2n + 1 < 16$ and $u(F) < 16$, then ψ is similar to φ .*

Proof of Theorem 5.3(i). For $2n + 1 = 5$, ψ is similar to φ by the definition of relatives. We only need to prove Theorem 5.3(i) for dimension 7.

We write X and Y for the (5-dimensional) quadrics of φ and ψ . Besides, we write X' and Y' for the (3-dimensional) quadrics of the anisotropic parts φ' and ψ' of $\varphi_{F(X)}$ and $\psi_{F(Y)}$. Note that $\psi'_{F(X)}$ is a relative of the essential form $\varphi'_{F(Y)}$ (over the field $F(Y)(X) = F(X)(Y)$); therefore the two 5-dimensional forms are similar, i.e., the quadrics $Y'_{F(X)}$ and $X'_{F(Y)}$ are isomorphic. Besides, by Lemma 5.1, their motives are indecomposable.

Assume that the form $\varphi_{F(Y)(Y')}$ is anisotropic. Then its first Witt index is 2, which is impossible by [1, Corollary 79.6].

We conclude that $\varphi_{F(Y)(Y')}$ is isotropic. Then it actually has the Witt index 2. Assume now that the form $\varphi_{F(Y)}$ is anisotropic. Then for any field extension $L/F(Y)$, the Witt index of φ_L is at least 2 if and only if ψ'_L is isotropic. By [6, Proposition 4.6] (which is a generalization of [8, Theorem 4.15] to more general than quadrics varieties, but, what is important here, avoids its characteristic restriction), this implies that the motive of Y' , shifted by 1, is a direct summand in the motive of $X_{F(Y)}$. The complimentary summand is then given by a Rost projector. Since $\dim \varphi = 7$, this contradicts Lemma 5.2.

We therefore conclude that the form $\varphi_{F(Y)}$ is isotropic. Then it follows by [1, Theorem 76.1] that the form $\psi_{F(X)}$ is also isotropic, i.e., the quadrics X and Y are stably birationally equivalent. Since the motives of X and of Y are indecomposable, we conclude that the two motives are isomorphic. Finally, by [1, Theorems 93.1 and 27.3], the quadrics X and Y are isomorphic, i.e., the forms φ and ψ are similar. $\square$

Proof of Theorem 5.3(ii). We may assume from the very beginning that the forms φ and ψ have the same discriminant. In this case they are isomorphic if they are similar.

We use notation of the previous proof and start with the case $2n + 1 = 9$. The quadrics X' and Y' are now 5-dimensional; by Theorem 5.3(i) they are isomorphic. The same way as in the previous proof, we show that the form $\varphi_{F(Y)(Y')}$ is isotropic.

Now let us assume that $\varphi_{F(Y)}$ is anisotropic. From this we deduce the same way as before that the motive of Y' , shifted by 1, is a direct summand in the motive of $X_{F(Y)}$. The

quadratic form $\pi := \varphi_{F(Y)} \perp - \psi'$ becomes hyperbolic over the function field of $\varphi_{F(Y)}$. By [1, Theorem 27.1] we conclude that π is similar to an anisotropic 4-Pfister form. However, since $u(F) < 16$, the anisotropic part of the 18-dimensional form $\varphi \perp - \psi$ is at most 14-dimensional, which implies that π is isotropic.

Therefore the form $\varphi_{F(Y)}$ is isotropic. It follows that the forms φ and ψ are stably birational equivalent. Moreover, they satisfy the condition of [1, Theorem 27.3] and therefore are similar.

In the remaining case of $2n + 1 \in \{11, 13, 15\}$, we show as in the proof of Theorem 5.3(i) that the forms φ and ψ are similar provided that the forms $\varphi^{(1)}$ and $\psi^{(1)}$ are so, where $\varphi^{(1)} := \varphi'_{F(Y)}$ and $\psi^{(1)} := \psi'_{F(X)}$. Using this repeatedly, we eventually come back to the case of 9-dimensional forms considered in the very beginning, namely the forms $\varphi^{(n-4)}$ and $\psi^{(n-4)}$. But the current circumstances are slightly different: we do not dispose of the bound on the u -invariant for the base field of these two forms anymore, we have to get along with the bound $u(F) < 16$ for the initial base field.

Well, assuming that the form $\varphi^{(n-4)}_{F(\psi^{(n-4)})}$ is anisotropic, we still see that the form

$$\pi := \varphi^{(n-4)}_{F(\psi^{(n-4)})} \perp - \psi^{(n-5)}$$

(living over the function field of the quadratic form $\psi^{(n-4)}$) becomes hyperbolic over the function field of the first orthogonal summand and conclude that π is similar to an anisotropic 4-Pfister form. Since $u(F) < 16$, the anisotropic part of the $2(2n + 1)$ -dimensional form $\varphi \perp - \psi$ is at most 14-dimensional. This implies that π is isotropic. The contradiction obtained proves that the form $\varphi^{(n-4)}_{F(\psi^{(n-4)})}$ is isotropic and so we deduce as before that the forms $\varphi^{(n-4)}$ and $\psi^{(n-4)}$ are similar. $\square$

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