# AN ULTIMATE PROOF OF HOFFMANN-TOTARO'S CONJECTURE

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Abstract. We prove the last open case of the conjecture on the possible values of the first isotropy index of an anisotropic quadratic form over a field. It was initially stated by Detlev Hoffmann for fields of characteristic $\neq 2$ and then extended to arbitrary characteristic by Burt Totaro. The initial statement was proven by the author in 2002. In characteristic 2, the case of a totally singular quadratic form was done by Stephen Scully in 2015 and the nonsingular case by Eric Primozic in early 2019.

## Contents



## 0. Introduction

Let $F$ be an arbitrary field, $V$ a finite dimensional vector space over $F$, $\varphi\colon V \to F$ a quadratic form, [1, Definition 7.1]. A subspace $W \subset V$ is *totally isotropic*, if $\varphi(W) = 0$. *Isotropy index* $i_0(\varphi)$ of $\varphi$ is the maximum of dimension of a totally isotropic subspace.

Assume that $\varphi$ is anisotropic (i.e., $i_0(\varphi) = 0$) and $\dim(\varphi) := \dim(V)$ is at least 2. The projective quadric $X$ given by $\varphi$ is then an integral variety. The *first isotropy index* $i_1(\varphi)$ is a positive integer defined as the isotropy index of $\varphi$ over the function field $F(X)$.

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The following conjecture was initially formulated by Detlev Hoffmann (orally to Oleg Izhboldin) in 1998 for fields of characteristic $\neq 2$ (see [13, Conjecture 7.13] or [9, Conjecture 1.4]) and then extended to arbitrary characteristic by Burt Totaro (see [12, Page 596]):

**Conjecture 0.1.** *The first isotropy index* $i_1(\varphi)$ *does not exceed the highest* 2*-power dividing* $\dim(\varphi) - i_1(\varphi)$.

(We refer to [1, Remark 79.5 and Theorem 79.9] for reformulations of the above statement and for examples realising – for any given dimension of $\varphi$ – all those values of $i_1(\varphi)$ which are not forbidden by the statement.)

The initial (characteristic $\neq 2$) version of Conjecture 0.1 was proven in [8] with a help of Steenrod operations on the modulo 2 Chow groups (see [8, Introduction] for a list of older partial contributions). Such operations were not available in characteristic 2 at the time. Recently, Eric Primozic constructed the Steenrod operations in characteristic 2 (and, more generally, for any prime $p$, the operations in motivic cohomology with coefficients in $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ of smooth quasi-projective varieties over a field of characteristic $p$, [10]) and extended the proof of Conjecture 0.1 to nonsingular $\varphi$ in any characteristic. Note that all anisotropic quadratic forms in characteristic $\neq 2$ are nonsingular (meaning that the associated quadrics are smooth varieties). But in characteristic 2 there are singular and even totally singular anisotropic forms (the associated quadrics are nowhere smooth in the latter case).

Several years ago, Olivier Haution provided constructions of (some weak versions of) some Steenrod operations on Chow groups modulo 2 in characteristic 2: in [3] (weak first Steenrod operation), in [2] (actual first one), and in [4] (weak second and third ones). As an application, he proved the part [4, Theorem 4.8] of Conjecture 0.1 for nonsingular forms in characteristic 2, where the difference $\dim(\varphi) - i_1(\varphi)$ is not divisible by 4. (If the difference is not divisible by $2^{s+1}$, then only the Steenrod operations up to $2^s$th are needed in the proof of Conjecture 0.1.)

The case of totally singular forms was done (using completely different methods) by Stephen Scully in [11].

The remaining case (of singular, not totally singular $\varphi$ in characteristic 2) is treated here (see §2 and §7). With this final step, it is shown that Conjecture 0.1 holds in full generality. If instead of [10] one makes use of the first and second Steenrod operations of [3] and [4] together with the approach developed here, one extends the statement of [4, Theorem 4.8] to arbitrary – possibly singular quadratic forms.

Conjecture 0.1 is a wholesome companion of the statement on existence of rational maps out of anisotropic projective quadrics into arbitrary complete varieties without closed points of odd degree in terms of the first isotropy index and their dimensions, generalizing a conjecture due to Oleg Izhboldin. This statement has been proved in characteristic $\neq 2$ in [7, Theorem 3.1], then extended in [1, Theorem 76.1] to smooth quadrics in any characteristic, and finally proved in [12, Theorem 5.2] in the nonsmooth case as well.

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## 1. Preliminaries

In the remainder of this text, $\varphi$ is a quadratic form on a finite dimensional vector space $V$ over a field $F$ of characteristic 2.

By *variety*, we simply mean a separated scheme of finite type over a field so that the projective quadric $X$, given by $\varphi$, is a variety. If the form $\varphi$ is zero (i.e., $\varphi(V) = 0$), $X$ is the projective space $\mathbb{P}(V)$ of lines in $V$ (nonempty iff $\dim(\varphi) := \dim(V) \geq 1$). Otherwise, it is a hypersurface in $\mathbb{P}(V)$ (nonempty iff $\dim(\varphi) \geq 2$).

We write $V_{\mathrm{ts}}$ for the radical of the bilinear form associated to $\varphi$. The restriction $\varphi_{\mathrm{ts}}$ of $\varphi$ to $V_{\mathrm{ts}}$ is the *totally singular part* of $\varphi$. The quadric $X_{\mathrm{ts}}$, given by $\varphi_{\mathrm{ts}}$, is a closed subvariety in $X$. Assume that $\varphi$ is nonzero. Then $X_{\mathrm{ts}} \subset X$ is the nonsmooth locus of $X$. In particular, $X$ is smooth if and only if $\dim(\varphi_{\mathrm{ts}}) = 0$ or $\dim(\varphi_{\mathrm{ts}}) = 1$ and $\varphi_{\mathrm{ts}}$ is nonzero (cf. [1, Proposition 22.1]). We say that $\varphi$ is *nonsingular*[1] in this case; otherwise, we say that $\varphi$ is singular. We say that $\varphi$ is *totally singular* if $\varphi = \varphi_{\mathrm{ts}}$ or, equivalently, if $X$ is nowhere smooth. (A totally singular quadratic form $\varphi$ satisfies $\varphi(av + bw) = a^2\varphi(v) + b^2\varphi(w)$ for $a, b \in F$ and $v, w \in V$. So, in coordinates, it has the form $\varphi(x_1, ..., x_n) = a_1x_1^2 + ... + a_nx_n^2$ for some $a_1, ..., a_n$ in $F$.)

Any $\varphi$ can be decomposed in the orthogonal sum of the zero form of some dimension $m \geq 0$ (the *radical* of $\varphi$), some anisotropic quadratic form $\varphi_{\mathrm{an}}$ (called *anisotropic part* of $\varphi$), and the orthogonal sum of $n \geq 0$ hyperbolic planes $\mathbb{H}\colon F^2 \to F$, $\mathbb{H}(x, y) = xy$ ([1, Theorem 8.5]). The integer $n$ is the *Witt index* $i_W(\varphi)$ of $\varphi$ and the integer $n + m$ is the isotropy index $i_0(\varphi)$ from Introduction. So, we have $i_0(\varphi) \geq i_W(\varphi)$ and, of course, in general, there is no equality.

However, if $\varphi$ is anisotropic and not totally singular, then the two indexes coincide for the form $\varphi_{F(X)}$ (see [6, Corollary 2.7(2)] or [12, Lemma 2.3]) so that the first isotropy index $i_1(\varphi)$ can actually be defined using any of them (and therefore may also be called the *first Witt index*). Also note that $i_1(\varphi)$ is the minimum of $i_W(\varphi_L)$ over all field extensions $L/F$ with $i_W(\varphi_L) > 0$, [6, Corollary 2.21].

It is also the minimum of $i_0(\varphi_L)$ over all field extensions $L/F$ with $i_0(\varphi_L) > 0$, as noticed in [11, Page 1092]. Indeed, if $i_0(\varphi_L) > 0$ for some $L/F$, the variety $X$ has an $L$-point. If an $L$-point can be found on the smooth locus of $X$, there is an $F$-place $F(X)$ to $L$ ([1, §103]); therefore any projective $F$-variety with an $F(X)$-point has an $L$-point ([1, §103]). Applying this fact to the Grassmannian of totally isotropic subspaces in $V$ of dimension $i_1(\varphi)$, we get that $i_1(\varphi) \leq i_0(\varphi_L)$. If the smooth locus of $X$ has no $L$-points, then for some intermediate fields $F \subset K \subset K' \subset L$ with $K'/K$ quadratic inseparable, $\varphi$ is anisotropic over $K$, the totally singular part of $\varphi$ is isotropic over $K'$, and the variety $X_K$ has a regular closed point with residue field $K'$. We get a $K$-place $K(X)$ to $K'$ ([1, §103]). It follows that $i_1(\varphi) \leq i_1(\varphi_K) \leq i_0(\varphi_{K'}) \leq i_0(\varphi_L)$.

Note that a nowhere smooth anisotropic projective quadric of any given dimension (and over an appropriate field) can be but not always is a regular variety, see [12, Lemma 6.6] for a criterion of its regularity.

[1] Nonsingular forms are called *nondegenerate* in [1].

## 2. Main Observations

We now explain the main point of the paper.

Let $X$ be the projective quadric given by an anisotropic quadratic form $\varphi$ over $F$. Let $\bar{F}$ be a field extension of $F$ containing an algebraic closure of $F$. For any $r \geq 1$ we write $X^r$ for the direct product of $r$ copies of $X$. Given an $F$-variety $Y$, we write $\bar{Y}$ for the $\bar{F}$-variety $Y_{\bar{F}}$.

We write $\mathrm{CH}(Y)$ for the Chow group with integer coefficients of a variety $Y$ and $\mathrm{Ch}(Y)$ for $\mathrm{CH}(Y)/2\,\mathrm{CH}(Y)$. The groups $\mathrm{CH}(Y)$ and $\mathrm{Ch}(Y)$ are graded; we write $\mathrm{CH}_i(Y)$ (resp., $\mathrm{Ch}_i(Y)$) for the $i$th component – the group of classes of dimension $i$ algebraic cycles. We also set $\mathrm{CH}^i(Y) := \mathrm{CH}_{\dim Y - i}(Y)$ and $\mathrm{Ch}^i(Y) := \mathrm{Ch}_{\dim Y - i}(Y)$ for equidimensional $Y$. For smooth $Y$, intersection product turns $\mathrm{CH}^*(Y)$ and $\mathrm{Ch}^*(Y)$ into graded rings.

The proof of Conjecture 0.1 in the case of smooth $X$ makes use of multiplication and the (cohomological) Steenrod operation on the image of the change of field (ring) homomorphism $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ (mainly for $r \leq 2$) and on the image $\mathrm{Ch}(X^r_{F(X)}) \to \mathrm{Ch}(\bar{X}^r_{\bar{F}(X)}) = \mathrm{Ch}(\bar{X}^r)$ (also mainly for $r \leq 2$). Note that unlike the group $\mathrm{Ch}(X^r)$ itself, the group $\mathrm{Ch}(\bar{X}^r)$ and the operations on it are easy to compute because the variety $\bar{X}$ is cellular. Of course, the latter group is easy to compute in the nonsmooth case as well but a priori we do not have the ring and Steenrod structures to play with.

Here are three Main Observations of this paper (see Theorem 5.1) regarding a singular, possibly isotropic $\varphi$ with anisotropic totally singular part:

(1) The group homomorphism $\mathrm{res}_{\bar{F}/F}\colon \mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ factors through $\mathrm{Ch}(U^r)$, where $U$ is the nonsingular locus of $X$.
(2) The kernel of $\mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ is an ideal of the ring $\mathrm{Ch}(U^r)$.
(3) The kernel of $\mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ is stable under the total cohomological Steenrod operation $\mathrm{Ch}(U^r) \to \mathrm{Ch}(U^r)$.

Main Observations show that the image of $\mathrm{res}_{\bar{F}/F}$ inherits the ring structure and Steenrod operation from $\mathrm{Ch}(U^r)$. This makes possible to transfer the proof of Conjecture 0.1 from the nonsingular to the singular, not totally singular case. Note that we cannot replace the destination $\mathrm{Ch}(\bar{X}^r)$ by $\mathrm{Ch}(\bar{U}^r)$ because the latter is too small. Going to it would result in loss of the information we are interested in.

Main Observations generalize the approach of [12, Proof of Theorem 3.1], where the degree map for the Chow group modulo 2 of the noncomplete variety $U$ has been successfully considered. The existence of the degree map for $U^r$ allows one to define numerical equivalence on $\mathrm{Ch}(U^r)$. Main Observations (2) and (3) are proved by showing that, in the case of separably closed $F$, the kernel of $\mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ coincides with the ideal of numerically trivial elements (cf. Lemma B.2).

## 3. Totally singular form

In this section we assume that $\varphi$ is totally singular and nonzero.

For $r \geq 1$, let $X^r$ (resp., $\mathbb{P}(V)^r$) be the direct product of $r$ copies of $X$ (resp., $\mathbb{P}(V)$). We write *in* for the closed embedding $X^r \hookrightarrow \mathbb{P}(V)^r$.

**Lemma 3.1.** *For anisotropic totally singular $\varphi$, the push-forward homomorphism*

$$in_*\colon \operatorname{Ch}(X^r) \to \operatorname{Ch}(\mathbb{P}(V)^r)$$

*is zero.*

*Proof.* The degree of any closed point on $X$ is even by Springer's theorem [1, Corollary 71.3]. It follows that the degree of any closed point on $X^r$ is even as well. Since the degree homomorphism $\operatorname{Ch}_0(\mathbb{P}(V)^r) \to \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ is an isomorphism, we already see that $in_*$ is zero on $\operatorname{Ch}_0(X^r)$.

In fact, to check that an element of $\operatorname{Ch}(\mathbb{P}(V)^r)$ is trivial, it suffices to check that it is numerically trivial. For $i > 0$, any $\alpha \in \operatorname{Ch}_i(X^r)$, and any $\beta \in \operatorname{Ch}^i(\mathbb{P}(V)^r)$, the product

$$in_*(\alpha) \cdot \beta \in \operatorname{Ch}_0(\mathbb{P}(V)^r)$$

is represented by a 0-cycle on $X^r$ ([1, Proposition 56.11]). It follows that the product is zero and consequently $in_*(\alpha) = 0$. $\square$

Let $\bar{F}$ be a field extension of $F$ containing an algebraic closure of $F$. As we already did before, given an $F$-variety $Y$, we write $\bar{Y}$ for the $\bar{F}$-variety $Y_{\bar{F}}$.

**Corollary 3.2.** *The change of field homomorphism $\operatorname{Ch}(X^r) \to \operatorname{Ch}(\bar{X}^r)$ is zero.*

*Proof.* The variety $\bar{X}$ is a double hyperplane in $\bar{\mathbb{P}}(V)$. The associated reduced variety (which has the same Chow group) is a hyperplane. It follows that $\operatorname{Ch}(\bar{X}^r) \to \operatorname{Ch}(\bar{\mathbb{P}}(V)^r)$ is an injection. The composition $\operatorname{Ch}(X^r) \to \operatorname{Ch}(\bar{X}^r) \to \operatorname{Ch}(\bar{\mathbb{P}}(V)^r)$ is zero by Lemma 3.1 because it coincides with the composition $\operatorname{Ch}(X^r) \to \operatorname{Ch}(\mathbb{P}(V)^r) \to \operatorname{Ch}(\bar{\mathbb{P}}(V)^r)$. $\square$

Let $V_{\mathrm{rad}}$ be the radical of $\varphi$. We write $V_{\mathrm{an}}$ for the quotient vector space $V/V_{\mathrm{rad}}$, $\varphi_{\mathrm{an}}$ for the (anisotropic totally singular) quadratic form induced by $\varphi$ on $V_{\mathrm{an}}$, and $X_{\mathrm{an}}$ for the projective quadric of $\varphi_{\mathrm{an}}$.

**Lemma 3.3.** *For any $F$-variety $Z$, the graded group $\operatorname{Ch}_*(X \times Z)$ is a direct sum of $\operatorname{Ch}_*(\mathbb{P}(V_{\mathrm{rad}}) \times Z)$ and $\operatorname{Ch}_{*-\dim(V_{\mathrm{rad}})}(X_{\mathrm{an}} \times Z)$.*

*Proof.* The complement of the closed subvariety $\mathbb{P}(V_{\mathrm{rad}})$ in $X$ is an *affine bundle* (see Appendix B) over $X_{\mathrm{an}}$, where the morphism $X \setminus \mathbb{P}(V_{\mathrm{rad}}) \to X_{\mathrm{an}}$ is the base change of the morphism $\mathbb{P}(V) \setminus \mathbb{P}(V_{\mathrm{rad}}) \to \mathbb{P}(V_{\mathrm{an}})$ induced by the projection $V \twoheadrightarrow V_{\mathrm{an}}$. By localization [1, Proposition 57.9] and homotopy invariance [1, Theorem 57.13] for Chow groups we have an exact sequence

$$\operatorname{Ch}_*(\mathbb{P}(V_{\mathrm{rad}}) \times Z) \to \operatorname{Ch}_*(X \times Z) \to \operatorname{Ch}_{*-\dim(V_{\mathrm{rad}})}(X_{\mathrm{an}} \times Z) \to 0. \tag{3.4}$$

The first arrow is a split monomorphism because its composition with the push-forward $\operatorname{Ch}_*(X \times Z) \to \operatorname{Ch}_*(\mathbb{P}(V) \times Z)$ is the push-forward

$$\operatorname{Ch}_*(\mathbb{P}(V_{\mathrm{rad}}) \times Z) \to \operatorname{Ch}_*(\mathbb{P}(V) \times Z)$$

which is a split monomorphism. $\square$

**Remark 3.5.** In the above proof, a splitting of the short exact sequence (3.4) (with an additional zero on the left) is constructed. Let us check that the image of the splitting of the epimorphism is contained in the image of the push-forward

$$\operatorname{Ch}((X \times Z) \times X_{\mathrm{an}}) \to \operatorname{Ch}(X \times Z) \tag{3.6}$$

with respect to the first projection.

Recall that $X \setminus \mathbb{P}(V_{\mathrm{rad}})$ is an affine bundle over $X_{\mathrm{an}}$. For $\alpha \in \mathrm{Ch}(X_{\mathrm{an}} \times Z)$, its image in $\mathrm{Ch}(X \times Z)$ is obtained in two steps. First we choose any $\beta \in \mathrm{Ch}(X \times Z)$ restricting to $\mathrm{Ch}\Big(\big(X \setminus \mathbb{P}(V_{\mathrm{rad}})\big) \times Z\Big)$ to the pull-back of $\alpha$. By Lemma B.1, such $\beta$ can be found inside of the image of (3.6).

As the second (and final) step, we subtract from $\beta$ its image under the composition

$$\mathrm{Ch}(X \times Z) \to \mathrm{Ch}(\mathbb{P}(V_{\mathrm{rad}}) \times Z) \to \mathrm{Ch}(X \times Z). \tag{3.7}$$

It is easy to check that when we apply (3.7) to any element in the image of (3.6) then the result is still in the image of (3.6). Indeed, for every $i = 0, \dots, \dim(\mathbb{P}(V_{\mathrm{rad}}))$, let us fix an $i$-dimensional linear subspace $\mathbb{P}^i \subset \mathbb{P}(V_{\mathrm{rad}})$. The composition (3.7) is then the sum of the homomorphisms

$$\begin{aligned}(3.8)\quad \mathrm{Ch}(X \times Z) \to \mathrm{Ch}(\mathbb{P}(V) \times Z) \xrightarrow{e^i} \mathrm{Ch}(\mathbb{P}(V) \times Z) \xrightarrow{pr_*}& \\ \mathrm{Ch}(Z) \xrightarrow{pr^*} \mathrm{Ch}(\mathbb{P}^i \times Z) \to \mathrm{Ch}(\mathbb{P}(V_{\mathrm{rad}}) \times Z) \to \mathrm{Ch}(X \times Z),&\end{aligned}$$

where $e$ is the Euler class given by the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V)$. Using [1, Proposition 53.3], one sees that the image of the composition

$$\mathrm{Ch}(X \times Z \times X_{\mathrm{an}}) \to \mathrm{Ch}(X \times Z) \to \mathrm{Ch}(\mathbb{P}(V) \times Z) \xrightarrow{e^i} \mathrm{Ch}(\mathbb{P}(V) \times Z) \to \mathrm{Ch}(Z)$$

is inside of the image of $\mathrm{Ch}(Z \times X_{\mathrm{an}}) \xrightarrow{pr_*} \mathrm{Ch}(Z)$. It follows that the image of the composition of (3.6) with (3.8) is inside of the image of (3.6).

## 4. Totally singular anisotropic part

In this section we assume that the anisotropic part $\varphi_{\mathrm{an}}$ of $\varphi$ is totally singular and nonzero. We set $n := i_W(\varphi)$ and $m := \dim(V_{\mathrm{ts}})$, so that $\dim(V) = m + 2n$.

The space $V$ contains a totally isotropic subspace $V'$ of dimension $n$ intersecting $V_{\mathrm{ts}}$ trivially. Write $W$ for $V_{\mathrm{ts}} \oplus V'$, $\psi$ for $\varphi$ restricted to $W$, and $Y$ for the projective quadric of $\psi$. Then $Y$ is a (nowhere smooth) closed subvariety of $X$. We have a fibered square of closed embeddings

$$\begin{array}{ccc} \mathbb{P}(W) & \longrightarrow & \mathbb{P}(V) \\ \uparrow & & \uparrow \\ Y & \longrightarrow & X \end{array}$$

i.e., $Y$ is the intersection of $X$ and $\mathbb{P}(W)$ inside $\mathbb{P}(V)$. The composition

$$X \setminus Y \to \mathbb{P}(V) \setminus \mathbb{P}(W) \to \mathbb{P}(V/W)$$

is an affine bundle. Note that $\dim Y = m + n - 2$ whereas the rank of the affine bundle is

$$\dim X - \dim \mathbb{P}(V/W) = (m + 2n - 2) - (n - 1) = m + n - 1 = \dim Y + 1.$$

It follows by [1, Proof of Theorem 66.2] that for any $F$-variety $Z$ one has

**Lemma 4.1.** $\mathrm{Ch}_*(X \times Z) \simeq \mathrm{Ch}_*(Y \times Z) \oplus \mathrm{Ch}_{*-(m+n-1)}(\mathbb{P}^{n-1} \times Z)$. □

Now we assume that the anisotropic part of $\varphi$ is 1-dimensional. (This will be automatically the case for any singular $\varphi$ over $F$ after we extend the scalars from $F$ to a field extension $\bar{F}$ containing an algebraic closure of $F$ in the next section.) Then the variety $Y$ is a double hyperplane in the projective space $\mathbb{P}(W)$. The corresponding reduced variety $Y_{\mathrm{red}}$ is isomorphic to $\mathbb{P}^{m+n-2}$. Since $\mathrm{Ch}_*(Y) = \mathrm{Ch}_*(Y_{\mathrm{red}})$, it follows that for any $i$ with $0 \le i \le \dim(X)$, the Chow group $\mathrm{Ch}_i(X)$ is of order 2.

Let $A$ be the polynomial ring $\mathbb{F}_2[h,l]$ in two variables $h$ and $l$ modulo two relations: $h^n = 0$ and $l^2 = 0$. We introduce an upper grading $A^*$ on $A$ by declaring $h \in A^1$ and $l \in A^{m+n-1}$. (One can also consider the lower grading $A_*$ with $A_i := A^{\dim X - i}$; then $l \in A_{n-1}$.) Note that the graded group $A^*$ is uniquely identified with a subgroup in $\mathrm{Ch}^*(X)$: $h^i$ is the nonzero element of $\mathrm{Ch}^i(X)$ and $l_i := lh^{n-1-i}$ is the nonzero element of $\mathrm{Ch}_i(X)$ for $i = 0, \dots, n-1$. Let $S\colon A \to A$ be a (non-homogeneous) ring homomorphism determined by $S(h) = (1+h)h$ and $S(l) = (1+h)^{m+n} l$.

It follows from Lemmas 4.1 and 3.3 that for any $r \ge 1$ the external product homomorphism $\mathrm{Ch}(X)^{\otimes r} \to \mathrm{Ch}(X^r)$ is an isomorphism. Therefore the group $A^{\otimes r}$ is identified with a subgroup in $\mathrm{Ch}(X^r)$.

The ring endomorphism $S$ extends and extends uniquely to $A^{\otimes r}$ the way that all $r$ embeddings $A \hookrightarrow A^{\otimes r}$, $a \mapsto 1 \otimes \dots \otimes 1 \otimes a \otimes 1 \otimes \dots \otimes 1$ are respected.

## 5. Anisotropic totally singular part

In this section $\varphi$ is any quadratic form such that its totally singular part $\varphi_{\mathrm{ts}}$ is anisotropic and nonzero. As before, $\bar{F}$ is a field extension of $F$ containing an algebraic closure of $F$. We refer to $A \subset \mathrm{Ch}(\bar{X})$ defined in the previous section.

Here is the main observation:

**Theorem 5.1.** *For any $r \ge 1$, the image of the change of field homomorphism $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ is a subring of $A^{\otimes r} \subset \mathrm{Ch}(\bar{X}^r)$ stable under $S$.*

*Proof.* We first assume that $F$ is separably closed. Then the totally singular part of $\varphi$ coincides with its anisotropic part. In particular, we are in the situation of §4. Applying Lemmas 4.1 and 3.3, we decompose $\mathrm{Ch}(X^r)$ in a direct sum, where the summands are the Chow groups of products of projective spaces and copies of the quadric $X_{\mathrm{ts}}$. By Corollary 3.2, the summands containing at least one copy of $X_{\mathrm{ts}}$ disappear over $\bar{F}$. (The sum of these summands is precisely the kernel because the sum of the remaining summands maps injectively.) It follows that the image of $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ is the entire subgroup $A^{\otimes r} \subset \mathrm{Ch}(\bar{X}^r)$. This already proves Theorem 5.1 for $F$ separably closed. But, to pass to arbitrary fields, we need to discuss (still for $F$ separably closed) the geometric meaning of the ring structure and the endomorphism $S$ we defined on $A^{\otimes r}$.

By Corollary 3.2 once again, the image of the push-forward $\mathrm{Ch}(X_{\mathrm{ts}} \times X^{r-1}) \to \mathrm{Ch}(X^r)$ disappears over $\bar{F}$. By localization, it follows that the homomorphism $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ factors through the surjective homomorphism $\mathrm{Ch}(X^r) \to \mathrm{Ch}(U^r)$, where $U := X \setminus X_{\mathrm{ts}}$. Since $X_{\mathrm{ts}}$ is the nonsmooth locus of $X$, the variety $U$ is smooth so that $\mathrm{Ch}(U^r)$ is a ring and the total Steenrod operation $S\colon \mathrm{Ch}(U^r) \to \mathrm{Ch}(U^r)$ of [10] is defined.

**Lemma 5.2.** *The kernel of the group homomorphism $f^r\colon \mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ is an ideal of the ring $\mathrm{Ch}(U^r)$.*

*Proof.* As the degree homomorphism $\deg\colon \mathrm{Ch}_0(X^r) \to \mathbb{F}_2$ factors through $\mathrm{Ch}_0(U^r)$, we can consider numerical equivalence on the (non-complete) variety $U^r$: $\alpha \in \mathrm{Ch}_i(U^r)$ is declared numerically trivial if $\deg(\alpha \cdot \beta) = 0$ for all $\beta \in \mathrm{Ch}^i(U^r)$. Clearly, the set $N^r$ of numerically trivial elements in $\mathrm{Ch}(U^r)$ is an ideal of this ring. To prove Lemma 5.2, we check that the $\mathrm{Ker}(f^r) = N^r$.

By Remark 3.5, the kernel of $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ is equal to the image of the push-forward $pr_*\colon \mathrm{Ch}(X^r \times X_{\mathrm{ts}}) \to \mathrm{Ch}(X^r)$. Therefore the kernel of $f^r$ is equal to the image of the push-forward $pr_*\colon \mathrm{Ch}(Y) \to \mathrm{Ch}(U^r)$, where $Y := U^r \times X_{\mathrm{ts}}$. Because of the second projection, all closed points on $Y$ are of even degree. Consequently, the latter image is numerically trivial by [1, Proposition 56.11] applied to the compositions $Z \hookrightarrow Y \xrightarrow{pr} U^r$, where $Z$ runs over closed equidimensional subvarieties of $Y$.

We proved the inclusion $\mathrm{Ker}(f^r) \subset N^r$. Now we prove the opposite inclusion by induction on $r$. For $r = 0$ (let's allow this value here) we have $U^r = \mathrm{Spec}\, F$ and the statement is trivial. To deal with $r \geq 1$, for $i = 0, \dots, i_W(\varphi) - 1$, abusing notation, we write $h^i \in \mathrm{Ch}^i(U)$ and $l_i \in \mathrm{Ch}_i(U)$ for the elements, defined in §6, whose images under $f = f^1$ are $h^i, l_i \in A$. Together these elements form a subring in $\mathrm{Ch}(U)$ which maps isomorphically onto $A$. Given a numerically trivial $\alpha \in \mathrm{Ch}(U^r)$, by adding to $\alpha$ an appropriate element of $\mathrm{Ker}(f^r)$, we come to the situation where $\alpha$ is (still numerically trivial and) equal to a sum of external products with each factor being $h^i$ or $l_i$ for some $i$. For any $i$, the push-forwards of the products $(h^i \times [U^{r-1}])\alpha$ and $(l_i \times [U^{r-1}])\alpha$ with respect to the projection $U \times U^{r-1} \to U^{r-1}$ are numerically trivial and therefore zero by the induction hypothesis. It follows that $\alpha = 0$. $\square$

**Lemma 5.3.** *The kernel of the group homomorphism $f^r\colon \mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ is stable under the total Steenrod operation $S\colon \mathrm{Ch}(U^r) \to \mathrm{Ch}(U^r)$.*

*Proof.* We prove it like Lemma B.2. The only missing ingredient is Formula (B.3). Let us prove this formula. We have a closed embedding $in\colon U^r \hookrightarrow (\mathbb{P}(V) \setminus X_{\mathrm{ts}})^r$ and an open embedding $(\mathbb{P}(V) \setminus X_{\mathrm{ts}})^r \hookrightarrow \mathbb{P}(V)^r$. By [10, Proposition 7.1],[2]

$$\deg(c(-T_{U^r})S(\alpha)) = \deg(in_*(c(-T_{U^r})S(\alpha))) = \deg(c(-T_{(\mathbb{P}(V)\setminus X_{\mathrm{ts}})^r})S(in_*(\alpha))),$$

where $T_Y$ for a smooth variety $Y$ is its tangent vector bundle and $c(-T_Y)$ is the Segre class of $T_Y$. So, it suffices to prove Formula (B.3) for the variety $(\mathbb{P}(V) \setminus X_{\mathrm{ts}})^r$. The pull-back $j^*\colon \mathrm{Ch}(\mathbb{P}(V)^r) \to \mathrm{Ch}((\mathbb{P}(V) \setminus X_{\mathrm{ts}})^r)$ is surjective and commutes with $S$. Therefore Formula (B.3) for $\mathbb{P}(V)^r$ implies the same formula for the open subvariety. $\square$

It follows by Lemmas 5.2 and 5.3 that the image of $\mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ (which, as we already know, is equal to $A^{\otimes r} \subset \mathrm{Ch}(\bar{X}^r)$) is a ring with an endomorphism induced by the Steenrod operation on $\mathrm{Ch}(U^r)$. Let us note that $l_i \in A_i$ is the image of the class in $\mathrm{Ch}(X)$ of a linear subspace $\mathbb{P}^i \subset X$ whereas $h^i \in A^i$ is the image of the pull-back to $\mathrm{Ch}(X)$ of the class in $\mathrm{Ch}(\mathbb{P}(V))$ of a codimension $i$ linear subspace in $\mathbb{P}(V)$. It will be verified in §6 that the ring structure on $A^{\otimes r}$ we get coincides with the one we introduced earlier,

[2] The assumption in [10, Propositions 7.1 and 7.2] of projectivity of the varieties is superfluous and not used in the proof; it can be replaced by the assumption of projectivity of the morphism between the varieties.

and that the total Steenrod operations $S\colon \mathrm{Ch}(U^r) \to \mathrm{Ch}(U^r)$ of [10] corresponds to the operation $S\colon A^{\otimes r} \to A^{\otimes r}$. These verifications finish the discussion of separably closed $F$.

Now we allow $F$ to be arbitrary and write $\tilde{F}$ for the separable closure of $F$ inside of $\bar{F}$. Note that the totally singular part of $\varphi$ – as any anisotropic totally singular quadratic form over $F$ – remains anisotropic over $\tilde{F}$ (see [5, Proposition 5.3] or [12, Lemma 2.1]). Still by Corollary 3.2, the homomorphism $\mathrm{Ch}(X^r) \to \mathrm{Ch}(\bar{X}^r)$ factors through $\mathrm{Ch}(U^r)$, where $U$ is the smooth variety $X \setminus X_{\mathrm{ts}}$. And the homomorphism $\mathrm{Ch}(U^r) \to \mathrm{Ch}(\bar{X}^r)$ factors through the ring homomorphism $\mathrm{Ch}(U^r) \to \mathrm{Ch}(\tilde{U}^r)$ (respecting the total Steenrod operation – [10, Corollary 2.4]), where $\tilde{U} := U_{\tilde{F}}$. □

## 6. Computation of operations

In this section, $\varphi$ is such that its totally singular part is nonzero and coincides with its anisotropic part. We set $n := i_W(\varphi)$ and assume that $n \geq 1$. We are computing the multiplication and the Steenrod operation on $A := \mathrm{Im}(\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X}))$ induced by those on the Chow group of the smooth variety $U = X \setminus X_{\mathrm{ts}}$.

For any $i = 0, \ldots, n-1$, the element $h^i \in A^i$ is the image of the pull-back of $H^i \in \mathrm{Ch}^i(\mathbb{P}(V))$ with respect to the embedding $U \hookrightarrow X \hookrightarrow \mathbb{P}(V)$, where $H \in \mathrm{Ch}^1(\mathbb{P}(V))$ is the hyperplane class and $H^i$ is its $i$th power. It follows that $h^i$ is also $i$th power of $h$ ($h^0$ is the unity 1 of the ring $A$). We have $h^n = 0$ simply because $h^n \in A^n = 0$.

Let $W \subset V$ be a totally isotropic subspace of dimension $i+1$ for $i = 0, \ldots, n-1$. Then $\mathbb{P}(W)$ is a closed subvariety of $X$ and $l_i \in A_i$ (defined as the only nonzero element in $A_i$) is the image of its class $[\mathbb{P}(W)] \in \mathrm{Ch}_i(X)$. For $l = l_{n-1}$ we have $l^2 = 0$ simply because $l^2 \in A_{-\dim(\varphi_{\mathrm{ts}})} = 0$. Considering the pull-back of $[\mathbb{P}(W)] \in \mathrm{Ch}(X)$ to $\mathrm{Ch}(U)$ and applying Projection Formula [1, Proposition 56.9] to the embedding $U \hookrightarrow \mathbb{P}(V) \setminus X_{\mathrm{ts}}$, we get that $hl_i = l_{i-1}$ for $i = 1, \ldots, n-1$.

At this stage we have verified that the multiplication on $A$ induced by multiplication of the ring $\mathrm{Ch}(U)$ is indeed as described in §5. Now we turn to computation of the total Steenrod operation $S\colon A \to A$.

Since $S$ is a ring homomorphism (see [10, Proposition 5.1]), we only need to compute it on the generators $h, l \in A$. Since $h \in A^1$, we have $S(h) = h + h^2$ by [10, Corollaries 6.3 and 6.5]).

In order to compute $S(l)$, we apply [10, Proposition 7.1] (see also [10, Proposition 7.2])[2] to the closed embedding $in\colon \mathbb{P}(W) \cap U \hookrightarrow U$. We get that $S(l) = in_*(c(N))$, where $c(N) \in \mathrm{Ch}(\mathbb{P}(W) \cap U)$ is the total Chern class of the normal bundle $N$ of the embedding. The total Chern class $c(T_U)$ of the tangent bundle $T_U$ of $U$ is computed – similarly to [1, Lemma 78.1] – as the pull-back of $(1+H)^{\dim(V)} \in \mathrm{Ch}(\mathbb{P}(V))$, where $H$ is the hyperplane class for $\mathbb{P}(V)$. The total Chern class $c(T_{\mathbb{P}(W)\cap U})$ is the pull-back of $c(T_{\mathbb{P}(W)}) \in \mathrm{Ch}(\mathbb{P}(W))$ computed, e.g., in [1, Example 61.16], as $(1+H')^{\dim(W)} \in \mathrm{Ch}(\mathbb{P}(W))$, where $H'$ is the hyperplane class for $\mathbb{P}(W)$. Arguing like in [1, Corollary 78.2], we obtain that $c(N)$ is the pull-back of $(1+H')^{\dim(V)-\dim(W)}$. It follows that $S(l) = (1+h)^{\dim(V)-\dim(W)} l$ which is the desired formula.

## 7. Proof of Conjecture 0.1

With Theorem 5.1 at our disposal, we prove the remaining open case of Conjecture 0.1 (dealing with singular but not totally singular forms over a field of characteristic 2) similarly to [1, Proof of Proposition 79.4] using the yoga of cycles on $X^r$ and especially on $X^2$ similar to [1, §72 and §73].

Let us fix a singular not totally singular anisotropic quadratic form $\varphi$ and set $n := i_W(\bar{\varphi})$, where, as usual, $\bar{F}$ is an extension field of $F$, containing an algebraic closure of $F$, and $\bar{\varphi}$ is $\varphi$ over $\bar{F}$. The assumption that $\varphi$ is not totally singular means that $n \geq 1$. The nonzero homogeneous elements $h^i, l_i$ ($i = 0, \ldots, n-1$) of $A$ form a basis of $A$ viewed as a vector space over $\mathbb{F}_2$. Their pairwise tensor products form a basis of $A^{\otimes 2}$. For any $\alpha \in A^{\otimes 2}$ we can look at its coordinate at any of the basis elements. The coordinate can only be 0 or 1, and we say that $\alpha$ *contains* the basis element if the coordinate is 1 and that $\alpha$ does not contain it otherwise.

For any $r \geq 1$, let $B(r)$ be the image of $\mathrm{Ch}(X^r)$ in $A^{\otimes r}$ and let $C(r)$ be the image of $\mathrm{Ch}(X^r_{F(X)})$ in $A^{\otimes r}$. (Here we identify $\mathrm{Ch}(\bar{X}^r)$ and $\mathrm{Ch}(\bar{X}^r_{\bar{F}(\bar{X})})$ via the change of field isomorphism.) Obviously, $B(r) \subset C(r) \subset A^{\otimes r}$. (For the second inclusion note that since $\varphi$ is not totally singular, the field extension $F(X)/F$ is separable.) By Theorem 5.1, $B(r)$ and $C(r)$ are subrings of $A^{\otimes r}$ preserved by the Steenrod operation on $A^{\otimes r}$. Also (for varying $r$) these subrings are preserved by pull-backs and push-forwards with respect to partial projections.

Sometimes we will consider the elements of $A^{\otimes 2}$ as correspondences and take their compositions. Composing correspondences only involves pull-backs and push-forwards with respect to partial projections and multiplication. Consequently, a composition of correspondences of $B(2)$ is again in $B(2)$. Basis elements in $A^{\otimes 2}$ are easy to compose. For instance, $(h^i \times l_i) \circ (h^j \times l_j)$ is $h^i \times l_i$ if $i = j$ and is 0 if $i \neq j$.

By definition of $i_1(\varphi)$, $V_{F(X)}$ contains a totally isotropic subspace of dimension $i_1(\varphi)$. Therefore $X_{F(X)}$ contains a linear subspace $\mathbb{P}^m$ with $m := i_1(\varphi) - 1$. Note that $i_1(\varphi) \leq n$ so that $m \leq n-1$. It follows that $l_m \in C(1)_m$. Since the pull-back $\mathrm{Ch}(X^2) \to \mathrm{Ch}(X_{F(X)})$ is surjective, there exists $\alpha \in B(2)_{d+m}$ containing $h^0 \otimes l_m$, where $d := \dim X$ (cf. [1, Lemma 73.18]).

Assume that $\alpha$ contains $h^i \times l_{m+i}$ for some $i \leq m$. Note that $l_i \in C(1)_i$ for $i \leq m$. Considering $\alpha$ as a correspondence and applying it to $l_i$, we get $l_{m+i}$ showing that the isotropy index of $\varphi_{F(X)}$ is greater than $m+i$. Therefore $i = 0$ and $\alpha$ does not contain any of $h^i \times l_{m+i}$ with $1 \leq i \leq m$ (cf. [1, Lemma 73.12]).

For any $i \geq 0$, we have $h^i \in B(1)^i$, i.e., $h^i$ is the image of an element of $\mathrm{Ch}^i(X)$ (namely, of the class of the section by a linear subspace in $\mathbb{P}(V)$ of codimension $i$). Therefore $\beta := \alpha \cdot (h^0 \otimes h^m)$ is in $B(2)_d$. Note that $\beta$ contains $h^0 \otimes l_0$, does not contain any of $h^i \otimes l_i$ for $i = 1, \ldots, m$, and does not contain any of $l_i \otimes h^i$ for $i = m-1, \ldots, 0$. (In the end we will see that it also contains $l_m \otimes h^m$.)

Now we assume that $\varphi$ does not satisfy Conjecture 0.1: for the highest 2-power $2^s$ dividing $\dim(\varphi) - i_1(\varphi)$, we have $i_1(\varphi) > 2^s$. The binomial coefficient $\binom{\dim(\varphi)-i_1(\varphi)}{2^s}$ is odd (see [1, Lemma 78.6]). Therefore $S^{2^s}(l_m) = l_{m-2^r}$, where $S^{2^s}$ is the $2^s$th homogeneous component of the (Steenrod) operation $S$. We apply $S^{2^s}$ to $\alpha$ and consider the element

$$\gamma := S^{2^s}(\alpha) \cdot (h^0 \otimes h^{m-2^s}) \in B(2)_d$$

which has the following properties: it contains $h^0 \otimes l_0$ and does not contain $l_m \otimes h^m$. Consider the composition of correspondences $\delta := \gamma \circ \beta \in B(2)_d$. Among all $h^i \otimes l_i$ and $l_i \otimes h^i$ with $i = 0, \ldots, m$, the element $\delta$ contains and only contains $h^0 \otimes l_0$.

We are almost done. One more observation we need is that the number of all basis elements contained in an element of $B(2)_d$ is even (cf. [1, Lemma 73.14]). Because if it is odd, then the image of the element under the homomorphism $A^{\otimes 2} \to A$, given by the product in $A$, is $l_0$ and is in $B(1)_0$. This contradicts anisotropy of $\varphi$.

Note that the homomorphism $A^{\otimes 2} \to A$, we just served of, corresponds to the diagonal pull-back $\mathrm{Ch}(U^2) \to \mathrm{Ch}(U)$ and therefore maps $B(2)$ to $B(1)$.

Here is the final step. Let $X'$ be the projective quadric over $F(X)$ given by the anisotropic part $\varphi'$ of $\varphi_{F(X)}$ and set $d' := \dim X'$. Let $B'$ be the analogue of $B$ for $X'$. Using Proposition A.2, one shows that for any element of $\xi \in B(2)_d$ there is an element $\xi' \in B'(2)_{d'}$ such that the number of basis elements contained in $\xi'$ coincides with the number of basis element contained in $\xi$ and different from $h^i \otimes l_i$ and $l_i \otimes h^i$ with $i = 0, \ldots, m$ (cf. [1, Lemma 72.3, Corollary 72.4, and Illustration 73.25]). Applying this fact to $\xi = \delta$, we see that the number of basis elements contained in $\delta$ is odd. This is the contradiction finishing the proof.

Let us indicate an alternative ending of the above proof. After we constructed the element $\delta$, let us consider the composition $\delta \circ \alpha$. It contains $h^0 \otimes l_m$ and does not contain $l_m \otimes h^0$. Let $\lambda$ be an element in $\mathrm{Ch}_{d+m}(U^2)$ mapping to $\delta \circ \alpha$. Let $\varphi' \subset \varphi$ be a codimension $m$ subform, $X' \subset X$ the corresponding subquadric, and $U' := X' \cap U$. Let $\lambda' \in \mathrm{Ch}((U')^2)$ be the pull-back of $\lambda$ with respect to the embedding $(U')^2 \hookrightarrow U^2$. The degree of the correspondence $\lambda'$ over the first factor is 1 and its degree over the second factor is 0. This contradicts to [12, Theorem 3.1] because $i_1(\varphi') = 1$ by [12, Theorem 5.2].

## Appendix A. Adding hyperbolic planes

In this appendix we prove Proposition A.2, used in §7.

Let $X'$ be the projective quadric given by an arbitrary quadratic form $\varphi' \colon V' \to F$ over an arbitrary field $F$ (of any characteristic). For some $n \geq 0$, let $W$ be a vector space of dimension $n$ over $F$, $W^\#$ its dual, and $\mathbb{H}(W) \colon W \oplus W^\# \to F$ the hyperbolic quadratic form given by $W$ (isomorphic to the orthogonal sum of $n$ hyperbolic planes $\mathbb{H} = \mathbb{H}(F)$), [1, §7]. Let $\varphi \colon V \to F$ be the orthogonal sum of $\varphi'$ and $\mathbb{H}(W)$. In particular, the vector space $V$ is the direct sum $V' \oplus W \oplus W^\#$. As usually, we write $X$ for the projective quadric given by $\varphi$.

The following statement is well-known in the case of nonsingular $\varphi'$:

**Lemma A.1.** *For any $F$-variety $Z$, there is a decomposition*

$$\mathrm{CH}_*(X \times Z) = \mathrm{CH}_*(\mathbb{P}(W) \times Z) \oplus \mathrm{CH}_{*-n}(X' \times Z) \oplus \mathrm{CH}_{*-n-\dim(X')-1}(\mathbb{P}(W^\#) \times Z).$$

*Proof.* The proof is a combination of slightly generalized proofs of Lemmas 4.1 and 3.3. Let $Y$ be the quadric given by the restriction of $\varphi$ to the subspace $V' \oplus W \subset V$. Then $Y$ is a closed subvariety in $X$ of pure dimension $n + \dim(X')$ whose complement is an affine bundle over the (smooth) variety $\mathbb{P}(W^\#)$. By [1, Proof of Theorem 66.2] we get a decomposition (cf. Lemma 4.1)

$$\mathrm{Ch}_*(X \times Z) \simeq \mathrm{Ch}_*(Y \times Z) \oplus \mathrm{Ch}_{*-\dim(Y)-1}(\mathbb{P}(W^\#) \times Z).$$

Next, $\mathbb{P}(W)$ is a closed subvariety of $Y$ whose complement is an affine bundle over the (possibly singular) variety $X'$. We get an exact sequence (cf. (3.4))

$$\mathrm{CH}_*(\mathbb{P}(W) \times Z) \to \mathrm{CH}_*(Y \times Z) \to \mathrm{CH}_{*-n}(X' \times Z) \to 0.$$

By the same argument as in the proof of Lemma 3.3, the first map here is a split monomorphism. □

Below we assume that $\operatorname{char} F = 2$ and that the totally singular part of $\varphi'$ is anisotropic and nonzero. For any $r \geq 1$, as in §7, we consider the image $B(r)$ of $\mathrm{Ch}(X^r)$ in $A^{\otimes r}$ and we consider the similar rings $B'(r) \subset (A')^{\otimes r}$ given by $X'$. We define a group homomorphism $g\colon A \to A'$ by mapping $h^i$ to $h^{i-n}$ and $l_i$ to $l_{i-n}$. (The elements with negative indexes are defined to be 0.) It gives rise to a homomorphism $g^{\otimes r}\colon A^{\otimes r} \to (A')^{\otimes r}$.

**Proposition A.2.** *$g^{\otimes r}(B(r)) \subset B'(r)$.*

*Proof.* Since the decomposition of Lemma A.1 holds with integer coefficients, it holds for any coefficient ring, including $\mathbb{F}_2$. Also, it holds over any extension field of $F$ and the change of field homomorphism respects it. Over a field $\bar{F}$ containing an algebraic closure of $F$, the projection $\mathrm{Ch}(\bar{X}) \to \mathrm{Ch}(\bar{X}')$ maps $h^i$ to $h^{i-n}$ and $l_i$ to $l_{i-n}$ for $i \geq n$ because these are the only nonzero elements of the corresponding graded components. From the decomposition given in Lemma A.1, we obtain a decomposition of $\mathrm{Ch}(X^r)$ (again over any extension field) containing $\mathrm{Ch}((X')^r)$ as a summand. And the projection $\mathrm{Ch}(X^r) \to \mathrm{Ch}((X')^r)$ commutes with the change of field homomorphism for $\bar{F}/F$. □

## Appendix B. Miscellaneous

In this appendix we prove Lemma B.1, used in Remark 3.5, and Lemma B.2, used in the proof of Lemma 5.3. The field $F$ here is arbitrary (of any characteristic).

Let $X, Y$ be varieties (not necessarily smooth) over a field $F$, $U \subset X$ an open subset, $U \to Y$ an *affine bundle* (of constant rank) in the sense of [1, §52.G], i.e., a flat morphism whose all fibers are isomorphic to affine spaces of the same dimension. Then we have a surjective homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ – the composition of the surjection $\mathrm{CH}(X) \to \mathrm{CH}(U)$ and the inverse of the pull-back isomorphism $\mathrm{CH}(Y) \xrightarrow{\sim} \mathrm{CH}(U)$.

**Lemma B.1.** *The composition* $\mathrm{CH}(X \times Y) \xrightarrow{pr_*} \mathrm{CH}(X) \to \mathrm{CH}(Y)$ *is also surjective.*

*Proof.* For a closed integral subvariety $Y' \subset Y$, let $U' \to Y'$ be the affine bundle given by the base change of $U \to Y$ and let $X'$ be the closure of $U'$ in $X$. Let $\Gamma$ be the closure in $X' \times Y'$ of the graph of the affine bundle $U' \to Y'$. Then the class $[Y'] \in \mathrm{CH}(Y')$ is the image of the class $[\Gamma] \in \mathrm{CH}(X' \times Y')$. The commutative diagram

$$\begin{array}{ccccccc}
\mathrm{CH}(X' \times Y') & \longrightarrow & \mathrm{CH}(X') & \longrightarrow & \mathrm{CH}(U') & \longrightarrow & \mathrm{CH}(Y') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{CH}(X \times Y) & \longrightarrow & \mathrm{CH}(X) & \longrightarrow & \mathrm{CH}(U) & \longrightarrow & \mathrm{CH}(Y)
\end{array}$$

finishes the proof. □

**Lemma B.2.** *Let $X$ be a smooth projective $F$-variety. The ideal of numerically trivial elements in* $\mathrm{Ch}(X)$ *is stable under the total cohomological Steenrod operation*

$$S\colon \mathrm{Ch}(X) \to \mathrm{Ch}(X).$$

*Proof.* We recall that the degree homomorphism $\deg\colon \mathrm{Ch}(X) \to \mathbb{F}_2 = \mathrm{Ch}(\operatorname{Spec} F)$ is the push-forward with respect to the structure morphism $X \to \operatorname{Spec} F$ of the projective variety $X$. For any $i > 0$ and any $\gamma \in \mathrm{Ch}_i(X)$, by [10, Proposition 7.1] (for $\operatorname{char} F \neq 2$ see [1, Chapter XI]), we have

$$\deg(c(-T_X)S(\gamma)) = \deg(\gamma) = 0, \tag{B.3}$$

where $T_X$ is the tangent vector bundle on $X$ and $c(-T_X)$ is its Segre class.

Let $\alpha \in \mathrm{Ch}(X)$ be a homogeneous numerically trivial element. We prove that $S^i(\alpha)$ is numerically trivial using induction on $i$. The case of $i = 0$ is trivial. Assume that the statement holds for all $i < n$ for some $n \geq 1$ and take any homogeneous $\beta \in \mathrm{Ch}(X)$. The relation $\deg(S^n(\alpha)\beta) = 0$ follows from (B.3) with $\gamma = \alpha\beta$ and Cartan formula [10, Proposition 5.1] (for $\operatorname{char} F \neq 2$ see [1, Corollary 61.15]). □

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