FIELDS OF ANY u-INVARIANT BUT A 2-POWER MINUS 1 OR 3

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Abstract. The u-invariant of a field is the highest dimension of a non-degenerate anisotropic quadratic form over this field. Let u be a natural number not of the form a 2-power minus 1 or a 2-power minus 3. Given any field, we find its overfield of the u-invariant u.

Contents

1. Introduction

The u-invariant of a field is the highest dimension of a non-degenerate anisotropic quadratic form over this field. As known since the 50es (see [5], raising the question), the set of possible finite values of the u-invariant starts with 1, excludes 3, 5, 7, and includes all 2-powers (see [9, §6 of Chapter XI] for a modern exposition). It was shown by Alexander Merkurjev in the end of the 80es (see [11]) that this set contains 6 and – a couple of years later – all positive even integers (see [12] or [3, Theorem 38.4]). Oleg Izhboldin proved by the end of the 90es that 9 is also there (see [4]). In the second half of the 00s, this result has been extended to all larger numbers of the form a 2-power plus 1 by Alexander Vishik (see [16]).

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The core result here is Theorem 7.2. The techniques showing up in the proof are mostly in the spirit of [16]. The most advanced ingredient is given by the Steenrod operations on the modulo 2 Chow groups invented by Vladimir Voevodsky ([17]), which became available over arbitrary fields due to Patrick Brosnan ([1], see also [3, Chapter XI]) and Eric Primozic ([14]). Although the symmetric operations [15] of Alexander Vishik in algebraic cobordism [10] of Marc Levine and Fabien Morel do not show up directly in the proof, they provide the ideology behind it. Finally, the most important classical ingredient of the proof is Index Reduction Formula for quadrics ([12], see also [3, Theorem 30.5]), discovered by Alexander Merkurjev.

By a standard argument (as in [3, §38] or in [16, §4]), Theorem 7.2 implies (for odd u) the title result:

Theorem 1.1. *For any given field F_0 and for any natural number u not of the form a 2-power minus 1 or a 2-power minus 3, there is an overfield $F \supset F_0$ of the u-invariant u.*

For $u \geqslant 9$ a 2-power plus 1, the proof of Theorem 1.1 given here is simpler than the previously available ones of [4] and [16].

Notation and results we are using are introduced "on the go".

2. Chow groups and quadrics

Let φ be a non-degenerate quadratic form over a field F. We consider the projective quadric $X = X_1$ defined by φ and write $\bar{X}$ for X over an algebraic closure $\bar{F}$ of F. Recall (see, e.g., [3, §68]) that the Chow group $\mathrm{CH}(\bar{X})$ is free with the basis

$$h^0, \ldots, h^n, l_n, \ldots, l_0,$$

where n is such that $\dim X = \dim \varphi - 2$ equals $2n$ or $2n+1$. The element $h \in \mathrm{CH}^1(\bar{X})$ is the hyperplane section class, $h^i \in \mathrm{CH}(\bar{X})$ is its ith power, and $l_i \in \mathrm{CH}_i(\bar{X})$ is the class of an i-dimensional projective space lying on $\bar{X}$. One has

$$h^{n+1} = 2l_{\dim X - n - 1} \quad \text{and} \quad hl_i = l_{i-1},$$

where $l_{-1} := 0$.

We are going to work with the modulo 2 Chow group

$$\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$$

and write $\overline{\mathrm{Ch}}(X) \subset \mathrm{Ch}(\bar{X})$ for the image of the change of field homomorphism

$$\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X}).$$

(This notation will be applied not only for quadrics.)

Note that the inclusion $l_i \in \overline{\mathrm{Ch}}(X)$ holds for some given i if and only if the Witt index $\mathrm{ind}(\varphi)$ is at least $i+1$ (see [3, Corollary 72.6]).

3. Orthogonal grassmannians

Let F be a field and let φ be a non-degenerate quadratic F-form of dimension $2n+1$ with $n \geqslant 1$. For $m \in \{1, \ldots, n\}$, let X_m be the mth grassmannian of φ, i.e., the variety of its m-dimensional totally isotropic subspaces. In particular, X_1 is the projective quadric $\varphi = 0$.

We write $c_1, \dots, c_{2n-m+1} \in \mathrm{CH}(X_m)$ for the Chern classes of the quotient of the rank $2n+1$ trivial vector bundle by the tautological (rank m) vector bundle on X_m. Since $c_{2n-m+1} = 0$, this last Chern class does not show up below.

We write $\bar{X}_m$ for the variety X_m over an algebraic closure $\bar{F}$ of F.

As shown in [16, §2] (see also [2]), the images of some of the above Chern classes in $\mathrm{CH}(\bar{X}_m)$ (for which we use the same notation)[1] are divisible by 2:

$$c_i = 2e_i \quad \text{for } i \geqslant n - m + 1,$$

where the elements $e_{n-m+1}, \dots, e_{2n-m}$ are images (respectively) of the elements $l_{n-1}, \dots, l_0 \in \mathrm{CH}(\bar{X}_1)$ (introduced in §2) under the composition

$$\mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_{1\subset m}) \to \mathrm{CH}(\bar{X}_m)$$

of the pullback followed by pushforward with respect to the projections

$$\bar{X}_1 \leftarrow \bar{X}_{1\subset m} \to \bar{X}_m,$$

where $X_{1\subset m} \subset X_1 \times X_m$ is the variety of 2-flags.

Note that the elements

$$e_{n-m+1} \in \mathrm{CH}^{n-m+1}(\bar{X}_m), \dots, e_{2n-m} \in \mathrm{CH}^{2n-m}(\bar{X}_m)$$

are indexed by their codimensions whereas the elements

$$l_{n-1} \in \mathrm{CH}_{n-1}(\bar{X}_1), \dots, l_0 \in \mathrm{CH}_0(\bar{X}_1)$$

– by their dimensions. For $m = 1$ we have

$$e_n = l_{n-1}, \dots, e_{2n-1} = l_0.$$

Recall that we are writing $\overline{\mathrm{CH}}(X_m)$ for the image of the change of field homomorphism

$$\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m).$$

Note that the index of the even Clifford algebra $C_0(\varphi)$ is 2^{n-r} for some $r \in \{0, \dots, n\}$.

Proposition 3.1. *If* $\operatorname{ind} C_0(\varphi) = 2^{n-r}$*, then* $\overline{\mathrm{CH}}(X_m) \subset \mathrm{CH}(\bar{X}_m)$ *is contained in the subring* $A_r \subset \mathrm{CH}(\bar{X}_m)$ *generated by*

$$c_1, \dots, c_{2n-m-r}, e_{2n-m-r+1}, \dots, e_{2n-m}.$$

(For $r = 0$*, this list consists of the Chern classes only; for* $r = 1$*, the list consists of the single* e_{2n-m} *and the Chern classes.)*

Proof. By Index Reduction Formula for quadrics [3, Theorem 30.5], we can find a field extension L/F such that the Witt index of φ_L is r and the index of the even Clifford algebra $C_0(\varphi_L)$ is still 2^{n-r}. Indeed, one may produce L by a chain of function fields of quadrics, starting with X_1, or simply take for L the function field of the variety X_r.

Since $\overline{\mathrm{CH}}(X_m) \subset \overline{\mathrm{CH}}((X_m)_L)$, where the groups $\mathrm{CH}((X_m)_{\bar{F}})$ and $\mathrm{CH}((X_m)_{\bar{L}})$ are identified by the change of field isomorphism, it suffices to show that $A_r = \overline{\mathrm{CH}}((X_m)_L)$. The inclusion $A_r \subset \overline{\mathrm{CH}}((X_m)_L)$ is obvious. We prove the equality by showing that the indexes of the subgroups A_r and $\overline{\mathrm{CH}}((X_m)_L)$ in their common overgroup $\mathrm{CH}(\bar{X}_m)$ coincide.

[1] The abuse can be partially justified by [8, Theorem 2.1] affirming that these images satisfy the exactly same relations as the original elements.

Recall that by [13], the Grothendieck group $K(X_m)$ is a direct sum of several copies of $K(F) = \mathbb{Z}$ and several copies of $K(C_0(\varphi)) = 2^{n-r}\mathbb{Z}$. Moreover, the number c of copies of $C_0(\varphi)$ depends on $\dim\varphi$ only. We determine c by taking a *generic* $(2n+1)$-dimensional φ, where $r = 0$ and the group $\mathrm{CH}(X_m) = A_0$ is free with a basis given by all products

$$c_1^{\alpha_1} \dots c_{2n-m}^{\alpha_{2n-m}}$$

satisfying $\alpha_1 + \dots + \alpha_{2n-m} \leqslant m$ and $\alpha_i \leqslant 1$ for $i \geqslant n-m+1$ (see [8, Theorem 2.1]). A basis of the group $\mathrm{CH}(\bar{X}_m)$ is given by the similar products with c_i replaced by e_i for all $i \geqslant n-m+1$. It follows that

$$\log_2[\mathrm{CH}(\bar{X}_m) : \mathrm{CH}(X_m)] \;=\; 1 \cdot \tbinom{n}{1} \cdot f_{m-1} + 2 \cdot \tbinom{n}{2} \cdot f_{m-2} + \dots + m \cdot \tbinom{n}{m} \cdot f_0,$$

where f_i is the number of monomials $c_1^{\alpha_1} \dots c_{n-m}^{\alpha_{n-m}}$ with $\alpha_1 + \dots + \alpha_{n-m} \leqslant i$. Since the group $\mathrm{CH}(X_m)$ is free of torsion, the canonical surjective homomorphism of $\mathrm{CH}(X_m)$ to the graded ring associated with the topological filtration on $K(X_m)$ is an isomorphism. Therefore

$$\log_2[\mathrm{CH}(\bar{X}_m) : \mathrm{CH}(X_m)] = \log_2[K(\bar{X}_m) : K(X_m)] = cn$$

(cf. [6, Proposition 2]), and we conclude that

$$c = \tfrac{1}{n}\Big(1 \cdot \tbinom{n}{1} \cdot f_{m-1} + 2 \cdot \tbinom{n}{2} \cdot f_{m-2} + \dots + m \cdot \tbinom{n}{m} \cdot f_0\Big) = \\ \tbinom{n-1}{0} \cdot f_{m-1} + \tbinom{n-1}{1} \cdot f_{m-2} + \dots + \tbinom{n-1}{m-1} \cdot f_0.$$

For arbitrary (not necessarily generic) φ with $r = 0$, we have

$$cn = [\mathrm{CH}(\bar{X}_m : A_0)] \geqslant [\mathrm{CH}(\bar{X}_m) : \overline{\mathrm{CH}}(X_m)] \geqslant [K(\bar{X}_m) : K(X_m)] = cn,$$

where the second inequality comes from [6, Proposition 2]. Therefore all the inequalities in the chain are in fact equalities. In particular, $A_0 = \overline{\mathrm{CH}}(X_m)$ which is the statement of Proposition 3.1 for $r = 0$.

For arbitrary r, we have $[K(\bar{X}_m) : K(X_m)] = c(n-r)$. For $r \geqslant 1$, a direct computation of the index $[A_r : A_{r-1}]$ yields c. Using this computation and inducting on r, we show that $c(n-r) = [\mathrm{CH}(\bar{X}_m) : A_r]$. Thus

$$c(n-r) = [\mathrm{CH}(\bar{X}_m) : A_r] \geqslant [\mathrm{CH}(\bar{X}_m) : \overline{\mathrm{CH}}((X_m)_L)] \geqslant [K(\bar{X}_m) : K((X_m)_L)] = c(n-r).$$

Consequently, $A_r = \overline{\mathrm{CH}}((X_m)_L)$. □

4. Isotropic forms

Here we prove Proposition 4.2 which will be used later on to prove Proposition 6.3 and Theorem 7.2.

Let F be a field and let V be a vector F-space of odd dimension $2n+1$. Let $\varphi\colon V \to F$ be a non-degenerate isotropic quadratic form over F. We fix an isotropic line $L \subset V$ and let φ' be the quadratic form on $V' := L^\perp/L$ induced by φ, where $L^\perp$ is the orthogonal complement of L with respect to the polar bilinear form of φ. Thus φ' is a Witt-equivalent to φ non-degenerate quadratic form of dimension $2n-1$ over F.

For $m \in \{0, \dots, n\}$, let X_m be the mth grassmannian of φ. In particular, $X_0 = \operatorname{Spec} F$ and X_1 is the projective quadric of φ.

For $m \in \{1, \dots, n\}$, let $X_m^1 \subset X_m$ be the closed subvariety of m-dimensional totally isotropic subspaces in $L^\perp$ and let $X_m^2 \subset X_m^1$ be the closed subvariety of those of them that contain L. The variety X_m^2 is identified with the $(m-1)$st grassmannian X'_{m-1} of φ' via $U \mapsto U/L$.

Let us consider the homomorphism of graded groups

$$i^*\colon\ \mathrm{CH}^*(X_m) \to \mathrm{CH}^*(X'_{m-1})$$

given by the pull-back with respect to the closed embedding

$$i\colon X'_{m-1} = X_m^2 \hookrightarrow X_m,$$

or, in slightly different terms, induced by the correspondence

$$\iota^t\colon X_m \rightsquigarrow X'_{m-1}$$

given by the transpose of the graph ι of i.

The incidence correspondence $\alpha\colon X'_1 \rightsquigarrow X_1$ is defined as the variety of pairs $(U'/L, U)$ of totally isotropic lines in V' and V with $U \subset U'$. By [3, Lemma 72.3], the homomorphism

$$\alpha^*\colon\ \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}'_1),$$

given by α^t, maps l_i to l'_{i-1} for all i, where $l'_{-1} := 0$. The notation $\bar{X}_1$ stands for the variety X_1 with scalars extended to an algebraic closure of the base field; $l_i \in \mathrm{CH}_i(\bar{X}_1)$ (and $l'_i \in \mathrm{CH}_i(\bar{X}'_1)$) is the class of an i-dimensional projective space lying on $\bar{X}_1$ (respectively on $\bar{X}'_1$).

The incidence correspondence $\gamma\colon X_1 \rightsquigarrow X_m$ is defined as the variety of pairs (U_1, U_m) with $U_1 \subset U_m$. The homomorphism

$$\gamma_*\colon\ \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_m)$$

maps the elements $l_{n-1}, \dots, l_0$ respectively to $e_{n-m+1}, \dots, e_{2n-m}$. This is the definition of the latter set of elements; they are indexed by codimension: $e_i \in \mathrm{CH}^i(\bar{X}_m)$ for all i, whereas the elements of the former set are indexed by their dimensions: $l_i \in \mathrm{CH}_i(\bar{X}_1)$.

Similarly, the homomorphism

$$\gamma'_*\colon\ \mathrm{CH}(\bar{X}'_1) \to \mathrm{CH}(\bar{X}'_{m-1})$$

induced by the incidence correspondence $\gamma'\colon X_1 \rightsquigarrow X_m$ maps the elements $l'_{n-2}, \dots, l'_0$ respectively to $e'_{n-m+1}, \dots, e'_{2n-m-1}$.

Lemma 4.1 (c.f. [3, Lemma 86.7]). *For any $m \in \{1, \dots, n\}$, the square of correspondences*

$$\begin{tikzcd} X_1 \arrow[r, rightsquigarrow, "\alpha^t"] \arrow[d, rightsquigarrow, "\gamma"'] & X'_1 \arrow[d, rightsquigarrow, "\gamma'"'] \\ X_m \arrow[r, rightsquigarrow, "\iota^t"] & X'_{m-1} \end{tikzcd}$$

is commutative.

Lemma 4.1 results in the following proposition which is [3, Corollary 86.8] if $m = n$:

Proposition 4.2. *For any $m \in \{2, \dots, n\}$, the homomorphism*

$$i^*\colon \mathrm{CH}^*(\bar{X}_m) \to \mathrm{CH}^*(\bar{X}'_{m-1})$$

maps the elements $e_{n-m+1}, \dots, e_{2n-m-1}$ to $e'_{n-m+1}, \dots, e'_{2n-m-1}$ (whereas $e_{2n-m} \mapsto 0$). □

5. A computation

For a smooth projective quadric Q over a field F of dimension $2n$ or $2n+1$, recall (from, e.g., [3]) that the Chow group $\mathrm{CH}(\bar{Q})$ of the variety $\bar{Q}$ (which is Q with the scalars extended to an algebraic closure of F) is free with a basis

$$h^0, \dots, h^n, l_n, \dots, l_0.$$

The element $h \in \mathrm{CH}^1(\bar{Q})$ is the hyperplane section class, $h^i \in \mathrm{CH}^i(\bar{Q})$ is its ith power, and $l_i \in \mathrm{CH}_i(\bar{Q})$ is the class of an i-dimensional projective space lying on $\bar{Q}$.

We recall that by $\bar{X}$ for a smooth quasi-projective F-variety X, we mean X with the base field extended to an algebraic closure of F. We write $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$ for the modulo 2 Chow group of X; $\overline{\mathrm{Ch}}(X)$ stands for the image of the change of field homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X})$; similarly, $\overline{\mathrm{CH}}(X)$ is the image of $\mathrm{CH}(X) \to \mathrm{CH}(\bar{X})$.

For $i \geqslant 0$, we work with the cohomological Steenrod operations

$$\mathrm{St}^i\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^{*+i}(X)$$

on the modulo 2 Chow groups. Their history, construction, and basic properties are explained in [3, Chapter XI] for characteristic not 2 and in [14] for characteristic 2.

Proposition 5.1. *For an integer $n \geqslant 2$, let Q be a smooth projective quadric of dimension $2n$ over a field F. Let V be a projective homogeneous variety over F. For an element $\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$, we consider the decomposition*

$$\bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \dots + h^n \times \alpha_n + l_n \times \beta + a l_{n-1} \times [\bar{V}] \tag{5.2}$$

of its image $\bar{\alpha} \in \mathrm{Ch}^{n+1}(\bar{Q} \times \bar{V})$ with $\alpha_i \in \mathrm{Ch}^{n+1-i}(\bar{V})$, $\beta \in \mathrm{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$.

- **(i)** *If neither $n+1$ nor $n+2$ is a 2-power, then*

$$\alpha_0 + \sum_{i=1}^{[(n+1)/2]} a_i \,\mathrm{St}^i(\alpha_i) \in \overline{\mathrm{Ch}}^{n+1}(V)$$

  for some $a_i \in \mathbb{Z}/2\mathbb{Z}$, where $[(n+1)/2]$ is the integral part of $(n+1)/2$.
- **(ii)** *If $a = 0$ in decomposition (5.2) and n is even, then*

$$\alpha_0 + \mathrm{St}^1(\alpha_1) + \alpha_1\beta \in \overline{\mathrm{Ch}}^{n+1}(V).$$

Proof of Proposition 5.1(i). For every $i \geqslant 0$, let s^i be the image in $\mathrm{CH}^{n+1+i}(\bar{Q} \times \bar{V})$ of an element of the group $\mathrm{CH}^{n+1+i}(Q \times V)$ representing the value $\mathrm{St}^i(\alpha) \in \mathrm{Ch}^{n+1+i}(Q \times V)$ of the Steenrod operation. We will refer to these elements s^i later in the proof.

For $d := 2^r - 1$, where 2^r is the highest 2-power not exceeding n, we choose a d-dimensional smooth subquadric P of Q. Writing in for the imbedding $P \times Y \hookrightarrow Q \times Y$ and applying [3, Theorem 61.9 and Proposition 61.10] (along with their characteristic 2 counterparts from [14]), we have

$$pr^P_* \sum_{i=0}^{d} c_i(-T_P)\; in^*\, \mathrm{St}^{d-i}\alpha = \mathrm{St}^d\, pr^P_*\; in^*\,\alpha \in \mathrm{Ch}^{n+1}(V), \tag{5.3}$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \to V$. The summation on the left side in (5.3) is from 0 to d only because $c_i(-T_P) = 0$ for $i < 0$ whereas $\mathrm{St}^{d-i}\alpha = 0$ for $i > d$. If $d > n+1-d$, the right side in (5.3) vanishes because

$$\mathrm{St}^d\, pr^P_*\, in^*\,\alpha \in \mathrm{St}^d\, \mathrm{Ch}^{n+1-d}(V)$$

and $\mathrm{St}^d\,\mathrm{Ch}^{n+1-d}(V) = 0$ (see [3, Theorem 61.13] and [14, Corollary 7.5]). However the relation $d > n+1-d$ fails if (and only if) n has the highest possible value with respect to $d = 2^r - 1$. Since n is not a 2-power minus 1 / minus 2, the highest possible value of n is $n = 2^{r+1} - 3$, in which case $d = n+1-d$ and the operation St^d on $\mathrm{Ch}^{n+1-d}(V)$ is the squaring (by [3, Theorem 61.13] and [14, Corollary 7.3]).

Rewriting the left side of (5.3) with a help of the projection formula [3, Proposition 56.9], we obtain

$$pr_* \sum_{i=0}^{d} in_*(c_i(-T_P))\,\mathrm{St}^{d-i}\alpha,$$

where pr is the projection $Q \times V \to V$. By the computation [3, Lemma 78.1] of the Chern classes of the tangent bundle, it follows from (5.3) that

$$\sum_{i=0}^{d} \binom{-d-2}{i} \cdot pr_*(h^{2n-d+i} \cdot s^{d-i}) \in 2\,\overline{\mathrm{CH}}^{n+1}(V) \subset \mathrm{CH}^{n+1}(\bar V) \tag{5.4}$$

provided that $n \neq 2^{r+1} - 3$. For $n = 2^{r+1} - 3$, we obtain relation (5.4) with $2\,\overline{\mathrm{CH}}^{n+1}(V)$ on the right replaced by $4\,\mathrm{CH}^{n+1}(\bar V)$. So, in both cases the following combined relation holds:

$$\sum_{i=0}^{d} \binom{-d-2}{i} \cdot pr_*(h^{2n-d+i} \cdot s^{d-i}) \in 2\,\overline{\mathrm{CH}}^{n+1}(V) + 4\,\mathrm{CH}^{n+1}(\bar V) \subset \mathrm{CH}^{n+1}(\bar V). \tag{5.5}$$

The sum showing up in (5.5) is a linear combination of the elements

$$pr_*(h^{2n-d}s^d),\ pr_*(h^{2n-d+1}s^{d-1}),\ \dots,\ pr_*(h^{2n}s^0) \in \overline{\mathrm{CH}}^{n+1}(V) \subset \mathrm{CH}^{n+1}(\bar V).$$

Let us compute the ith element $pr_*(h^{2n-d+i}\cdot s^{d-i}) \in \mathrm{CH}^{n+1}(\bar V)$ modulo $4\,\mathrm{CH}^{n+1}(\bar V)$. Since $2n-d+i > n$, the factor $h^{2n-d+i} \in \mathrm{CH}(\bar Q)$ is divisible by 2. The other factor modulo 2 is $\mathrm{St}^{d-i}(\bar\alpha)$ and it follows that

$$pr_*(h^{2n-d+i} \cdot s^{d-i}) \equiv 2 \sum_{k \geqslant 0} \binom{k}{d-i-k} \varepsilon_k \pmod{4\,\mathrm{CH}^{n+1}(\bar V)},$$

where $\varepsilon_k \in \mathrm{CH}^{n+1}(\bar V)$ is an integral representative of $\mathrm{St}^k(\alpha_k) \in \mathrm{Ch}^{n+1}(\bar V)$ which in the case of $k > n+1-k$ we choose to be 0 taking into account that $\mathrm{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \mathrm{Ch}^{n+1-k}(\bar Q \times \bar V)$. So, the sum over k runs up to the integral part of $(n+1)/2$ only.

Since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i \in \{0, 1, \dots, d\}$ (the binomial coefficient $\binom{-d-2}{i} = \binom{d+1+i}{i}$ is easy to compute modulo 2 using [3, Lemma 78.6]), we deduce from (5.5) that

$$2 \sum_{i=0}^{d} \sum_{k=0}^{[(n+1)/2]} \binom{k}{d-i-k} \varepsilon_k \equiv 2\gamma \pmod{4\,\mathrm{CH}^{n+1}(\bar V)} \tag{5.6}$$

for some $\gamma \in \overline{\mathrm{CH}}^{n+1}(V)$. The coefficient at ε_0 is 2. Therefore

$$2\varepsilon_0 + 2\sum_{k=1}^{[(n+1)/2]} a_k\varepsilon_k \equiv 2\gamma \pmod{4\,\mathrm{CH}^{n+1}(\bar V)}$$

for some integers a_k. Dividing by 2 (we have the right to divide by 2 because the group $\mathrm{CH}^{n+1}(\bar V)$ for a projective homogeneous variety V is free of torsion), we obtain the congruence

$$\varepsilon_0 + \sum_{k=1}^{[(n+1)/2]} a_k\varepsilon_k \equiv \gamma \pmod{2\,\mathrm{CH}^{n+1}(\bar V)}$$

meaning that

$$\alpha_0 + \sum_{k=1}^{[(n+1)/2]} a_k \operatorname{St}^k \alpha_k \quad\in\quad \overline{\mathrm{Ch}}^{n+1}(V). \qquad \square$$

Proof of Proposition 5.1(ii). For every $i = 0, 1, \dots, n$, let s^i be the image in the group $\mathrm{CH}^{n+1+i}(\bar Q \times \bar V)$ of an element of $\mathrm{CH}^{n+1+i}(Q \times V)$ representing $\operatorname{St}^i(\alpha) \in \mathrm{Ch}^{n+1+i}(Q \times V)$. We also set $s^i := 0$ for $i > n+1$ as well as for $i < 0$. Finally, we set $s^{n+1} := (s^0)^2$.

Note that we have

$$s^0 := h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^n \times \alpha_n + l_n \times \beta + b l_{n-1} \times [\bar V] \in \mathrm{CH}^{n+1}(\bar Q \times \bar V)$$

with some $\alpha_i \in \mathrm{CH}^{n+1-i}(\bar V)$ for $i = 0, 1, \dots, n$, $\beta \in \mathrm{CH}^1(\bar V)$, and an integer b. Since

$$s^0 \mod 2\,\mathrm{CH}^{n+1}(\bar Q \times \bar V) = \bar\alpha,$$

the integer b is even. Since the element $2l_{n-1} = h^{n+1}$ is in $\overline{\mathrm{CH}}^{n+1}(Q) \subset \mathrm{CH}^{n+1}(\bar Q)$, we remove the last summand from the above decomposition of s^0.

Let d be any integer with $n \leqslant d \leqslant 2n$. For a d-dimensional smooth subquadric P of Q, writing in for the imbedding $P \times V \hookrightarrow Q \times V$ and applying [3, Theorem 61.9 and Proposition 61.10]), we have

$$pr^P_* \sum_{i=d-n-1}^{d} c_i(-T_P)\, in^* \operatorname{St}^{d-i} \alpha = \operatorname{St}^d pr^P_*\, in^*\, \alpha \in \mathrm{Ch}^{n+1}(V),$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \to V$. The summation on the left side is from $d-n-1$ to d only because $\operatorname{St}^{d-i}\alpha = 0$ outside of this interval, see [3, Theorem 61.13]. The right side is actually zero since

$$\operatorname{St}^d pr^P_*\, in^*\, \alpha \in \operatorname{St}^d \mathrm{Ch}^{n+1-d}(V)$$

and $\operatorname{St}^d \mathrm{Ch}^{n+1-d}(V) = 0$ because $d > 1 \geqslant n+1-d$. Rewriting the left side with a help of the projection formula [3, Proposition 56.9], we obtain the relation

$$pr_* \sum_{i=d-n-1}^{d} in_*(c_i(-T_P)) \operatorname{St}^{d-i} \alpha = 0 \in \mathrm{Ch}^{n+1}(V),$$

where pr is the projection $Q \times V \to V$. By the computation [3, Lemma 78.1] of the Chern classes of the tangent bundle, it follows that

$$\sum_{i=d-n-1}^{d} \binom{-d-2}{i} \cdot pr_*(h^{2n-d+i} \cdot s^{d-i}) \in 2\,\overline{\mathrm{CH}}^{n+1}(V) \subset \mathrm{CH}^{n+1}(\bar V). \tag{5.7}$$

We are going to use (5.7) for various values of d. Note that when d varies, the sum showing up in (5.7) is a linear combination of always the same elements

$$pr_*(h^{n-1}s^{n+1}),\ pr_*(h^ns^n),\ \dots,\ pr_*(h^{2n}s^0) \in \overline{\mathrm{CH}}^{n+1}(V).$$

However, the coefficients of this linear combination vary with d.

Let us compute the ith element $pr_*(h^{2n-d+i}\cdot s^{d-i}) \in \mathrm{CH}^{n+1}(\bar V)$ modulo $4\,\mathrm{CH}^{n+1}(\bar V)$. For $i \geqslant d-n+1$, we have $2n-d+i \geqslant n+1$ so that the factor $h^{2n-d+i} \in \mathrm{CH}(\bar Q)$ is divisible by 2. The other factor modulo 2 is $\mathrm{St}^{d-i}(\bar\alpha)$ and it follows that

$$pr_*(h^{2n-d+i}\cdot s^{d-i}) \equiv 2\sum_{k\geqslant 0}\binom{k}{d-i-k}\varepsilon_k \pmod 4 \qquad \text{for } i \geqslant d-n+1,$$

where $\varepsilon_k \in \mathrm{CH}^{n+1}(\bar V)$ is an integral representative of $\mathrm{St}^k(\alpha_k) \in \mathrm{Ch}^{n+1}(\bar V)$ which in the case of $k > n+1-k$ we choose to be 0 taking into account that $\mathrm{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \mathrm{Ch}^{n+1-k}(\bar Q \times \bar V)$. So, the sum over k runs up to $n/2$ only (recall that n is even). Besides, we choose $\varepsilon_0 = \alpha_0$.

For $i = d-n-1$, the ith summand is

$$\begin{aligned}pr_*(h^{n-1}s^{n+1}) &= pr_*\Big(h^{n-1}\cdot(h^0\times\alpha_0 + h^1\times\alpha_1 + \cdots + h^n\times\alpha_n + l_n\times\beta)^2\Big)\\ &\equiv 2\,pr_*\Big(h^{n-1}\cdot(h^1\times\alpha_1)\cdot(l_n\times\beta)\Big) = 2\alpha_1\beta,\end{aligned}$$

where the congruence is modulo $4\,\mathrm{CH}^{n+1}(\bar V)$. Here again we use the assumption that n is even: for odd n the answer would be $2\alpha_1\beta + 2\alpha^2_{(n+1)/2}$.

There is no need to compute the last remaining summand $pr_*(h^ns^n)$ which occurs for $i = d-n$.

With the above computations in hand, we are going to use (5.7) for the following two values of d: for $d = 2^r - 1$ and for $d = 2^r$, where $2^r \leqslant 2n$ is the highest 2-power not exceeding $2n$. For the first choice of d, since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i = 0, 1, \dots, d$, we get that

$$2\alpha_1\beta + pr_*(h^ns^n) + 2\sum_{i=d-n+1}^{d}\sum_{k=0}^{n/2}\binom{k}{d-i-k}\varepsilon_k \equiv 2a \pmod{4\,\mathrm{CH}^{n+1}(\bar V)}$$

for some $a \in \overline{\mathrm{CH}}^{n+1}(\bar V)$. For $k < n/2$, the coefficient at ε_k is twice the sum of all binomial coefficients $\binom{k}{\cdot}$ and therefore is divisible by 4 for $0 < k < n/2$. The coefficient at ε_0 is 2. The coefficient at $\varepsilon_{n/2}$ is twice the sum of all binomial coefficients $\binom{n/2}{\cdot}$ except $\binom{n/2}{n/2} = 1$ and so is congruent to 2 modulo 4. Therefore the congruence we get with the first choice of d is

$$2\alpha_1\beta + pr_*(h^ns^n) + 2\varepsilon_0 + 2\varepsilon_{n/2} \equiv 2a \pmod{4\,\mathrm{CH}^{n+1}(\bar V)}. \tag{5.8}$$

For the second choice of d (namely, $d = 2^r$), the binomial coefficient $\binom{-d-2}{i}$ with $i \in \{0, 1, \dots, d\}$ is odd for even $i < d$ and is even for the remaining i. Since the integer $d-n-1$ is odd, we get that

$$pr_*(h^ns^n) + 2\sum_{\substack{i=d-n+2\\ i \text{ is even}}}^{d-2}\sum_{k=0}^{n/2}\binom{k}{d-i-k}\varepsilon_k \equiv 2b \pmod{4\,\mathrm{CH}^{n+1}(\bar V)}$$

for some $b \in \overline{\mathrm{CH}}^{n+1}(\bar{V})$.

The coefficient at ε_0 is 0 here. Since for any $k \geqslant 1$, we have

$$\sum_{\text{even } l} \binom{k}{l} = 2^{k-1} = \sum_{\text{odd } l} \binom{k}{l},$$

only the coefficients at ε_1 and $\varepsilon_{n/2}$ survive modulo 4 (the coefficient at $\varepsilon_{n/2}$ survives because the binomial coefficient $\binom{n/2}{n/2}$ is missing). Therefore the congruence we get with our second choice of d is

$$pr_*(h^n s^n) + 2\varepsilon_1 + 2\varepsilon_{n/2} \equiv 2b \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}. \tag{5.9}$$

Adding together (5.8) and (5.9), we get

$$2\alpha_1\beta + 2\,pr_*(h^n s^n) + 2\varepsilon_0 + 2\varepsilon_1 \equiv 2(a+b) \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}.$$

Dividing by 2 (we have the right to do so because the group $\mathrm{CH}^{n+1}(\bar{V})$ is free of torsion), we obtain the congruence

$$\alpha_1\beta + \varepsilon_0 + \varepsilon_1 \equiv a + b - pr_*(h^n s^n) \pmod{2\,\mathrm{CH}^{n+1}(\bar{V})} \tag{5.10}$$

giving the statement of Proposition 5.1(ii) because $a + b - pr_*(h^n s^n) \in \overline{\mathrm{CH}}^{n+1}(V)$. ☐

Remark 5.11. Proposition 5.1(ii) is a remake of [16, Proposition 3.5] in the spirit of [7, Proposition 5.3].

6. Highest Schur index

Given a non-degenerate quadratic form φ of dimension $2n+1$ over a field F, we consider the following two conditions:

$$\operatorname{ind} C_0(\varphi) = 2^n, \tag{6.1}$$

where $\operatorname{ind} C_0(\varphi)$ is the Schur index of the even Clifford algebra $C_0(\varphi)$ of φ (note that the index always is a 2-power, 2^n is its highest possible value for quadratic forms of such dimension), and

$$e_{n+1} \notin \overline{\mathrm{Ch}}^{n+1}(X_{n-1}), \tag{6.2}$$

where X_{n-1} is the almost highest grassmannian of φ and $e_{n+1} \in \mathrm{Ch}^{n+1}(\bar{X}_{n-1})$ is the defined in §4 element, corresponding to the rational point class $l_0 \in \overline{\mathrm{Ch}}(\bar{X}_1)$ on the quadric $\bar{X}_1$ of φ (over an algebraic closure of F).

Note that by Proposition 3.1, condition (6.1) implies condition (6.2).

Proposition 6.3. *Let φ be a non-degenerate quadratic form over a field F. Assume that the dimension of φ is at least 5 and congruent to 1 modulo 4:*

$$\dim\varphi = 2n + 1 \geqslant 5$$

with even n. Let ψ be a non-degenerate quadratic form of dimension one higher: $\dim\psi = \dim\varphi + 1$. Assume that φ satisfies condition (6.1). Then $\varphi_{F(\psi)}$ over the function field $F(\psi)$ of the projective quadric of ψ satisfies condition (6.2).

Proof. It suffices to consider the case where ψ is anisotropic.

If the discriminant of ψ is nontrivial, the even Clifford algebra $C_0(\psi)$ is simple. In particular, any homomorphism $C_0(\psi) \to C_0(\varphi)$ is injective. Since $\dim_F C_0(\psi) > \dim_F C_0(\varphi)$, there is no such homomorphism and it follows by Index Reduction Formula for quadrics that $C_0(\varphi)_{F(\psi)}$ is still a division algebra, i.e., $\varphi_{F(\psi)}$ still satisfies (6.1). Therefore, by Proposition 3.1, $\varphi_{F(\psi)}$ also satisfies (6.2).

So, we only need to consider the case where the discriminant of ψ is trivial.

Let Q be the projective quadric of ψ and let $V := X_{n-1}$ be the almost last grassmannian of φ. Assume that condition (6.2) fails for $\varphi_{F(\psi)}$ and consider a lift $\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$ of an element in $\mathrm{Ch}^{n+1}(V_{F(Q)})$ representing $e_{n+1} \in \overline{\mathrm{Ch}}^{n+1}(V_{F(Q)})$. Note that $\alpha_0 = e_{n+1}$ in decomposition (5.2) of the image $\bar{\alpha} \in \mathrm{Ch}^{n+1}(\bar{Q} \times \bar{V})$ of α. Since the discriminant of ψ is trivial, we may assume that $a = 0$. Indeed, if $a \neq 0$, we have $l_n \in \overline{\mathrm{Ch}}(Q_{F(V)})$. Lifting this element to the group $\overline{\mathrm{Ch}}(Q \times V)$, we get some α' of the form

$$\alpha' = h^0 \times \alpha'_0 + h^1 \times \alpha'_1 + \cdots + h^{n-1} \times \alpha'_{n-1} + l_n \times [\bar{V}].$$

Subtracting from α the product

$$(h^1 \times [\bar{V}]) \cdot \alpha' = h^1 \times \alpha'_0 + h^2 \times \alpha'_1 + \cdots + h^n \times \alpha'_{n-1} + l_{n-1} \times [\bar{V}] \in \overline{\mathrm{Ch}}(Q \times V),$$

we get rid of the term $l_{n-1} \times [\bar{V}]$ in α.

Thus, we can apply Proposition 5.1(ii) and conclude that

$$\alpha_0 + \mathrm{St}^1(\alpha_1) + \alpha_1 \beta \in \overline{\mathrm{Ch}}^{n+1}(V). \tag{6.4}$$

By Proposition 3.1, $be_{n+1} \in \overline{\mathrm{Ch}}^{n+1}(V)$, where $b \in \mathbb{Z}/2\mathbb{Z}$ is the coefficient at e_{n+1} in (6.4). The third summand of (6.4) does not provide any contribution to b. (Note that β is a multiple of c_1 because c_1 generates the group $\mathrm{Ch}^1(\bar{V}) \ni \beta$.) By [16, Proposition 2.9], the contribution of $\mathrm{St}^1(e_n)$ is also trivial: the coefficient $\binom{n}{1} = n$ at e_{n+1} in this formula is even by the assumption on n. Therefore $b = 1$ and $e_{n+1} \in \overline{\mathrm{Ch}}^{n+1}(V)$ as desired. $\square$

7. The core result

Given a non-degenerate quadratic form φ of dimension $u = 2n + 1$ over a field F, we consider the following modification of condition (6.1):

$$\operatorname{ind} C_0(\varphi) \geqslant 2^{n-1}. \tag{7.1}$$

Theorem 7.2. *Assume that an odd natural number $u = 2n + 1$ is neither a 2-power minus 1 nor a 2-power minus 3. Let ψ be a non-degenerate quadratic form over F of dimension $u + 1$. Assume that φ satisfies conditions (7.1) and (6.2). Then $\varphi_{F(\psi)}$ also satisfies conditions (7.1) and (6.2).*

Proof. Condition (7.1) is satisfied by $\varphi_{F(\psi)}$ according to Index Reduction Formula for quadrics. We just need to check condition (6.2).

It suffices to consider the case where ψ is anisotropic. We induct on $\dim \varphi \geqslant 9$.

Let Q be the projective quadric of ψ and let $V := X_{n-1}$ be the almost last grassmannian of φ. Assume that condition (6.2) fails for $\varphi_{F(\psi)} = \varphi_{F(Q)}$ and consider a lift

$$\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$$

of an element in $\mathrm{Ch}^{n+1}(V_{F(Q)})$ representing $e_{n+1} \in \overline{\mathrm{Ch}}^{n+1}(V_{F(Q)})$. Note that $\alpha_0 = e_{n+1}$ in the decomposition (5.2).

By Proposition 5.1(i), we have

$$\alpha_0 + \sum_{i=1}^{[(n+1)/2]} \mathrm{St}^i(\alpha_i) \in \overline{\mathrm{Ch}}^{n+1}(V). \tag{7.3}$$

For $i = 1, \ldots, n$, let X_i be the ith grassmannian of φ and let Y_i be the ith grassmannian of ψ. By Index Reduction Formula for quadrics (the function field extension $F(Y_i)/F$ in the next formula can be replaced by a chain of function fields of quadrics), we have

$$\operatorname{ind} C_0((\varphi)_{F(Y_i)}) \geqslant 2^{n-i} \quad \text{for } i \in \{1, \ldots, n\}. \tag{7.4}$$

Viewing α as a correspondence $Q \rightsquigarrow V$ and applying the homomorphism α_* to $l_i \in \overline{\mathrm{Ch}}(Q_{F(Y_{i+1})})$, we get

$$\alpha_i \in \overline{\mathrm{Ch}}^{n+1-i}(V_{F(Y_{i+1})})$$

for $i < n$. Therefore by Proposition 3.1 we may assume that for every $i \in \{1, \ldots, n-1\}$, the element α_i is e_{n+1-i} or 0.

Let us choose some $i \in \{1, \ldots, n-1\}$ and consider the anisotropic part φ' of the quadratic form $\varphi_{F(X_i)}$. We have

$$\dim \varphi' = \dim \varphi - 2i = 2(n-i) + 1$$

and $C_0(\varphi')$ is a division algebra. Let X'_{n-i-1} be the almost highest grassmannian of φ', and let $e'_{n-i+1} \in \mathrm{Ch}^{n-i+1}(\bar{X}'_{n-i-1})$ be the last of the special elements from §4. Since $C_0(\varphi')$ is a division algebra, it follows by the induction hypothesis of our proof that

$$e'_{n-i+1} \notin \overline{\mathrm{Ch}}^{n-i+1}((X'_{n-i-1})_{F(X_i)(Y_{i+1})} \tag{7.5}$$

provided that $n-i$ neither $n-i+1$ nor $n-i+2$ is a 2-power. If $n-i+2$ is a 2-power and $n-i \geqslant 2$, we still have (7.5): this time by Proposition 6.3.

By Proposition 4.2, (7.5) implies that

$$e_{n-i+1} \notin \mathrm{Ch}^{n-i+1}((X_{n-1})_{F(X_i)(Y_{i+1})}) = \mathrm{Ch}^{n-i+1}(V_{F(X_i)(Y_{i+1})})$$

and, in particular,

$$e_{n-i+1} \notin \mathrm{Ch}^{n-i+1}(V_{F(Y_{i+1})}).$$

Consequently, $\alpha_i = 0$ in (7.3) provided that $n-i$ is at least 2 and not a 2-power minus 1.

Since n is at least 4 in our setting and the sum over i in (7.3) runs up to $[n+1]/2$ only, the difference $n-i$ is at least 2 for all i appearing in (7.3). But if $n-i$ is a 2-power minus 1, we cannot conclude that $\alpha_i = 0$. So, let us assume that $\alpha_i = e_{n-i+1}$. By [16, Proposition 2.9], providing a formula for $\mathrm{St}^i(e_{n-i+1})$, the coefficient at e_{n+1} is the binomial coefficient $\binom{n-i+1}{i}$. Note that $i < n-i+1$: otherwise $i = n-i+1$, i.e., $i = (n+1)/2$ and $n-i+1 = (n+1)/2$ is a 2-power – a contradiction with the assumption that n is not a 2-power minus 1. It follows that $\binom{n-i+1}{i} \equiv 0 \pmod 2$. Thus (7.3) yields

$$e_{n+1} = \alpha_0 \in \overline{\mathrm{Ch}}^{n+1}(V)$$

meaning that φ does not satisfy (7.3). $\square$

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