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Chow Groups of Quadrics And The Stabilization Conjecture

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§0. Introduction

Let F be an arbitrary field of characteristic different from 2 and φ be a nondegenerate quadratic form of dimension n over F. By X_φ we denote the projective quadric corresponding to φ (i.e., the hypersurface in $\mathbb{P}_F^{n-1}$ defined by the equation $\varphi = 0$). Let $\mathrm{CH}^*(X_\varphi)$ be the Chow group of a variety X (i.e., the group of cycles on X modulo rational equivalence) graded by codimension of cycles. The problem of computing $\mathrm{CH}^*(X_\varphi)$ raised by Swan [S1], [S2] was found hopeless [KM]. However, there is a conjecture, namely the stabilization conjecture stated below which perhaps can be proved.

Let $h^p \in \mathrm{CH}^p(X_\varphi)$ be the class of a p-codimensional linear section (i.e., the inverse image of the class of a p-codimensional linear subspace in $\mathbb{P}_F^{n-1}$ with respect of the closed imbedding $X_\varphi \hookrightarrow \mathbb{P}_F^{n-1}$). It is obvious that the element h^p has infinite order. In its weakest form the conjecture of stabilization is

CONJECTURE 0.1. *For any p, if n is sufficiently large, then*

$$\mathrm{CH}^p(X_\varphi) = \mathbb{Z} \cdot h^p.$$

A more exact wording is given just below:

CONJECTURE 0.2. *If $n > 4p$ for some p, then $\mathrm{CH}^p(X_\varphi) = \mathbb{Z} \cdot h^p$.*

An easy computation of $\mathrm{CH}^1(X_\varphi)$ [K1] shows Conjecture 0.2 is true for $p = 1$.

Let $T\,\mathrm{CH}^p(X_\varphi) \subset \mathrm{CH}^p(X_\varphi)$ be the torsion subgroup. It is easy to verify

PROPOSITION 0.3 [K1], [S2]. *The composition*

$$T\,\mathrm{CH}^p(X_\varphi) \hookrightarrow \mathrm{CH}^p(X_\varphi) \twoheadrightarrow \mathrm{CH}^p(X_\varphi)/(\mathbb{Z} \cdot h^p)$$

is an isomorphism for $p < (\dim X_\varphi)/2 = (n-2)/2$.

So conjecture 0.2 can be formulated for $p > 1$ in an equivalent way as follows:

CONJECTURE 0.4. *If* $n > 4p$ *for some* $p > 1$, *then* $T\,\mathrm{CH}^p(X_\varphi) = 0$.

Recall that $\langle a_1, \ldots, a_n \rangle$ denotes the quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$, a form of the kind $\langle 1, a_1 \rangle \otimes \ldots \otimes \langle 1, a_r \rangle$ is called an *r-fold Pfister form* [L]. We say two forms φ_1 and φ_2 over F are *proportional* iff $\varphi_1 \simeq c\varphi_2$ for some $c \in F^*$.

For $p = 2$ Conjecture 0.4 follows from

THEOREM 0.5 [K1]. *If* φ *is proportional to a subform of an anisotropic 3-fold Pfister form and* $\dim \varphi > 4$, *then*

$$T\,\mathrm{CH}^2(X_\varphi) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Otherwise

$$T\,\mathrm{CH}^2(X_\varphi) = 0.$$

From $p = 3$ the proof of 0.4 is unknown. There is only the following information on $T\,\mathrm{CH}^3(X_\varphi)$:

THEOREM 0.6 [K2]. *For any* φ *the group* $T\,\mathrm{CH}^3(X_\varphi)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$ *or equal to* 0.

Note that such a statement is not true for $T\,\mathrm{CH}^p(X_\varphi)$ if $p \geqslant 4$: such a group can be infinite (for every $p \geqslant 4$ and a suitable F and φ over F) [KM].

There is another argument supporting 0.4. That is Rost's computation of $\mathrm{CH}^*(X_\varphi)$ for a Pfister form φ [R]. It is natural to expect that Chow groups of such quadrics are the "worst". However Rost's result yields, in particular, the following statement:

THEOREM 0.7. *If* φ *is an anisotropic r-fold Pfister form, then*

$$\min\{p : T\,\mathrm{CH}^p(X_\varphi) \neq 0\} = 2^{r-2}.$$

This fact is nice for 0.4 and shows that the number $4p$ in 0.4 cannot be decreased if p is a power of 2 because the dimension of a r-fold Pfister form equals $2^r = 4 \cdot 2^{r-2}$.

The aim of this paper is getting proof of Theorem 2.4 which shows that for any p (not only for a power of 2) the number $4p$ in 0.4 cannot be decreased.

§1. The Grothendieck group

Let $\mathrm{K}(X)$ denote the Grothendieck group $\mathrm{K}_0'(X)$ of a variety X. Consider the filtration by codimension of support (see [Q])

$$\mathrm{K}(X) = \mathrm{K}(X)^{(0)} \supset \mathrm{K}(X)^{(1)} \supset \cdots .$$

The quotient $\mathrm{K}(X)^{(p/p+1)}$ will be denoted by $G^p\,\mathrm{K}(X)$ and the torsion in it by $TG^p\,\mathrm{K}(X)$.

PROPOSITION 1.1. *The kernel of the natural epimorphism*

$$\mathrm{CH}^p(X) \twoheadrightarrow G^p\,\mathrm{K}(X)$$

is contained in $T\,\mathrm{CH}^p(X)$.

PROOF. The Chern classes $\mathrm{K}(X) \to \mathrm{CH}^*(X)$ induce a homomorphism $G^p\,\mathrm{K}(X) \to \mathrm{CH}^p(X)$ for each p, and the composition

$$\mathrm{CH}^p(X) \twoheadrightarrow G^p\,\mathrm{K}(X) \to \mathrm{CH}^p(X)$$

coincides with multiplication by $(-1)^{p-1}(p-1)!$. □

COROLLARY 1.2. *If* $TG^p\,\mathrm{K}(X) \neq 0$ *for some* p *then* $T\,\mathrm{CH}^p(X) \neq 0$. □

The aim of this section is to formulate the part of the results of [K1] on $G^p\,\mathrm{K}(X_\varphi)$ which will be used in §2. Recall some notations: $C_0(\varphi)$ is the even part of the Clifford algebra of φ, $I(F)$ is the ideal of even dimensional forms in the Witt ring $W(F)$ of nondegenerate quadratic forms over F [L].

PROPOSITION 1.3 [L]. *The algebra* $C_0(\varphi)$ *has dimension* 2^{n-1} *over* F *(*$n = \dim\varphi$*). If* $\varphi \notin I^2(F)$, *then* $C_0(\varphi)$ *is a simple algebra; if* $\varphi \in I^2(F)$ *(i.e.,* $n \vdots 2$ *and* $(-1)^{n/2}\det\varphi \in F^{*2}$ [L]*), then* $C_0(\varphi)$ *is isomorphic to the Cartesian square of a simple algebra.*

DEFINITION 1.4. We define the number $s(\varphi)$ as follows. In view of 1.3, if $\varphi \notin I^2(F)$, then $C_0(\varphi) \simeq M_{2^s}(D)$ for some division algebra D and integer nonnegative s, and if $\varphi \in I^2(F)$, then $C_0(\varphi) \simeq M_{2^s}(D) \times M_{2^s}(D)$ for some D, s. In both cases we put $s(\varphi) = s$.

THEOREM 1.5 [K1]. *If* φ *is anisotropic and* $\varphi \notin I^2(F)$, *then* $TG^p\,\mathrm{K}(X_\varphi)$ *is isomorphic to* $\mathbb{Z}/2\mathbb{Z}$ *or equal to* 0 *for each* p*; moreover*

$$\mathrm{card}\{\, p : TG^p\,\mathrm{K}(X_\varphi) \neq 0 \,\} = s(\varphi).$$

The description of $TG^* \mathrm{K}(X_\varphi)$ for $\varphi \in I^2(F)$ is more intricate. Here we shall need the case $s(\varphi) = 0$ only.

THEOREM 1.6 [K1]. *For any φ, if $s(\varphi) = 0$, then $TG^* \mathrm{K}(X_\varphi) = 0$.*

We denote the image of $h^p \in \mathrm{CH}^p(X_\varphi)$ in $G^p \mathrm{K}(X_\varphi)$ by the same symbol. The next statement is not complicated and is also true for Chow groups.

PROPOSITION 1.7 [K1], [S2]. *For any anisotropic φ, the quotient*

$$G^p \mathrm{K}(X_\varphi)/TG^p \mathrm{K}(X_\varphi)$$

is generated by the class of h^p if $p \neq (\dim X_\varphi)/2$.

§2. Main theorem

LEMMA 2.1. *Let U be affine quadric over the field $F(t)$ defined by the equation $\varphi_{F(t)} = t$, where φ is a quadratic form over F, t is an independent variable. Then $\mathrm{CH}^p U = 0$ for all $p > 0$.*

PROOF. We consider the morphism $\mathbb{A}^n_F \to \mathbb{A}^1_F$ defined by φ, where $n = \dim\varphi$. The fiber over the generic point of $\mathbb{A}^1_F$ is isomorphic to U, thus we obtain an epimorphism $\mathrm{CH}^p(\mathbb{A}^n_F) \twoheadrightarrow \mathrm{CH}^p(U)$ [KM]. Now it is sufficient to recall that $\mathrm{CH}^p(\mathbb{A}^n_F) = 0$ for $p > 0$. □

COROLLARY 2.2. *If $F = k(t_1, \dots, t_n)$, $\varphi = \langle t_1, \dots, t_n\rangle$, where k is a field and t_i are independent variables, then $\mathrm{CH}^p(X_\varphi) = \mathbb{Z} \cdot h^p$ for all p.*

PROOF. Let U be the affine quadric $\psi = -t_n$ over the field F, where $\psi = \langle t_1, \dots, t_{n-1}\rangle$. There is a closed imbedding $X_\psi \hookrightarrow X_\varphi$ and $U \simeq X_\varphi \setminus X_\psi$, thus we obtain an exact sequence

$$\mathrm{CH}^{p-1}(X_\psi) \longrightarrow \mathrm{CH}^p(X_\varphi) \longrightarrow \mathrm{CH}^p(U) \longrightarrow 0.$$

The last group in the sequence equals 0 in view of 2.1. Thus $\mathrm{CH}^p(X_\varphi)$ is generated by the class of a linear section if so is $\mathrm{CH}^{p-1}(X_\psi)$, and one can prove the corollary using induction on n. □

PROPOSITION 2.3. *Let ψ be an anisotropic quadratic form of dimension $2p$ over a field k. $\psi \in I^2(k)$ and $s(\psi) = 0$ (for example one can have: $k = k_0(t_1, \dots, t_{2p-1})$ and $\psi = \langle t_1, \dots, t_{2p-1}, (-1)^p t_1, \dots, t_{2p-1}\rangle$ where*

k_0 is a field, t_i are independent variables, $p > 1$). If one puts $F = k(t)$, $\varphi = \langle t\rangle \perp \psi_F$ then $T\,\mathrm{CH}^p(X_\varphi) \neq 0$.

PROOF. Abusing notations, we denote ψ_F by ψ without a special sign. In view of 2.1, the first homomorphism of the exact sequence

$$\mathrm{CH}^{i-1}(X_\psi) \longrightarrow \mathrm{CH}^i(X_\varphi) \longrightarrow \mathrm{CH}^i(U) \longrightarrow 0$$

is surjective. Considering the commutative square

$$\begin{array}{ccc} \mathrm{CH}^{i-1}(X_\psi) & \twoheadrightarrow & \mathrm{CH}^i(X_\varphi) \\ \downarrow & & \downarrow \\ G^{i-1}\,\mathrm{K}(X_\psi) & \rightarrow & G^i\,\mathrm{K}(X_\varphi) \end{array}$$

we see that the lower arrow is surjective too. Since $s(\psi) = 0$, Theorem 1.6 says that $TG^{i-1}\,\mathrm{K}(X_\psi) = 0$ for all i. In view of 1.7, the last statement implies that $G^{i-1}\,\mathrm{K}(X_\psi)$ is generated by h^{i-1} for $i - 1 \neq (\dim X_\psi)/2 = p - 1$. Thus $G^i\,\mathrm{K}(X_\varphi)$ is generated by h^i and therefore has no torsion for $i \neq p$.

On the other hand, φ is anisotropic and $s(\varphi) = s(\psi) + 1 = 1$. Consequently Theorem 1.5 implies $TG^i\,\mathrm{K}(X_\varphi) \neq 0$ for some i, and because of 1.2 the proof is over. □

THEOREM 2.4. *For any $p > 1$, there exist a $4p$-dimensional quadratic form φ (over a suitable field F) such that $T\,\mathrm{CH}^p(X_\varphi) \neq 0$.*

CONSTRUCTION. Let k be an arbitrary field and $t_0, t_1, \ldots, t_{2p-1}$ be independent variables where $p > 1$. We put $F = k(t_0, t_1, \ldots, t_{2p-1})$, $\rho = \langle t_0, t_1, \ldots, t_{2p-1}\rangle$ and finally $\varphi = \rho \otimes \langle 1, (-1)^p \det\rho\rangle$. We claim that

$$T\,\mathrm{CH}^p(X_\varphi) \neq 0.$$

PROOF. A reduced notation $d\rho$ for the term $(-1)^p \det\rho$ will be used. It is sufficient to show that $\mathrm{CH}^p(U) \neq 0$, where U is the affine quadric defined over F by the equation $\rho \perp \langle d\rho\rangle \otimes \langle t_1, \ldots, t_{2p-1}\rangle = -(d\rho)t_0$. Projecting on the coordinates which are terms of the second summand in the left part of the equation, we obtain a flat morphism $U \to \mathbb{A}_F^{2p-1}$. Because $\mathrm{CH}^p(U) \twoheadrightarrow \mathrm{CH}^p(\widetilde{U})$, where $\widetilde{U}$ is the fiber over the generic point, is epimorphic, it sufficies to prove that $\mathrm{CH}^p(\widetilde{U}) \neq 0$.

The variety $\widetilde{U}$ is the affine quadric over $E = F(x_1, \ldots, x_{2p-1})$ defined by the equation $\rho_E = -(d\rho)\sigma$, where $\sigma = t_0 + t_1 x_1^2 + \cdots + t_{2p-1} x_{2p-1}^2$.

Denote by $\widetilde{X}$ the projective closure of $\widetilde{U}$ and set $\widetilde{Y} = \widetilde{X} \setminus \widetilde{U}$. The projective quadric $\widetilde{Y}$ is determined by the form ρ_E and $\mathrm{CH}^{p-1}(\widetilde{Y}) = \mathbb{Z} \cdot h^{p-1}$ in view of 2.2. We have a sequence consisting of isomorphisms and an equality:

$$\mathrm{CH}^p(\widetilde{U}) \simeq \mathrm{CH}^p(\widetilde{X})/\operatorname{Im}\mathrm{CH}^{p-1}(\widetilde{Y}) = \mathrm{CH}^p(\widetilde{X})/(\mathbb{Z}\cdot h^p) \simeq T\,\mathrm{CH}^p(\widetilde{X}).$$

We shall prove that the last group is nontrivial.

The projective quadric $\widetilde{X}$ corresponds to the form $\rho_E \perp \langle (d\rho)\sigma \rangle$. Since ρ_E represents σ, one can find elements $f_1, \dots, f_{2p-1} \in E^*$ such that $\rho_E \simeq \langle \sigma, f_1, \dots, f_{2p-1} \rangle$ [L]. Comparison of determinants shows that

$$\rho_E \perp \langle (d\rho)\sigma \rangle \simeq \langle \sigma, f_1, \dots, f_{2p-1}, (-1)^p f_1 \dots f_{2p-1} \rangle.$$

In view of Proposition 2.5 stated below, the elements $\sigma, f_1, \dots, f_{2p-1}$ are algebraically independent over $l = k(x_1, \dots, x_{2p-1})$ and generate E over l if the choice of f_i was suitable. So $\widetilde{X}$ answers the conditions of 2.3 and therefore $T\,\mathrm{CH}^p(\widetilde{X}) \neq 0$. □

The last step is to state and prove

PROPOSITION 2.5. *Suppose that l is a field, $t_0, t_1, \dots, t_n$ are independent variables, $E = l(t_0, t_1, \dots, t_n)$, $\sigma = t_0 x_0^2 + t_1 x_1^2 + \dots + t_n x_n^2$ for some $x_i \in l^*$. There exist $f_1, \dots, f_n \in E^*$ such that:*

(1) $\langle t_0, t_1, \dots, t_n \rangle \simeq \langle \sigma, f_1, \dots, f_n \rangle$,
(2) $l(\sigma, f_1, \dots, f_n) = E$

(in particular $\sigma, f_1, \dots, f_n$ are algebraically independent over l because the number of these elements coincides with $\operatorname{deg\,tr} E/l$ *).*

PROOF. Replacing t_i by $t_i x_i^2$ reduces the proof to the case $x_i = 1$ for all i.

Let $e_0, e_1, \dots, e_n$ be the orthogonal basis of the form $\langle t_0, t_1, \dots, t_n \rangle$ for which $(e_i, e_i) = t_i$. Note that σ is equal to the square of $e = e_0 + e_1 + \dots + e_n$. To prove the proposition, it is sufficient to find a diagonal form $\langle f_1, \dots, f_n \rangle$ on the orthogonal complement $e^\perp$ satisfying condition 2. The amily of vectors $e_i' = e_0/t_0 - e_i/t_i$, $i = 1, \dots, n$, is a basis of $e^\perp$. We claim the orthogonalization of this basis will give (after a small correction) what is required.

To prove this, let us consider the Gramm matrix of $\{e_i'\}$ and compute its principal minors. Since $(e_i', e_i') = 1/t_0 + 1/t_i$, $(e_i', e_j') = 1/t_0$ for $i \neq j$, the i-th principal minor M_i equals $(1/t_0^i)$ multiplied by the determinant

$$\left|\begin{pmatrix} (1+g_1) & 1 & 1 & \dots & 1 \\ 1 & (1+g_2) & 1 & \dots & 1 \\ 1 & 1 & (1+g_3) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & (1+g_i) \end{pmatrix}\right|$$

where $g_i = t_0/t_i$. This determinant equals

$$g_1 g_2 \cdots g_i (1 + 1/g_1 + 1/g_2 + \cdots + 1/g_i),$$

therefore $M_i = (t_0 + t_1 + \cdots + t_i)/(t_0 t_1 \dots t_i)$. Consequently, orthogonalizing $\{e_i'\}$, we obtain a basis whose elements have squares of the form

$$M_i/M_{i-1} = 1/(t_0 + t_1 + \cdots + t_{i-1}) + 1/t_i.$$

For $i = 1, \dots, n$, we put $f_i = t_i^2 \cdot M_i/M_{i-1}$.

The last problem remaining is to show that $t_0, t_1, \dots, t_n \in E'$, where $E' = l(\sigma, f_1, \dots, f_n)$. We prove the statements $\sigma_i, t_i \in E'$, where $\sigma_i = t_0 + t_1 + \cdots + t_i$, by using inverse induction on i and the equality

$$t_i = \sigma_i f_i/(\sigma_i + f_i) \qquad (*)$$

We have $\sigma_n = \sigma \in E'$, and $(*)$ shows $t_n \in E'$. If $\sigma_{i+1}, t_{i+1} \in E'$, then $\sigma_i = \sigma_{i+1} - t_{i+1} \in E'$ and therefore $t_i \in E'$ in view of $(*)$. Finally $t_0 = \sigma_0 = \sigma_1 - t_1 \in E'$. □

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