Chow Groups of Quadrics
And The Stabilization Conjecture

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§0. Introduction

Let \( F \) be an arbitrary field of characteristic different from 2 and \( \varphi \)
be a nondegenerate quadratic form of dimension \( n \) over \( F \). By \( X_{\varphi} \) we
denote the projective quadric corresponding to \( \varphi \) (i.e., the hypersurface in
\( \mathbb{P}^{n-1}_F \) defined by the equation \( \varphi = 0 \)). Let \( \text{CH}^*(X_{\varphi}) \)
be the Chow group of a variety \( X \) (i.e., the group of cycles on \( X \) modulo rational equivalence)
graded by codimension of cycles. The problem of computing \( \text{CH}^*(X_{\varphi}) \)
raised by Swan [S1], [S2] was found hopeless [KM]. However, there is a
conjecture, namely the stabilization conjecture stated below which perhaps
can be proved.

Let \( h^* \in \text{CH}^p(X_{\varphi}) \) be the class of a \( p \)-codimensional linear section
(i.e., the inverse image of the class of a \( p \)-codimensional linear subspace
in \( \mathbb{P}^{n-1}_F \) with respect of the closed imbedding \( X_{\varphi} \hookrightarrow \mathbb{P}^{n-1}_F \)). It is obvious
that the element \( h^* \) has infinite order. In its weakest form the conjecture
of stabilization is

**Conjecture 0.1.** For any \( p \), if \( n \) is sufficiently large, then
\[ \text{CH}^p(X_{\varphi}) = \mathbb{Z} \cdot h^*. \]

A more exact wording is given just below:

**Conjecture 0.2.** If \( n > 4p \) for some \( p \), then \( \text{CH}^p(X_{\varphi}) = \mathbb{Z} \cdot h^* \).

An easy computation of \( \text{CH}^1(X_{\varphi}) \) [K1] shows Conjecture 0.2 is true
for \( p = 1 \).

Let \( T \text{CH}^p(X_{\varphi}) \subset \text{CH}^p(X_{\varphi}) \) be the torsion subgroup. It is easy to verify

**Proposition 0.3** [K1], [S2]. The composition
\[ T \text{CH}^p(X_{\varphi}) \hookrightarrow \text{CH}^p(X_{\varphi}) \twoheadrightarrow \text{CH}^p(X_{\varphi})/(\mathbb{Z} \cdot h^*) \]
is an isomorphism for \( p < (\dim X_*)/2 = (n - 2)/2 \).

So conjecture 0.2 can be formulated for \( p > 1 \) in an equivalent way as follows:

**Conjecture 0.4.** If \( n > 4p \) for some \( p > 1 \), then \( T \mathrm{CH}^p(X_*) = 0 \).

Recall that \((a_1, \ldots, a_n)\) denotes the quadratic form \( a_1x_1^2 + \cdots + a_nx_n^2\), a form of the kind \((1, a_1) \otimes \cdots \otimes (1, a_n)\) is called an \( r \)-fold Pfister form [L]. We say two forms \( \varphi_1 \) and \( \varphi_2 \) over \( F \) are proportional if \( \varphi_1 \simeq c\varphi_2 \) for some \( c \in F^* \).

For \( p = 2 \) Conjecture 0.4 follows from

**Theorem 0.5 [K1].** If \( \varphi \) is proportional to a subform of an anisotropic 3-fold Pfister form and \( \dim \varphi > 4 \), then

\[
T \mathrm{CH}^2(X_*) \simeq \mathbb{Z}/2\mathbb{Z}.
\]

Otherwise

\[
T \mathrm{CH}^2(X_*) = 0.
\]

From \( p = 3 \) the proof of 0.4 is unknown. There is only the following information on \( T \mathrm{CH}^3(X_*) \):

**Theorem 0.6 [K2].** For any \( \varphi \) the group \( T \mathrm{CH}^3(X_*) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) or equal to 0.

Note that such a statement is not true for \( T \mathrm{CH}^p(X_*) \) if \( p \geq 4 \): such a group can be infinite (for every \( p \geq 4 \) and a suitable \( F \) and \( \varphi \) over \( F \)) [KM].

There is another argument supporting 0.4. That is Rost's computation of \( \mathrm{CH}^*(X_*) \) for a Pfister form \( \varphi \) [R]. It is natural to expect that Chow groups of such quadrics are the "worst". However Rost's result yields, in particular, the following statement:

**Theorem 0.7.** If \( \varphi \) is an anisotropic \( r \)-fold Pfister form, then

\[
\min \{ \varphi : T \mathrm{CH}^p(X_*) \neq 0 \} = 2^{r-2}.
\]

This fact is nice for 0.4 and shows that the number \( 4p \) in 0.4 cannot be decreased if \( p \) is a power of 2 because the dimension of a \( r \)-fold Pfister form equals \( 2^r = 4 \cdot 2^{r-2} \).

The aim of this paper is getting proof of Theorem 2.4 which shows that for any \( p \) (not only for a power of 2) the number \( 4p \) in 0.4 cannot be decreased.
§1. The Grothendieck group

Let $K(X)$ denote the Grothendieck group $K'(X)$ of a variety $X$. Consider the filtration by codimension of support (see [Q])

$$K(X) = K(X)^{(0)} \supset K(X)^{(1)} \supset \cdots.$$ 

The quotient $K(X)^{(p/p+1)}$ will be denoted by $G^pK(X)$ and the torsion in it by $TG^pK(X)$.

**Proposition 1.1.** The kernel of the natural epimorphism

$$\text{CH}^p(X) \to G^pK(X)$$

is contained in $T\text{CH}^p(X)$.

**Proof.** The Chern classes $K(X) \to \text{CH}^*(X)$ induce a homomorphism $G^pK(X) \to \text{CH}^p(X)$ for each $p$, and the composition

$$\text{CH}^p(X) \to G^pK(X) \to \text{CH}^p(X)$$

coincides with multiplication by $(-1)^{p-1}(p-1)!$. □

**Corollary 1.2.** If $TG^pK(X) \neq 0$ for some $p$ then $T\text{CH}^p(X) \neq 0$. □

The aim of this section is to formulate the part of the results of [K1] on $G^pK(X)$ which will be used in §2. Recall some notations: $C_0(\varphi)$ is the even part of the Clifford algebra of $\varphi$, $I(F)$ is the ideal of even dimensional forms in the Witt ring $W(F)$ of nondegenerate quadratic forms over $F$ [L].

**Proposition 1.3** [L]. The algebra $C_0(\varphi)$ has dimension $2^{n-1}$ over $F$ ($n = \text{dim } \varphi$). If $\varphi \notin I^2(F)$, then $C_0(\varphi)$ is a simple algebra; if $\varphi \in I^2(F)$ (i.e., $n:2 \text{ and } (-1)^{n/2} \det \varphi \in F^{*2}$ [L]), then $C_0(\varphi)$ is isomorphic to the Cartesian square of a simple algebra.

**Definition 1.4.** We define the number $s(\varphi)$ as follows. In view of 1.3, if $\varphi \notin I^2(F)$, then $C_0(\varphi) \simeq M_2(D)$ for some division algebra $D$ and integer nonnegative $s$, and if $\varphi \in I^2(F)$, then $C_0(\varphi) \simeq M_2(D) \times M_2(D)$ for some $D$, $s$. In both cases we put $s(\varphi) = s$.

**Theorem 1.5** [K1]. If $\varphi$ is anisotropic and $\varphi \notin I^2(F)$, then $TG^pK(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or equal to 0 for each $p$; moreover

$$\text{card}\{p : TG^pK(X) \neq 0\} = s(\varphi).$$
The description of $T G^* K(X_\varphi)$ for $\varphi \in \Omega^2(F)$ is more intricate. Here we shall need the case $s(\varphi) = 0$ only.

**Theorem 1.6 [K1].** For any $\varphi$, if $s(\varphi) = 0$, then $T G^* K(X_\varphi) = 0$.

We denote the image of $h^p \in CH^p(X_\varphi)$ in $G^p K(X_\varphi)$ by the same symbol. The next statement is not complicated and is also true for Chow groups.

**Proposition 1.7 [K1], [S2].** For any anisotropic $\varphi$, the quotient

$G^p K(X_\varphi)/T G^p K(X_\varphi)$

is generated by the class of $h^p$ if $p \neq (\dim X_\varphi)/2$.

§2. Main theorem

**Lemma 2.1.** Let $U$ be affine quadric over the field $F(t)$ defined by the equation $\varphi_{F(t)} = t$, where $\varphi$ is a quadratic form over $F$, $t$ is an independent variable. Then $CH^p U = 0$ for all $p > 0$.

**Proof.** We consider the morphism $A^n_F \to A^1_F$ defined by $\varphi$, where $n = \dim \varphi$. The fiber over the generic point of $A^1_F$ is isomorphic to $U$, thus we obtain an epimorphism $CH^p(A^n_F) \to CH^p(U)$ [KM]. Now it is sufficient to recall that $CH^p(A^n_F) = 0$ for $p > 0$. □

**Corollary 2.2.** If $F = k(t_1, \ldots, t_n)$, $\varphi = (t_1, \ldots, t_n)$, where $k$ is a field and $t_i$ are independent variables, then $CH^p(X_\varphi) = \mathbb{Z} \cdot h^p$ for all $p$.

**Proof.** Let $U$ be the affine quadric $\psi = -t_n$ over the field $F$, where $\psi = (t_1, \ldots, t_{n-1})$. There is a closed imbedding $X_\varphi \hookrightarrow X_\psi$ and $U \simeq X_\psi \setminus X_\varphi$, thus we obtain an exact sequence

$$CH^{p-1}(X_\varphi) \to CH^p(X_\varphi) \to CH^p(U) \to 0.$$ 

The last group in the sequence equals 0 in view of 2.1. Thus $CH^p(X_\varphi)$ is generated by the class of a linear section if so is $CH^{p-1}(X_\varphi)$, and one can prove the corollary using induction on $n$. □

**Proposition 2.3.** Let $\psi$ be an anisotropic quadratic form of dimension $2p$ over a field $k$. $\psi \in \Omega^2(k)$ and $s(\psi) = 0$ (for example one can have: $k = k_0(t_1, \ldots, t_{2p-1})$ and $\psi = (t_1, \ldots, t_{2p-1}, (-1)^p t_1, \ldots, t_{2p-1})$ where
$k_0$ is a field, $t_i$ are independent variables, $p > 1$). If one puts $F = k(t)$, $\varphi = (t) \perp \psi_F$ then $T \text{CH}^{p}(X_\varphi) \neq 0$.

Proof. Abusing notations, we denote $\psi_F$ by $\psi$ without a special sign. In view of 2.1, the first homomorphism of the exact sequence

$$
\text{CH}^{i-1}(X_\psi) \to \text{CH}^i(X_\psi) \to \text{CH}^i(U) \to 0
$$

is surjective. Considering the commutative square

$$
\begin{array}{ccc}
\text{CH}^{i-1}(X_\psi) & \to & \text{CH}^i(X_\psi) \\
\downarrow & & \downarrow \\
G^{i-1}K(X_\psi) & \to & G^iK(X_\psi)
\end{array}
$$

we see that the lower arrow is surjective too. Since $s(\psi) = 0$, Theorem 1.6 says that $TG^{i-1}K(X_\psi) = 0$ for all $i$. In view of 1.7, the last statement implies that $G^{i-1}K(X_\psi)$ is generated by $h^{i-1}$ for $i - 1 \neq (\dim X_\psi)/2 = p - 1$. Thus $G^iK(X_\psi)$ is generated by $h^i$ and therefore has no torsion for $i \neq p$.

On the other hand, $\varphi$ is anisotropic and $s(\varphi) = s(\psi) + 1 = 1$. Consequently Theorem 1.5 implies $TG^iK(X_\psi) \neq 0$ for some $i$, and because of 1.2 the proof is over. □

Theorem. For any $p > 1$, there exist a $4p$-dimensional quadratic form $\varphi$ (over a suitable field $F$) such that $T \text{CH}^{p}(X_\varphi) \neq 0$.

Construction. Let $k$ be an arbitrary field and $t_0, t_1, \ldots, t_{2p-1}$ be independent variables where $p > 1$. We put $F = k(t_0, t_1, \ldots, t_{2p-1})$, $\rho = (t_0, t_1, \ldots, t_{2p-1})$ and finally $\varphi = \rho \circ (1, (-1)^p \det \rho)$. We claim that

$$
T \text{CH}^{p}(X_\varphi) \neq 0.
$$

Proof. A reduced notation $d\rho$ for the term $(-1)^p \det \rho$ will be used. It is sufficient to show that $\text{CH}^{p}(U) \neq 0$, where $U$ is the affine quadric defined over $F$ by the equation $\rho \perp (d\rho) \otimes (t_1, \ldots, t_{2p-1}) = -(d\rho)t_0$. Projecting on the coordinates which are terms of the second summand in the left part of the equation, we obtain a flat morphism $U \to \mathbb{A}^{2p-1}_k$. Because $\text{CH}^{p}(U) \to \text{CH}^{p}(\tilde{U})$, where $\tilde{U}$ is the fiber over the generic point, is epimorphic, it suffices to prove that $\text{CH}^{p}(\tilde{U}) \neq 0$.

The variety $\tilde{U}$ is the affine quadric over $E = F(x_1, \ldots, x_{2p-1})$ defined by the equation $\rho_E = -(d\rho)\sigma$, where $\sigma = t_0 + t_1 x_1^2 + \cdots + t_{2p-1} x_{2p-1}^2$.  

Denote by $\tilde{X}$ the projective closure of $\tilde{U}$ and set $\tilde{Y} = \tilde{X} \setminus \tilde{U}$. The projective quadric $\tilde{Y}$ is determined by the form $\rho_\tilde{X}$ and $\text{CH}^{p-1}(\tilde{Y}) = \mathbb{Z} \cdot h^{p-1}$ in view of 2.2. We have a sequence consisting of isomorphisms and an equality:

$$
\text{CH}^p(\tilde{U}) \simeq \text{CH}^p(\tilde{X})/\text{Im} \text{CH}^{p-1}(\tilde{Y}) = \text{CH}^p(\tilde{X})/(\mathbb{Z} \cdot h^p) \simeq T \text{CH}^p(\tilde{X}).
$$

We shall prove that the last group is nontrivial.

The projective quadric $\tilde{X}$ corresponds to the form $\rho_\tilde{X} \perp (d\rho)\sigma$. Since $\rho_\tilde{X}$ represents $\sigma$, one can find elements $f_1, \ldots, f_{2p-1} \in E^*$ such that $\rho_\tilde{X} \simeq (\sigma, f_1, \ldots, f_{2p-1})$ [L]. Comparison of determinants shows that

$$
\rho_\tilde{X} \perp (d\rho)\sigma \simeq (\sigma, f_1, \ldots, f_{2p-1}, (-1)^p f_1 \ldots f_{2p-1}).
$$

In view of Proposition 2.5 stated below, the elements $\sigma, f_1, \ldots, f_{2p-1}$ are algebraically independent over $l = k(x_1, \ldots, x_{2p-1})$ and generate $E$ over $l$ if the choice of $f_i$ was suitable. So $\tilde{X}$ answers the conditions of 2.3 and therefore $T \text{CH}^p(\tilde{X}) \neq 0$. □

The last step is to state and prove

**Proposition 2.5.** Suppose that $l$ is a field, $t_0, t_1, \ldots, t_n$ are independent variables, $E = l(t_0, t_1, \ldots, t_n)$, $\sigma = t_0 x_0^2 + t_1 x_1^2 + \cdots + t_n x_n^2$ for some $x_i \in l^*$. There exist $f_1, \ldots, f_n \in E^*$ such that:

1. $(t_0, t_1, \ldots, t_n) \simeq (\sigma, f_1, \ldots, f_n)$,
2. $l(\sigma, f_1, \ldots, f_n) = E$

(in particular $\sigma, f_1, \ldots, f_n$ are algebraically independent over $l$ because the number of these elements coincides with $\text{deg tr } E/l$).

**Proof.** Replacing $t_i$ by $t_i x_i^2$ reduces the proof to the case $x_i = 1$ for all $i$.

Let $e_0, e_1, \ldots, e_n$ be the orthogonal basis of the form $(t_0, t_1, \ldots, t_n)$ for which $(e_i, e_i) = t_i$. Note that $\sigma$ is equal to the square of $e = e_0 + e_1 + \cdots + e_n$. To prove the proposition, it is sufficient to find a diagonal form $(f_1, \ldots, f_n)$ on the orthogonal complement $e^\perp$ satisfying condition 2. The family of vectors $e'_i = e_0/t_0 - e_i/t_i$, $i = 1, \ldots, n$, is a basis of $e^\perp$. We claim the orthogonalization of this basis will give (after a small correction) what is required.

To prove this, let us consider the Gramm matrix of $\{e'_i\}$ and compute its principal minors. Since $(e'_i, e'_j) = 1/t_0 + 1/t_i$, $(e'_i, e'_j) = 1/t_0$ for $i \neq j$.
the $i$-th principal minor $M_i$ equals $(1/t_i^2)$ multiplied by the determinant
\[
\begin{vmatrix}
(1 + g_1) & 1 & 1 & \ldots & 1 \\
1 & (1 + g_2) & 1 & \ldots & 1 \\
1 & 1 & (1 + g_3) & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \ldots & (1 + g_i)
\end{vmatrix}
\]
where $g_i = t_0/t_i$. This determinant equals
\[g_1 g_2 \ldots g_i (1 + 1/g_1 + 1/g_2 + \ldots + 1/g_i),\]
therefore $M_i = (t_0 + t_1 + \ldots + t_i)/(t_0 t_1 \ldots t_i)$. Consequently, orthogonalizing \(\{e_i\}\), we obtain a basis whose elements have squares of the form
\[M_i/M_{i-1} = 1/(t_0 + t_1 + \ldots + t_{i-1}) + 1/t_i.\]
For $i = 1, \ldots, n$, we put $f_i = t_i^2 \cdot M_i/M_{i-1}$.

The last problem remaining is to show that $t_0, t_1, \ldots, t_n \in E'$, where $E' = l(\sigma, f_1, \ldots, f_n)$. We prove the statements $\sigma_i, t_i \in E'$, where $\sigma_i = t_0 + t_1 + \ldots + t_i$, by using inverse induction on $i$ and the equality
\[t_i = \sigma_i f_i / (\sigma_i + f_i) \quad (\ast)\]
We have $\sigma_n = \sigma \in E'$, and $(\ast)$ shows $t_n \in E'$. If $\sigma_{i+1}, t_{i+1} \in E'$, then $\sigma_i = \sigma_{i+1} - t_{i+1} \in E'$ and therefore $t_i \in E'$ in view of $(\ast)$. Finally $t_0 = \sigma_0 = \sigma_i - t_i \in E'$.

References


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