Canonical Dimension
of (semi-)spinor Groups of Small Ranks

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Abstract: We show that the canonical dimension $\text{cd} \text{Spin}_{2n+1}$ of the spinor group $\text{Spin}_{2n+1}$ has an inductive upper bound given by $n + \text{cd} \text{Spin}_{2n-1}$. Using this bound, we determine the precise value of $\text{cd} \text{Spin}_n$ for all $n \leq 16$ (previously known for $n \leq 10$). We also obtain an upper bound for the canonical dimension of the semi-spinor group $\text{cd} \text{Spin}_n^\sim$ in terms of $\text{cd} \text{Spin}_{n-2}$. This bound determines $\text{cd} \text{Spin}_n^\sim$ for $n \leq 16$; for any $n$, assuming a conjecture on the precise value of $\text{cd} \text{Spin}_{n-2}$, this bound determines $\text{cd} \text{Spin}_n^\sim$.

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1. Introduction

Let $X$ be a smooth algebraic variety over a field $F$. A field extension $L/F$ is called a splitting field of $X$, if $X(L) \neq \emptyset$. A splitting field $E$ of $X$ is called generic, if it has an $F$-place $E \rightarrow L$ to any splitting field $L$ of $X$. Given a prime number $p$, a splitting field $E$ of $X$ is called $p$-generic, if for any splitting field $L$ of $X$ there exists an $F$-place $E \rightarrow L'$ to some finite extension $L'/L$ of degree prime to $p$. Note that since $X$ is smooth, the function field $F(X)$ is a generic splitting field of $X$; besides, any generic splitting field of $X$ is $p$-generic for any $p$. 

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The canonical dimension $\text{cd}(X)$ of the variety $X$ is defined as the minimum of $\text{tr. deg}_F E$, where $E$ runs over the generic splitting fields of $X$; the canonical $p$-dimension $\text{cd}_p(X)$ of $X$ is defined as the minimum of $\text{tr. deg}_F E$, where $E$ runs over the $p$-generic splitting fields of $X$. For any $p$, one evidently has $\text{cd}_p(X) \leq \text{cd}(X)$.

Let $G$ be an algebraic group over $F$. The notion of canonical dimension $\text{cd}(G)$ of $G$ is introduced in [1]: $\text{cd}(G)$ is the maximum of $\text{cd}(T)$, where $T$ runs over the $G$-torsors over all field extensions $K/F$. The notion of canonical $p$-dimension $\text{cd}_p(G)$ of $G$ is introduced in [3]: $\text{cd}_p(G)$ is the maximum of $\text{cd}_p(T)$, where $T$ runs over the $G$-torsors over all field extensions $K/F$. For any $p$, one evidently has $\text{cd}_p(G) \leq \text{cd}(G)$.

A recipe of computation of $\text{cd}_p(G)$ for an arbitrary $p$ and an arbitrary split simple algebraic group $G$ is given in [3]; the value of $\text{cd}_p(G)$ is determined there for all $G$ of classical type (the remaining types are treated in [4]).

Let $G$ be a split simple algebraic group over $F$ and let $p$ be a prime. As follows from the definition of the canonical $p$-dimension, $\text{cd}_p(G) \neq 0$ if and only if $p$ is a torsion prime of $G$. It is shown in [2], that $\text{cd}(G) = \text{cd}_p(G)$ for any $G$ possessing a unique torsion prime $p$ with the exception of the case where $G$ is a spinor or a semi-spinor group.

According to [3], for any $n \geq 1$ one has

$$\text{cd}_2(\text{Spin}_{2n+1}) = \text{cd}_2(\text{Spin}_{2n+2}) = n(n+1)/2 - 2^l + 1,$$

where $l$ is the smallest integer such that $2^l \geq n + 1$ (the prime 2 is the unique torsion prime of the spinor group). As shown in [1], $\text{cd}(\text{Spin}_{2n+1}) = \text{cd}(\text{Spin}_{2n+2})$ for any $n$ and $\text{cd}(\text{Spin}_n) = \text{cd}_2(\text{Spin}_n)$ for all $n \leq 10$.

We note that the Spin$_{10}$-torsors are related to the 10-dimensional quadratic forms of trivial discriminant and trivial Clifford invariant, and that the value of $\text{cd}(\text{Spin}_{10})$ is obtained due to a theorem of Pfister on those quadratic forms.

In [2], an upper bound on $\text{cd}(\text{Spin}_{2n+1})$ given by $n(n - 1)/2$ is established. If $n + 1$ is a power of 2, this upper bound coincides with the lower bound given by the known value of $\text{cd}_2(\text{Spin}_{2n+1})$. Therefore $\text{cd}(\text{Spin}_n) = \text{cd}_2(\text{Spin}_n)$, if $n$ or $n + 1$ is a 2 power.

In the current note, we establish for an arbitrary $n$ the following inductive upper bound on $\text{cd}(\text{Spin}_{2n+1})$ (see Theorem 2.2):

$$\text{cd}(\text{Spin}_{2n+1}) \leq n + \text{cd}(\text{Spin}_{2n-1}).$$

This bound together with the computation of $\text{cd}(\text{Spin}_n)$ for $n \leq 10$, cited above, shows (see Corollary 2.4) that $\text{cd}(\text{Spin}_n) = \text{cd}_2(\text{Spin}_n)$ for any $n \leq 16$ (the really new cases are $n \in \{11, 12, 13, 14\}$). More generally, if $\text{cd}(\text{Spin}_{2m+1}) = \text{cd}_2(\text{Spin}_{2m+1})$ for some positive integer $m$, then our inductive bound shows that
\( \mathfrak{c}(\text{Spin}_n) = \mathfrak{c}_2(\text{Spin}_n) \) for any \( n \) lying in the interval \([2^m + 1, 2^{m+1}]\) (see Corollary 2.3).

Note that \( \mathfrak{c}_2(\text{Spin}_{2n+1}) = \mathfrak{c}_2(\text{Spin}_{2n}) \). Therefore the crucial statement needed for a further progress on \( \mathfrak{c}(\text{Spin}_n) \) is the statement that \( \mathfrak{c}(\text{Spin}_{17}) = \mathfrak{c}(\text{Spin}_{16}) \). As mentioned above, the similar equality \( \mathfrak{c}(\text{Spin}_9) = \mathfrak{c}(\text{Spin}_8) \), concerning the previous 2 power, is a consequence of the Pfister theorem.

We finish the introduction by discussing the semi-spinor group \( \text{Spin}_n^\sim \). Here \( n \) is a positive integer divisible by 4. To see the parallels with the spinor case, it is more convenient to speak on \( \text{Spin}_{2n+2}^\sim \) with \( n \) odd. The lower bound on \( \mathfrak{c}_2(\text{Spin}_{2n+2}^\sim) \) given by the canonical 2-dimension (the prime 2 is the unique torsion prime of the semi-spinor group) is calculated in [3] as

\[
\mathfrak{c}_2(\text{Spin}_{2n+2}^\sim) = \frac{n(n+1)}{2} + 2^k - 2^l,
\]

where \( k \) is the largest integer such that \( 2^k \) divides \( n+1 \) (and \( l \) is still the smallest integer with \( 2^l \geq n+1 \)). The upper bound \( \mathfrak{c}(\text{Spin}_{2n+2}^\sim) \leq \frac{n(n-1)}{2} + 2^k - 1 \), established in [2], shows that the canonical 2-dimension is the value of the canonical dimension if \( n+1 \) is a power of 2. In particular, \( \mathfrak{c}(\text{Spin}_n^\sim) = \mathfrak{c}_2(\text{Spin}_n) \) for \( n \in \{4, 8, 16\} \).

In the current note we establish the following general upper bound on the canonical dimension of the semi-spinor group in terms of the canonical dimension of the spinor group (see Theorem 3.1):

\[
\mathfrak{c}(\text{Spin}_{2n+2}^\sim) \leq n - 1 + 2^k + \mathfrak{c}(\text{Spin}_{2n})
\]

(with \( k \) as above). This bound together with the computation of \( \mathfrak{c}(\text{Spin}_{10}) \) shows (see Corollary 3.3) that \( \mathfrak{c}(\text{Spin}_{12}) = \mathfrak{c}_2(\text{Spin}_{12}^\sim) = 11 \); therefore the formula \( \mathfrak{c}(\text{Spin}_n) = \mathfrak{c}_2(\text{Spin}_n^\sim) \) holds for all \( n \leq 16 \) (where the only new case is \( n = 12 \)).

In general, if \( \mathfrak{c}(\text{Spin}_{2n}) = \mathfrak{c}_2(\text{Spin}_{2n}) \) for some (odd) \( n \), then our upper bound on \( \mathfrak{c}(\text{Spin}_{2n+2}^\sim) \) shows that \( \mathfrak{c}(\text{Spin}_{2n+2}^\sim) = \mathfrak{c}_2(\text{Spin}_{2n+2}^\sim) \) for this \( n \) (see Corollary 3.2).

2. The spinor group

Our main tool is the following general observation made in [2]. Let \( G \) be a split semisimple algebraic group over a field \( F \), \( P \) a parabolic subgroup of \( G \), \( P' \) a special parabolic subgroup of \( G \) sitting inside of \( P \). Saying special, we mean that any \( P' \)-torsor over any field extension \( K/F \) is trivial.

For any \( G \)-torsor \( T \) over \( F \), let us write \( \mathfrak{c}'(T/P) \) for \( \min\{\dim X\} \), where \( X \) runs over all closed subvarieties of the variety \( T/P \) admitting a rational morphism \( F(T/P') \to X \).
Lemma 2.1 ([2, lemma 5.3]). In the above notation, one has
\[ \text{cd}(T) \leq \text{cd}'(T/P) + \max_Y \text{cd}(Y), \]
where \( Y \) runs over all fibers of the projection \( T/P' \to T/P \).

In this section, we apply Lemma 2.1 in the following situation: \( G = \text{Spin}_{2n+1} = \text{Spin}(\varphi) \), where \( \varphi : F^{2n+1} \to F \) is a split quadratic form; \( P \) is the stabilizer of a rational point \( x \) under the standard action of \( G \) on the variety of 1-dimensional totally isotropic subspaces of \( \varphi \); \( P' \subset P \) is the stabilizer of a rational point \( x' \), lying over \( x \), under the standard action of \( G \) on the variety of parabolics consisting of a 1-dimensional totally isotropic subspace sitting inside of an \( n \)-dimensional (maximal) totally isotropic subspace of \( \varphi \).

The parabolic subgroup \( P' \) of \( G \) is clearly special.

Let \( T \) be a \( G \)-torsor over \( F \) and let \( \psi : F^{2n+1} \to F \) be a quadratic form such that the similarity class of \( \psi \) is the class corresponding to \( T \) in the sense of [3, §8.2]. Note that the even Clifford algebra of \( \psi \) is trivial.

The algebraic variety \( T/P \) is identified with the projective quadric of \( \psi \); in particular, \( \dim(T/P) = 2n - 1 \). The variety \( T/P' \) is identified with the variety of flags consisting of a 1-dimensional totally isotropic subspace sitting inside of an \( n \)-dimensional (maximal) totally isotropic subspace of \( \psi \). The morphism \( T/P' \to T/P \) is identified with the natural projection of the flag variety onto the quadric.

Let \( X \subset T/P \) be an arbitrary subquadric of dimension \( n \) (\( X \) is the quadric of the restriction of \( \psi \) onto an \( (n + 2) \)-dimensional subspace of \( F^{2n+1} \)). Since over the function field \( F(T/P') \) the quadratic form \( \psi \) becomes split, the variety \( X_{F(T/P')} \) has a rational point, or, in other words, there exists a rational morphism \( T/P' \dashrightarrow X \). Therefore \( \text{cd}'(T/P) \leq \dim X = n \).

Any fiber \( Y \) of the projection \( T/P' \to T/P \) is the variety of \( n \)-dimensional (maximal) totally isotropic subspaces of \( \psi \), containing a fixed 1-dimensional subspace \( U \). The latter variety is identified with the variety of \( (n - 1) \)-dimensional (maximal) totally isotropic subspaces of the quotient \( U^\perp/U \). Note that we have \( \dim U^\perp/U = 2n - 1 \); besides, the quadratic form on \( U^\perp/U \), induced by the restriction of \( \psi \), is Witt-equivalent to \( \psi \) and, in particular, its even Clifford algebra is trivial. Since \( \text{cd}(\text{Spin}_{2n-1}) \) is the maximum of the canonical dimension of the variety of maximal totally isotropic subspaces of a \( (2n - 1) \)-dimensional quadratic forms with trivial even Clifford algebra, it follows that \( \text{cd}(Y) \leq \text{cd}(\text{Spin}_{2n-1}) \).

Applying Lemma 2.1, we get our main inequality for the spinor group:

**Theorem 2.2.** For any \( n \), one has \( \text{cd}(\text{Spin}_{2n+1}) \leq n + \text{cd}(\text{Spin}_{2n-1}) \). \( \blacksquare \)
Corollary 2.3. Assume that \(c_0(\text{Spin}_{2m+1}) = c_0(\text{Spin}_{2m+1})\) for some positive integer \(m\). Then \(c_0(\text{Spin}_n) = c_0(\text{Spin}_n)\) for any \(n\) lying in the interval \([2^m + 1, 2^{m+1}]\).

Proof. Let \(n\) be such that \(2n \pm 1 \in [2^m, 2^{m+1}]\) and \(c_0(\text{Spin}_{2n-1}) = c_0(\text{Spin}_{2n-1})\). Then

\[
c_0(\text{Spin}_{2n+1}) \leq n + c_0(\text{Spin}_{2n-1}) = n + n(n - 1)/2 - 2^m + 1 = n(n + 1)/2 - 2^m + 1 = c_0(\text{Spin}_{2n+1}) \leq c_0(\text{Spin}_{2n+1}).
\]

Consequently, \(c_0(\text{Spin}_{2n+1}) = c_0(\text{Spin}_{2n+1})\).

Since \(c_0(\text{Spin}_n) = c_0(\text{Spin}_n)\) for \(n \leq 10\) (see [1, example 12.2]), the assumption of Corollary 2.3 holds for \(m = 3\), and we get

Corollary 2.4. The equality \(c_0(\text{Spin}_n) = c_0(\text{Spin}_n)\) holds for any \(n \leq 16\). □

3. The semi-spinor group

In this section, we apply Lemma 2.1 in the following situation: \(G = \text{Spin}_{2n+2} = \text{Spin}^\sim(\varphi)\), where \(\varphi : F^{2n+2} \to F\) is a hyperbolic quadratic form; \(P\) is the stabilizer of a rational point \(x\) under the standard action of \(G\) on the variety of 1-dimensional totally isotropic subspaces of \(\varphi\); \(P' \subset P\) is the stabilizer of a rational point \(x'\), lying over \(x\), under the standard action of \(G\) on the scheme of flags consisting of a 1-dimensional totally isotropic subspace sitting inside of an \((n + 1)\)-dimensional (maximal) totally isotropic subspace of \(\varphi\).

The parabolic subgroup \(P'\) of \(G\) is clearly special.

Let \(T\) be a \(G\)-torsor over \(F\) and let \(\pi\) be a quadratic pair on a degree \(2n + 2\) central simple \(F\)-algebra \(A\) such that the isomorphism class of \(\pi\) corresponds to \(T\) in the sense of [3, §8.4]. Note that the discriminant and a component of the Clifford algebra of \(\pi\) are trivial.

The quotient \(T/P\) is identified with the variety of rank 1 isotropic ideals of \(\pi\); in particular, \(\dim(T/P) = 2n\). The quotient \(T/P'\) is identified with a component of the scheme of flags consisting of a rank 1 ideal sitting inside of a rank \((n + 1)\) (maximal) isotropic ideal of \(\pi\). The morphism \(T/P' \to T/P\) is identified with the natural projection.

The index of the degree \(2n + 2\) central simple algebra \(A\) is a 2 power dividing \(2n + 2\). Therefore \(A\) is Brauer-equivalent to a central simple algebra \(A'\) of degree \(n + 1 + 2^k\), where \(k\) is the largest integer such that \(2^k\) divides \(n + 1\). Let \(\pi'\) be the adjoint quadratic pair on \(A'\) and let \(X\) be the variety of rank 1 isotropic
ideals of $\pi'$. The variety $X$ is a closed subvariety of the quotient $T/P$. Over the function field $F(T/P')$ the variety $T/P$ becomes a hyperbolic quadric and the closed subvariety $X$ becomes its subquadric; since $\dim X > \dim(T/P)$, the variety $X_{F(T/P')}$ has a rational point, or, in other words, there exists a rational morphism $T/P' \dashrightarrow X$. Therefore $\cd'(T/P) \leq \dim X = n - 1 + 2^k$.

Let $y$ be a point of $T/P$. The algebra $A_{F(y)}$ is isomorphic to the algebra of $(2n + 2) \times (2n + 2)$ matrices over $F(y)$. Let $\psi : F(y)^{2n+2} \rightarrow F(y)$ be the adjoint quadratic form. Note that the discriminant and the Clifford algebra of $\psi$ are trivial.

The fiber $Y$ of the projection $T/P' \rightarrow T/P$ over the point $y$ is a component of the scheme of rank $n + 1$ (maximal) isotropic ideals of $\pi$, containing a fixed rank 1 isotropic ideal. Therefore $Y$ is identified with a component of the scheme of $(n + 1)$-dimensional (maximal) totally isotropic subspaces of $\psi$, containing a fixed 1-dimensional subspace $U$. The latter variety is identified with a component of the scheme of $n$-dimensional (maximal) totally isotropic subspaces of the quotient $U^1/U$. Note that $\dim U^1/U = 2n$; besides, the quadratic form on $U^1/U$, induced by the restriction of $\psi$, is Witt-equivalent to $\psi$ and, in particular, its discriminant and Clifford algebra are trivial.

Since $\cd(Spin_{2n})$ is the maximum of the canonical dimension of a component of the scheme of maximal totally isotropic subspaces of a $2n$-dimensional quadratic form with trivial discriminant and Clifford algebra, it follows that $\cd(Y) \leq \cd(Spin_{2n})$. Applying Lemma 2.1, we get our main inequality for the semi-spinor group:

**Theorem 3.1.** For any odd $n$, one has $\cd(Spin^{\sim}_{2n+2}) \leq n - 1 + 2^k + \cd(Spin_{2n})$. □

**Corollary 3.2.** Assume that $\cd(Spin_{2n}) = \cd_2(Spin_{2n})$ for some odd $n$. Then

$$\cd(Spin^{\sim}_{2n+2}) = \cd_2(Spin^{\sim}_{2n+2})$$

for this $n$.

**Proof.** Let $l$ be the smallest integer such that $2^l \geq n + 1$. Since $n$ is odd, $l$ is also the smallest integer such that $2^l \geq n$, therefore $\cd(Spin_{2n}) = \cd_2(Spin_{2n}) = n(n - 1)/2 - 2^l + 1$. By Theorem 3.1 we have

$$\cd(Spin^{\sim}_{2n+2}) \leq (n - 1 + 2^l) + (n(n - 1)/2 - 2^l + 1) = n(n + 1)/2 + 2^k - 2^l = \cd_2(Spin^{\sim}_{2n+2}) \leq \cd(Spin^{\sim}_{2n+2}).$$

Consequently, $\cd(Spin^{\sim}_{2n+2}) = \cd_2(Spin^{\sim}_{2n+2})$. □
Since the assumption of Corollary 3.2 holds for \( n \leq 8 \) (see Corollary 2.4), we get

**Corollary 3.3.** The equality \( c_d(\text{Spin}_n) = c_d(\text{Spin}_n^\sim) \) holds for any \( n \leq 16 \). \( \square \)

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