ON GENERIC FLAG VARIETIES FOR ODD SPIN GROUPS

NIKITA A. KARPENKO

Abstract. For the spin group $G = \mathrm{Spin}(2n+1)$ with arbitrary n, a generic G-torsor E over a field, and a parabolic subgroup $P \subset G$, we consider the generic flag variety E/P and describe its Chow ring modulo torsion. This description determines the index of E/P, completing results of [2], where the index has been determined for most P.

Contents

1. Introduction

We consider the split spin group $G = \mathrm{Spin}(2n+1)$ with arbitrary n over an arbitrary field. A *generic G-torsor E* can be defined as the generic fiber of the quotient map $\mathrm{GL}(N) \to \mathrm{GL}(N)/G$ given by any embedding of G into a general linear group $\mathrm{GL}(N)$ with some N. Of course, different choices of the embedding produce different E. However, our object of interest – the Chow ring $\mathrm{CH}(E/P)$ for a fixed parabolic subgroup $P \subset G$ – is canonic, [8, Lemma 2.1].

Understanding $\mathrm{CH}(E/P)$ allows one, in particular, to compute the index $\mathrm{ind}(E/P)$ – the greatest common divisor of degrees of closed points on the variety E/P . In fact, it is enough to know the quotient $\bar{\mathrm{CH}}(E/P)$ of the ring $\mathrm{CH}(E/P)$ by the ideal of the elements of finite order.

Date: 5 May 2021. Revised: 23 May 2021.

Key words and phrases. Quadratic forms over fields; algebraic groups; spin groups; torsors; classifying spaces; Chow groups. *Mathematical Subject Classification (2010):* 20G15; 14C25.

Author's work has been supported by a Discovery Grant from the National Science and Engineering Research Council of Canada.

Let us fix an extension field $\bar{F}$ of the base field F of E trivializing E (e.g., an algebraic closure). Since the Chow ring of the cellular variety G/P is not affected by base field extensions, the change of field homomorphism $\mathrm{CH}(G/P) \to \mathrm{CH}(G/P)_{\bar{F}}$ is an isomorphism. Choosing a trivialization of the G-torsor $E_{\bar{F}}$, we identify $\mathrm{CH}(E/P)_{\bar{F}}$ with $\mathrm{CH}(G/P)_{\bar{F}}$. Since G acts trivially on $\mathrm{CH}(G/P)$ (see [6, Corollary 4.2]), the identification is canonical, i.e., does not depend on the choice of trivialization. Summarizing, we get a homomorphism

$$\mathrm{CH}(E/P) \to \mathrm{CH}(G/P).$$

Since its kernel is exactly the ideal of torsion elements, it identifies $\bar{\mathrm{CH}}(E/P)$ with a subring in $\mathrm{CH}(G/P)$.

The indexes $\mathrm{ind}(E/P)$ have been computed in [2] for many P. The starting point there was the upper bound on $\bar{\mathrm{CH}}(E/P)$ given by the image of the homomorphism

$$\mathsf{S}(\hat{T})^W \to \mathrm{CH}(G/P),$$

defined in [2, Remark 2.3], where $T \subset P$ is a split maximal torus, $\hat{T}$ is the group of characters of T endowed with the action of the Weyl group W of P, $\mathsf{S}(\hat{T})$ is the symmetric ring, and $\mathsf{S}(\hat{T})^W$ is its subring of the W-invariant elements. An important part of [2] was construction of generators for the ring of A-invariants $\mathsf{S}(\hat{T})^A$ for certain subgroup $A \subset W$.

There is a natural ring homomorphism $\mathrm{CH}(\mathsf{B}P) \to \mathsf{S}(\hat{T})^W$ and a natural surjective ring homomorphism $\mathrm{CH}(\mathsf{B}P) \twoheadrightarrow \mathrm{CH}(E/P)$ (see [2, §2]), both departing from the Chow ring $\mathrm{CH}(\mathsf{B}P)$ of the classifying space $\mathsf{B}P$ of P (see [13]). The precise value of $\bar{\mathrm{CH}}(E/P)$ is given by the image of the composition

$$\mathrm{CH}(\mathsf{B}P) \to \mathsf{S}(\hat{T})^W \to \mathrm{CH}(G/P)$$

simply because it coincides with the composition

$$\mathrm{CH}(\mathsf{B}P) \twoheadrightarrow \mathrm{CH}(E/P) \to \mathrm{CH}(G/P).$$

Unfortunately, in most cases, we do not understand the Chow ring $\mathrm{CH}(\mathsf{B}P)$ well enough because its description involves mysterious and complicated $\mathrm{CH}(\mathsf{B}\,\mathrm{Spin}(l))$ with $l > 8$. (For $l < 7$, $\mathrm{CH}(\mathsf{B}\,\mathrm{Spin}(l))$ is well understood; descriptions for $l = 7$ and $l = 8$ are given in [5] and [12].) By this reason, a precise determination of $\bar{\mathrm{CH}}(E/P)$ for general P seemed to be out of reach.

Quite surprisingly, it turns out that the above upper bound coincides with $\bar{\mathrm{CH}}(E/P)$! We will prove it here by listing certain generators for the ring $\mathsf{S}(\hat{T})^W$ and then showing that their images are in $\bar{\mathrm{CH}}(E/P)$ (for a non-related to $\mathrm{CH}(\mathsf{B}P)$ reason: they turn out to be Chern classes of certain elements in the Grothendieck group of E/P). This way we get a very handy system of generators for the ring $\bar{\mathrm{CH}}(E/P)$ and remove the hindrance to computation of $\mathrm{ind}(E/P)$ for arbitrary P.

Our main result here is Theorem 3.6 describing $\bar{\mathrm{CH}}(E/P)$ in the case of maximal P: the study of $\mathrm{CH}(E/P)$ (and determination of $\mathrm{ind}(E/P)$) for arbitrary P is easily reduced to the case of maximal P (see [2]). The description is particularly simple in the situation of Corollary 3.7, explaining and providing a more conceptual proof for [2, Theorem 4.2].

Theorem 4.1 is the second main result. It gives a formula and an algorithm for determination of the indexes: in every concrete case the concrete value can be then calculated by computer (having enough computer time and power).

As an example of application of Theorem 4.1, we do the calculation in some cases. To formulate the answers, let us first recall that the conjugacy classes of maximal parabolic subgroups in G are indexed by the n vertices of the Dynkin diagram of G. Given $m \in \{1, \dots, n\}$, we write P_m for the mth standard maximal parabolic subgroup in the standard realization of $G = \mathrm{Spin}(2n+1)$ as in [2, §4] and we write X_m for the variety E/P_m. The G-torsor E yields a non-degenerate $(2n+1)$-dimensional quadratic form q of trivial discriminant and Clifford invariant. The variety X_m is identified with the variety of m-dimensional totally isotropic subspaces of q. In particular, X_1 is the projective quadric.

Let us mention that the index of the highest orthogonal grassmannian X_n is computed in [14]. For all m, the indexes $\mathrm{ind}(X_m)$ have been computed so far for $n \leq 7$ (i.e., $\dim q < 17$) only (see [7]). In §5 and §7, this boundary is pushed further away. As a byproduct, we also get some new information on the even spin group Spin(18) and Spin(20) (see §6 and §8).

We conclude with some general remarks. For arbitrary n and m, it is easy to check that $\mathrm{ind}(X_m)$ is a 2-power 2^{i_m}. Moreover, $i_{m-1} \leq i_m \leq i_{m-1} + 1$, where $i_0 := 0$.

To demonstrate an interest of knowing $\mathrm{ind}(X_m)$, we indicate that it gives a (sharp) upper bound on the value of $\mathrm{ind}\, X'_m$, where X'_m is the m-th orthogonal grassmannian of any $(2n+1)$-dimensional non-degenerate quadratic form q' with trivial discriminant and Clifford invariant over a field F' (when varying F' and q'). For those new in the subject, let us also note that the variety X'_m has a rational point if and only if the Witt index of q' is at least m; consequently, $\mathrm{ind}\, X'_m$ is the greatest common divisor of degrees of finite field extensions L/F' such that the Witt index of q'_L is at least m.

Acknowledgements. I am grateful to Alexander Merkurjev for stimulating conversations.

2. Invariants

Let T be the standard split maximal torus of G contained in $P := P_m$. In order to determine $S(\hat{T})^W$, where W is the Weyl group of P, we need a modification of [2, Proposition 3.3].

We consider the polynomial ring $R = \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l]$ over the integers $\mathbb{Z}$ in the variables $x_1, \dots, x_m$ and $y_1, \dots, y_l$, where $m + l = n$. Let $A := (\mathbb{Z}/2\mathbb{Z})^{\times l}$ be the direct product of l copies of the group $\mathbb{Z}/2\mathbb{Z}$ acting on R as follows: for any $i = 1, \dots, l$, the ith copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of y_i and trivially on the remaining variables.

Instead of A, considered in [2, Proposition 3.3], we are going to work with larger groups. We start with the Weyl group W' of the spin group $\mathrm{Spin}(2l+1)$ which is a semidirect product of A and the symmetric group S_l. The action of W' on R we are interested in is the (unique) extension of the action of A, defined above, and the action of S_l by permutation of $y_1, \dots, y_l$. We will also consider the action of S_m by permutation of $x_1, \dots, x_m$ and the resulting action of $W = S_m \times W'$ on R. The latter action extends (uniquely) to an action of W on $R[z]$, where – as in [2, §3] – $R[z]$ is an R-algebra with a generator z subject to the relation

$$2z = x_1 + \cdots + x_m + y_1 + \cdots + y_l.$$

The ring $S(\hat{T})$ is identified with $R[z]$ and the action of the Weyl group W of P on $S(\hat{T})$ is the action of W on $R[z]$ just defined.

As in [2, §3], we define an element $\tilde{z} \in R[z]^A$ as the product of all elements in the A-orbit of z. Since the A-orbit of z coincides with its W-orbit, the element $\tilde{z}$ is actually W-invariant.

We borrow from [2, §3] the construction of A-invariant elements $f_k \in R[z]$, $k \geq 0$. We set

$$f_0 := 2z - y_1 - \cdots - y_l = x_1 + \cdots + x_m \in R^A.$$

Assume that for some $k > 0$ the element f_k is already constructed and has the shape

$$f_k = 2z \cdot g_k + a_1 + \cdots + a_s, \tag{2.1}$$

where g_k is a polynomial with integer coefficients in $z, y_1, \ldots, y_l$ and where $a_1, \ldots, a_s$ for some $s \geq 0$ are monomials in $y_1, \ldots, y_l$. Then we define f_{k+1} as one half of the difference

$$f_k^2 - (a_1^2 + \cdots + a_s^2) = 2\Big(2z\big(zg_k^2 + (a_1 + \cdots + a_s)g_k\big) + \sum_{i \neq j} a_i a_j\Big). \tag{2.2}$$

Note that the new element f_{k+1} has the shape (2.1) allowing to continue the procedure.

Lemma 2.3. *For any $k \geq 0$, the element f_k is W-invariant.*

Proof. By construction, the element f_k is in the subring $\mathbb{Z}[z, y_1, \ldots, y_l] \subset R[z]^{S_m}$. Therefore, f_k is S_m-invariant. Since f_k is A-invariant as well, it remains to check that f_k is S_l-invariant.

The element $f_0 = x_1 + \cdots + x_m$ is S_l-invariant. So, let us assume f_k is S_l-invariant for some $k \geq 0$ and let us then check that f_{k+1} is also S_l-invariant. To do this, we view $\mathbb{Z}[z, y_1, \ldots, y_l]$ as a polynomial ring in z over $\mathbb{Z}[y_1, \ldots, y_l]$. Note that z is an independent generator and S_l acts trivially on z. So, a polynomial in $\mathbb{Z}[z, y_1, \ldots, y_l] = \mathbb{Z}[y_1, \ldots, y_l][z]$ is S_l-invariant if and only if all its coefficients are. From the formula (2.1) we see that the sum $a_1 + \cdots + a_s$ is the constant term of the polynomial f_k. Therefore this sum is S_l-invariant. Now it follows by formula (2.2) that f_{k+1} is also S_l-invariant. □

Proposition 2.4. *The R^W-algebra $R[z]^W$ is generated by the elements $f_1, \ldots, f_{l-1}, \tilde{z}$.*

Proof. As a first step, acting as in the proof of [2, Proposition 6.1], we prove that the

$$\mathbb{Z}[x_1 + \cdots + x_m, y_1, \ldots, y_l]^{W'}\text{-algebra} \quad \mathbb{Z}[y_1, \ldots, y_l][z]^{W'}$$

is generated by the indicated elements. As a second (and final) step we apply [10, Lemma 8.1]. □

3. Images of invariants

We continue using the settings of §2. We also let $B \subset G$ be the standard Borel subgroup; we have $T \subset B \subset P$.

We are going to prove that the image in $\mathrm{CH}(G/P)$ of $S(\hat{T})^W$ lies in

$$\bar{\mathrm{CH}}(X_m) = \bar{\mathrm{CH}}(E/P) \subset \mathrm{CH}(G/P).$$

We start with the easiest part of $S(\hat{T})^W$ whose image is in the subring $C \subset \bar{\mathrm{CH}}(E/P)$ generated by the Chern classes of the tautological (rank m) vector bundle $\mathcal{T}$ on X_m. Note that one can also view or define $\mathcal{T}$ as the tautological vector bundle on the split orthogonal grassmannian G/P.

Proposition 3.1. *The image in* $\mathrm{CH}(G/P)$ *of* $R^W \subset S(\hat{T})^W$ *lies in* $C \subset \bar{\mathrm{CH}}(E/P)$*.*

Proof. The images of $x_1, \dots, x_m$ in $\mathrm{CH}(G/B)$ are the roots of the vector bundle $\mathcal{T}$ (pulled back to G/B along the projection $G/B \to G/P$). The roots of the vector bundle $\mathcal{T}^\perp$, given by the orthogonal complement, are the images of $x_1, \dots, x_m$ along with the images of $\pm y_1, \dots, \pm y_l$ and 0. Finally, the roots of the trivial vector bundle V given by the vector space of definition of q are the images of all $\pm x_1, \dots, \pm x_m$, $\pm y_1, \dots, \pm y_l$, and 0 all together. (Concerning the root 0, see [3, Proof of Proposition 86.13].)

The ring R^W is easily seen to be generated by the elementary symmetric polynomials in $x_1, \dots, x_m$ together with the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$. The images in $\mathrm{CH}(G/P)$ of the first ones are the Chern classes of $\mathcal{T}$. The images of the second ones are the Chern classes of the quotient $\mathcal{T}^\perp/\mathcal{T}$. The isomorphism $V/\mathcal{T}^\perp = \mathcal{T}^\vee$, where $\mathcal{T}^\vee$ is the dual vector bundle, shows that the Chern classes of $\mathcal{T}^\perp$ are polynomials in the Chern classes of $\mathcal{T}$. □

Proposition 3.2. *For any* $i \geq 0$*, the images in* $\mathrm{CH}(G/P)$ *of* $f_i \in S(\hat{T})^W$ *also lie in* $C \subset \bar{\mathrm{CH}}(E/P)$*.*

Proof. The pull-back ring homomorphism $\mathrm{CH}(G/P) \to \mathrm{CH}(G/B)$ is injective and the quotient

$$\mathrm{CH}(G/B)/\,\mathrm{CH}(G/P)$$

is a free abelian group (see [2, Proof of Lemma 2.2]).

The variety G/B is the variety of complete flags of totally isotropic subspaces of q. Let $C_B \subset \mathrm{CH}(G/B)$ be the subring generated by the Chern classes of all (from rank 1 to rank n) tautological vector bundles on G/B. Then C is a subring of C_B and the quotient C_B/C is also a free abelian group. The claim on the quotient can be shown by identifying respectively C and C_B with the Chow rings of the two varieties: the variety Y_m of m-dimensional totally isotropic subspaces and the variety Y of complete flags of totally isotropic subspaces of a $(2n)$-dimensional non-degenerate alternating bilinear form (see [9, Remark 2.6] and Remark 3.3): there is such an identification for which the respective Chern classes of the respective tautological vector bundles correspond to each other. The quotient $\mathrm{CH}(Y)/\,\mathrm{CH}(Y_m)$ is free abelian by the argument of [2, Proof of Lemma 2.2] once again.

It has been shown in [2, Lemma 3.5] that for every $i \geq 0$, the image in $\mathrm{CH}(G/B)$ of f_i is in C_B. Since $2^i f_i \in R$, the image of $2^i f_i$ is in C. It follows that the image of f_i is in C. □

Remark 3.3 (Geometric interpretation of the homomorphism $\mathrm{CH}(Y_m) \to C$, cf. [14, §4])**.** The existence of the isomorphism $\mathrm{CH}(Y_m) = C$, used in the above proof, is justified in [9, Remark 2.6] by information about relations on the generators. So, its geometric construction, described below is not actually needed (but still interesting to look at). Note that both rings are independent of the base field and, in particular, of its characteristic.

In characteristic 2, defining Y_m by the associated (alternating) bilinear form b of q on the vector space V modulo the (1-dimensional) radical $\mathrm{Rad}(b) \subset V$, we get a morphism of varieties $X_m \to Y_m$, mapping every m-dimensional totally isotropic subspace of V (viewed as a point of X_m) to its image in the quotient $V/\mathrm{Rad}(b)$ (which is an m-dimensional totally isotropic subspace giving a point of Y_m). Since $\mathcal{T}$ is the pull-back of the tautological vector bundle $\mathcal{T}'$ on Y_m and since the Chow ring of Y_m is generated by the Chern classes of $\mathcal{T}'$, the pull-back homomorphism $\mathrm{CH}(Y_m) \to \mathrm{CH}(X_m)$ lands in $C \subset \mathrm{CH}(X_m)$ and is the one we are looking for.

Proposition 3.4. *The image in* $\mathrm{CH}(G/P)$ *of the generator* $\tilde{z} \in \mathcal{S}(\hat{T})^W$ *lies in* $\bar{\mathrm{CH}}(E/P)$*.*

Proof. Since the group G is simply connected, the Grothendieck group $K(E/P)$ coincides with $K(G/P)$, [11].

Let us consider the group ring $\mathbb{Z}[\hat{T}]$. Since the addition in $\hat{T}$ becomes multiplication in $\mathbb{Z}[\hat{T}]$, we use the exponential notation $\chi \in \hat{T} \mapsto \exp(\chi) \in \mathbb{Z}[\hat{T}]$ for the embedding $\hat{T} \hookrightarrow \mathbb{Z}[\hat{T}]$. There is a (surjective) ring homomorphism

$$\mathbb{Z}[\hat{T}] \to K(G/B),$$

mapping the exponent $\exp(\chi) \in \mathbb{Z}[\hat{T}]$ of any character $\chi \in \hat{T}$ to the class of the line bundle on G/B given by χ. Restricting to the W-invariants, we get a ring homomorphism

$$\mathbb{Z}[\hat{T}]^W \to K(G/P) \subset K(G/B).$$

The image in $\mathrm{CH}(G/P)$ of the generator $\tilde{z}$ is the 2^lth Chern class of the image in $K(E/P) = K(G/P)$ of the element

$$\sum_{I \subset \{1,\dots,l\}} \exp(z - \sum_{i \in I} y_i) \in \mathbb{Z}[\hat{T}]^W. \qquad \square$$

Remark 3.5. Propositions 3.1, 3.2, and 3.4 show that the ring $\bar{\mathrm{CH}}(E/P)$ is generated by Chern classes (of elements of $K(E/P)$). Actually, R^W and $\tilde{z}$ are already in the subring of $\mathcal{S}(\hat{T})^W$ generated by Chern classes (of elements of $\mathbb{Z}[\hat{T}]^W$). However, in the process of showing that the images of $f_1, \dots, f_{l-1}$ are in C, certain relations are used which occur only after $\mathcal{S}(\hat{T})^W$ is mapped to $\bar{\mathrm{CH}}(E/P)$.

The ring C (which depends only on n) is well understood. In particular, the relations on its generators – the Chern classes (or rather the Segre classes) of $\mathcal{T}$ – are well known (see, e.g., [9]). As we just proved,

Theorem 3.6. *The* C*-algebra* $\bar{\mathrm{CH}}(E/P)$ *is generated by the image of* $\tilde{z}$ *in* $\bar{\mathrm{CH}}^{2^l}(E/P)$*.* $\square$

The index $\mathrm{ind}(E/P)$ has been computed in [2] in the situation where $2^l > \dim(E/P)$. This situation is simpler by the following reason:

Corollary 3.7. *We have* $\bar{\mathrm{CH}}(E/P) = C$ *provided that* $2^l > \dim(E/P)$*.* $\square$

4. How to compute $\operatorname{ind}(X_m)$

We keep notation of the previous section and provide an algorithm computing $\operatorname{ind}(X_m)$.

Since the element $2^{2^l}\tilde{z}$ is in R, it yields an element $\tilde{c} \in C$. The additive group of the ring C is free abelian of finite rank ([9, Theorem 2.1]). For every integer $j \geq 0$, let 2^{k_j} be the highest 2-power diving $\tilde{c}^j$ in C. Here we define $\tilde{c}^0$ to be 1 and therefore $k_0 = 0$. Let k be the maximum of $j2^l - k_j$ over all $j \geq 0$ with $j2^l \leq \dim X_m$.

Theorem 4.1. $\operatorname{ind}(X_m) = 2^{m-k}$.

Example 4.2. If $2^l > \dim X_m$, then $k = 0$ and we recover [2, Theorem 4.2].

Example 4.3. In the case of $m = n$, Theorem 4.1 is [14, Lemma 4.1].

Proof of Theorem 4.1. Recall that C is the Chow ring of the cellular variety Y_m defined in the proof of Proposition 3.2. Let j be such that $k = j2^l - k_j$. Then $\tilde{c}^j = 2^{k_j}d$ for some $d \in C$ non-divisible by 2. Therefore, by Poincaré duality (see [14, §4] or [10, Remark 5.6]) there exists $d' \in C$ such that dd' has an odd degree e on Y_m. Since the class of a rational point in $\mathrm{CH}(Y_m)$ equals 2^m times the class of a rational point in $\mathrm{CH}(G/P) \supset \bar{\mathrm{CH}}(X_m)$, the product $dd' \in \bar{\mathrm{CH}}(X_m)$ has degree $2^m \cdot e$ on X_m and is divisible by 2^k in $\bar{\mathrm{CH}}(X_m)$. It follows that $\operatorname{ind}(X_m)$ divides 2^{m-k}.

For the opposite, applying Theorem 3.6, write the class in $\bar{\mathrm{CH}}(X_m)$ of a 0-cycle of degree $\operatorname{ind}(X_m)$ on X_m as a polynomial in $2^{-2^l}\tilde{c}$ over C. The polynomial contains a monomial $M = 2^{-j2^l}c\tilde{c}^j$ (with some $c \in C$ and some j) of degree an odd multiple of $\operatorname{ind}(X_m)$. Then $2^{j2^l-k_j}M$ is in C and has degree an odd multiple of $2^{j2^l-k_j-m+i}$ on Y_m, with i such that $2^i = \operatorname{ind}(X_m)$. It follows that $j2^l - k_j - m + i \geq 0$ so that $i \geq m - (j2^l - k_j) \geq m - k$. □

5. Spin(17)

Note that for any n the index $\operatorname{ind}(X_n)$ is known (due to [14]) and coincides with $\operatorname{ind}(X_{n-1})$ and $\operatorname{ind}(X_{n-2})$.

All indexes are known for q of dimension lower than 17 (see [7]). For q of dimension 17 we have $n = 8$. Let $n = 8$ and $m = 5$.

A computation (made on Maple 2021), using the Chow ring package (Version 4.0) by S. Nikolenko, V. Petrov, N. Semenov, and K. Zainoulline, shows that the image $\tilde{c}^3$ of $(2^{2^3}\tilde{z})^3 \in R$ in $C \subset \bar{\mathrm{CH}}(X_5)$ is not divisible by $2^{3\cdot 2^3-1}$. It follows by Theorem 4.1 that $\operatorname{ind}(X_5)$ divides 2^3. Since $\operatorname{ind}(X_6) = 2^4$, we conclude that $\operatorname{ind}(X_5) = 2^3$ (see §1).

If $\operatorname{ind}(X_3)$ would be at most 2^2, we could find a finite extension field L of the base field of degree not divisible by 2^3 such that the anisotropic part of q_L would have dimension at most 11. Then q_L splits completely over a finite field extension of degree dividing 2, a contradiction to $\operatorname{ind}(X_8) = 2^4$. It follows that $\operatorname{ind}(X_3) = 2^3$ implying $\operatorname{ind}(X_m) = 2^m$ for $m < 3$ as well (the latter being also confirmed by [2, Theorem 4.2] as well as by [1, Theorem 4.2]).

Summarizing, we get the whole list of indexes of $\operatorname{ind}(X_m)$ for Spin(17):

$$\operatorname{ind}(X_m) = 2^m \text{ for } m \leq 3, \ \boxed{\operatorname{ind}(X_m) = 2^3 \text{ for } m \in \{4, 5\}}, \text{ and } \operatorname{ind}(X_m) = 2^4 \text{ for } m \geq 6,$$

where the box marks the values which were not known before.

For more credibility, we provide further details on the computation with the Chow package in §A.

6. Spin(18)

Let q be a generic quadratic form of dimension 18 of trivial discriminant and Clifford invariant (given by a generic Spin(18)-torsor). The result of the previous section allows one to determine the index of mth orthogonal grassmannian X_m (i.e., the variety of totally isotropic m-planes) of q for all m.

Let q' be a 1-codimensional subform of q and let X'_m be the mth orthogonal grassmannian of q'. Then we have $\operatorname{ind}(X'_m) \geq \operatorname{ind}(X_m)$ for $m = 1, \ldots, 8$ and $\operatorname{ind}(X'_m)$ has the upper bound given by the index of §5. We also have

$$\operatorname{ind}(X_9) = \operatorname{ind}(X_8) = \operatorname{ind}(X_7) = \operatorname{ind}(X_6) = 2^4.$$

Besides, by the same argument as in the previous section, we have $\operatorname{ind}(X_3) = 2^3$, implying $\operatorname{ind}(X_m) = 2^m$ for $m \leq 3$.

Summarizing, we get the whole list of indexes of $\operatorname{ind}(X_m)$ for Spin(18):

$$\operatorname{ind}(X_m) = 2^m \text{ for } m \leq 3, \boxed{\operatorname{ind}(X_m) = 2^3 \text{ for } m \in \{4, 5\}}, \text{ and } \operatorname{ind}(X_m) = 2^4 \text{ for } m \geq 6.$$

7. Spin(19)

Here we start to work out the case of $n = 9$. First of all, we have $\operatorname{ind}(X_m) = 2^m$ for $m = 1, 2, 3$ by [2, Theorem 4.2] because the condition $2^{n-m} > \dim X_m$ of [2, Theorem 4.2] is satisfied for $m = 3$:

$$2^{n-m} = 2^6 = 64 > \dim X_m = m(m-1)/2 + m(2n - 2m + 1) = 42.$$

For $m = 4$, [2, Theorem 4.2] does not work anymore because

$$2^{n-m} = 2^5 = 32 \leq \dim X_4 = 50.$$

A computation with the Chow ring package (see §B) shows that the image $\tilde{c}$ of $2^{2^5}\tilde{z} \in R$ in $C \subset \bar{\mathrm{C}}\mathrm{H}(X_4)$ is not divisible by 2^{2^5} inside of C. (It is divisible by 2^{2^5-1} though.) It follows by Theorem 4.1 that $\boxed{\operatorname{ind}(X_4) = 2^3}$.

8. Spin(20)

We do not expect that knowledge of indexes for Spin($2n-1$) always allows one to determine the indexes for Spin($2n$). This happens with the highest orthogonal grassmannians by the very special reason that they are isomorphic to each other. It looks like coincidence that in §6 we were able to determine all indexes for Spin(18) using the information on Spin(17).

For Spin(20) and $\operatorname{ind}(X_4)$, the information on Spin(19) helps again.

First of all, $\operatorname{ind}(X_m) = 2^m$ for Spin(20) and $m = 1, 2, 3$ by [2, Theorem 7.2] because for $m = 3$ we have the inequality

$$2^{n-m-1} = 2^{10-3-1} = 64 > \dim X_m = m(m-1)/2 + 2m(n-m) = 45.$$

It follows that $\operatorname{ind}(X_4)$ is 2^3 or 2^4, but for precise determination, [2, Theorem 7.2] does not help anymore since

$$2^{n-m-1} = 2^5 = 32 \leq \dim X_m = 54$$

for $m = 4$. We are going to use the result of §7 instead.

Let q' be a 1-codimensional subform of q and let X'_4 be the 4th orthogonal grassmannian of q'. Since $\dim q' = 19$, we know from §7 that $\operatorname{ind}(X'_4) \leq 2^3$. Since $\operatorname{ind}(X'_4) \geq \operatorname{ind}(X_4)$, we conclude that $\boxed{\operatorname{ind}(X_4) = 2^3}$.

Appendix A. Programming Spin(17)

Everyone with access to Maple, can download the Chow package from

```
mathematik.uni-muenchen.de/~semenov/software/chowring5.txt
```

and verify the computation of §5. The algorithm used in the package is described in [4, §5].

Open a Maple Worksheet and load the package with

```
read("C:/Packages/chowring5.txt");
```

indicating your way to the package file. You should receive the message

```
Chow ring package v. 4.0 loaded
```

In Maple 2021, there will be a Warning on an implicitly local variable t, which can be ignored.[1]

Run the following definitions:

```
x1:=omega[8];           x2:=omega[7]-omega[8]; x3:=omega[6]-omega[7];
x4:=omega[5]-omega[6]; x5:=omega[4]-omega[5]; y1:=omega[3]-omega[4];
y2:=omega[2]-omega[3]; y3:=omega[1]-omega[2];
```

This defines our elements $x_1, \ldots, x_5, y_1, y_2, y_3$ in the ring $R = \mathbb{Z}[x_1, \ldots, x_5, y_1, y_2, y_3]$ of §2, which we view as the symmetric ring of the group of characters of the standard split maximal torus of the symplectic group Sp(16) (of type $\mathbf{C}_8$). The simple roots are numbered backwards in the Chow package and `omega`$[i]$ is the notation for the ith fundamental weight, used in the package.

Next step is the construction of the element $2^3\tilde{z} \in R$, denoted a here:

```
x:=x1+x2+x3+x4+x5;
a:=(x+y1+y2+y3)*(x-y1+y2+y3)*(x+y1-y2+y3)*(x+y1+y2-y3)*
    (x-y1-y2+y3)*(x-y1+y2-y3)*(x+y1-y2-y3)*(x-y1-y2-y3);
```

(The Maple Warning on multi-line expression can be ignored; to avoid it, put the definition of a in a single line.)

Now we compute the image $\tilde{c}$ of a in $\mathrm{CH}(Y_5)$. This is done with the procedure `c_func` of Chow package. The element $\tilde{c}$ is denoted just c for simplicity:

```
c:=c_func([1,2,3,5,6,7,8],C8,a);
```

[1] To get rid of the warning, it suffices to add t to the list of local variables in the second line of the definition of the procedure "fundam_invariant" in "chowring5.txt". (Thanks to Nikita Semenov for this information.)

The first argument $[1,2,3,5,6,7,8]$ of the procedure `c_func` indicates the parabolic subgroup we are interested in. (Recall that the simple roots are numbered backwards. In the usual numbering, our maximal parabolic subgroup is obtained by erasing the 5th root, not the 4th.) The second argument is the Dynkin type and the third argument can be any W-invariant element of the ring R. We take a for the third argument. Maple output, coming almost immediately, is:

```
c:=128Z[5, 4, 3, 2, 1, 2, 3, 4] + 128Z[4, 3, 2, 1, 2, 3, 5, 4]+
   128Z[3, 2, 1, 2, 4, 3, 5, 4] + 128Z[2, 1, 2, 4, 3, 6, 5, 4]+
   128Z[2, 1, 3, 2, 4, 3, 5, 4] + 128Z[1, 3, 2, 4, 3, 6, 5, 4]+
   128Z[1, 2, 5, 4, 3, 6, 5, 4] + 128Z[1, 2, 4, 3, 7, 6, 5, 4]
```

where `Z[...]` stand for certain Schubert classes in $\mathrm{CH}(Y_5)$ constituting its $\mathbb{Z}$-basis.

To simplify, we divide by 128

```
c:=c/128;
```

and compute the cube in $\mathrm{CH}(Y_5)$ of the result, using the procedure `chow_expand` of the Chow package:

```
c3:=chow_expand([1,2,3,5,6,7,8],C8,c^3);
```

In the end, we divide by 2 and reduce modulo 2:

```
c3/2 mod 2;
```

The output is

```
Z[2, 1, 2, 3, 2, 1, 2, 4, 3, 7, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 2, 4, 3, 2, 1, 2, 5, 4, 3, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 2, 3, 2, 1, 2, 5, 4, 3, 7, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 3, 2, 1, 4, 3, 2, 5, 4, 3, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 2, 1, 4, 3, 2, 5, 4, 3, 7, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 2, 1, 3, 2, 6, 5, 4, 3, 7, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 2, 3, 4, 6, 5, 4, 3, 2, 1, 2, 3, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4] +
Z[1, 2, 3, 5, 4, 3, 2, 1, 2, 3, 6, 5, 4, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4]
```

All computations run almost immediately on my small laptop with an exception of the last one, taking a bit longer.

Appendix B. Programming Spin(19)

Here is the input used in §7. We switch to notation $x_{m+1},\dots,x_n$ for $y_1,\dots,y_l$ so that the polynomial ring R is simply $\mathbb{Z}[x_1,\dots,x_n]$. We are computing $\mathrm{ind}(X_4)$ for Spin(19) so that we have $n=9$, $m=4$, and $l=n-m=5$.

We are defining $x_1,\dots,x_9$ in terms of the fundamental weights, next defining $a=2^{2^5}\tilde{z}\in R$ in terms of $x_1,\dots,x_9$, and finally computing $c=\tilde{c}\in C=\mathrm{CH}(Y_4)$:

```
x[1]:=omega[9];
for i from 2 to 9 do x[i]:=omega[10-i]-omega[11-i]  od;
```

```
a:=1:    for s5 from -1 by 2 to 1 do
         for s6 from -1 by 2 to 1 do
         for s7 from -1 by 2 to 1 do
         for s8 from -1 by 2 to 1 do
         for s9 from -1 by 2 to 1 do
a:=a*(x[1]+x[2]+x[3]+x[4]+s5*x[5]+s6*x[6]+s7*x[7]+s8*x[8]+s9*x[9])
od; od; od; od; od; a;

c:=c_func([1,2,3,4,5,7,8,9],C9,a);
```

The computation of the last line takes about 30 minutes.
Below is the value of `c/2^31 mod 2;`

```
Z[2, 1, 2, 3, 5, 4, 3, 2, 1, 2, 3, 6, 5, 4, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 8, 7, 6, 5, 9, 8, 7, 6] +
Z[1, 2, 4, 3, 5, 4, 3, 2, 1, 2, 3, 6, 5, 4, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 8, 7, 6, 5, 9, 8, 7, 6] +
Z[2, 1, 3, 2, 4, 3, 2, 1, 2, 5, 4, 3, 6, 5, 4, 3, 2, 1, 2, 3, 7, 6, 5, 4, 8, 7, 6, 5, 9, 8, 7, 6] +
Z[5, 4, 6, 5, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 5, 4, 7, 6, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[1, 2, 3, 4, 3, 2, 1, 2, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[2, 3, 5, 4, 3, 2, 1, 2, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[4, 3, 5, 4, 3, 2, 1, 2, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[2, 1, 2, 3, 2, 1, 2, 4, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[1, 2, 3, 2, 1, 2, 5, 4, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 9, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6] +
Z[4, 3, 5, 4, 6, 5, 4, 3, 2, 1, 2, 3, 4, 7, 6, 5, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 9, 8, 7, 6] +
Z[1, 2, 4, 3, 2, 1, 2, 5, 4, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 9, 8, 7, 6] +
Z[3, 2, 4, 3, 2, 1, 2, 5, 4, 3, 6, 5, 4, 7, 6, 5, 8, 7, 6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 9, 8, 7, 6]
```

References

[1] Chernousov, V., and Merkurjev, A. Essential dimension of spinor and Clifford groups. *Algebra Number Theory 8*, 2 (2014), 457–472.

[2] Devyatov, R. A., Karpenko, N. A., and Merkurjev, A. S. Maximal indexes of flag varieties for spin groups. *Forum of Mathematics, Sigma 9* (2021), e34.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[4] Garibaldi, S., Petrov, V., and Semenov, N. Shells of twisted flag varieties and the Rost invariant. *Duke Math. J. 165*, 2 (2016), 285–339.

[5] Guillot, P. The Chow rings of G_2 and Spin(7). *J. Reine Angew. Math. 604* (2007), 137–158.

[6] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.

[7] Karpenko, N. A. Around 16-dimensional quadratic forms in I_q^3. *Math. Z. 285*, 1-2 (2017), 433–444.

[8] Karpenko, N. A. Chow ring of generic flag varieties. *Math. Nachr. 290*, 16 (2017), 2641–2647.

[9] Karpenko, N. A. On generic quadratic forms. *Pacific J. Math. 297*, 2 (2018), 367–380.

[10] Karpenko, N. A., and Merkurjev, A. S. Canonical p-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.

[11] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.

[12] Rojas, L. A. M. The chow ring of the classifying space of Spin_8. Ph.D. Thesis (2006), 71 pages. Available at `matfis.uniroma3.it/Allegati/Dottorato/TESI/molina/TesiMolina.pdf`.

[13] Totaro, B. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, vol. 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[14] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: `karpenko@ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`