ON SPECIAL CLIFFORD GROUPS AND THEIR CHARACTERISTIC CLASSES

NIKITA A. KARPENKO

Abstract. We prove a conjecture on the Chow ring of characteristic classes for the special Clifford groups. This conjecture was the only obstacle for obtaining an algorithm computing the maximal indexes of twisted spin grassmannians.

Contents

1. Introduction

The index $i(X)$ of an algebraic variety X over a field F is defined as the g.c.d. of the degrees of its closed points. In slightly different terms, $i(X)$ is the g.c.d. of the degrees of finite field extensions of F over which X acquires a rational point. In particular, $i(X) = 1$ provided that X has a rational point already over F.

Let X be a projective homogeneous variety under a split reductive group G over a field F_0. For any field F, containing F_0, and any G-torsor E over F, twisting X over F by E in the sense of [6, Proposition 2.12], we get an F-variety X^E. Fixing F_0, G, X and varying F and E, one achieves the maximal value of $i(X^E)$ on any *generic* G-torsor E, e.g., the G-torsor given by the generic fiber of the quotient morphism $\mathrm{GL}(N) \to \mathrm{GL}(N)/G$ for an embedding of G into the general linear group $\mathrm{GL}(N)$ with some $N \geq 1$ (see [11, §6] for a proof). We are interested in computing this maximal value for arbitrary F_0 and for certain G and X. Note that in the cases, where (an algorithm for getting) the answer is

Date: 19 Jul 2023. Revised: 4 December 2023.

Key words and phrases. Affine algebraic groups; spin groups; Clifford groups; projective homogeneous varieties; classifying spaces; Chow rings. *Mathematical Subject Classification (2020):* 20G15; 14C25.

Author's work has been supported by a Discovery Grant from the National Science and Engineering Research Council of Canada and initiated during his stay at the Institut des Hautes Etudes Scientifiques.

available (e.g. [20, Theorems 0.1 and 3.2] , [19, Theorem 4.1], [4, Theorems 4.2 and 7.2], [10, Theorem 4.1], [9, Theorems 6.6. and 7.1]), it does not depend on F_0.

The split spin groups $G = \mathrm{Spin}(2n+1)$ and $G = \mathrm{Spin}(2n)$ have been studied in [4], [10], [12] in this respect. Since any G-torsor becomes trivial over some finite base field extension of a 2-power degree, the index $i(X^E)$ is always a 2-power for such G. Note that the case of an arbitrary X for such G is easily reduced to the case where X is a spin grassmannian given by a standard maximal parabolic subgroup of G. For the standard maximal parabolic subgroup corresponding to any of the last three vertices of the Dynkin diagram of G, $i(X^E)$ is the torsion index of G, computed in [20, Theorem 0.1].

Let us add some details concerning the last sentence. The Dynkin diagram of the split spin group $G = \mathrm{Spin}(2n+1)$ is of type B_n, its vertices are numbered as shown in [14, §24.A]. If $P \subset G$ is the standard maximal parabolic subgroup corresponding to mth vertex, the spin grassmannian $X = G/P$ is the variety of totally isotropic m-planes in the split $(2n+1)$-dimensional quadratic form q,

$$q(a_1, b_1, \dots, a_n, b_n, c) = a_1b_1 + \dots + a_nb_n + c^2,$$

and P is the stabilizer of the rational point

$$a_{m+1} = \dots = a_n = b_1 = \dots = b_n = c = 0$$

on X. For any G-torsor E, the variety X^E is then the variety of totally isotropic m-planes in the generic $(2n+1)$-dimensional quadratic form q^E of trivial discriminant and Clifford invariant, given by E. The variety X^E has a rational point if and only if the Witt index of q^E is at least m. For $m \geq n-2$ this means that the anisotropic part of q^E has dimension at most 5 and therefore – due to the vanishing of its Clifford invariant – exactly 1. In other terms, the G-torsor E is trivial. This shows that $i(X^E)$ for $m \geq n-2$ and generic E is the index of E itself, i.e., the torsion index of G.

The Dynkin diagram of the split spin group $G = \mathrm{Spin}(2n)$ is of type D_n, its vertices are numbered as shown in [14, §24.A]. If $P \subset G$ is the standard maximal parabolic subgroup corresponding to mth vertex, the spin grassmannian $X = G/P$ is the variety of totally isotropic m-planes in the split $(2n)$-dimensional quadratic form q,

$$q(a_1, b_1, \dots, a_n, b_n) = a_1b_1 + \dots + a_nb_n,$$

provided that $m \leq n-2$, whereas what we get for $m = n-1$ and $m = n$ are the two (isomorphic to each other) connected components of the variety of totally isotropic n-planes. In the case of $m \leq n-2$, P is the stabilizer of the rational point

$$a_{m+1} = \dots = a_n = b_1 = \dots = b_n = 0$$

on X. Otherwise, P is the stabilizer of one of the two rational points

$$b_1 = \dots = b_n = 0, \quad b_1 = \dots = b_{n-1} = a_n = 0$$

on the nth grassmannian depending on which of them lies on X. For any G-torsor E, the twist X^E is then the analogues variety for the generic $(2n)$-dimensional quadratic form q^E of trivial discriminant and Clifford invariant, given by E. The variety X^E has a rational point if and only if the Witt index of q^E is at least the dimension of the planes. For $m \geq n-3$ this means that the anisotropic part of q^E has dimension at most 6 and therefore – due to the vanishing of its discriminant and Clifford invariant – just 0. In other terms, the G-torsor E is trivial. This shows that $i(X^E)$ for $m \geq n-3$ and generic E is the index of E itself, i.e., the torsion index of G.

For $G = \mathrm{Spin}(2n+1)$, an algorithm computing $i(X^E)$ for generic E has been obtained in [10, Theorem 4.1]. For $G = \mathrm{Spin}(2n)$ and $X = G/P$, where P is a maximal parabolic subgroup corresponding to mth vertex of Dynkin diagram with $m \leq n-2$, the same approach delivers a similar algorithm computing $i(X^E)$ provided Conjecture 1.2, formulated below, is positively solved for the split special Clifford group $\Gamma^+(2(n-m))$ (instead of $\Gamma^+(2n)$ in terms of which the conjecture will be formulated, see §5 for details).

To formulate Conjecture 1.2, let G be the split special Clifford group $\Gamma^+(2n)$ defined, e.g., in [14, §23.A] (under the name of an *even* Clifford group). This is a split reductive group with the semisimple part $\mathrm{Spin}(2n) \subset G$ and with the quotient $G/\mathrm{Spin}(2n)$ isomorphic to $\mathbb{G}_{\mathrm{m}}$. Moreover,

$$G = (\mathbb{G}_{\mathrm{m}} \times \mathrm{Spin}(2n))/\mu_2, \tag{1.1}$$

where μ_2 is embedded diagonally into the product of $\mathbb{G}_{\mathrm{m}}$ by the center of $\mathrm{Spin}(2n)$.

To complete the introduction of G, let us mention that the G-torsors are exactly the $2n$-dimensional quadratic forms with trivial discriminant and Clifford invariant. Such quadratic forms also arise from $\mathrm{Spin}(2n)$-torsors; however, non-isomorphic torsors here may give isomorphic quadratic forms. Let us also note that the groups G and $\mathrm{Spin}(2n)$ have the same torsion index.

For the standard split maximal torus $T \subset G$, let us consider the induced by the embedding homomorphism

$$\Phi\colon \mathrm{CH}(\mathsf{B}G) \to \mathrm{CH}(\mathsf{B}T)$$

of the Chow rings of the classifying spaces $\mathsf{B}G$ and $\mathsf{B}T$, defined in [21, §2.2]. In other terms, these are the rings of Chow characteristic classes for the corresponding groups – see [21, Theorem 2.8]. The destination ring $\mathrm{CH}(\mathsf{B}T)$ of Φ is known to be the symmetric $\mathbb{Z}$-algebra on the character group of T and as such can be identified with the polynomial ring $\mathbb{Z}[z, x_1, \ldots, x_n]$ in $n+1$ variables. The second projection from (1.1) to the special orthogonal group $\mathrm{O}^+(2n) = \mathrm{Spin}(2n)/\mu_2$ (called in [14] *the vector representation* of G), maps T to the standard maximal split torus of $\mathrm{O}^+(2n)$, and $x_1, \ldots, x_n$ correspond to its standard characters. Under the first projection $G \to \mathbb{G}_{\mathrm{m}}/\mu_2 = \mathbb{G}_{\mathrm{m}}$, the standard (tautological) character of $\mathbb{G}_{\mathrm{m}}$ corresponds to $2z - x_1 - \cdots - x_n$.

The elements in the image of Φ are invariant under the action of the Weyl group W of G with respect to T. To describe the group and its action, let $\tilde{W}$ be the subgroup in the group $\mathrm{Aut}\,\mathrm{CH}(\mathsf{B}T)$ of ring automorphisms of $\mathrm{CH}(\mathsf{B}T)$, generated by the permutations of $x_1, \ldots, x_n$ along with the automorphisms $\sigma_1, \ldots, \sigma_n$, where σ_i changes the sign of x_i and maps z to $z - x_i$. Then W is the subgroup in $\tilde{W}$, generated by the permutations of $x_1, \ldots, x_n$ along with the products of $\sigma_1, \ldots, \sigma_n$ having an even number of factors.

For the entire ring $\mathrm{CH}(\mathsf{B}T)^W$ of W-invariants, certain generators were found in [12]. One of them is the product

(e) $e := x_1 \ldots x_n$, called the *Euler class*.

The remaining generators are:

(p) the elementary symmetric polynomials $p_1, \ldots, p_n$ in $x_1^2, \ldots, x_n^2$ (called the *Pontryagin classes*);

(f) certain homogeneous elements $f_0, f_1, \dots, f_{n-2}$ with f_i of degree 2^i, constructed inductively as in [12, §4] starting with $f_0 := 2z - x_1 - \cdots - x_n$;

(t) the product

$$t := \prod_{\text{even } I \subset \{1,\dots,n\}} (z - \sum_{i \in I} x_i)$$

of the elements in the orbit of z, where the product is taken over all subsets I of $\{1, \dots, n\}$ with an even number of elements.

Conjecture 1.2. *For $n \geq 6$, the image of Φ is contained in the subring of $\mathrm{CH}(BT)^W$ generated by $2e$ along with generators* (p), (f), (t).

One difficulty of Conjecture 1.2 is that its statement is false for $n < 6$ (see the proof of Theorem 5.3). Therefore, proving it by induction on n, we cannot start with a group smaller than $\Gamma^+(12)$.

Here we prove Conjecture 1.2 (see Theorem 4.9) and, as the main application, describe the missing algorithm for the even spin groups (see Theorem 5.3).

It is not so difficult to reduce the proof of Conjecture 1.2 to the case of $n = 6$. This is done in §4. The hardest step is to prove the conjecture for $n = 6$, where in the end a much stronger Theorem 3.16 (in the spirit of [7, Theorem 1.1]) is obtained, providing a complete computation of the image of Φ. The proof of Theorem 3.16, given in §3, is based on the preparation work of §2.

For $n > 6$, we do not determine the image of Φ. It may be possible to do this by similar (though more complicated) computations with Steenrod operations, possibly using the recent result of [18] on the Nisnevich classifying spaces of the special Clifford groups.

2. A formal computation

Over the field $\mathbb{F} := \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ of 2 elements, let us consider the polynomial algebra $\mathbb{F}[z, x_1, \dots, x_n]$ in the $n+1$ variables and write

$$c_1, \dots, c_n \in \mathbb{F}[x_1, \dots, x_n] \subset \mathbb{F}[z, x_1, \dots, x_n]$$

for the elementary symmetric polynomials in $x_1, \dots, x_n$. The (formal) total Steenrod operation

$$\mathrm{St}\colon \mathbb{F}[z, x_1, \dots, x_n] \to \mathbb{F}[z, x_1, \dots, x_n]$$

is the (non-graded) ring endomorphism, satisfying $p \mapsto p + p^2$ for every degree 1 homogenous polynomial p. For every $i \geq 0$, the ith Steenrod operation St^i is the additive endomorphism given by the homogeneous component of St raising by i the degree of every homogeneous polynomial. In particular, St^0 is the identity. Since the total Steenrod operation is a ring homomorphism, its components St^i all together satisfy the Cartan formulas $\mathrm{St}^k(ab) = \sum_{i+j=k} \mathrm{St}^i(a)\,\mathrm{St}^j(b)$ with $k \geq 0$.

The subring $\mathbb{F}[z, c_1, \dots, c_n] \subset \mathbb{F}[z, x_1, \dots, x_n]$ is stable under the Steenrod operations, which can be defined directly on this subring by the simple classical formula as in [7, §4] (based on the formula stated for the first time in [23], proved in [2, Théorème 7.1], and simplified in [17, Proposition 3.1.12]), expressing $\mathrm{St}^i(c_j)$ for every $i, j > 0$ as a polynomial in $c_1, \dots, c_n$ over $\mathbb{F}$. This formula is convenient for the actual computations performed below (see Appendix A for the corresponding Maple code). Note that the ring $\mathbb{F}[z, c_1, \dots, c_n]$

is also a polynomial ring in $n+1$ variables; the variables are weighted however: z has degree 1 and c_i for $i = 1, \dots, n$ has degree i.

The ideal in $\mathbb{F}[z, c_1, \dots, c_n]$, generated by c_1, is stable under the Steenrod operations, which are therefore defined on the quotient. This quotient is a polynomial ring in n variables $z, c_2, \dots, c_n$. The mentioned above formulas for the Steenrod operations are adapted by removing the numerous monomials containing c_1 and become considerably simpler.

Assume that we are given a homogeneous element $y \in \mathbb{F}[z, c_2, \dots, c_n]$ satisfying

$$\mathrm{St}(y) \in \mathbb{F}[y, c_2, \dots, c_n].$$

Then the subring $\mathbb{F}[y, c_2, \dots, c_n] \subset \mathbb{F}[z, c_2, \dots, c_n]$ is stable under the Steenrod operations. If $y \notin \mathbb{F}[c_2, \dots, c_n]$, then $\mathbb{F}[y, c_2, \dots, c_n]$ is a polynomial ring in n variables $y, c_2, \dots, c_n$. Otherwise it coincides with $\mathbb{F}[c_2, \dots, c_n]$. Both situations are allowed in this section but the first situation will actually occur in the application of §3.

We are going to work with the subrings

$$B := \mathbb{F}[y^2, c_2^2, \dots, c_n^2] \subset A := \mathbb{F}[y^2, c_2, \dots, c_n] \subset \mathbb{F}[y, c_2, \dots, c_n]$$

which are also stable under the Steenrod operations. Note that A is a free B-module with the basis

$$\{c_I\}_{I \subset \{2, \dots, n\}}, \quad \text{where } c_I := \prod_{i \in I} c_i. \tag{2.1}$$

The Cartan formula for St^1 is the Leibniz rule $\mathrm{St}^1(aa') = \mathrm{St}^1(a)a' + a\,\mathrm{St}^1(a')$. It follows that $\mathrm{St}^1 \colon A \to A$ vanishes on $B \subset A$ and is a homomorphism of B-modules. In particular, its kernel is a B-submodule of A.

The following lemma is actually already implicitly proved in [7, §4]. The proof given here for completeness is more self-contained.

Lemma 2.2. *The kernel of* $\mathrm{St}^1 \colon A \to A$ *is the B-submodule of A generated by* 1, c_n, *and* $\mathrm{St}^1(c_I)$ *for all* $I \subset \{2, \dots, n\}$.

Proof. In the case with $y \notin \mathbb{F}[c_2, \dots, c_n]$, elements of A are polynomials in y^2 over the ring $\mathbb{F}[c_2, \dots, c_n]$ and St^1 acts on them coefficient-wise. Therefore it is enough to consider the case with $y \in \mathbb{F}[c_2, \dots, c_n]$, i.e., the case with $A = \mathbb{F}[c_2, \dots, c_n]$.

The composition of St^1 with itself vanishes, i.e., St^1 is a differential. This is easiest to see on the ring $\mathbb{F}[x_1, \dots, x_n] \supset A$, generated by degree 1 homogeneous elements p for which we have $\mathrm{St}^1(\mathrm{St}^1(p)) = \mathrm{St}^1(p^2) = 0$. Since $\mathrm{St}^1(\mathrm{St}^1(ab)) = \mathrm{St}^1(\mathrm{St}^1(a))b + a\,\mathrm{St}^1(\mathrm{St}^1(b))$ and the products of various p generate A additively, we get the statement.

It follows that the B-submodule of A, indicated in Lemma 2.2, is contained in the kernel of St^1. Note that this kernel is a ring and the image of St^1 is an ideal of this ring and so, the quotient $\operatorname{Ker}\mathrm{St}^1 / \operatorname{Im}\mathrm{St}^1$ is a ring as well, which can be called the St^1-cohomology ring. For the opposite inclusion in Lemma 2.2, it suffices to show that the homomorphism of $B[c_n]$ to the St^1-cohomology ring is onto.

To compute the quotient ring, we proceed as in [1, §9]. Since $\mathrm{St}^1(c_i)$ equals c_{i+1} for even $i < n$ and vanishes otherwise, the differential ring A decomposes, depending on the parity of n, either in the tensor product $\mathbb{F}[c_2, c_3] \otimes \cdots \otimes \mathbb{F}[c_{n-1}, c_n]$ or in the tensor product $\mathbb{F}[c_2, c_3] \otimes \cdots \otimes \mathbb{F}[c_{n-2}, c_{n-1}] \otimes \mathbb{F}[c_n]$. Let us describe the cohomology rings of the factors.

Since St^1 vanishes on $\mathbb{F}[c_n]$, the corresponding cohomology ring is the entire $\mathbb{F}[c_n]$. For even $i < n$, writing an element of $\mathbb{F}[c_i, c_{i+1}]$ as

$$a + b \cdot c_i + c \cdot c_{i+1} + d \cdot c_i c_{i+1}$$

with $a, b, c, d \in B_i := \mathbb{F}[c_i^2, c_{i+1}^2]$ and imposing vanishing of St^1 on it, we get vanishing of $b \cdot c_{i+1} + dc_{i+1}^2$, implying that $b = 0 = d$. Since $c_{i+1} = \mathrm{St}^1(c_i)$, the St^1-cohomology ring for $\mathbb{F}[c_i, c_{i+1}]$ coincides with the image of B_i. It follows by the Künneth Tensor Formula [15, Theorem 10.1 of Chapter 5] (applied to the differential $\mathbb{Z}$-graded $\mathbb{F}$-algebra A_* with $A_i := A$ for every $i \in \mathbb{Z}$) that the St^1-cohomology ring for A is the image of

$$(B_2 \otimes B_4 \otimes \dots)[c_n] = B[c_n]$$

(or even of just B). $\square$

Below in this section, we work with $n = 6$ only. The following proposition is actually already implicitly proved in [7, §5]. The proof given here for completeness is slightly different.

Proposition 2.3 (**The Formal Computation**). *For $n = 6$ and a graded B-submodule M of A, concentrated in even degrees, the following holds: if M is stable under all Steenrod operations, then $M \subset B$; if M is stable under St^i for $i < 8$, then M is contained in the B-submodule generated by 1 and $c_5^2 \cdot c_6 + c_3^2 \cdot c_4 c_6 + c_2 c_3 c_5 c_6$.*

Proof. To prove the second statement, let us assume that M is stable under St^i for $i < 8$. Since M is concentrated in even degrees, it vanishes under St^1. By Lemma 2.2, M is therefore contained in the B-submodule generated by 1, c_6, and all $\mathrm{St}^1(c_I)$ with $I \subset \{2, 3, 4, 5, 6\}$ such that c_I is of odd degree. There are 16 elements c_I of odd degree: the 8 square-free products of c_2, c_4, c_6 multiplied by c_3 and the same 8 products multiplied by c_5. Since

$$\begin{aligned} \mathrm{St}^1(c_3 c_2) &= c_3^2 \in B, & \mathrm{St}^1(c_3 c_4) &= c_3 c_5, \\ \mathrm{St}^1(c_3 c_2 c_4) &= c_3^2 \cdot c_4 + c_2 c_3 c_5, & \mathrm{St}^1(c_5 c_2) &= c_3 c_5, \\ \mathrm{St}^1(c_5 c_4) &= c_5^2 \in B, & \mathrm{St}^1(c_5 c_2 c_4) &= c_5^2 \cdot c_2 + c_3 c_4 c_5, \end{aligned}$$

and $\mathrm{St}^1(c_6) = 0$, any element of M has the form

$$\begin{aligned} (2.4) \quad & a \cdot 1 + b \cdot c_3 c_5 + c \cdot (c_3^2 \cdot c_4 + c_2 c_3 c_5) + d \cdot (c_5^2 \cdot c_2 + c_3 c_4 c_5) + \\ & \big(a_* + b_* \cdot c_3 c_5 + c_* \cdot (c_3^2 \cdot c_4 + c_2 c_3 c_5) + d_* \cdot (c_5^2 \cdot c_2 + c_3 c_4 c_5)\big) c_6 \end{aligned}$$

with $a, b, c, d, a_*, b_*, c_*, d_* \in B$.

We will not obtain any further restriction on M applying just the operation St^1: any linear combination of the form (2.4) vanishes under it. But from the assumption that M is stable under St^2 as well, it follows that

$$\begin{aligned} (2.5) \quad & a \,\mathrm{St}^2(1) + b \,\mathrm{St}^2(c_3 c_5) + c \,\mathrm{St}^2(c_3^2 \cdot c_4 + c_2 c_3 c_5) + d \,\mathrm{St}^2(c_5^2 \cdot c_2 + c_3 c_4 c_5) + \\ & a_* \,\mathrm{St}^2(c_6) + b_* \,\mathrm{St}^2(c_3 c_5 c_6) + c_* \,\mathrm{St}^2(c_3^2 \cdot c_4 c_6 + c_2 c_3 c_5 c_6) + d_* \,\mathrm{St}^2(c_5^2 \cdot c_2 c_6 + c_3 c_4 c_5 c_6) \end{aligned}$$

has the shape of (2.4) if $\alpha =$(2.4).

Since

$$\begin{gathered}
\mathrm{St}^2(1)=0, \qquad \mathrm{St}^2(c_3c_5)=c_5^2,\\
\mathrm{St}^2(c_3^2\cdot c_4+c_2c_3c_5)=c_2^2\cdot c_3c_5+c_3^2\cdot c_2c_4+c_5^2\cdot c_2+c_3^2\cdot c_6,\\
\mathrm{St}^2(c_5^2\cdot c_2+c_3c_4c_5)=(c_2c_5)^2+c_2c_3c_4c_5+c_5^2\cdot c_4+c_3c_5c_6,\\
\mathrm{St}^2(c_6)=c_2c_6, \qquad \mathrm{St}^2(c_3c_5c_6)=c_2c_3c_5c_6+c_5^2\cdot c_6,\\
\mathrm{St}^2(c_3^2\cdot c_4c_6+c_2c_3c_5c_6)=c_5^2\cdot c_2c_6+(c_3c_6)^2,\\
\mathrm{St}^2(c_5^2\cdot c_2c_6+c_3c_4c_5c_6)=c_6^2\cdot c_3c_5+c_5^2\cdot c_4c_6,
\end{gathered}$$

the coefficient at c_2c_4 in (2.5) is cc_3^2 implying that $c=0$. The coefficient at $c_2c_3c_4c_5$ is d so that $d=0$ as well. The coefficient at $c_3c_4c_5c_6$ is 0 and the coefficient at c_2c_6 is $a_*+c_*c_5^2$; therefore $a_*=c_*c_5^2$. Finally, the coefficients at $c_2c_3c_5c_6$ and at c_4c_6 are b_* and $d_*c_5^2$. Consequently, $d_*c_5^2=b_*c_3^2$ from where we conclude that b_* is divisible by c_5^2, say, $b_*=xc_5^2$ for some $x\in B$, and $d_*=xc_3^2$.

Now we know that any element of M has the form

$$a\cdot 1+b\cdot c_3c_5+c(c_5^2\cdot c_6+c_3^2\cdot c_4c_6+c_2c_3c_5c_6)+ d(c_5^2\cdot c_3c_5c_6+(c_3c_5)^2\cdot c_2c_6+c_3^2\cdot c_3c_4c_5c_6) \tag{2.6}$$

with $a,b,c,d\in B$, where we changed the notation of the coefficients c_* and x to c and d.

At this point, the operation St^2 cannot provide any additional restriction on M: the value of St^2 at any linear combination of the form (2.6) is again a linear combination like that. Since $\mathrm{St}^3=\mathrm{St}^1\circ\mathrm{St}^2$ as a particular case of the general formula $\mathrm{St}^{2i+1}=\mathrm{St}^1\circ\mathrm{St}^{2i}$ for any $i\geq 0$ (cf. [22, Lemma 9.6]), St^3 is not of help either. Let us proceed to St^4 and use the fact that for $\alpha\in M$, written in the form (2.6), $\mathrm{St}^4(\alpha)$ should also be of the shape (2.6). It follows that

$$b\cdot\mathrm{St}^4(c_3c_5)+c\,\mathrm{St}^4(c_5^2\cdot c_6+c_3^2\cdot c_4c_6+c_2c_3c_5c_6)+ d\,\mathrm{St}^4(c_5^2\cdot c_3c_5c_6+(c_3c_5)^2\cdot c_2c_6+c_3^2\cdot c_3c_4c_5c_6) \tag{2.7}$$

is of the shape (2.6). Since

$$\begin{gathered}
\mathrm{St}^4(c_3c_5)=c_2^2\cdot c_3c_5+c_5^2\cdot c_2+c_3^2\cdot c_6+c_3c_4c_5,\\
\mathrm{St}^4(c_5^2\cdot c_6+c_3^2\cdot c_4c_6+c_2c_3c_5c_6)=0,\\
\mathrm{St}^4(c_5^2\cdot c_3c_5c_6+(c_3c_5)^2\cdot c_2c_6+c_3^2\cdot c_3c_4c_5c_6)=\\
(c_2c_3c_5)^2\cdot c_2c_6+(c_2c_3)^2\cdot c_3c_4c_5c_6+(c_3c_6)^2\cdot c_2c_3c_5+\\
c_3^4c_6^2\cdot c_4+(c_3c_4)^2\cdot c_3c_5c_6+c_5^4\cdot c_2c_6+c_5^2\cdot c_3c_4c_5c_6,
\end{gathered}$$

the coefficient at c_2 in (2.7) is bc_5^2 implying that $b=0$. The coefficient at the basic element $c_2c_3c_5$ is $d(c_3c_6)^2$ implying that $d=0$.

Now we know that every element of M has the simple form

$$a+c(c_5^2\cdot c_6+c_3^2\cdot c_4c_6+c_2c_3c_5c_6) \tag{2.8}$$

with $a,c\in B$, cf. [7, Formula 5.5]. This finishes the proof of the second part of Proposition 2.3.

To prove the first part, from now on, we are assuming that M is stable under all Steenrod operations. It turns out that for any $a, c \in B$ and any positive $i < 8$, the value of St^i at the element (2.8) is in B. So, we have to proceed with a higher Steenrod operation. Applying St^8 to (2.8) and taking into account the formula

$$\mathrm{St}^8(c_5^2 \cdot c_6 + c_3^2 \cdot c_4c_6 + c_2c_3c_5c_6) = (c_2^2 + c_4)^2 \cdot c_2c_3c_5c_6 + ((c_2^2 + c_4)c_3)^2 \cdot c_4c_6 + \\ ((c_2^2 + c_4)c_5)^2 \cdot c_6 + (c_2c_6)^2 \cdot c_3c_5 + (c_3c_5)^2 \cdot c_2c_6 + (c_3^2c_6)^2 + c_3^2 \cdot c_3c_4c_5c_6 + c_5^2 \cdot c_3c_5c_6$$

with, for instance, nonzero coefficient at c_3c_5, we conclude that $c = 0$. Thus $M \subset B$ as claimed. □

3. The case of $\Gamma^+(12)$

In this section, we prove Theorem 3.16 on the standard split special Clifford group $G = \Gamma^+(12)$, implying Conjecture 1.2 for $n = 6$. For the sake of the next section, in the beginning here we work with arbitrary n and with $G = \Gamma^+(2n)$.

Recall that in §1 we identified the Chow ring $\mathrm{CH}(BT)$ of the standard split maximal torus $T \subset G$ with the polynomial ring $\mathbb{Z}[z, x_1, \ldots, x_n]$. So, the polynomial ring $\mathbb{F}[z, x_1, \ldots, x_n]$ is identified with the modulo 2 Chow ring

$$\mathrm{Ch}(BT) := \mathrm{CH}(BT)/2\,\mathrm{CH}(BT).$$

We are investigating the image of the homomorphism

$$\varphi\colon \mathrm{Ch}(BG) \to \mathrm{Ch}(BT) \ –$$

the modulo 2 reduction of Φ.

The Steenrod operations on the modulo 2 Chow groups, constructed in [3] and [16] for smooth quasi-projective varieties, are also defined for the classifying spaces of affine algebraic groups (not necessarily smooth) via their approximations by smooth varieties (see [21, §2.2]). On the polynomial ring $\mathrm{Ch}(BT)$ they coincide with the formal Steenrod operations considered in the previous section. They yield for every $i \geq 0$ a commutative square

$$\begin{array}{ccc} \mathrm{Ch}(BG) & \xrightarrow{\varphi} & \mathrm{Ch}(BT) \\ \downarrow{\scriptstyle \mathrm{St}^i} & & \downarrow{\scriptstyle \mathrm{St}^i} \\ \mathrm{Ch}(BG) & \xrightarrow{\varphi} & \mathrm{Ch}(BT) \end{array} \tag{3.1}$$

from which it follows that the image of φ is stable under St^i.

The Weyl group W acts on $\mathrm{CH}(BT)$ and on $\mathrm{Ch}(BT)$. Let us write $t \in \mathrm{Ch}(BT)$ as well for the modulo 2 reduction of the orbit product $t \in \mathrm{CH}(BT)$ from §1. The polynomial ring

$$\hat{A} := \mathbb{F}[t, c_1, \ldots, c_n]$$

coincides with the W-invariants of $\mathrm{Ch}(BT)$ (cf. [4, Lemma 3.2]). Therefore $\hat{A}$ is stable under the Steenrod operations and $\mathrm{Im}\,\varphi$ is a subring in $\hat{A}$. Note that for even n (e.g., for $n = 6$), the image of t in the quotient

$$A = \hat{A}/c_1\hat{A} = \mathbb{F}[t, c_2, \ldots, c_n] \subset \mathbb{F}[z, c_2, \ldots, c_n]$$

equals y^2, where

$$y = \prod_{\text{even } I \subset \{2,\dots,n\}} (z - \sum_{i\in I} x_i) \in \mathbb{F}[z, c_2, \dots, c_n].$$

The element y satisfies $\mathrm{St}(y) \in \mathbb{F}[y, c_2, \dots, c_n]$. This can be verified directly or by identifying the ring $\mathbb{F}[z, x_1, \dots, x_n]/(x_1+\dots+x_n)$ with the modulo 2 Chow ring of the standard split maximal torus of $\mathrm{Spin}(2n)$ and noticing that $\mathbb{F}[y, c_2, \dots, c_n]$ is its subring of Weyl invariants (see [7, Proposition 3.2]). With this information in hand, we will be in position to apply Proposition 2.3 from the previous section.

Returning to arbitrary (not necessarily even) n, one sees that the elements $t, c_1, c_2^2, \dots, c_n^2$ are in the image of φ, because they are the images under φ of the Chern classes of certain G-representations. Namely, the element t comes from the highest Chern class of a half-spin G-representation; $c_2^2, \dots, c_n^2$ come from Chern classes of the orthogonal G-representation – the composition of the vector representation with the standard representation of the special orthogonal group; finally, c_1 comes from the G-representation $G \to \mathbb{G}_{\mathrm{m}}$ mentioned in §1. It follows that

$$\mathrm{Im}\,\varphi \supset \hat{B} := \mathbb{F}[t, c_1, c_2^2, \dots, c_n^2].$$

Note that $\hat{A}$ as a $\hat{B}$-module is free with the basis (2.1).

As we already know, the ring $\mathrm{Im}\,\varphi$ is contained in a smaller than $\hat{A}$ ring $\hat{R} \subset \hat{A}$, which we define as the image in $\hat{A} = \mathrm{Ch}(BT)^W$ of the integral W-invariants $\mathrm{CH}(BT)^W$. The computation of the integral W-invariants, made in [12] and formulated here in §1, tells us that $\hat{R}$ as a $\hat{B}$-algebra is generated by the elements $f_1, \dots, f_{n-2}$ and the element c_n. Indeed, the modulo 2 reduction of $e, f_0 \in \mathrm{CH}(BT)^W$ are c_n and c_1. Note that the generators of the ring $\hat{R}$ (including the generators of $\hat{B}$) are homogeneous. Besides, for even n (e.g., for $n = 6$), all of them but c_1 are of even degrees.

The inductive definition of the integral elements $f_i \in \mathrm{CH}(BT)^W$, $i \ge 0$, given in [12, §4], implies that their modulo 2 reductions will not change if the definition is modified as follows:

$$f_0 := c_1 = x_1 + \dots + x_n \in \mathbb{Z}[x_1, \dots, x_n] \ \text{ and } \ f_{i+1} := (f_i^2 - f_i')/2 \in \mathbb{Z}[x_1, \dots, x_n],$$

where $f_i' = f_i'(x_1, \dots, x_n)$ is the polynomial $f_i(x_1^2, \dots, x_n^2)$. Note that the variable z, involved in the old definition, does not intervene in the new one.

Below we work with the modulo 2 reductions of f_i and, abusing notation, denote them the same way. One has

$$f_1 = c_2 \ \text{ and } \ f_2 = c_1c_3 - c_4 \in \mathbb{F}[x_1, \dots, x_n], \tag{3.2}$$

where with the new definition of f_i both formulas also hold integrally. The formulas for f_i with $i \ge 3$ are much more complicated.

Let us refer as Steenrod operation to any element of the $\mathbb{F}$-algebra generated by all St^i, $i \ge 0$. Let $R \subset \hat{R}$ be the subset of the elements whose image under every Steenrod operation is also in $\hat{R}$. It is easy to check that R is a subring in $\hat{R}$. Since $\mathrm{Im}\,\varphi$ is Steenrod stable, R contains $\mathrm{Im}\,\varphi$ as a subring.

Summarizing, we have defined a chain of subrings we are going to work with:

$$\boxed{\hat{B} \subset \mathrm{Im}\,\varphi \subset R \subset \hat{R} \subset \hat{A}}\,.$$

It is time to switch to the case $n = 6$. Our plan is to show that $R = \hat{B}$ and this will imply that $\operatorname{Im}\varphi = \hat{B}$.

We need the following formulas for the generators $f_3, f_4 \in \hat{A}$ of the $\hat{B}$-algebra $\hat{R}$ (see Appendix B for the corresponding Maple code):

$$f_3 \equiv c_1c_3c_4 + c_2c_3^2 + c_2c_6 + c_3c_5 + c_4^2, \quad \text{and}$$

$$\begin{aligned} f_4 \equiv c_1c_2c_3^3c_4 + c_1c_3^5 + c_1c_2c_3c_4c_6 + c_1c_3^2c_4c_5 + c_1c_3c_4^3 + c_1c_3c_6^2 + c_1c_5^3+ \\ c_2^4c_4^2 + c_2^2c_3^2c_6 + c_2c_3^3c_5 + c_3^4c_4 + c_2^2c_6^2 + c_2c_3c_5c_6+ \\ c_2c_4^2c_6 + c_2c_4c_5^2 + c_3^2c_4c_6 + c_3^2c_5^2 + c_3c_4^2c_5 + c_4c_6^2 + c_5^2c_6, \end{aligned}$$

where the congruences are modulo $c_1^2\hat{A}$. The formulas for the actual values of f_3 and especially of f_4 are more complicated, but fortunately we only need their reductions. Since we are only interested in describing the $\hat{B}$-algebra $\hat{R}$, generated by f_1, f_2, f_3, f_4 and c_6, we can do further simplifications.

Subtracting from f_3 the polynomial $c_2c_3^2 + c_2c_6 + c_4^2$ in $f_1 = c_2$ and c_6 over $\hat{B}$, we replace the old f_3 by the new – simpler one:

$$f_3 = c_1c_3c_4 + c_3c_5. \tag{3.3}$$

To simplify f_4, let us note that the image of f_4 in $A = \hat{A}/c_1\hat{A}$ is a polynomial in (the images of) f_1, f_2, f_3, c_6 over

$$B := \hat{B}/c_1\hat{B} = \mathbb{F}[t = y^2, c_2^2, \ldots, c_6^2].$$

So, subtracting from $f_4 \in \hat{A}$ a polynomial in f_1, f_2, f_3, c_6 over $\hat{B}$, we can make the new f_4 divisible by c_1: $f_4 = c_1 g$, where the image of $g \in \hat{A}$ in A is

$$c_3^2 \cdot (c_3c_6 + c_4c_5) + c_5^2 \cdot (c_2c_3 + c_5) \in A. \tag{3.4}$$

We now proceed to a further analysis of the rings $\hat{R}$ and R. Let $\check{R} \subset \hat{R}$ be the subring given by the direct sum of the homogeneous components of $\hat{R}$ of even degrees. Clearly, $\hat{R} = \check{R} \oplus c_1\check{R}$; in other terms, $c_1\check{R}$ is the direct sum of the homogeneous components of $\hat{R}$ of odd degrees. The ring $\check{R}$ is generated by the generators of $\hat{R}$ with c_1 replaced by c_1^2. Similarly, let $\check{B} \subset \hat{B}$ be the subring given by the direct sum of the homogeneous components of $\hat{B}$ of even degrees. Explicitly, $\check{B} = \mathbb{F}[t, c_1^2, c_2^2, \ldots, c_6^2]$.

Lemma 3.5. *$\check{R} \cap c_1^2\hat{A} = c_1^2\check{R}$.*

Remark 3.6. The equality of Lemma 3.5 with $\hat{R}$ in place of $\check{R}$ is false: the product c_1f_4 belongs to $\hat{R} \cap c_1^2\hat{A}$ and does not belong to $c_1^2\hat{R}$.

Proof of Lemma 3.5. We only need to prove the inclusion $\subset$ and so we take some $\alpha \in \check{R} \cap c_1^2\hat{A}$. Since $\alpha \in \check{R}$, it is a linear combination over $\check{B}$ of the square-free products of f_1, f_2, f_3, f_4, c_6. Since f_4 vanishes in A whereas f_1, f_2, f_3 map to c_2, c_4, c_3c_5 by (3.2) and (3.3), the image of α in A turns out to be a linear combination over B of the square-free products of c_2, c_4, c_3c_5, c_6. These products constitute a part of the basis (2.1) of A over B (which actually is the even degree part exactly). Since $\alpha \in c_1^2\hat{A} \subset c_1\hat{A}$, the image of α in A vanishes. It follows that modulo $c_1^2\check{R}$, α is f_4 times a linear combination of the square-free products of f_1, f_2, f_3, f_4, c_6. Subtracting from α an appropriate element of $c_1^2\check{R}$, we come to the situation, where $\alpha = f_4\beta$ and β is a linear combination of the square-free products of f_1, f_2, f_3, f_4, c_6. The image of α in $\hat{A}/c_1^2\hat{A}$ is then in $c_1\hat{A}/c_1^2\hat{A} \simeq A$, the corresponding element of A is the product by (3.4) of the image of β in A. Since the image of α actually vanishes, we conclude that $\beta \in c_1^2\check{R}$. Consequently $\alpha \in c_1^2\check{R}$. □

Corollary 3.7. *For any even $i \geq 0$, one has $\check{R} \cap c_1^i\hat{A} = c_1^i\check{R}$.*

Proof. We induct on even $i \geq 0$. Since $\check{R} \cap c_1^{i+2}\hat{A} \subset \check{R} \cap c_1^2\hat{A} = c_1^2\check{R}$, we have

$$\check{R} \cap c_1^{i+2}\hat{A} \subset c_1^2\check{R} \cap c_1^{i+2}\hat{A} = c_1^2(\check{R} \cap c_1^i\hat{A}) = c_1^2(c_1^i\check{R}) = c_1^{i+2}\check{R}.$$ □

Proposition 3.8. *For any even $i \geq 0$, one has*

$$R \cap c_1^i\check{R} \subset c_1^i\check{B} + R \cap c_1^{i+2}\check{R}.$$

Proof. By Corollary 3.7, we may replace $\check{R}$ by $\hat{A}$ on the right hand side of the inclusion. Furthermore, since $c_1^i\check{B} \subset R$, we may remove R from the right hand side. The inclusion we get then to check for every even $i \geq 0$ is

$$R \cap c_1^i\check{R} \subset c_1^i\check{B} + c_1^{i+2}\hat{A}. \tag{3.9}$$

We prove inclusion (3.9) in two steps. For the first step, we map the intersection $R \cap c_1^i\check{R} \subset c_1^i\hat{A}$ to the B-module $c_1^i\hat{A}/c_1^{i+1}\hat{A}$, which we identify with the B-module $A = \hat{A}/c_1\hat{A}$ via the isomorphism $\hat{A}/c_1\hat{A} \to c_1^i\hat{A}/c_1^{i+1}\hat{A}$ induced by multiplication by c_1^i. Note that this isomorphism respects the action of the Steenrod operations. To make sense to this remark, let us mention that $c_1^i\hat{A}$ and $c_1^{i+1}\hat{A}$ are stable under the Steenrod operations on $\hat{A}$ and therefore the Steenrod operations are defined on the quotient.

Let us consider the image N of $R \cap c_1^i\check{R}$ in A. By Corollary 3.7, N is the image of $R \cap \check{R} \cap c_1^i\hat{A}$, i.e., the image of the even degree part of $R \cap c_1^i\hat{A}$. Since the odd degree part vanishes in A, N coincides with the image of the entire $R \cap c_1^i\hat{A}$ and therefore is a graded B-submodule of A, concentrated in even degrees and stable under the Steenrod operations. By Proposition 2.3 it is contained in B. It follows that

$$R \cap c_1^i\check{R} \subset c_1^i\check{B} + c_1^{i+1}\hat{A}.$$

This inclusion is our frist step. Our second and final step will be the inclusion

$$R \cap \check{R} \cap c_1^{i+1}\hat{A} \subset c_1^{i+2}\hat{A}. \tag{3.10}$$

Note that

$$R \cap \check{R} \cap c_1^{i+1}\hat{A} \subset \check{R} \cap c_1^i\hat{A} = c_1^i\check{R}$$

by Corollary 3.7. The image of $c_1^i\check{R} \cap c_1^{i+1}\hat{A}$ in the quotient $c_1^{i+1}\hat{A}/c_1^{i+2}\hat{A} = A$ is the free B-module with the basis consisting of the square-free products of c_2, c_4, c_6, c_3c_5 multiplied by (3.4). In particular, this image is concentrated in odd degrees. The image M of $R \cap \check{R} \cap c_1^{i+1}\hat{A}$, we are interested in, coincides with the image of $R \cap c_1^{i+1}\hat{A}$ and is a Steenrod stable submodule here. Let M' be the B-submodule in $B[c_2, c_4, c_6, c_3c_5]$ such that M' multiplied by (3.4) is M. Then M' is concentrated in even degrees. One checks that for positive $i < 8$ the operation St^i vanishes on (3.4). It follows that M' is stable under St^i for such i. By the second part of Proposition 2.3, we conclude that every element of M' has the form

$$a + b(c_5^2 \cdot c_6 + c_3^2 \cdot c_4c_6 + c_2c_3c_5c_6) \tag{3.11}$$

with $a, b \in B$. This means that every element of M has the form (3.4)·(3.11)=

$$\boxed{a\alpha + b\beta} \text{ with } \alpha = c_3^2 \cdot (c_3c_6 + c_4c_5) + c_5^2 \cdot (c_2c_3 + c_5) \text{ and } \tag{3.12}$$
$$\beta = c_3^4c_6^2 \cdot (c_2c_5 + c_3c_4) + (c_2^2c_3^2c_5^2 + c_3^4c_4^2 + c_5^4) \cdot c_5c_6 + c_3^2c_5^2c_6^2 \cdot c_3.$$

Let μ be an element of M written in the form (3.12) with some coefficients $a, b \in B$. Since M is Steenrod stable, $\mathrm{St}^8(\mu)$ has to have the same form (with some other coefficients in B). As already mentioned after (2.8), for every positive $i < 8$, St^i maps (2.8)=(3.11) to B. Besides, (3.4) vanishes under St^i for those i. It follows that

$$a\,\mathrm{St}^8(\alpha) + b\,\mathrm{St}^8(\beta) = a'\alpha + b'\beta \tag{3.13}$$

for some $a', b' \in B$. One has

$$\mathrm{St}^8(\alpha) = (c_2^4 + c_4^2)\big(c_3^2 \cdot (c_3c_6 + c_4c_5) + c_5^2 \cdot (c_2c_3 + c_5)\big)+$$
$$c_3^2c_5^2 \cdot (c_2c_5 + c_3c_4) + c_3^4 \cdot c_5c_6 + c_5^4 \cdot c_3,$$
$$\mathrm{St}^8(\beta) = c_3^4c_6^2\big(c_3^2 \cdot (c_3c_6 + c_4c_5) + c_5^2 \cdot (c_2c_3 + c_5)\big)+$$
$$c_2^2c_3^2c_5^2c_6^2 \cdot (c_2c_5 + c_3c_4) + c_2^2c_3^4c_6^2 \cdot c_5c_6 + c_2^2c_5^4c_6^2 \cdot c_3.$$

Looking at the coefficient at c_3 in (3.13), we get

$$b'c_3^2c_6^2 = ac_5^2 + bc_2^2c_5^2c_6^2. \tag{3.14}$$

Looking at the coefficient at c_5c_6, we see

$$b'(c_2^2c_3^2c_5^2 + c_3^4c_4^2 + c_5^4) = ac_3^4 + bc_2^2c_3^4c_6^2.$$

Multiplying by $c_3^2c_6^2$ and substituting (3.14), we obtain the relation

$$ax = by \text{ with } x = c_5^2(c_2^2c_3^2c_5^2 + c_3^4c_4^2 + c_5^4) + c_3^6c_6^2 \text{ and } \tag{3.15}$$
$$y = c_2^2c_5^2c_6^2(c_2^2c_3^2c_5^2 + c_3^4c_4^2 + c_5^4) + c_2^2c_3^6c_6^4.$$

Let $d \in B$ be the g.c.d. of x and y. It follows that M is contained in the B-module $\tilde{M}$, generated by $y'\alpha + x'\beta$ with $x' := x/d$ and $y' := y/d$. One checks that the coefficients at c_3 and at c_5 in $y\alpha + x\beta = d(y'\alpha + x'\beta)$ are nonzero. Therefore every nonzero element of $\tilde{M}$ has the same property.

Let us take an arbitrary element $\mu \in M$ and write it as $\mu = c(y'\alpha + x'\beta)$ with $c \in B$. Then $M \ni d\mu = c(y\alpha + x\beta)$. One checks that $\mathrm{St}^i(y\alpha + x\beta) = 0$ for every positive $i < 16$. It follows that

$$M \ni \mathrm{St}^{16}(d\mu) = \big(\mathrm{St}^{16}(cy)\alpha + \mathrm{St}^{16}(cx)\beta\big) + c\big(y\,\mathrm{St}^{16}(\alpha) + x\,\mathrm{St}^{16}(\beta)\big).$$

The first summand here is in $\tilde{M} \supset M$. Therefore the second summand is also in $\tilde{M}$.

One checks that the coefficients at c_3 in $y\,\mathrm{St}^{16}(\alpha) + x\,\mathrm{St}^{16}(\beta)$ is zero and the coefficient at c_5 is not. It follows that $c = 0$, implying that $\mu = 0$. Therefore $M = 0$ and inclusion (3.10) is proved. □

We are in position to prove Conjecture 1.2 for $n = 6$ now. In fact, we get a result on $\Gamma^+(12)$ that goes far beyond the statement of Conjecture 1.2 for $n = 6$ and is similar to [7, Corollary 3.5] on Spin(12):

Theorem 3.16. *For $n = 6$, one has $R = \hat{B}$. For $G = \Gamma^+(12)$, the image of $\varphi\colon \operatorname{Ch}(BG) \to \operatorname{Ch}(BT)$ is the subring generated by $t, c_1, c_2^2, \dots, c_6^2$. The image of $\Phi\colon \operatorname{CH}(BG) \to \operatorname{CH}(BT)$ is the subring generated by $2\operatorname{CH}(BT)^W$ along with t, f_0, and the Pontryagin classes $p_1, \dots, p_6$.*

Proof. Employing the inclusion of Proposition 3.8 several times, we prove that every homogeneous element of R of a given even degree is in $\check{B} \subset \hat{B}$. For an odd degree homogeneous $\alpha \in R$ we then have $c_1\alpha \in \check{B}$ implying that $\alpha \in \hat{B}$. Thus we get the first statement of Theorem 3.16.

The equality $R = \hat{B}$ just proved, together with the inclusions $\hat{B} \subset \operatorname{Im}\varphi \subset R$, imply the equality $\operatorname{Im}\varphi = \hat{B}$ which is the second statement of Theorem 3.16.

Concerning Φ, the Pontryagin classes are in its image because, up to a sign, they are the images of Chern classes of the orthogonal G-representation. The element f_0 is in the image as well being the image of the Chern class of the character $G \to \mathbb{G}_{\mathrm{m}}$. And t is the image of the highest Chern class of a half-spin G-representation. Finally, since the torsion index of G is 2, $\operatorname{Im}\Phi \supset 2\operatorname{CH}(BT)^W$ by [20, Theorem 1.3(1)]. This directly proves one of the two inclusions. Since

$$2\operatorname{CH}(BT) \cap \operatorname{CH}(BT)^W = 2\operatorname{CH}(BT)^W,$$

the opposite inclusion follows from the second statement of Theorem 3.16. $\square$

4. Reduction to $\Gamma^+(12)$

In this section, we work with arbitrary $n \geq 6$. We do computations inside the polynomial rings

$$\operatorname{CH}(BT) = \mathbb{Z}[z, x_1, \dots, x_n] \quad \text{and} \quad \operatorname{Ch}(BT) = \mathbb{F}[z, x_1, \dots, x_n],$$

where T is the standard split maximal torus of the special Clifford group $G = \Gamma^+(2n)$. Recall from §1 that these rings are equipped with an action of the Weyl groups $W \subset \tilde{W}$ and with the Steenrod operations St^i, $i \geq 0$. We consider the $\mathbb{F}$-algebra generated by all St^i, and refer as Steenrod operation to any element of this algebra.

Recall from the previous section that we write $\hat{R}$ for the image of the composition

$$\operatorname{CH}(BT)^W \hookrightarrow \operatorname{CH}(BT) \twoheadrightarrow \operatorname{Ch}(BT) \tag{4.1}$$

of the embedding followed by the mod 2 reduction. As a ring, $\hat{R}$ is generated by $c_2^2, \dots, c_{n-1}^2$, $f_0 = c_1, f_1, \dots, f_{n-2}$, c_n, and t. We write $R \subset \hat{R}$ for the subring of the elements whose images under any Steenrod operation are in $\hat{R}$. It follows from the commutative square (3.1) that $\operatorname{Im}\varphi \subset R$.

By Theorem 3.16, the following conjecture holds for $n = 6$:

Conjecture 4.2. *For $n \geq 6$, R is contained in the subring of $\hat{R}$, generated by $c_2^2, \dots, c_{n-1}^2$, $f_0, \dots, f_{n-2}$, and t.*

Clearly, Conjecture 4.2 implies Conjecture 1.2. So, let's concentrate on proving Conjecture 4.2. As a first step, we are going to replace Conjecture 4.2 by a simpler statement, involving a ring smaller than $\hat{R}$. Let R' be the subring in $\mathbb{F}[c_1,\dots,c_n]$ generated by the squares and $f_0,\dots,f_{n-2}$. Note that $R'\subset\hat{R}$. We will also work with the ring $R'[c_n]=R'+R'\cdot c_n\subset\hat{R}$. The sum is actually direct:

Lemma 4.3. $R'[c_n]=R'\oplus R'\cdot c_n$.

Proof. The ring $R'[c_n]$ is the image under (4.1) of the ring generated by $p_1,\dots,p_n$, $f_0,\dots,f_{n-2}$, and e. Since $e^2=p_n$, the latter decomposes as $\tilde{R}+\tilde{R}\cdot e$, where $\tilde{R}\subset\mathrm{CH}(BT)^W$ is the subring generated just by $p_1,\dots,p_n$ and $f_0,\dots,f_{n-2}$. Note that $\tilde{R}$ consists of $\tilde{W}$-invariant elements whereas any element of $\tilde{W}\setminus W$ changes the sign of e. It follows that $\tilde{R}\cap(\tilde{R}\cdot e)=0$. This implies Lemma 4.3 because the image of $\tilde{R}$ under (4.1) is R'. □

Conjecture 4.4. *For $n\geq 6$, assume that an element $x\in R'[c_n]$ is such that the image of x under any Steenrod operation is again in $R'[c_n]$. Then $x\in R'$.*

Lemma 4.5. *Conjecture 4.4 is equivalent to Conjecture 4.2.*

Proof. Note that $\hat{R}$ is the polynomial ring in t over $R'[c_n]$.

Let us assume Conjecture 4.2 and take some $x\in R'[c_n]$ satisfying the condition of Conjecture 4.4. Then $x\in R$ and so $x\in R'[t]$ by Conjecture 4.2. Since the intersection $R'[t]\cap R'[c_n]$ is R', we get that $x\in R'$. This proves that Conjecture 4.2 implies Conjecture 4.4.

To prove that Conjecture 4.4 implies Conjecture 4.2, let us take an element of R and write it as a polynomial

$$a_mt^m+\cdots+a_1+a_0 \tag{4.6}$$

in t with coefficients $a_m,\dots,a_1,a_0\in R'[c_n]$. Let S be the Steenrod operation, given by the composition of several St^i with various $i\geq 0$. By induction on the degree j of the operation S, we show that the value of S at every coefficient of (4.6) is in $R'[c_n]$. For $j=0$, we have $S=\mathrm{id}$. For $j>0$, $S(a_mt^m+\cdots+a_1t+a_0)$ equals

$$S(a_m)t^m+\cdots+S(a_1)t+S(a_0) \tag{4.7}$$

plus remaining terms which are in $R'[c_n][t]$ by induction hypothesis and because $t\in R$. Therefore (4.7) is in $R'[c_n][t]$ which means that the coefficients of (4.7) are in $R'[c_n]$. It follows by Conjecture 4.4 that the coefficients of (4.6) are in R'. □

Proposition 4.8. *Conjectures 4.2 and 4.4 hold.*

Proof. Since we already know that Conjectures 4.2 and 4.4 are equivalent and hold for $n=6$, it suffices to prove Conjecture 4.4 for a given $n>6$.

Let $\tilde{W}_6$ be the subgroup of $\tilde{W}$ generated by the permutations of $x_1,\dots,x_6$ and the sign changes $\sigma_1,\dots,\sigma_6\in\tilde{W}$. The ring $\mathbb{Z}[z,x_1,\dots,x_n]^{\tilde{W}_6}$ is the polynomial ring over $\mathbb{Z}[z,x_1,\dots,x_6]^{\tilde{W}_6}$ in the variables $x_7,\dots,x_n$.

Besides of the subring $R'\subset\mathbb{F}[x_1,\dots,x_n]$, we have a similar subring in $\mathbb{F}[x_1,\dots,x_6]$ that will be denoted R'_6 below in the proof. Note that R' is contained in the polynomial ring $R'_6[x_7,\dots,x_n]$ over R' in the variables $x_7,\dots,x_n$. Indeed, R' is the image of

$\mathbb{Z}[z, x_1, \dots, x_n]^{\tilde{W}}$ in $\mathbb{F}[t, c_1, \dots, c_n]$ intersected with $\mathbb{F}[c_1, \dots, c_n]$. It coincides with the image of $\mathbb{Z}[z, x_1, \dots, x_n]^{\tilde{W}}$ in $\mathbb{F}[t, x_1, \dots, x_n]$ intersected with $\mathbb{F}[x_1, \dots, x_n]$ and contained in the image of $\mathbb{Z}[z, x_1, \dots, x_n]^{\tilde{W}_6}$ in $\mathbb{F}[t, x_1, \dots, x_n]$ intersected with $\mathbb{F}[x_1, \dots, x_n]$. The latter intersection is exactly $R'_6[x_7, \dots, x_n]$.

Assume that we are given an element $x \in R'[c_n]$ satisfying the condition of Conjecture 4.4. In view of Lemma 4.3, let us write x in the form $x = x' + x''c_n$ with $x', x'' \in R'$. Note that

$$c_n = (x_1 \dots x_6) \cdot (x_7 \dots x_n).$$

By the $n = 6$ case of Conjecture 4.4, $x \in R'_6[x_7, \dots, x_n]$. Therefore $x'' = 0$. □

Proposition 4.8 implies

Theorem 4.9 (**Main Theorem**). *Conjecture 1.2 holds.* □

5. The algorithm

In this section, G is the split spin group $\mathrm{Spin}(2n) = \mathrm{Spin}(q)$ with $n \geq 3$, given by the split $2n$-dimensional quadratic form q over a field, where

$$q(a_1, b_1, \dots, a_n, b_n) = a_1 b_1 + \dots + a_n b_n,$$

and $P \subset G$ is the standard maximal parabolic subgroup corresponding to the mth vertex of the Dynkin diagram of G, where $m \in \{1, \dots, n-2\}$. We describe an algorithm computing $i(X^E)$ for a generic G-torsor E, where $X := G/P$ is the variety of totally isotropic m-planes of q. For the similar problem related to the odd spin group $\mathrm{Spin}(2n+1)$, the corresponding algorithm is already provided in [10]. As already mentioned in §1, the answer for $m \in \{n-1, n\}$ is the same as for $m = n-2$ and has been already obtained in [20, Theorem 0.1].

We start by summarizing facts established in [4] and [12]. One has $i(X^E) = i(\tilde{X}^E)$ for $\tilde{X} := G/\tilde{P}$, where $\tilde{P} \subset G$ is the standard parabolic subgroup given by the first m vertices of the Dynkin diagram. The variety $\tilde{X}$ is the variety of flags of totally isotropic subspaces of dimensions $1, \dots, m$ of the quadratic form q.

The integer $i(\tilde{X}^E)$ is the positive generator of the image of the composition

$$\mathrm{CH}(B\tilde{P}) \longrightarrow \mathrm{CH}(G/\tilde{P}) = \mathrm{CH}(\tilde{X}) \xrightarrow{\deg} \mathbb{Z}. \tag{5.1}$$

In view of the interpretation [21, Theorem 2.8] of $\mathrm{CH}(B\tilde{P})$, the first homomorphism in the composition is given by evaluation at the $\tilde{P}$-torsor $G \to G/\tilde{P}$. The degree homomorphism deg is the push-forward to the Chow ring of the base field with respect to the structure morphism of the proper variety $\tilde{X}$. It vanishes on the components of the Chow group other than $\mathrm{CH}_0(\tilde{X})$.

The reductive part (i.e., the Levi subgroup)

$$\tilde{P}_{\mathrm{red}} = \big(\mathbb{G}_{\mathrm{m}}^{\times m} \times \mathrm{Spin}(2l)\big)/\mu_2$$

of $\tilde{P}$, where μ_2 is embedded diagonally into the product of $\mathbb{G}_{\mathrm{m}}$s and the center of $\mathrm{Spin}(2l)$, is isomorphic to the product of $m-1$ copies of $\mathbb{G}_{\mathrm{m}}$ by the special Clifford group $\Gamma := \Gamma^+(2l)$ with $l := n - m$. So, $\mathrm{CH}(B\tilde{P}_{\mathrm{red}})$ is the polynomial ring over $\mathrm{CH}(B\Gamma)$ in $m-1$ variables

$\mathbf{y} := \{y_1, \dots, y_{m-1}\}$, given by the standard characters of the split torus $\mathbb{G}_{\mathrm{m}}^{\times(m-1)}$. The homomorphism

$$\mathrm{CH}(B\tilde{P}) \to \mathrm{CH}(B\tilde{P}_{\mathrm{red}}) = \mathrm{CH}(B\Gamma)[\mathbf{y}],$$

induced by the embedding $\tilde{P}_{\mathrm{red}} \hookrightarrow \tilde{P}$, is an isomorphism. The first homomorphism in (5.1) decomposes as

$$\mathrm{CH}(B\tilde{P}) = \mathrm{CH}(B\Gamma)[\mathbf{y}] \xrightarrow{\Phi[\mathbf{y}]} \mathrm{CH}(BT)^W[\mathbf{y}] \xrightarrow{f} \mathrm{CH}(G/\tilde{P}), \tag{5.2}$$

where $T \subset \Gamma$ is the standard split maximal torus of the special Clifford group, W is the Weyl group, Φ is the homomorphism $\mathrm{CH}(B\Gamma) \to \mathrm{CH}(BT)^W$ induced by the embedding, and f is defined in [4, Lemma 2.2]. As a clarification for the reference, let us note that W is also the Weyl group of $\tilde{P}$ with respect to its standard split maximal torus $\mathbb{G}_{\mathrm{m}}^{\times(m-1)} \times T$, and $\mathrm{CH}(BT)[\mathbf{y}] = \mathrm{CH}(B(\mathbb{G}_{\mathrm{m}}^{\times(m-1)} \times T))$.

Recall that $\tilde{X}$ is the variety of flags of totally isotropic subspaces of dimensions $1, \dots, m$ in q and as such is equipped with the tautological vector bundles of ranks $1, \dots, m$. Let $C \subset \mathrm{CH}(\tilde{X})$ be the subring generated by their Chern classes. We also set

$$\varepsilon := f(e) \in \mathrm{CH}(\tilde{X}), \quad \tau := f(t) \in \mathrm{CH}(\tilde{X})$$

with $e, t \in \mathrm{CH}(BT)^W$ defined in §1, and consider the C-subalgebra $C[\tau] \subset \mathrm{CH}(\tilde{X})$ generated by τ. The image of the composition (5.2) contains $C[\tau]$ and is contained in $C[\tau, \varepsilon]$. In fact, by [12, §5.b], the image under (5.2) of $y_1, \dots, y_{m-1}$ and of every generator of $\mathrm{CH}(BT)^W$ other than t and e is in C. It follows by Theorem 4.9 that the image of the composition (5.2) is contained in $C[\tau, 2\varepsilon]$ for $l \geq 6$.

Although we don't know exactly what the image of (5.2) is equal to, the above approximate information turns out to be sufficient for determination of the image of (5.1) and therefore for determination of the index:

Theorem 5.3 (The Algorithm). *For $l \geq 6$, one has $\deg(C[\tau]) = \deg(C[\tau, 2\varepsilon])$ and $i(X^E) = \deg(C[\tau])$. For $l < 6$, one has $i(X^E) = \deg(C[\tau, \varepsilon])$.*

Proof. We use the identification $\mathrm{CH}(BT) = \mathbb{Z}[z, x_1, \dots, x_l]$ of §1. The involutional automorphism of the quadratic form q, given by the exchange of the vectors in the last hyperbolic pair of q, yields an involutional automorphisms of the group G, globally fixing the subgroups $T \subset \mathbb{G}_{\mathrm{m}}^{\times(m-1)} \times T \subset \tilde{P} \subset G$, and of $\tilde{X}$. Indeed, in coordinates, the automorphism of q just exchanges a_n and b_n. The parabolic subgroup $\tilde{P} \subset G$ is the stabilizer of the rational point on $\tilde{X}$ given by the standard full flag of the totally isotropic m-plane $a_{m+1} = \dots = a_n = 0$, $b_1 = \dots = b_n = 0$. The standard split maximal torus $\mathbb{G}_{\mathrm{m}}^{\times(m-1)} \times T$ of G is the preimage under the isogeny of $G \to \mathrm{O}^+(2n)$ of the torus of the diagonal matrices $\mathrm{diag}(\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1})$.

We write σ for the automorphism of $\mathrm{CH}(\tilde{X})$ given by the involutional automorphism of $\tilde{X}$. The induced automorphism of $\mathrm{CH}(BT)$ is the automorphism σ_l from §1, changing the sign of x_l and mapping z to $z - x_l$. The induced automorphism of $\mathrm{CH}(B(\mathbb{G}_{\mathrm{m}}^{\times(m-1)} \times T)) =$

$\mathrm{CH}(BT)^W[\mathbf{y}]$ is the extension $\sigma_l[\mathbf{y}]$ of σ_l to $\mathrm{CH}(BT)^W[\mathbf{y}]$ identical on $\mathbf{y}$. The square

$$\begin{array}{ccc}
\mathrm{CH}(BT)^W[\mathbf{y}] & \xrightarrow{f} & \mathrm{CH}(\tilde{X}) \\
{\scriptstyle\sigma_l}\big\downarrow & & {\scriptstyle\sigma}\big\downarrow \\
\mathrm{CH}(BT)^W[\mathbf{y}] & \xrightarrow{f} & \mathrm{CH}(\tilde{X})
\end{array}$$

commutes.

Since the automorphism σ of $\mathrm{CH}(\tilde{X})$ is induced by an automorphism of the variety $\tilde{X}$, we have $\deg(\sigma(x)) = \deg(x)$ for any $x \in \mathrm{CH}(\tilde{X})$.

The sum $\tau+\sigma(\tau)$ is in C (cf. [12, Proposition 5.6]). It follows that the $C[\tau\sigma(\tau)]$-module $C[\tau]$ is generated by 1 and τ. Since $\varepsilon^2 \in C$, the $C[\tau]$-module $C[\tau, 2\varepsilon]$ is generated by 1 and 2ε. Consequently, the $C[\tau\sigma(\tau)]$-module $C[\tau, 2\varepsilon]$ is generated by $1, 2\varepsilon, \tau, 2\varepsilon\tau$. Since $\sigma(\varepsilon) = -\varepsilon$ and σ is identical on $C[\tau\sigma(\tau)]$, we have $\deg(C[\tau\sigma(\tau)]2\varepsilon) = 0$. Since $2\tau \in C[\varepsilon]$ by [12, Proposition 5.6], we have $\deg(C[\tau\sigma(\tau)]2\varepsilon\tau) = \deg(C[\tau\sigma(\tau)])$. Therefore

$$\deg(C[\tau, 2\varepsilon]) \subset \deg(C[\tau\sigma(\tau)] + C[\tau\sigma(\tau)]\tau) = \deg(C[\tau]),$$

proving Theorem 5.3 for $l \geq 6$.

To prove Theorem 5.3 for $l < 6$, it suffices to show that the image of the homomorphism

$$\Phi\colon\ \mathrm{CH}(B\Gamma) \to \mathrm{CH}(BT)^W$$

contains the sum $e + \alpha$ of the Euler class e and certain polynomial α in the remaining generators of $\mathrm{CH}(BT)^W$. (This in particular means that the statement of Conjecture 1.2 is false for $n < 6$.)

If $l \leq 3$, then the torsion index of Γ is 1 implying that Φ is surjective (see [20, Theorem 1.3(1)]). Below we assume that l is 4 or 5.

Let us take $n = l + 1$. Then $m = 1$ and the variety $\tilde{X}_m = \tilde{X}_1 = X_1$ is the projective quadric of q. There are exactly two distinct rational equivalence classes

$$\lambda \neq \lambda' \in \mathrm{CH}^l(X_1)$$

of n-dimensional totally isotropic subspaces in q (see, e.g., [5, Proposition 68.2]). Their sum $\lambda' + \lambda$ is a power of the hyperplane section class, which lies in $C \subset \mathrm{CH}(X_1)$ and in the image of (5.2). Note that $n \leq 6$ meaning that $\dim q \leq 12$. It follows that 2λ is in the image of of the homomorphism

$$\mathrm{CH}(X_1^E) \to \mathrm{CH}(X_1), \tag{5.4}$$

obtained as the composition of the change of field homomorphism of $\mathrm{CH}(X_1^E)$, given by a splitting field of E, followed by the inverse of the change of field isomorphism of $\mathrm{CH}(X_1)$. By [11, Theorem 6.4(2)], the image of (5.4) coincides with the image of (5.2). Consequently, $\lambda' - \lambda = (\lambda' + \lambda) - 2\lambda$ is in the image of (5.2) as well.

We claim that $\lambda' - \lambda = \pm\varepsilon$. Since the whole group $\mathrm{CH}^l(X_1)$ is generated by λ' and λ, we a priori know that $\varepsilon = a'\lambda' + a\lambda$ for some integers a' and a. The involutional automorphism σ exchanges λ' with λ and changes the sign of ε. Therefore we have $a' = -a$. Finally, to see that $a = \pm 1$, note that the degree of t is $2^{l-1} > l$. Besides, the image under f of any of the generators of $\mathrm{CH}(BT)^W$ other than t and e is a power of the hyperplane section.

We conclude that $\operatorname{Im}\Phi$ contains an element of the form $ae+\alpha$, where α is a polynomial in the remaining (i.e., distinct from e) generators of $\mathrm{CH}(BT)^W$ and a is an odd integer. Since $l = 4, 5$, the torsion index of Γ is 2 so that $2e \in \operatorname{Im}\Phi$ and we can replace the appeared above odd integer a by 1. □

Theorem 5.3 provides an algorithm for computing $i(X^E)$ in the sense that it reduces determination of $i(X^E)$ to a computations of g.c.d. of degrees of certain elements in the Chow group of a split projective homogeneous variety (namely, the elements of $C[\tau, \varepsilon]$ in $\mathrm{CH}(\tilde{X})$). This is not yet an algorithm of a practical value, but it reduces the original difficult problem to an "elementary" computation. It also implies

Corollary 5.5. *The index $i(X^E)$ does not depend on the initial field F_0 and, in particular, on its characteristic.* □

Based on Theorem 5.3, a more efficient algorithm, computing $i(X^E)$, can be obtained using the duality result of [13] in the spirit of how the Poincaré duality had been used in [20, §4] (see also [10, §4]). However, it still needs further improvements. Since this work is not yet finished, we do not provide further details here. Also note that all indexes $i(X^E)$ for $\mathrm{Spin}(2n)$ with $n \le 9$ have already been determined without Theorem 5.3, see [8].

Appendix A. Computing Steenrod operations

To compute the Steenrod operations St^i for all $i \ge 0$ on the ring $\mathbb{F}[c_1, \dots, c_6]/(c_1)$, the following simple Maple code can be used:

```
c[0]:=1: c[1]:=0:
for i from 7 to 12 do c[i]:=0 od:
for j from 1 to 6 do Sc[j]:=0:
for i from 0 to j do for k from 0 to i do
Sc[j]:=Sc[j]+binomial(i-j,k)*c[i-k]*c[j+k]*t^i mod 2
od: od: od:
```

After it is run, `Sc[j]` for $j = 1, \dots, 6$ becomes the value of the total Steenrod operation on c_j (with its homogeneous components separated with a help of a variable t). Now, in order to compute $\mathrm{St}^i(P)$ for a polynomial P in c_j, one uses the command

```
expand(coeff(SP, t, i)) mod 2;
```

where `SP` is P with `Sc[j]` substituted for c_j.

Appendix B. Computing f_2, f_3, f_4

In order to compute the generators $f_2, f_3, f_4 \in \mathbb{F}[c_1, \dots, c_6]$, the following Maple code can be used. It requires the Maple Symmetric Polynomials package by Kahtan H. Alzubaidy loaded with

```
restart; with(ListTools); with(combinat);
with(PolynomialTools); with(Groebner);
es:=proc() local V, A, p, L, K;
V:=[seq(args[i],i=2..nargs)];A:=[seq(sigma[i],i=1..nargs-1)];
p:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
L := Reverse([seq((-1)^(r+nargs-1)*coeff(p, x_, r), r = 0..nargs-2)])-A;
```

```
K:=Basis(L,tdeg(V[])); NormalForm(args[1],K,tdeg(V[])); end proc;
```

This package expresses symmetric polynomials in terms of the elementary ones. Note that once such an expression for a given polynomial is determined, its correctness is easily verified (without employing the package).

The following simple code computes f_i, $i \leq 4$, as polynomials in $x_1, \ldots, x_6$:

```
f[0]:= x[1] + x[2] + x[3] + x[4] + x[5] + x[6]:
for i from 1 to 4 do
f[i]:=expand(f[i-1]^2-subs(x[1]=x[1]^2, x[2]=x[2]^2,
x[3]=x[3]^2,x[4]=x[4]^2,x[5]=x[5]^2,x[6]=x[6]^2, f[i-1]))/2; od:
```

To rewrite `f[i]` as polynomials in $c_1, \ldots, c_6$, use

```
for i from 1 to 4 do
f[i]:=es(f[i], x[1], x[2], x[3], x[4], x[5], x[6]) mod 2; od;
```

Warning: the computation of `f[4]` requires about 15 minutes of a standard computer time. The computation of `f[i]` for $i \leq 3$ is immediate. It is recommended to run the Repetition Statement `from 1 to 3` for testing first.

Appendix C. Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Benson, D. J., and Wood, J. A. Integral invariants and cohomology of $B\mathrm{Spin}(n)$. *Topology 34*, 1 (1995), 13–28.

[2] Borel, A. La cohomologie mod 2 de certains espaces homogènes. *Comment. Math. Helv. 27* (1953), 165–197.

[3] Brosnan, P. Steenrod operations in Chow theory. *Trans. Amer. Math. Soc. 355*, 5 (2003), 1869–1903 (electronic).

[4] Devyatov, R. A., Karpenko, N. A., and Merkurjev, A. S. Maximal indexes of flag varieties for spin groups. *Forum Math. Sigma 9* (2021), Paper No. e34, 12.

[5] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[6] Florence, M. On the essential dimension of cyclic p-groups. *Invent. Math. 171*, 1 (2008), 175–189.

[7] Karpenko, N. A. On characteristic classes modulo torsion for spin groups. Preprint (revised version of 23 Mar 2023, 13 pages). Available on author's webpage.

[8] Karpenko, N. A. On isotropy fields of Spin(18)-torsors. Preprint (28 Aug 2023, 7 pages). Available on author's webpage. To appear in *St. Petersburg Math. J.*

[9] Karpenko, N. A. Partially splitting PGO-torsors. Preprint (3 Mar 2023, 17 pages). Available on author's webpage.

[10] Karpenko, N. A. On generic flag varieties for odd spin groups. *Publ. Mat. 67*, 2 (2023), 743–756.

[11] Karpenko, N. A., and Merkurjev, A. S. Canonical p-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.

[12] Karpenko, N. A., and Merkurjev, A. S. Indexes of generic Grassmannians for spin groups. *Proc. Lond. Math. Soc. (3) 125*, 4 (2022), 825–840.

[13] Karpenko, N. A., and Merkurjev, A. S. Poincaré duality for tautological Chern subrings of orthogonal grassmannians. *Math. Scand. 128*, 2 (2022), 221–228.

[14] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[15] Mac Lane, S. *Homology*. Die Grundlehren der mathematischen Wissenschaften, Band 114. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[16] Primozic, E. Motivic Steenrod operations in characteristic p. *Forum Math. Sigma 8* (2020), Paper No. e52, 25.

[17] Smirnov, A., and Vishik, A. Subtle characteristic classes. arXiv:1401.6661v2 [math.AG] (4 Oct 2021), 32 pages.

[18] Tanania, F. Motivic cohomology of the Nisnevich classifying space of even Clifford groups. arXiv:2302.04772 [math.AG], 9 Feb 2023, 13 pages.

[19] Totaro, B. The torsion index of E_8 and other groups. *Duke Math. J. 129*, 2 (2005), 219–248.

[20] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.

[21] Totaro, B. *Group cohomology and algebraic cycles*, vol. 204 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2014.

[22] Voevodsky, V. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, 98 (2003), 1–57.

[23] Wu, W. T. On squares in Grassmannian manifolds. *Acta Sci. Sinica 2* (1953), 91–115.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: `karpenko@ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`