ON ISOTROPY FIELDS OF Spin(18)-TORSORS

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Abstract. We show that any 18-dimensional non-degenerate quadratic form of trivial discriminant and Clifford invariant acquires Witt index at least 5 over some finite base field extension of degree not divisible by 2^4. Based on previous research, we also establish a general formula on all possible similar statements for forms of arbitrary dimension.

1. Introduction

Given a split semisimple algebraic group G and a parabolic subgroup $P \subset G$, a G-torsor E over an extension of the base field of G is called *P-isotropic*, if the quotient variety E/P has a rational point, cf. [12]. As an example, for a Borel subgroup $B \subset G$, "B-isotropic" means the same as "trivial".

Let E be a *generic G-torsor*, i.e., the generic fiber of the quotient morphism

$$\mathrm{GL}(N) \to \mathrm{GL}(N)/G$$

given by an embedding of G into a general linear group $\mathrm{GL}(N)$ for some $N \geq 1$. It is interesting to know the g.c.d. of degrees of finite extensions L of its base field which are *P-isotropy fields* of E, i.e., the torsor E_L is P-isotropic. This number is the *index* of the variety E/P, defined (for any algebraic variety) as the g.c.d. of degrees of its closed points. For instance, for $P = B$ what we get is the well-studied *torsion index* of the group G (see [13, Theorem 1.1]).

Our motivation to consider a generic G-torsor E relies on the fact that any G-torsor E' over an extension of the base field of G is a specialization of E. By that reason, the index of E'/P is a divisor of the index of E/P (see [9, Theorem 6.4]).

Below we take for G the split spin group $\mathrm{Spin}(d)$ with some $d \geq 3$. We write n for the integer satisfying $d = 2n + 1$ or $d = 2n + 2$. The torsion index of G is a 2-power 2^t, depends only on n, and has been computed in [13]; the exponent t is called the *torsion exponent* here.

Any G-torsor E yields a d-dimensional non-degenerate quadratic form q of trivial discriminant and Clifford invariant. (The corresponding map of the sets of isomorphism classes is surjective.) We say that q is *generic* if E is so. Determination of the index of E/P for any P easily reduces to the case, where E/P is isomorphic to the variety X_m of totally isotropic m-planes of q for some $m = 0, \ldots, n$. (The variety X_0 is just the

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base point and has index 1.) We call X_m the mth grassmannian of q. Its index is easily seen to be a 2-power, and we write i_m for the exponent. We have $i_0 = 0$, $i_n = t$, and $i_{m-1} \leq i_m \leq i_{m-1} + 1$ for any $m = 1, \ldots, n$. It follows that $i_m = \min\{m, t\}$ for any m provided that $i_t = t$. By this reason, we say that the tth exponent i_t is *critical*.

For generic forms of dimension $d \leq 12$, it is easy to see that $i_t = t$ always except $d = 10$, where $t = 1$ and $i_1 = 0$. For generic forms of dimensions 13 and 14, the equality $i_t = t$ has been simultaneously and independently shown in [4] and [5]. In dimensions 15 and 16, the equality has been obtained in [7]. In the light of these results, the exception of 10-dimensional forms seemed to be a special low-dimensional effect. It has been shown later in [8] that $i_t \neq t$ for dimensions 17–20, but the proofs relied on computer computations.

For a generic form of arbitrary dimension, it has been proven in [10] (based on earlier [1]) that $i_t \in \{t-1,\ t\}$. With Theorem 4.1 here, we give a mathematical proof that $i_t = t - 1 = 3$ for generic forms of dimension 17. This implies the same and actually determines all exponents for dimension 18 (see Corollary 4.4).

We start in §2 by summarizing available results and establishing in Theorem 2.3 a simple general formula on all possible similar statements for forms of arbitrary dimension and any m. The case of $m = t$ for forms of arbitrary odd dimension is discussed more extensively in §3. In final §4, the specific computation for dimension 17 is made.

2. Exponents for arbitrary dimension

Clearly, for any m, the mth exponent i_m satisfies

$$\deg(\mathrm{CH}(X_m)) = 2^{i_m} \cdot \mathbb{Z}. \tag{2.1}$$

The left-hand side of the formula is the image of the degree homomorphism

$$\deg\colon\ \mathrm{CH}(X_m) \to \mathbb{Z},$$

of the Chow group given by the push-forward with respect to the structure morphism of the projective variety X_m. Since the generic quadratic form q, defining X_m, splits over some field extension, we have a ring homomorphism

$$\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m), \tag{2.2}$$

where $\bar{X}_m$ is the mth grassmannian of a split non-degenerate d-dimensional quadratic form. Since degree does not change under field extensions, we can replace $\mathrm{CH}(X_m)$ in (2.1) by the image $\bar{\mathrm{CH}}$ of (2.2) and replace the degree map of X_m by the degree map of $\bar{X}_m$. These replacements provide a simplification because the subring $\bar{\mathrm{CH}} \subset \mathrm{CH}(\bar{X}_m)$ is isomorphic to $\mathrm{CH}(X_m)$ modulo torsion and because a computation of the ring $\mathrm{CH}(\bar{X}_m)$, unlike $\mathrm{CH}(X_m)$, is available.

There is a further simplification, which relies on the fact that q is generic. Unlike the previous one, it is highly non-trivial. Let $\mathrm{CC} \subset \mathrm{CH}(\bar{X}_m)$ be the subring generated by the Chern classes of all vector bundles on $\bar{X}_m$, or, equivalently, by the Chern classes of virtual vector bundles – the elements of the Grothendieck group $K_0(\bar{X}_m)$. Since the group G is simply connected, the homomorphism $K_0(X_m) \to K_0(\bar{X}_m)$ is an isomorphism implying that CC is a subring in $\bar{\mathrm{CH}}$.

Theorem 2.3. *One has*

$$\deg(\mathrm{CC}) = 2^{i_m} \cdot \mathbb{Z}$$

except, possibly, the case where d is even and $n-m \leq 4$.

Proof. The proof is simpler for odd d, where the ring CC happens to coincide with $\bar{\mathrm{CH}}$: by [8, Theorem 3.6], $\bar{\mathrm{CH}} = \mathrm{CC}^s$, where $\mathrm{CC}^s \subset \mathrm{CC}$ is the subring, generated by the Chern classes of the tautological (rank m) vector bundle $\mathcal{T}$ on $\bar{X}_m$ together with the 2^{n-m}th Chern class τ of certain (rank 2^{n-m}) virtual vector bundle, described in [8, Proof of Proposition 3.4].

Now assume that d is even. Here again the subring $\mathrm{CC}^s \subset \mathrm{CC}$, generated by the Chern classes of $\mathcal{T}$ and certain additional element $\tau \in \mathrm{CH}^{2^{n-m}}(\bar{X}_m)$, plays an important role. The needed element τ is defined in [10, Proposition 5.5] (see also [6, §5]), where it is shown to be the 2^{n-m}th Chern class of certain (rank 2^{n-m}) virtual vector bundle. The equalities $\mathrm{CC}^s = \mathrm{CC} = \bar{\mathrm{CH}}$ do not hold anymore. However

$$\deg(\mathrm{CC}^s) = \deg(\mathrm{CC}) = \deg(\bar{\mathrm{CH}})$$

by [6, Theorem 5.3] provided that $n-m \geq 5$: the elements outside CC^s do not contribute to the image of the degree map. □

Remark 2.4. Even though $\bar{\mathrm{CH}} = \mathrm{CC}$ for odd d and any m, it is not clear if (and rather not to expect that) the corresponding Chow ring $\mathrm{CH}(X_m)$ is generated by Chern classes.

Remark 2.5. The proof of Theorem 2.3 yields the formula

$$\deg(\mathrm{CC}^s) = 2^{i_m} \cdot \mathbb{Z},$$

which makes eventual computation of i_m more accessible than does the formula with CC.

Remark 2.6. Let us consider the situation of even $d = 2n+2$ with $n-m \leq 4$. Since any non-degenerate quadratic form with trivial discriminant and Clifford invariant and of even dimension ≤ 6 is hyperbolic, we have

$$i_{n-2} = i_{n-1} = i_n = t.$$

Since any such form of dimension 10 is isotropic, we have

$$i_{n-4} = i_{n-3} \in \{t-1, t\}.$$

We are going to show that the exponents i_{n-4} and i_{n-3} can be computed using Theorem 2.3 for the odd dimension $d' := 2n+1$. Let $i'_0, \dots, i'_n$ be the exponents for dimension d'. By [10, Lemma 2.3], we have $i_{n-4} \leq i'_{n-4} \leq i_{n-3}$ and it follows that $i_{n-4} = i'_{n-4} = i_{n-3}$. For the sake of Remark 2.7, note that $i_{n-m} = i'_{n-m}$ for $n-m = 0,1,2,4$ with an exception of $n-m = 3$. (Actually, we do not dispose of any example with $i_m \neq i'_m$ aside from $m = 1$ for $d = 10$.)

Remark 2.7. The mth grassmannian $\bar{X}'_m$ of a non-degenerate d'-dimensional subform in the d-dimensional quadratic form defining $\bar{X}_m$, is a closed subvariety of $\bar{X}_m$. The pull-back of the extra generator $\tau \in \mathrm{CH}(\bar{X}_m)$ is the corresponding extra generator $\tau' \in \mathrm{CH}(\bar{X}'_m)$. Using this observation, one can show that the formula

$$\deg(\mathrm{CC}) = \deg(\mathrm{CC}^s) = 2^{i_m} \cdot \mathbb{Z}$$

also holds for even d and $n-m \leq 4$ except, possibly, the case where $n-m = 3$ and $i_{n-3} \neq i'_{n-3}$ (in the notation of Remark 2.6). Note that for $d = 10$ and $n-m = 3$, the formula still holds despite that $i_{n-3} \neq i'_{n-3}$.

Remark 2.8. Note that the rings $\mathrm{CH}(\bar{X}_m)$ and $K_0(\bar{X}_m)$ do not depend on the base field. Therefore Theorem 2.3 and Remark 2.6 imply that for any m the exponent i_m does not depend on the base field. This fact has been already observed: for odd d in [8], for even d in [6].

It is not difficult to show that $\deg(\mathrm{CC}\mathcal{T}) = 2^m$, where $\mathrm{CC}\mathcal{T} \subset \mathrm{CC}^s$ is the subring generated by the Chern classes of $\mathcal{T}$ alone. This 2-power is the index of the grassmannian given by a d-dimensional non-degenerate generic quadratic form (without any restriction on its discriminant and Clifford invariant). To see how much i_m is lower than m, one needs to understand the contribution of the additional generator of CC^s. This is what the next section does in the case of the critical exponent and odd dimension.

Controlling contribution of τ simplifies due to certain duality property of the ring $\mathrm{CC}\mathcal{T}$, see [11]: the contribution turns out to be determined by the orders of powers of τ modulo $\mathrm{CC}\mathcal{T}$. (This has been first notices and used for $m = n$ in [13].) One shows that the order for τ^{2^i} divides 2 for any i. Because of that, the maximal contribution has to come from τ^{2^i-1} for some i, c.f. [13, §4]. In the case of the critical exponent, τ^3 vanish by dimension reason (see [10, Proposition A.1]), so that only the order of the class of τ itself needs to be determined. This explains the statement of Proposition 3.2 below.

3. Critical exponent for odd dimension

Let us describe the determination algorithm of the critical exponent, established in [10], which works for any odd dimension d.

We fix some $d = 2n + 1$ and consider the highest grassmannian $X := \bar{X}_n$ of a split d-dimensional quadratic form. Recall from [2, §86] (the result was originally obtained in [14]) that the Chow ring $\mathrm{CH}(X)$ is generated by elements $e_i \in \mathrm{CH}^i(X)$, $i = 1, \dots, n$, subject to the relations

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^{i-1}2e_1e_{2i-1} + (-1)^i e_{2i} = 0.$$

The additive group of $\mathrm{CH}(X)$ is free, a basis is given by the products

$$e_I := \prod_{i \in I} e_i, \quad I \subset \{1, \dots, n\}.$$

For every i, the element $(-1)^i 2e_i$ is the ith Chern class c_i of the tautological (rank n) vector bundle $\mathcal{T}$ on X.

Let Y be the variety of complete flags in $\mathcal{T}$. It comes equipped with the tautological vector bundles $\mathcal{T}_1, \dots, \mathcal{T}_n$, where $\mathcal{T}_i$ is of rank i and $\mathcal{T}_n$ on Y comes from $\mathcal{T}$ on X. We write $x_i \in \mathrm{CH}^i(Y)$ for the first Chern class of the line bundle $\mathcal{T}_i/\mathcal{T}_{i-1}$, where $\mathcal{T}_0 := 0$. The morphism $\pi\colon Y \to X$ makes $\mathrm{CH}(Y)$ a $\mathrm{CH}(X)$-algebra, generated by the elements $x_1, \dots, x_n$ subject to the relations

$$\sigma_i = \pi^*(c_i), \tag{3.1}$$

where σ_i is the ith elementary symmetric polynomial in $x_1, \dots, x_n$ (see [3, Example 3.3.5]). As a $\mathrm{CH}(X)$-module, $\mathrm{CH}(Y)$ is free with a basis given by the products $x_1^{a_1} \dots x_n^{a^n}$ satisfying the conditions $a_i < i$ for $i = 1, \dots, n$ (see, e.g., [1, Lemma 4.5]).

Let us consider the product

$$e := \prod_{I\subset\{t+1,\dots,n\}} (e_1 - \sum_{i\in I} x_i) \in \mathrm{CH}^{2^{n-t}}(Y),$$

where t is the torsion exponent. The element e is the image in $\mathrm{CH}(Y)$ of the extra generator τ (for $m = t$) from the proof of Theorem 2.3. By [10, Proposition 4.4], $2e$ is in the subring $C \subset \mathrm{CH}(Y)$, generated by $x_1, \dots, x_n$. This subring is an analogue of the subring $\mathrm{CC}\mathcal{T} \subset \mathrm{CH}(\bar{X}_m)$, considered in §2.

Proposition 3.2 ([8, Theorem 3.6])**.** *One has $i_t = t$ if and only if $e \in C$.*

Remark 3.3. Replacing t by any $m = 0, \dots, n$ in the definition of e, one can define an element $e_{[m]} \in \mathrm{CH}^{2^{n-m}}(Y)$ (which will be also the image in $\mathrm{CH}(Y)$ of the extra generator τ from the proof of Theorem 2.3) and show that $2e_{[m]} \in C$. One has $e_{[m]} \notin C$ for $m > t$ and one has $e_{[m]} \in C$ for $m < t-1$. Moreover, $e_{[t-1]} \in C$ if and only if $i_{t-1} = t-1$; otherwise $i_{t-1} = t-2$. We do not dispose of any actual example with $i_{t-1} = t-2$.

Remark 3.4. We need some observations on the formula defining e. Let us replace e_1 by a formal variable x and let us view $x_1, \dots, x_n$ as formal variables as well. For any $m = 0, \dots, n$, let us define the polynomial

$$f_m(x) := \prod_{I\subset\{m+1,\dots,n\}} (x - \sum_{i\in I} x_i) \in \mathbb{Z}[x_{m+1}, \dots, x_n][x].$$

Then $f_n(x) = x$ and $f_{m-1}(x) = f_m(x) \cdot f_m(x - x_m)$ for any positive m. It follows by descending induction on $m = n, n-1, \dots, 0$ that, modulo 2, $f_m(x)$ is a sum of monomials of 2-power degrees (in x). It follows then that

$$f_m(x) \equiv f_{m+1}(x)^2 + f_{m+1}(x) \cdot f_{m+1}(x_{m+1}) \pmod 2 \tag{3.5}$$

and that x_{m+1} appears in $f_m(x)$ mod 2 only with 2-power exponents as well. By symmetry of $f_m(x)$ in the variables $x_{m+1}, \dots, x_n$, each of them also appears with 2-power exponents only.

4. Critical exponent for dimension 17

Here we apply Proposition 3.2 to calculate the critical exponent for dimension $d = 17$. Note that $t = 4$ for this d.

Theorem 4.1. *For $d = 17$ we have $i_t = t - 1$.*

Proof. The modulo 2 Chow group $\mathrm{Ch}(Y) := \mathrm{CH}(Y)/2\,\mathrm{CH}(Y)$ has an $\mathbb{Z}/2\mathbb{Z}$-basis given by the products $x_1^{a_1} \dots x_8^{a_8} e_I$ with $a_i < i$ and $I \subset \{1, \dots, 8\}$. The $\mathbb{Z}/2\mathbb{Z}$-subspace $C \subset \mathrm{Ch}(Y)$ is generated by the part of the basis without e_I (i.e., with $I = \emptyset$).

By Remark 3.4, e, as a polynomial in e_1, contains monomials of 2-power degrees only. For $e_1 \in \mathrm{Ch}(Y)$ one has $e_1^{2^i} = e_{2^i}$ for any $i \geq 0$ with the agreement $e_i := 0$ for $i > n$. To prove that $e \notin C$, it suffices to find a nonzero term with some $i > 0$. We choose to take $i = 3$, i.e., we look at the term with $e_1^{2^3} = e_1^8 = e_8$.

The coefficient at e_8 is a sum of

$$x_5^{a_5} x_6^{a_6} x_7^{a_7} x_8^{a_8} \tag{4.2}$$

with 2-powers a_5, a_6, a_7, a_8 satisfying $a_5 + a_6 + a_7 + a_8 = 8$. Since

$$x_8^8 = c_1 x_8^7 + c_2 x_8^6 + \cdots + c_7 x_8 + c_8 \in \mathrm{Ch}(Y)$$

and $c_i = 0 \in \mathrm{Ch}(X)$ for $i > 0$, we have $x_8^8 = 0$. By symmetry in $x_1, \ldots, x_8 \in \mathrm{Ch}(Y)$ of relations (3.1), we have $x_i^8 = 0$ for every $i = 1, \ldots, 8$. It follows that the nonzero terms (4.2) have exponents $a_5, a_6, a_7, a_8 \leq 4$. Therefore they belong to the above basis. At least one term (4.2) actually appears as, for instance, the term $x_7^4 x_8^4$: using formula (3.5) from Remark 3.4 one sees that $f_6(x)$ contains the monomial $x_7 x_8 x^2$ and therefore $f_4(x)$ contains $(x_7 x_8 \cdot x^2)^4 = x_7^4 x_8^4 \cdot x^8$. Consequently $e \notin C$.

Proposition 3.2 terminates the proof. □

Remark 4.3. The proving method of Theorem 4.1 does not extend to higher dimensions: for $d = 19$ as well as for most higher odd d, the element e vanishes modulo 2. This does not allow to detect if e is in C the way as in the proof of Theorem 4.1.

Corollary 4.4. *For $d = 18$, the exponents are computed as follows:*

i_0	i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8
0	1	2	3	3	3	4	4	4

Proof. Since $n = 8$ for $d = 18$, Remark 2.6 and Theorem 4.1 yield a computation of $i_4 = i_{n-4}$ and $i_5 = i_{n-3}$ for $d = 18$. The remaining exponents for this dimension have been computed in [10]. □

In particular, "any 18-dimensional non-degenerate quadratic form of trivial discriminant and Clifford invariant acquires Witt index at least 5 over some finite base field extension of degree not divisible by 2^4", as claimed in Abstract. Since $i_3 = 3$, the result is best possible.

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