ON CHARACTERISTIC CLASSES
MODULO TORSION FOR SPIN GROUPS

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Abstract. We study the ring of Chow characteristic classes (also called the Chow ring of the classifying space) for the split spin group $\mathrm{Spin}(n)$ with n odd or divisible by 4. For such n up to 12, we determine this ring modulo torsion.

Contents

1. Introduction

For an affine algebraic group G over a field, the (graded) ring $\mathrm{CH}(BG)$ of its Chow characteristic classes (also called the Chow ring of the classifying space of G) has been introduced in [22].

Assume that G is reductive and has a split maximal torus T. The kernel of the ring homomorphism $\Phi\colon \mathrm{CH}(BG) \to \mathrm{CH}(BT)$, given by the inclusion $T \hookrightarrow G$, is precisely the ideal of the elements of finite order so that the ring $\mathrm{CH}(BG)$ modulo torsion is identified with the image of Φ. Every element in the image is invariant under the action of the Weyl group W of G and the quotient $\mathrm{CH}(BT)^W/\operatorname{Im}\Phi$ (as well as the kernel of Φ) is killed by the torsion index of G, [23, Theorem 1.3(1)]. Note that the ring $\mathrm{CH}(BT)$ is know to be the symmetric $\mathbb{Z}$-algebra on the character group of T.

Let G be the split spin group $\mathrm{Spin}(n)$ over some field (on which we don't put any restriction). For arbitrary n, unlike as in topology, where the cohomology of the classifying space of G is well-understood (see [18] and [1]), its Chow ring in algebraic geometry is *"notoriously difficult to study"* ([21, Page 43]).

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For $n \le 6$ however, the torsion index of G is 1 so that $\mathrm{CH}(BG) = \mathrm{Im}\,\Phi = \mathrm{CH}(BT)^W$. Note that the subring $\mathrm{CH}(BT)^W \subset \mathrm{CH}(BT)$ of the W-invariants has been computed for arbitrary n (in the topological context) in [1, Theorem 7.1]. One can also mention [6, Table 16] showing for $n \le 6$ that G is isomorphic to a classical group for which $\mathrm{CH}(BG)$ is computed in [22] and [16]. We do not consider these values of n any further in the paper.

For $n \ge 13$, the torsion index is divisible by 4. Note that the torsion index of $\mathrm{Spin}(n)$ has been determined for any n in [23, Theorem 0.1].

For the remaining values $7, \dots, 12$ of n, the torsion index is 2. Determination of $\mathrm{Im}\,\Phi$ in this case is equivalent to determination of the image for the modulo 2 reduction

$$\varphi\colon\ \mathrm{Ch}(BG) \to \mathrm{Ch}(BT)$$

of Φ, where $\mathrm{Ch}(-)$ is the Chow ring $\mathrm{CH}(-)/2\,\mathrm{CH}(-)$ with coefficients $\mathbb{Z}/2\mathbb{Z}$. Our main result affirms that for $n \ne 10$ the image of φ is the subring

$$(\mathrm{Ch}(BT)^W)^2 := \{a^2,\ a \in \mathrm{Ch}(BT)^W\} \subset \mathrm{Ch}(BT)^W$$

of squares in $\mathrm{Ch}(BT)^W$:

Theorem 1.1. *For the algebraic group $G = \mathrm{Spin}(n)$ with $n = 7, 8, 9, 11, 12, 13$ and its split maximal torus T, the image of $\varphi\colon \mathrm{Ch}(BG) \to \mathrm{Ch}(BT)$ is the ring $(\mathrm{Ch}(BT)^W)^2$ of squares of the W-invariants.*

The consequence for the integral Chow group is captured in Corollary 3.5.

The value $n = 13$ is included into Theorem 1.1 because it does not require additional efforts.

The value 10 of n is excluded because the statement of Theorem 1.1 fails for every $n \ge 7$ which is 2 modulo 4: the image under φ of the highest Chern class in $\mathrm{Ch}(BG)$ of the half-spin representation of G is not a square in $\mathrm{Ch}(BT)^W$ (cf. the proof of Proposition 3.4). At the same time, the odd degree Euler class (defined in Proposition 2.1), occurring for such n, creates additional complications for our approach to determination of the image of φ. For $n = 10$, the Euler class is in the image of Φ (see Lemma A.1) and, viewed modulo 2, yields another example of a non-square element in $\mathrm{Im}\,\varphi$; applying appropriate Steenrod operations to it (see §4), one can enlarge the number of such examples even further.

The ring $\mathrm{Ch}(BT)^W$ of W-invariants in the Chow ring $\mathrm{Ch}(BT)$ with coefficients $\mathbb{Z}/2\mathbb{Z}$ is easy to compute for arbitrary n – see Proposition 3.2. Note that the image of $\mathrm{CH}(BT)^W$ under the reduction modulo 2 homomorphism $\mathrm{CH}(BT) \to \mathrm{Ch}(BT)$ is in general smaller than $\mathrm{Ch}(BT)^W$ (see Remark 3.3).

Since the entire ring $\mathrm{CH}(B\,\mathrm{Spin}(n))$ (over a field of characteristic different from 2) is computed for $n = 7$ in [7] and [19] as well as for $n = 8$ in [19], the statement of Theorem 1.1 in these cases is not new.

Concerning the proof of Theorem 1.1, it is easy to check that $(\mathrm{Ch}(BT)^W)^2 \subset \mathrm{Im}\,\varphi$ for any n – see Proposition 3.4: the job is done by computing the images in $\mathrm{Ch}(BT)$ of the Chern classes in $\mathrm{Ch}(BG)$ for the (half-)spin and the orthogonal representations of G.

The opposite inclusion (now for the specified in Theorem 1.1 values of n) is obtained by combining the methods of [11] (summarized in §2) and [10] (pushed further in §4), where the second approach makes use of the Steenrod operations St^i, $i \ge 0$ on the modulo 2

Chow groups. These operations extend to the Steenrod operations Sq^{2i}, $i \geq 0$ in the motivic cohomology with coefficients $\mathbb{Z}/2\mathbb{Z}$, where, like as well in topology, the Steenrod algebra has one more generator – the Bockstein homomorphism Sq^1. We observe a formal similarity between Sq^1 and St^1. The similarity observed allows us to apply to our setting some topological techniques and computations related to the Bockstein cohomology (see the proof of Lemma 4.2). This results in a new bound on $\operatorname{Im}\varphi$ valid for arbitrary $n \geq 7$ which is odd or divisible by 4 (see Proposition 4.3). Adding on top the restrictions provided by the even Steenrod operations St^2, St^4, and St^8, we achieve the proof of Theorem 1.1.

Theorem 1.1 has been recently applied in [8] to prove a conjecture on the Chow ring of characteristic classes for the special Clifford groups. This conjecture was the only obstacle for obtaining an algorithm computing the maximal indexes of twisted spin grassmannians.

2. Known restrictions on $\operatorname{Im}\Phi$

Here we describe a stronger than $\mathrm{CH}(BT)^W$ upper bound on $\operatorname{Im}\Phi$ obtained in [11].

Depending on parity, we write the integer $n \geq 7$ in the form $n = 2l + 1$ or in the form $n = 2l$ (with an integer l) and identify the graded ring $\mathrm{CH}(BT)$ with the polynomial ring $\mathbb{Z}[z, x_1, \ldots, x_l]$ in the $l + 1$ variables modulo the homogeneous relation $2z = x_1 + \ldots + x_l$. For odd $n = 2l + 1$, the Weyl group W is a semidirect product of the symmetric group S_l and the direct product $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ of l copies of $\mathbb{Z}/2\mathbb{Z}$. The action of W on $\mathrm{CH}(BT)$ is induced by its action on the polynomial ring, in which S_l acts trivially on z and permutes $x_1, \ldots, x_l$, whereas the ith copy of $\mathbb{Z}/2\mathbb{Z}$ acts by $x_i \mapsto -x_i$, $z \mapsto z - x_i$, and trivially on the remaining variables. We let $\tilde{z}$ to be the product of the elements in the orbit of z:

$$\tilde{z} = \prod_{I \subset \{1,\ldots,l\}} (z - \sum_{i \in I} x_i) \in \mathbb{Z}[z, x_1, \ldots, x_l].$$

For even $n = 2l$, the Weyl group W is a semidirect product of S_l and the subgroup in $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ of the elements with an even number of nonzero components, acting by restriction of the odd case action. We let $\check{z}$ to be the product of the elements in the orbit of z:

$$\check{z} = \prod_{\text{even } I \subset \{1,\ldots,l\}} (z - \sum_{i \in I} x_i),$$

where an *even* subset is a subset with even number of elements.

It is shown in [11] that $\operatorname{Im}\Phi$ is contained in the image of $\mathbb{Z}[z, x_1, \ldots, x_l]^W$ (which, in general, is strictly smaller than $\mathrm{CH}(BT)^W$). Moreover, the ring $\mathbb{Z}[z, x_1, \ldots, x_l]^W$ is computed in [12, Proposition 2.4] and [14, Proposition 5.1] (see also [5]). As a part of this computation, for $n = 2l + 1$ and every $i \geq 0$, certain homogeneous W-invariant element $f_i \in \mathbb{Z}[z, x_1, \ldots, x_l]^W$ of degree 2^i is constructed. The element f_0 equals $2z - (x_1 + \ldots + x_l)$ and vanishes in $\mathrm{CH}(BT)$. As a result, we get

Proposition 2.1 ([11, Theorems 2.2 and 3.2]). *Let $G = \mathrm{Spin}(n)$ with $n \geq 7$ and let $S \subset \mathbb{Z}[z, x_1, \ldots, x_l]^W$ be the subring of symmetric polynomials in the squares $x_1^2, \ldots, x_l^2$. For $n = 2l + 1$, the image of $\Phi\colon \mathrm{CH}(BG) \to \mathrm{CH}(BT)$ is contained in the S-subalgebra of $\mathrm{CH}(BT)$ generated by $f_1, \ldots, f_{l-1}$ and the orbit product $\tilde{z}$ of z (of degree 2^l). The generator $\tilde{z}$ equals $\tilde{z}_1^2$, where $\tilde{z}_1 \in \mathrm{CH}(BT)^W$ is defined below.*

In the case of $n = 2l$, the image of Φ is contained in the S-subalgebra generated by $f_1, \dots, f_{l-2}$, the orbit product $\check{z}$ of z (now of degree 2^{l-1}), and the element $e := x_1 \dots x_l$ (called the Euler class). If n is divisible by 4 (i.e., l is even), the generator $\check{z}$ equals $\check{z}_1^2$, where $\check{z}_1 \in \mathrm{CH}(BT)^W$ is defined below.

Remark 2.2. The subring $S \subset \mathrm{CH}(BT)$ is the image of the composition

$$\mathrm{CH}(B\,\mathrm{O}(n)) \longrightarrow \mathrm{CH}(BG) \xrightarrow{\Phi} \mathrm{CH}(BT),$$

where $\mathrm{O}(n)$ is the standard split orthogonal group. More precisely, the elementary symmetric polynomials in the squares $x_1^2, \dots, x_l^2$ are, up to a sign, the images of the even Chern classes in $\mathrm{CH}(B\,\mathrm{O}(n))$ of the standard representation $\mathrm{O}(n) \hookrightarrow \mathrm{GL}(n)$. In particular, $S \subset \mathrm{Im}\,\Phi$. By [22] (see also [16]), the Chern classes of the standard representation generate the ring $\mathrm{CH}(B\,\mathrm{O}(n))$. The odd ones have exponent 2 and vanish in $\mathrm{CH}(BT)$.

Remark 2.3. The group $G = \mathrm{Spin}(n)$ is defined (e.g., in [15, §23A]) as a subgroup in $\mathrm{GL}_1(C_0(n))$, where $C_0(n)$ is the even Clifford algebra of the standard split n-dimensional quadratic form. For even n, the algebra $C_0(n)$ is the product of two copies of a split central simple algebra $C^+(n)$. The two representations $G \to \mathrm{GL}_1(C^+(n))$, given by the two projections $C_0(n) \to C^+(n)$, are irreducible and called the half-spin representations of G. There sum is the spin representation $G \to \mathrm{GL}_1(C_0(n))$. The image in $\mathrm{CH}(BT)$ of the highest Chern class in $\mathrm{CH}(BG)$ of one half-spin representation is equal to $\check{z}$. (The other half-spin representation yields $\prod_{\text{odd } I \subset \{1,\dots,l\}} (z - \sum_{i \in I} x_i)$.)

For odd n, $C_0(n)$ is a split central simple algebra and $G \to \mathrm{GL}_1(C_0(n))$ is the spin representation. This representation is irreducible and the image in $\mathrm{CH}(BT)$ of its highest Chern class is $\tilde{z}$.

The upper bound on $\mathrm{Im}\,\Phi$ described in Proposition 2.1 is in general smaller than the ring $\mathrm{CH}(BT)^W$, computed in [1]. For odd n, let us define

$$\tilde{z}_1 := \prod_{I \subset \{2,\dots,l\}} (z - \sum_{i \in I} x_i) \in \mathrm{CH}(BT).$$

Because of the relation $2z = x_1 + \ldots + x_l$, which holds in $\mathrm{CH}(BT)$, the element $\tilde{z}_1$ is W-invariant and $\tilde{z}_1^2 = \tilde{z}$.

Similarly, for n divisible by 4, let us define

$$\check{z}_1 := \prod_{\text{even } I \subset \{2,\dots,l\}} (z - \sum_{i \in I} x_i) \in \mathrm{CH}(BT).$$

Then $\check{z}_1$ is W-invariant and $\check{z}_1^2 = \check{z}$.

Proposition 2.4 ([1, Theorem 7.1])**.** *Assume that $n \geq 7$. For odd n, the S-algebra $\mathrm{CH}(BT)^W$ is generated by $f_1, \dots, f_{l-2}$ and $\tilde{z}_1$. For n divisible by 4, the S-algebra $\mathrm{CH}(BT)^W$ is generated by $e, f_1, \dots, f_{l-3}$ and $\check{z}_1$. For even n not divisible by 4, the S-algebra $\mathrm{CH}(BT)^W$ is generated by $e, f_1, \dots, f_{l-2}$ and $\check{z}$.*

Remark 2.5. Instead of the generators $f_1, \dots, f_{l-2}$, some different generators $q_1, \dots, q_{l-2}$ (homogeneous of degrees $2^1, \dots, 2^{l-2}$ as well) are used in [1]. However, as shown in [11, Lemma 2.3], both generate the same subring in $\mathrm{CH}(BT)$.

3. Computation of $\mathrm{Ch}(\mathsf{B}T)^W$

To decode the statement of Theorem 1.1, we provide a description of W-invariants $\mathrm{Ch}(\mathsf{B}T)^W$ for the modulo 2 Chow ring. First of all, the ring $\mathrm{Ch}(\mathsf{B}T)$ itself is the polynomial ring $\mathbb{F}[z, x_1, \dots, x_l]$ in the $l+1$ variables modulo the relation $x_1 + \ldots + x_l = 0$, where $\mathbb{F} := \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. So, $\mathrm{Ch}(\mathsf{B}T)$ is isomorphic to the polynomial ring $\mathbb{F}[z, x_2, \dots, x_l]$ in the variables other than x_1. The elementary symmetric polynomials $c_1, \dots, c_l$ in $x_1, \dots, x_l$ (where $c_1 = 0$ in $\mathrm{Ch}(\mathsf{B}T)$) are W-invariant.

Example 3.1. The quotient ring $R := \mathbb{Z}[x_1, \dots, x_l]/(x_1 + \ldots + x_l)$ can be viewed as the symmetric $\mathbb{Z}$-algebra of the character group of the standard split maximal torus in the special linear group $\mathrm{SL}(l)$. The Weyl group of $\mathrm{SL}(l)$ is the symmetric group S_l acting on R by permutations of $x_1, \dots, x_l$. By [13, Lemma 8.1] we have $R^{S_l} = \mathbb{Z}[c_2, \dots, c_n]$. It follows by [4, Théorème] that $(R \otimes \mathbb{F})^{S_l} = R^{S_l} \otimes \mathbb{F} = \mathbb{F}[c_2, \dots, c_n]$.

Proposition 3.2. *The $\mathbb{F}[c_2, \dots, c_l]$-algebra $\mathrm{Ch}(\mathsf{B}T)^W$ is generated by the following single element: $\tilde{z}_1$ for odd n, $\check{z}_1$ for n divisible by 4, $\check{z}$ for even n not divisible by 4.*

Proof. Since the group $(\mathbb{Z}/2\mathbb{Z})^{\times l}$ acts on $\mathrm{Ch}(\mathsf{B}T)$ trivially, the invariants under its intersection with W contain the $\mathbb{F}[x_1, \dots, x_l]$-subalgebra of $\mathrm{Ch}(\mathsf{B}T)$ generated by the orbit product of $z \in \mathrm{Ch}(\mathsf{B}T)$ which is equal – depending on $n \pmod 4$ – to $\tilde{z}_1$, $\check{z}_1$, or $\check{z}$. Since the linear factors of each orbit product are distinct primes of the polynomial ring $\mathbb{F}[z, x_2, \dots, x_l]$, the inclusion is actually an equality (cf. [5, Proof of Lemma 3.2]). Taking additionally into account the action of $S_l \subset W$ (trivial on the above orbit products) and Example 3.1, we come to the announced answer. ☐

Remark 3.3. Under the reduction modulo 2 homomorphism

$$\mathbb{Z}[z, x_1, \dots, x_l] \to \mathbb{F}[z, x_1, \dots, x_l]$$

of the polynomial rings, the images of the generators $f_0, f_1, \dots$ are determined as follows: the image of f_0 is $c_1 = x_1 + \ldots + x_l$ and for every $i \geq 0$ the image of f_{i+1} is the sum of pairwise products of distinct monomials in the image of f_i. In particular, these images are symmetric polynomials in $x_1, \dots, x_l$ (the variable z does not intervene). The element f_0 vanishes in $\mathrm{Ch}(\mathsf{B}T)$, whereas f_1 and f_2 map respectively to c_2 and c_4, where $c_i := 0$ for $i > l$. The formulas for f_i with $i \geq 3$ are more complicated.

We can already prove the easy inclusion of Theorem 1.1:

Proposition 3.4. *For any n, the image of φ contains $(\mathrm{Ch}(\mathsf{B}T)^W)^2$.*

Proof. For odd n, $\tilde{z}_1^2 = \tilde{z} \in \mathrm{CH}(\mathsf{B}T)$ is the image under Φ of the highest Chern class of the spin representation of G (see Remark 2.3). For even n, $\check{z} \in \mathrm{CH}(\mathsf{B}T)$ is the image under Φ of highest Chern class of a half-spin representation of G and $\check{z}_1^2 = \check{z}$ for n divisible by 4 (see Remark 2.3). Finally, the squares $c_1^2, \dots, c_n^2 \in \mathrm{Ch}(\mathsf{B}T)$ are the images under φ of the even Chern classes of the orthogonal representation of G (see Remark 2.2). ☐

Thus, Theorem 1.1 yields

Corollary 3.5. *We set $t := \tilde{z}$ for odd n and we set $t := \check{z}$ for even n. Let $S' \subset \mathrm{CH}(\mathsf{B}T)$ be the subring generated by t, S, and $2\,\mathrm{CH}(\mathsf{B}T)^W$. Then $\operatorname{Im}\Phi = S'$ for $n = 7, 8, 9, 11, 12$ and $\operatorname{Im}\Phi \subset S'$ for $n = 13$.*

Proof. For any n as in Theorem 1.1, any element of $\operatorname{Im}\Phi$ is a sum of an element of S' with an element $\alpha \in 2\operatorname{CH}(BT)$. It follows that $\alpha \in 2\operatorname{CH}(BT) \cap \operatorname{CH}(BT)^W = 2\operatorname{CH}(BT)^W$. If $n \leq 12$, the torsion index of G is 2 so that $2\operatorname{CH}(BT)^W \subset \operatorname{Im}\Phi$. $\square$

4. Restrictions on $\operatorname{Im}\varphi$

In this section, we discuss restrictions on the image of $\varphi\colon \operatorname{Ch}(BG) \to \operatorname{Ch}(BT)$, where $G = \operatorname{Spin}(n)$ with arbitrary $n \geq 7$. First of all, an upper bound on $\operatorname{Im}\varphi$ is given by the image of the subring described in Proposition 2.1. Another restriction, already considered in [10] and pushed further below, is given by the action of the modulo 2 Steenrod algebra. Combining the two restrictions will be our ultimate strategy.

We have a commutative square

(4.1)
$$\begin{tikzcd}
\operatorname{Ch}(BG) \arrow[r, "\varphi"] \arrow[d, "\operatorname{St}"] & \operatorname{Ch}(BT) \arrow[d, "\operatorname{St}"] \\
\operatorname{Ch}(BG) \arrow[r, "\varphi"] & \operatorname{Ch}(BT)
\end{tikzcd}$$

where St is the total cohomological Steenrod operation, constructed for smooth algebraic varieties in characteristic $\neq 2$ in [3] and in characteristic 2 in [17]. It is also defined for classifying spaces of affine algebraic groups via their approximations by algebraic varieties introduced in [22]. The operation St is a (nonhomogeneous) ring homomorphism, determined in the case of $\operatorname{Ch}(BT) = \mathbb{F}[z, x_1, \ldots, x_l]$ by the rule $\operatorname{Ch}^1(BT) \ni a \mapsto a + a^2$.

It follows from (4.1) that $\operatorname{Im}\varphi$ is stable under St. Moreover, being graded, the image of φ is stable for every $i \geq 0$ under the ith graded component St^i of St, rasing the degree by i. (The negative graded components of St are trivial.)

The image of φ is contained in the subring $\mathbb{F}[z, c_2, \ldots, c_l] \subset \mathbb{F}[z, x_1, \ldots, x_l] = \operatorname{Ch}(BT)$ which is also stable under St. The subring $\mathbb{F}[c_2, \ldots, c_l]$ is stable under St as well. For any $i, j \geq 0$, a formula for $\operatorname{St}^i(c_j)$ (where $c_0 := 1$) is provided in [2, Théorème 7.1] and applied here below. Note that $\operatorname{St}^i(c_j)$ vanishes for $i > j$, equals c_j^2 for $i = j$, and is equal to

$$\operatorname{St}^i(c_j) = \sum_{k=0}^{i} \binom{i-j}{k} c_{i-k} c_{j+k}$$

otherwise. The binomial coefficient in this simplified formula (borrowed from [20, Proposition 3.1.12]) is taken modulo 2 and has a negative upper entry.

We remind that c_1 is trivial in our setting.

Note that the additive map $\operatorname{St}^1\colon \mathbb{F}[c_2, \ldots, c_l] \to \mathbb{F}[c_2, \ldots, c_l]$ vanishes on $\mathbb{F}[c_2^2, \ldots, c_l^2]$ and therefore is a homomorphism of $\mathbb{F}[c_2^2, \ldots, c_l^2]$-modules. The $\mathbb{F}[c_2^2, \ldots, c_l^2]$-module $\mathbb{F}[c_2, \ldots, c_l]$ is free with the basis consisting of $c_I := \prod_{i \in I} c_i$ with $I \subset \{2, \ldots, l\}$.

Here is the key observation of the section:

Lemma 4.2. *The kernel of* $\operatorname{St}^1\colon \mathbb{F}[c_2, \ldots, c_l] \to \mathbb{F}[c_2, \ldots, c_l]$ *is the* $\mathbb{F}[c_2^2, \ldots, c_l^2]$*-module generated by* 1, c_l *and all* $\operatorname{St}^1(c_I)$.

Proof. We have $\operatorname{St}^1(c_i) = c_{i+1}$ for even i (with the agreement $c_{l+1} := 0$) and $\operatorname{St}^1(c_i) = 0$ for odd i. Since $\operatorname{St}^1(ab) = \operatorname{St}^1(a)b + a\operatorname{St}^1(b)$, the above rules determine the additive map $\operatorname{St}^1\colon \mathbb{F}[c_2, \ldots, c_l] \to \mathbb{F}[c_2, \ldots, c_l]$. Note that $\operatorname{St}^1 \circ \operatorname{St}^1 = 0$ so that the kernel of St^1 contains its image. The kernel is a ring, containing the squares $\mathbb{F}[c_2^2, \dots, c_l^2]$ and c_l. The image is an ideal in this ring. The quotient is known to be the ring generated by the squares for odd l; for even l, it is generated by c_l and the squares (see, e.g., [1, §9] dealing with the topological Sq^1 in place of St^1). ☐

Proposition 4.3. *Assume that $n \geq 7$ is odd or divisible by 4. If l is odd, then the image of φ is contained in the $(\mathrm{Ch}(\mathsf{B}T)^W)^2$-submodule of $\mathrm{Ch}(\mathsf{B}T)^W$ generated by 1 and all $\mathrm{St}^1(c_I)$ with c_I of odd degree. If l is even, then the image of φ is contained in the $(\mathrm{Ch}(\mathsf{B}T)^W)^2$-submodule of $\mathrm{Ch}(\mathsf{B}T)^W$ generated by 1, c_l, and all $\mathrm{St}^1(c_I)$ with c_I of odd degree.*

Proof. By Proposition 2.4, the assumption on n ensures that the graded ring $\operatorname{Im}\varphi$ is concentrated in even degrees. It follows that $\operatorname{Im}\varphi$ vanishes under the first Steenrod operation $\mathrm{St}^1\colon \mathrm{Ch}(\mathsf{B}T) \to \mathrm{Ch}(\mathsf{B}T)$.

By Proposition 2.1 and Remark 3.3, any element in $\operatorname{Im}\varphi$ is a polynomial in t^2 with coefficients in $\mathbb{F}[c_2, \dots, c_l]$, where $t := \tilde{z}_1$ for odd n and $t := \check{z}_1$ for n divisible by 4. Note that t is divisible by z in $\mathbb{F}[z, c_2, \dots, c_l]$.

Let $a \in \mathbb{F}[t^2, c_2, \dots, c_l]$ be any polynomial in t^2 with coefficients in $\mathbb{F}[c_2, \dots, c_l]$ satisfying $\mathrm{St}^1(a) = 0$. To prove Proposition 4.3 for odd l, it suffices to show that the coefficients of a are linear combinations with coefficients in $\mathbb{F}[c_2^2, \dots, c_l^2]$ of 1 and all $\mathrm{St}^1(c_I)$ with c_I of odd degrees. For even l, is suffices to show that the coefficients of a are linear combinations with coefficients in $\mathbb{F}[c_2^2, \dots, c_l^2$ of 1, c_l, and all $\mathrm{St}^1(c_I)$ with c_I of odd degrees. We prove that the coefficients of a have the required form by induction on degree of a.

If a is constant (i.e., $a \in \mathbb{F}[c_2, \dots, c_l]$), the statement follows by Lemma 4.2. Otherwise, we have $a = a't^2 + b$, where a' is a polynomial in t^2 of smaller degree and b is the constant term of a. We have $0 = \mathrm{St}^1(a) = \mathrm{St}^1(a')t^2 + \mathrm{St}^1(b)$ implying that $\mathrm{St}^1(a') = 0 = \mathrm{St}^1(b)$. It follows that b and the coefficients of a' have the required form. ☐

5. Proof of Theorem 1.1

This section is the proof of Theorem 1.1. More precisely, since we already proved Proposition 3.4, we prove here that $\operatorname{Im}\varphi \subset (\mathrm{Ch}(\mathsf{B}T)^W)^2$ for the values of n listed in the statement of Theorem 1.1. We do this by employing the upper bound on $\operatorname{Im}\varphi$ given in Proposition 4.3. Besides, we continue to employ the fact that $\operatorname{Im}\varphi$ is stable under the Steenrod operations St^i on $\mathrm{Ch}(\mathsf{B}T)$. (To get Proposition 4.3, we only used St^1.) Note that the subring $\mathrm{Ch}(\mathsf{B}T)^W \subset \mathrm{Ch}(\mathsf{B}T)$ is stable under the Steenrod operations because W acts on $\mathrm{Ch}(\mathsf{B}T)$ through automorphisms of approximations of $\mathsf{B}T$.

Continuing the analogy between the operation St^1 on $\mathrm{Ch}(\mathsf{B}T)$ and the Bockstein operation Sq^1 in the motivic cohomology, let us note that every odd operation St^{2i+1} on $\mathrm{Ch}(\mathsf{B}T)$ is the composition $\mathrm{St}^1 \circ \mathrm{St}^{2i}$. (See [24, Lemma 9.6] for the corresponding property of Sq^1.) Since we already exhausted (in Proposition 4.3) stability of $\operatorname{Im}\varphi$ under St^1, the additional restrictions on $\operatorname{Im}\varphi$ will come from the action of the even Steenrod operations. More exactly, we will be using St^2 and St^4 only.

Recall that the image of φ is a subring of the ring $B := \mathrm{Ch}(\mathsf{B}T)^W = \mathbb{F}[t, c_2, \dots, c_l]$, where $t := \tilde{z}_1$ for odd n and $t := \check{z}_1$ for n divisible by 4. (We do not consider the values of n congruent to 2 modulo 4 because they do not appear in Theorem 1.1.) The generators

$t, c_2, \dots, c_l$ of B are algebraically independent. By Proposition 2.1, $\operatorname{Im}\varphi$ is actually inside the smaller ring $A := \mathbb{F}[t^2, c_2, \dots, c_l]$. Note that A is stable under the Steenrod operations on B: for any $i \geq 0$, $\mathrm{St}^{2i+1}(t^2)$ vanishes and $\mathrm{St}^{2i}(t^2) = \mathrm{St}^i(t)^2 \in \mathbb{F}[t^2, c_2^2, \dots, c_l^2]$. By Proposition 3.4, $\operatorname{Im}\varphi$ contains the subring B^2 of squares in B, which is also stable under the Steenrod operations. As an B^2-module, A is free with the basis given by the 2^{l-1} products $c_I = \prod_{i\in I} c_i$, where I runs over the subsets in $\{2, \dots, l\}$.

$\mathbf{n = 7}$. Here we have $l = 3$ and we apply Proposition 4.3. There are only two elements c_I of odd degree: c_3 and c_2c_3. They satisfy $\mathrm{St}^1(c_3) = 0 \in B^2$ and $\mathrm{St}^1(c_2c_3) = c_3^2 \in B^2$. The statement under proof follows.

$\mathbf{n = 8, 9}$. We have $\mathrm{Ch}(BT)^W = \mathbb{F}[t, c_2, c_3, c_4] = B$ and $A = \mathbb{F}[t^2, c_2, c_3, c_4]$. By Proposition 4.3, $\operatorname{Im}\varphi$ is contained in the B^2-submodule of A generated by 1, c_4, and the elements

$$\begin{aligned}
\mathrm{St}^1(c_3) &= 0 \in B^2,\\
\mathrm{St}^1(c_3c_2) &= c_3^2 \in B^2,\\
\mathrm{St}^1(c_3c_4) &= 0 \in B^2,\\
\mathrm{St}^1(c_3c_2c_4) &= c_3^2 \cdot c_4 \in B^2 \cdot c_4.
\end{aligned}$$

Therefore any element α of $\operatorname{Im}\varphi$ has the form $\alpha = a^2 + b^2 \cdot c_4$ with $a, b \in B$. We have

$$B^2[c_4] \ni \mathrm{St}^2(\alpha) = (\mathrm{St}^1(a))^2 + (\mathrm{St}^1(b))^2 \cdot c_4 + b^2 \cdot c_2c_4$$

because $\mathrm{St}^2(c_4) = c_2c_4$. It follows that $b^2 \cdot c_2c_4 \in B^2[c_4]$ and therefore $b = 0$ meaning that $\alpha \in B^2$.

$\mathbf{n = 11}$. We have $\mathrm{Ch}(BT)^W = \mathbb{F}[t, c_2, c_3, c_4, c_5] = B$ and $A = \mathbb{F}[t^2, c_2, c_3, c_4, c_5]$. By Proposition 4.3, $\operatorname{Im}\varphi$ is contained in the B^2-module generated by

$$\begin{aligned}
\mathrm{St}^1(c_3c_2) &= c_3^2 \in B^2,\\
\mathrm{St}^1(c_3c_4) &= c_3c_5,\\
\mathrm{St}^1(c_3c_2c_4) &= c_3^2 \cdot c_4 + c_2c_3c_5,\\
\mathrm{St}^1(c_5c_2) &= c_3c_5,\\
\mathrm{St}^1(c_5c_4) &= c_5^2 \in B^2,\\
\mathrm{St}^1(c_5c_2c_4) &= c_5^2 \cdot c_2 + c_3c_4c_5.
\end{aligned} \tag{5.1}$$

So, any element of $\operatorname{Im}\varphi$ has the form

$$a^2 + b^2 \cdot c_3c_5 + c^2 \cdot (c_3^2 \cdot c_4 + c_2c_3c_5) + d^2 \cdot (c_5^2 \cdot c_2 + c_3c_4c_5) \tag{5.2}$$

with $a, b, c, d \in B$. The value of St^2 at $\alpha \in \operatorname{Im}\varphi$ should also have such a form. In particular, the value $\mathrm{St}^2(\alpha) \in A$ should vanish in the quotient A/A', where $A' \subset A$ is the B^2-submodule with the basis consisting of all c_I showing up in (5.2): 1, c_3c_5, c_4, $c_2c_3c_5$, c_2, $c_3c_4c_5$. (One can take the smaller A', generated by 1, c_3c_5, $c_4 + c_2c_3c_5$, $c_2 + c_3c_4c_5$, but this will only bring unnecessary complications.)

Recall that A is a free B^2-module with the basis $\{c_I\}_{I\subset\{2,3,4,5\}}$. Therefore A/A' is free with the basis consisting of all c_I not included in the basis of A'. Let us compute the image of $\mathrm{St}^2(\alpha) \in A$ in the quotient A/A'. For the computation, recall that St^1 vanishes on $B^2 \subset A$ as well as on every summand of (5.2). Concerning St^2, the formulas we need are

$$\begin{gathered}\mathrm{St}^2(c_3c_5) = c_5^2 \in B^2,\\ \mathrm{St}^2(c_3^2 \cdot c_4 + c_2c_3c_5) = c_2^2 \cdot c_3c_5 + c_3^2 \cdot c_2c_4 + c_5^2 \cdot c_2,\\ \mathrm{St}^2(c_5^2 \cdot c_2 + c_3c_4c_5) = (c_2c_5)^2 + c_2c_3c_4c_5 + c_5^2 \cdot c_4.\end{gathered}$$

Note that unlike St^1, the additive map $\mathrm{St}^2\colon A \to A$ is *not* a homomorphism of B^2-modules: $\mathrm{St}^2(b^2 \cdot a)$ is the sum of $b^2 \cdot \mathrm{St}^2(a)$ with the additional term $\mathrm{St}^1(b)^2 \cdot a$. However the value of St^2 at (5.2), considered in the quotient A/A', is just

$$a^2 + b^2 \cdot \mathrm{St}^2(c_3c_5) + c^2 \cdot \mathrm{St}^2(c_3^2 \cdot c_4 + c_2c_3c_5) + d^2 \cdot \mathrm{St}^2(c_5^2 \cdot c_2 + c_3c_4c_5).$$

It follows that

$$\mathrm{St}^2(\alpha) \mod A' = (cc_3)^2 \cdot c_2c_4 + d^2 \cdot c_2c_3c_4c_5$$

for $\alpha \in \operatorname{Im}\varphi$ written in the form (5.2). The coefficients $(cc_3)^2$ and d^2 have to vanish and therefore $c = 0 = d$.

So, any element α of $\operatorname{Im}\varphi$ actually has the simpler form $\alpha = a^2 + b^2 \cdot c_3c_5$ with $a, b \in B$. Since $\mathrm{St}^4(\alpha)$ also has such a form, we get that $b^2\,\mathrm{St}^4(c_3c_5)$ is in $B^2[c_3c_5]$. If follows by the formula

$$\begin{aligned}\mathrm{St}^4(c_3c_5) = \mathrm{St}^2(c_3) \cdot \mathrm{St}^2(c_5) + c_3 \cdot \mathrm{St}^4(c_5) =&\\ (c_2c_3 + c_5) \cdot (c_2c_5) + c_3 \cdot (c_4c_5) &= c_2^2 \cdot c_3c_5 + c_5^2 \cdot c_2 + c_3c_4c_5\end{aligned}$$

that $b = 0$. Therefore $\alpha \in B^2$.

$\mathbf{n = 12, 13}$. Here we have

$$\mathrm{Ch}(BT)^W = \mathbb{F}[t, c_2, c_3, c_4, c_5, c_6] = B \quad \text{and} \quad A = \mathbb{F}[t^2, c_2, c_3, c_4, c_5, c_6].$$

By Proposition 4.3, $\operatorname{Im}\varphi$ is contained in the B^2-module generated by 1, c_6 along with the elements outside B^2 from (5.1) and their products with c_6. So, any element of $\operatorname{Im}\varphi$ has the form

$$\begin{aligned}(5.3)\quad a^2 + b^2 \cdot c_3c_5 + c^2 \cdot (c_3^2 \cdot c_4 + c_2c_3c_5) + d^2 \cdot (c_5^2 \cdot c_2 + c_3c_4c_5) +&\\ \big(a_*^2 + b_*^2 \cdot c_3c_5 + c_*^2 \cdot (c_3^2 \cdot c_4 + c_2c_3c_5) + d_*^2 \cdot (c_5^2 \cdot c_2 + c_3c_4c_5)\big)&c_6\end{aligned}$$

with $a, b, c, d, a_*, b_*, c_*, d_* \in B$. The value of St^2 at $\alpha \in \operatorname{Im}\varphi$ should also have such a form. In particular, $\mathrm{St}^2(\alpha) \in A$ should vanish in the quotient A/A', where $A' \subset A$ is the B^2-submodule with the basis consisting of all c_I showing up in (5.2): 1, c_3c_5, c_4, $c_2c_3c_5$, c_2, $c_3c_4c_5$, and their products with c_6.

Recall that A is a free B^2-module with the basis $\{c_I\}_{I \subset \{2,\dots,6\}}$. Therefore A/A' is free with the basis consisting of all c_I not included in the basis of A'. Let us compute the image of $\mathrm{St}^2(\alpha) \in A$ in the quotient A/A'. For the computation, recall that St^1 vanishes on $B^2 \subset A$ as well as on every summand of (5.3). Concerning St^2, here are the formulas we need:

$$\begin{aligned}
\mathrm{St}^2(c_3c_5) &= c_5^2 \equiv 0,\\
\mathrm{St}^2(c_3^2\cdot c_4 + c_2c_3c_5) &= c_2^2\cdot c_3c_5 + c_3^2\cdot c_2c_4 + c_5^2\cdot c_2 + c_3^2\cdot c_6 \equiv c_3^2\cdot c_2c_4,\\
\mathrm{St}^2(c_5^2\cdot c_2 + c_3c_4c_5) &= (c_2c_5)^2 + c_2c_3c_4c_5 + c_5^2\cdot c_4 + c_3c_5c_6 \equiv c_2c_3c_4c_5,\\
\mathrm{St}^2(c_6) = c_2c_6 \equiv 0, &\qquad \mathrm{St}^2(c_3c_5c_6) = c_2c_3c_5c_6 + c_5^2\cdot c_6 \equiv 0,\\
\mathrm{St}^2(c_3^2\cdot c_4c_6 + c_2c_3c_5c_6) &= c_5^2\cdot c_2c_6 + (c_3c_6)^2 \equiv 0,\\
\mathrm{St}^2(c_5^2\cdot c_2c_6 + c_3c_4c_5c_6) &= c_6^2\cdot c_3c_5 + c_5^2\cdot c_4c_6 \equiv 0
\end{aligned}$$

with the congruences modulo A'. It follows that

$$\mathrm{St}^2(\alpha) \mod A' = (cc_3)^2\cdot c_2c_4 + d^2\cdot c_2c_3c_4c_5$$

for any $\alpha\in\operatorname{Im}\varphi$ written in the form (5.3). We conclude that $c=0=d$. This means that any element of $\operatorname{Im}\varphi$ has the form (5.3) with $c=0=d$.

Now we modify the submodule A' by removing from its basis the elements c_4, $c_2c_3c_5$, c_2, and $c_3c_4c_5$. (Their products with c_6 are kept.) Note that for any α of the form (5.3) with $c=0=d$, we have $\mathrm{St}^2(\alpha)\in A'$. So, we are going to exploit the next condition that $\mathrm{St}^4(\alpha)$ has to be in A' as well provided that $\alpha\in\operatorname{Im}\varphi$. The formula for computing $\mathrm{St}^4(\alpha)$ mod A' is just like if St^4 were a homomorphism of B^2-modules:

$$\begin{aligned}
&b^2\cdot\mathrm{St}^4(c_3c_5) + a_*^2\cdot\mathrm{St}^4(c_6) + b_*^2\cdot\mathrm{St}^4(c_3c_5c_6) +\\
&\qquad c_*^2\cdot\mathrm{St}^4(c_3^2\cdot c_4c_6 + c_2c_3c_5c_6) + d_*^2\cdot\mathrm{St}^4(c_5^2\cdot c_2c_6 + c_3c_4c_5c_6).
\end{aligned}$$

We have

$$\begin{aligned}
\mathrm{St}^4(c_3c_5) &= c_2^2\cdot c_3c_5 + c_5^2\cdot c_2 + c_3c_4c_5 + c_3^2\cdot c_6 \equiv c_5^2\cdot c_2 + c_3c_4c_5,\\
\mathrm{St}^4(c_6) &= c_4c_6 \equiv 0,\\
\mathrm{St}^4(c_3c_5c_6) &= c_2^2\cdot c_3c_5c_6 + (c_3c_6)^2 \equiv 0,\\
\mathrm{St}^4(c_3^2\cdot c_4c_6 + c_2c_3c_5c_6) &= (c_2c_5)^2\cdot c_6 + c_5^2\cdot c_4c_6 \equiv 0,\\
\mathrm{St}^4(c_5^2\cdot c_2c_6 + c_3c_4c_5c_6) &= c_6^2\cdot c_2c_3c_5 + (c_3c_6)^2\cdot c_4 + c_4^2\cdot c_3c_5c_6 + (c_5c_6)^2 \equiv\\
&\qquad c_6^2\cdot c_2c_3c_5 + (c_3c_6)^2\cdot c_4,
\end{aligned}$$

where the congruences are modulo A'. Therefore the image of $\mathrm{St}^4(\alpha)$ in A/A' looks as follows:

$$(bc_5)^2\cdot c_2 + b^2\cdot c_3c_4c_5 + (d_*c_6)^2\cdot c_2c_3c_5 + (d_*c_3c_6)^2\cdot c_4.$$

We conclude that b and d_* vanish.

What remains of (5.3) is just the sum of the four terms

$$a^2 + a_*^2\cdot c_6 + b_*^2\cdot c_3c_5c_6 + c_*^2\cdot(c_3^2\cdot c_4c_6 + c_2c_3c_5c_6), \tag{5.4}$$

and this what we now know about how every element of $\operatorname{Im}\varphi$ looks like.

As the next step, we take any $\alpha\in\operatorname{Im}\varphi$, now written in the form (5.4), and we look at $\mathrm{St}^2(\alpha)$ in the quotient A/A'', where the B^2-submodule $A''\subset A'$ is generated by 1, c_6, $c_3c_5c_6$, and $c_3^2\cdot c_4c_6 + c_2c_3c_5c_6$. This quotient is a free B^2-module with the basis consisting of all c_I other than 1, c_6, $c_3c_5c_6$, and $c_2c_3c_5c_6$. Note that $c_2c_3c_5c_6 = c_3^2 \cdot c_4c_6$ in A/A''. What we see is

$$a_*^2 \cdot (c_2c_6) + (b_*c_3)^2 \cdot c_4c_6 + (c_*c_5)^2 \cdot (c_2c_6).$$

Therefore $b_* = 0$ and $a_* = c_*c_5$ in (5.4) which becomes

$$a^2 + c_*^2 \cdot (c_5^2 \cdot c_6 + c_3^2 \cdot c_4c_6 + c_2c_3c_5c_6). \tag{5.5}$$

It turns out that for any positive $i < 8$, any element of the form (5.5) is mapped by St^i to B^2. So, we have to proceed with a higher Steenrod operation. And St^8 makes it:

$$\mathrm{St}^8(c_5^2 \cdot c_6 + c_3^2 \cdot c_4c_6 + c_2c_3c_5c_6) = (c_2^2 + c_4)^2 \cdot c_2c_3c_5c_6 + ((c_2^2 + c_4)c_3)^2 \cdot c_4c_6 + \\ ((c_2^2 + c_4)c_5)^2 \cdot c_6 + (c_2c_6)^2 \cdot c_3c_5 + (c_3c_5)^2 \cdot c_2c_6 + (c_3^2c_6)^2 + c_3^2 \cdot c_3c_4c_5c_6 + c_5^2 \cdot c_3c_5c_6.$$

Appendix A. Erratum to [10]

It is claimed in [10, Proof of Theorem 3] that for any even $n = 2l \geq 8$, the modulo 2 Euler class c_l is outside the image of $\mathrm{Ch}(BG) \to \mathrm{Ch}(BT)$, where T is the standard split maximal torus in $G := \mathrm{Spin}(n)$. But the proof of this claim, given there, is only valid for n divisible by 4. Lemma A.1 shows that the claim actually fails for $n = 10$. For all $n \neq 10$ however, the claim holds. To see it, assume that $n = 2l > 10$ with odd l. In particular, $l \geq 7$. By Proposition 2.4, any odd degree homogeneous element in $\mathrm{CH}(BT)^W$ is divisible by e in $\mathrm{CH}(BT)^W$. Assume that $c_l \in \operatorname{Im}\varphi$. Then $\mathrm{St}^6(c_l) = c_6c_l \in \operatorname{Im}\varphi$ and it follows that c_6 is in the image of $\mathrm{CH}(BT)^W \to \mathrm{Ch}(BT)$. However, in degree up to 6, this image is generated by c_2, c_4, and c_3^2 (see Remark 3.3). Therefore $c_l \notin \operatorname{Im}\varphi$.

Lemma A.1. *For $G = \mathrm{Spin}(10)$, the image of the homomorphism*

$$\Phi\colon\ \mathrm{CH}(BG) \to \mathrm{CH}(BT) = \mathbb{Z}[z, x_1, \dots, x_5]/(2z - x_1 - \dots - x_5)$$

contains the Euler class $e = x_1x_2x_3x_4x_5 \in \mathrm{CH}(BT)^W$.

Proof. Let P be the standard parabolic subgroup in $G' := \mathrm{Spin}(12)$ such that the quotient variety G'/P is the projective quadric X given by the standard split quadratic form q of dimension 12. The group P contains the standard split maximal torus T' of G'. The group G is the semisimple part of (the reductive part of) P. It contains T' and has the same as P Weyl group W acting on the polynomial ring $\mathrm{CH}(BT') = \mathbb{Z}[z, x_1, \dots, x_5]$ in the 6 (independent) variables the way described in §2. As already mentioned in §2, generators of the ring of W-invariants $\mathrm{CH}(BT')^W$ are constructed in [12, Proposition 2.4] and [14, Proposition 5.1] (see also [5]). One of them is the Euler class $e' := x_1 \dots x_5 \in \mathrm{CH}(BT')^W$. One other – the orbit product $\check{z}$ of z.

In view of the commutative square

$$\begin{tikzcd}
\mathrm{CH}(BP) \arrow[r, "\Phi'"] \arrow[d] & \mathrm{CH}(BT')^W \arrow[d] \\
\mathrm{CH}(BG) \arrow[r] & \mathrm{CH}(BT)^W
\end{tikzcd}$$

we prove Lemma A.1 by finding in the image of Φ' an element mapped e. Note that e' is mapped to e.

A homomorphism of graded rings $\Psi\colon \operatorname{CH}(BT')^W \to \operatorname{CH}(X)$ is constructed in [5, Lemma 2.2]. It is uniquely determined by the property that the composition $\Psi\circ\Phi'$ is (a particular case of) the homomorphism $\operatorname{CH}(BP) \to \operatorname{CH}(G/P)$ considered in [13, §6]. It is shown in [14, Propositions 5.3 and 5.4] that all the generators of $\operatorname{CH}(BT')^W$ other than e and $\check{z}$ are mapped to the subring in $\operatorname{CH}(X)$ generated by the class $h \in \operatorname{CH}^1(X)$ of a hyperplane section of the quadric. The image of $f_0 = 2z - (x_1 + \ldots + x_6) \in \operatorname{Im}\Phi'$ under Ψ equals h so that the subring generated by h is inside the image of the composition $\Psi\circ\Phi'$. By [13, Theorem 6.4], the cokernel of $\Psi\circ\Phi'$ is killed by the torsion index of G' equaling 2. Therefore the image of $\Psi\circ\Phi'$ contains $2\operatorname{CH}(X)$. The cokernel of Φ' is killed by the torsion index of P, which is also equal to 2, and so, $\operatorname{Im}\Phi' \supset 2\operatorname{CH}(BT')^W$.

Since the degree 2^4 of $\check{z}$ is higher that the degree 5 of the Euler class, we do not care about $\check{z}$. The image in $\operatorname{CH}^5(X)$ of the Euler class $e \in \operatorname{CH}^5(BT')$ is the difference $\lambda - \lambda'$ of two distinct classes of maximal totally isotropic subspaces of q. Since $h^5 = \lambda + \lambda'$ (see, e.g., [9, §2.1]), we have $\lambda - \lambda' = 2\lambda - h^5 \in \operatorname{Im}(\Psi\circ\Phi')$. It follows that $\operatorname{Im}\Phi'$ contains an element of the form $e + a$, where $a \in \operatorname{CH}^5(BT')^W$ is a polynomial in the generators of $\operatorname{CH}(BT')^W$ of degree < 5. Since f_0 is the only generator of odd degree < 5, a is divisible by f_0 and therefore vanishes in $\operatorname{CH}(BT)^W$. $\square$

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