ON GENERIC FLAG VARIETIES OF SPIN(11) AND SPIN(12)

NIKITA A. KARPENKO

ABSTRACT. Let X be the variety of Borel subgroups of a split semisimple algebraic group G over a field, twisted by a generic G-torsor. Conjecturally, the canonical epimorphism of the Chow ring $\mathrm{CH}\,X$ onto the associated graded ring $GK(X)$ of the topological filtration on the Grothendieck ring $K(X)$ is an isomorphism. We prove the new cases $G = \mathrm{Spin}(11)$ and $G = \mathrm{Spin}(12)$ of this conjecture. On an equivalent note, we compute the Chow ring $\mathrm{CH}\,Y$ of the highest orthogonal grassmannian Y for the generic 11- and 12-dimensional quadratic forms of trivial discriminant and Clifford invariant. In particular, we describe the torsion subgroup of the Chow group $\mathrm{CH}\,Y$ and determine its order which is equal to 16 777 216. On the other hand, we show that the Chow group $\mathrm{CH}_0\,Y$ of 0-cycles on Y is torsion-free.

CONTENTS

1. INTRODUCTION

Let G be a split semisimple algebraic group over a field k. A (standard) generic G-torsor E is defined as the generic fiber of the quotient map $\pi : \mathrm{GL}(N) \to \mathrm{GL}(N)/G$ for an integer $N \geq 1$ and an embedding of G into the general linear group $\mathrm{GL}(N)$. Thus E is a G-torsor over the function field $F := k(\mathrm{GL}(N)/G)$. The above quotient map π is a G-torsor with the (called versal) property that every G-torsor over a field extension of k is isomorphic to a fiber of π, [11, §5.3]. This explains the interest to E as to the "most generic" G-torsor over a field. More specifically, we are interested in the variety $X := E/B$, where $B \subset G$ is a Borel subgroup, which is a twisted (by the generic torsor E) form of the flag variety G/B.

Date: 20 September 2017. Revised: 28 November 2017.
Key words and phrases. Quadratic forms over fields; algebraic groups; generic torsors; projective homogeneous varieties; Chow groups. *Mathematical Subject Classification (2010):* 20G15; 14C25.
This work has been supported by a Discovery Grant from the National Science and Engineering Research Council of Canada.

Conjecture 1.1 ([6]). *For the above variety $X = E/B$, the canonical epimorphism*

$$\mathrm{CH}\, X \twoheadrightarrow GK(X)$$

of the Chow ring $\mathrm{CH}\, X$ *onto the associated graded ring* $GK(X)$ *of the topological filtration on the Grothendieck ring* $K(X)$ *is an isomorphism.*

We recall that the above epimorphism is an edge effect of the Brown-Gersten-Quillen spectral sequence; its kernel consists of torsion elements (but is not necessarily the whole torsion subgroup of $\mathrm{CH}\, X$).

Conjecture 1.1, being already proven for simple G of type A and of type C, for special orthogonal groups as well as for some exceptional groups (see [6] for the references and some proofs), is still widely open in the remaining cases, especially in the case of the spinor groups $G = \mathrm{Spin}(n)$. For $n \leq 6$ the statement is trivial since the group $\mathrm{Spin}(n)$ is *special* for such n meaning that every $\mathrm{Spin}(n)$-torsor over a field is trivial. Therefore X is isomorphic to G/B (with the scalars extended to F) for $n \leq 6$. In particular, X is cellular and the group $\mathrm{CH}\, X$ is torsion-free forcing the epimorphism in question to be an isomorphism.

Up to now, the only known nontrivial cases were $n = 7, 8, 9, 10$, where the proof was based on a decomposition of the Chow motive of X in a direct sum of shifts of the Rost motive associated to a 3-fold quadratic Pfister forms. The Rost motive in question is a direct summand of the motive of a 3-dimensional quadric Y, the epimorphism $\mathrm{CH}\, Y \twoheadrightarrow GK(Y)$ is easily seen to be an isomorphism, and this gives the similar statement for X. Note that the above argument goes through in the more general case where X is given by an arbitrary $\mathrm{Spin}(n)$-torsor, not necessarily the generic one.

The main result of this paper is Theorem 3.1 which settles the cases of $n = 11$ and $n = 12$. Although a generalization of the above motivic decomposition is available for any n (see [9]), the summands are shifts of a so-called generalized Rost motive; for $n \geq 11$ it is not related to quadrics anymore and is not a direct summand of any variety of dimension < 8. As a matter of fact, in this paper we do not use motives at all. Our method for treating $n = 11, 12$ actually works for $n = 7, 8$ as well, delivering a new and "motivically-free" proof.

Note that in general the destination ring $GK(X)$ of the epimorphism in Conjecture 1.1 is explicit at least in the sense that in each concrete case it can, in principle, be calculated by computer. The reason for that is, first of all, that the Grothendieck ring $K(X)$ is computed for an arbitrary projective homogeneous variety X ([8]). This computation alone is not enough, because the topological filtration on $K(X)$ is still quite mysterious for general X. However, for X as in Conjecture 1.1, it coincides with the explicit gamma filtration (see [3, Corollary 7.4]).

As to the Chow ring $\mathrm{CH}\, X$ in the situation of Conjecture 1.1, it comes with a finite system of generators given by Chern classes of certain (linear) bundles on X. Conjecture 1.1 simply means that all relations between the images of these very explicit generators in the explicit ring $GK(X)$ hold already in $\mathrm{CH}\, X$.

It is worthy to mention that although the construction of the generic torsor E depends on the choice of the embedding $G \hookrightarrow \mathrm{GL}(N)$, each of the rings $\mathrm{CH}\, X$ and $GK(X)$ is canonically the same for whatever choice of it (see [6, Lemma 2.1]).

2. Preliminaries

We recall that any $\mathrm{Spin}(n)$-torsor over a field gives rise to an n-dimensional non-degenerate quadratic form over the field with the property that its discriminant and Clifford invariant are trivial. A generic $\mathrm{Spin}(n)$-torsor E gives rise to a generic quadratic form q with the above property. Note that we can avoid quadratic forms over fields of characteristic 2, because by [6, Proposition 3.2] we may work over fields of characteristic 0 when proving Conjecture 1.1.

For any given split semisimple G and any special parabolic $P \subset G$, Conjecture 1.1 for G has an equivalent version, where the Borel subgroup B is replaced by P, [7, Lemma 4.2]. For $G = \mathrm{Spin}(n)$, we may choose P the way that the variety $X = E/P$ becomes a connected component of the highest orthogonal grassmannian of q. (For odd n, the highest orthogonal grassmannian is connected, but for even n it consists of two connected components isomorphic to each other.) Namely, P is a maximal parabolic subgroup whose conjugacy class is given by the subset of vertices of the Dynkin diagram of $\mathrm{Spin}(n)$ obtained by throwing away the very last vertex. We prove Conjecture 1.1 for $G = \mathrm{Spin}(11), \mathrm{Spin}(12)$ by showing that the epimorphism $\mathrm{CH}\, X \twoheadrightarrow GK(X)$ is an isomorphism for this new choice of X.

The input of the proof consists only of the following two properties of q and X. The first property is specific for $n \leq 12$ but does not require q to be generic: by Pfister's theorem [10], any non-degenerate quadratic form of dimension at most 12 with trivial discriminant and Clifford invariant completely splits over some quadratic field extension of the base field. In contrast, the second property holds for arbitrary $n = 2m+1, 2m+2$ and generic q:

Proposition 2.1. *For X as above, the Chow ring $\mathrm{CH}\, X$ is generated by the Chern classes $c_i \in \mathrm{CH}^i X$, $i = 1, \dots, m$ of the tautological (rank-m) vector bundle on X together with an additional generator $e \in \mathrm{CH}^1 X$ satisfying the relation $2e = c_1$.*

Proof. We recall that $X = E/P$, where E is a generic $\mathrm{Spin}(n)$-torsor and $P \subset \mathrm{Spin}(n)$ is a maximal parabolic subgroup whose conjugacy class is given by the subset of vertices of the Dynkin diagram of $\mathrm{Spin}(n)$ obtained by throwing away the very last vertex. For the variety $Y := E/B$, the ring $\mathrm{CH}\, Y$ is generated by $\mathrm{CH}^1 Y$, [5, Example 2.4]. The variety Y, considered over X via the projection $Y \to X$, is the variety of complete flags of the tautological vector bundle on X. It follows by [7, Lemma 4.3] that the ring $\mathrm{CH}\, X$ is generated by $\mathrm{CH}^1 X$ together with $c_1, \dots, c_m$. Finally, the group $\mathrm{CH}^1 X$ is infinite cyclic and its subgroup generated by c_1 is of index 2. □

3. Main result

Theorem 3.1. *Let q be the quadratic form over a field F corresponding to a generic $\mathrm{Spin}(n)$-torsor with $n = 11$ or $n = 12$ and let X be a connected component of its highest orthogonal grassmannian. Then the epimorphism $\mathrm{CH}\, X \twoheadrightarrow GK(X)$ is an isomorphism.*

Remark 3.2. Since the Grothendieck group $K(X)$ is torsion-free, Theorem 3.1 implies that the Chow group of 0-cycles $\mathrm{CH}_0 X$ is torsion-free. In fact, a proof of this statement will appear already on an earlier stage of the proof of Theorem 3.1 (see Corollary 3.4). For every $n \geq 13$, the similar statement is open. On the other hand, no example is known to the author, where the highest orthogonal grassmannian of a quadratic form possesses nontrivial torsion in its Chow group of 0-cycles.

By Proposition 2.1, the ring $\operatorname{CH} X$ is generated by the Chern classes $c_i \in \operatorname{CH}^i X$, $i = 1, \dots, 5$ of the tautological (rank-5) vector bundle on X and an additional element $e \in \operatorname{CH}^1 X$ satisfying the relation $2e = c_1$. The complete list of relations for the elements $c_1, \dots, c_5$ is as follows (see [2]):

$$c_i^2 - 2c_{i-1}c_{i+1} + 2c_{i-2}c_{i+2} - \cdots + (-1)^i 2c_0c_{2i} = 0 \quad \text{for all } i \geq 1.$$

Therefore, in order to compute the ring $\operatorname{CH} X$, we only need to find the relations involving the remaining generator e.

The group $\operatorname{CH} X$ is (additively) generated by the products of powers of the generators $e, c_2, \dots, c_5$, where the exponent of the power of each of $c_2, \dots, c_5$ is at most 1. In order to eliminate higher powers of $c_2, \dots, c_5$ one uses the above relations and the argument like in [2, proof of Theorem 2.1].

Let us fix a quadratic field extension L/F such that the quadratic form q_L is split. The abelian group $\operatorname{CH} X_L$ is free and the ring $\operatorname{CH} X_L$ is generated by the elements $e_i \in \operatorname{CH}^i X_L$, $i = 1, \dots, 5$ satisfying $2e_i = c_i$, where $c_i \in \operatorname{CH}^i X_L$ is the ith Chern class of the tautological vector bundle on X_L and therefore is the image of $c_i \in \operatorname{CH}^i X$, considered previously, under the change of field homomorphism $\operatorname{CH}^i X \to \operatorname{CH}^i X_L$. (Our elements e_i coincide up to their signs with the elements e_i of [1, §86].) In particular, $e_1 \in \operatorname{CH}^1 X_L$ is the image of $e \in \operatorname{CH}^1 X$.

The above relations on c_i imply (via division by 4) that

$$e_i^2 - 2e_{i-1}e_{i+1} + 2e_{i-2}e_{i+2} - \cdots + (-1)^{i-1}2e_1e_{2i-1} + (-1)^i e_{2i} = 0,$$

where e_i is defined to be 0 for $i > 5$, and this is a complete list of relations (see [12] – the original proof – or [1, Proposition 86.16]). An additive basis of $\operatorname{CH} X_L$ consists of the 2^5 products $\prod_{i \in I} e_i$, where I runs over the subsets of $\{1, \dots, 5\}$, [1, Theorem 86.12]. In particular, the product $e_1 \dots e_5 \in \operatorname{CH}^{15} X_L = \operatorname{CH}_0 X_L$ is the class of a rational point (and 15 is the dimension of the variety X).

Let us consider the norm homomorphism $N = N_{L/F} : \operatorname{CH} X_L \to \operatorname{CH} X$. This is a group (and not a ring) homomorphism satisfying the projection formula $N(\operatorname{res}(x)y) = xN(y)$ with respect to the change of field (ring) homomorphism $\operatorname{res} = \operatorname{res}_{L/F} : \operatorname{CH} X \to \operatorname{CH} X_L$, [1, Proposition 56.9]. In particular, the image of N is an ideal of the ring $\operatorname{CH} X$. Since the composition

$$N \circ \operatorname{res} : \operatorname{CH} X \to \operatorname{CH} X$$

is the multiplication by 2, the ideal $N \operatorname{CH} X_L \subset \operatorname{CH} X$ contains $2 \operatorname{CH} X$. Besides, the restriction of the inverse order composition

$$\operatorname{res} \circ N : \operatorname{CH} X_L \to \operatorname{CH} X_L$$

to the subgroup $\operatorname{res} \operatorname{CH} X \subset \operatorname{CH} X_L$ is also the multiplication by 2. Since $2^5 \operatorname{CH} X_L \subset \operatorname{res} \operatorname{CH} X$ and the group $\operatorname{CH} X_L$ is torsion-free, the composition $\operatorname{res} \circ N$ itself (on the whole $\operatorname{CH} X_L$) is the multiplication by 2. In particular, we have the stronger inclusion $2 \operatorname{CH} X_L \subset \operatorname{res} \operatorname{CH} X$, which can also be checked directly.

Here comes the starting observation which is crucial to the whole computation:

Lemma 3.3. *The quotient ring* $\mathrm{CH}\, X/N\, \mathrm{CH}\, X_L$ *is*

(1) *generated by the classes of* e, c_2, *and* c_4;
(2) *additively generated by the products* e^i, $e^i c_2$, $e^i c_4$, $e^i c_2 c_4$ *with* $i = 0, \dots, 7$;
(3) *a vector space over* $\mathbb{F}_2$ *of dimension at most* 2^5.

We will see later (in Corollary 3.5) that the upper bound in (3) is the precise value of the dimension so that the generators in (2) form a basis.

Proof of Lemma 3.3. (1): We need to exclude the generators c_3 and c_5. In order to exclude c_3, let us consider the element $x := N(e_3) \in \mathrm{CH}^3 X$. As the group $\mathrm{CH}^3 X$ is generated by e^3, ec_2, c_3, we have $x = \alpha e^3 + \beta e c_2 + \gamma c_3$ for some integers α, β, γ. It follows that $\mathrm{res}(x) = (\alpha + 2\beta) e_1 e_2 + 2\gamma e_3$. On the other hand, $\mathrm{res}(x) = (\mathrm{res} \circ N)(e_3) = 2e_3$ so that $\gamma = 1$ and $\alpha = -2\beta$. We get that

$$N(e_3) = c_3 + \beta e(c_2 - 2e^2)$$

showing that the class of c_3 in the quotient ring $\mathrm{CH}\, X/N\, \mathrm{CH}\, X_L$ is a multiple of ec_2. This excludes the generator c_3.

To exclude c_5, we consider $x := N(e_5) \in \mathrm{CH}^5 X$. As the group $\mathrm{CH}^5 X$ is generated by $e^5, e^3 c_2, e^2 c_3, e c_4, c_2 c_3, c_5$, we have

$$x = \alpha_5 e^5 + \alpha_3 e^3 c_2 + \alpha_2 e^2 c_3 + \alpha_1 e c_4 + \alpha_0 c_2 c_3 + \alpha c_5$$

for some integers $\alpha_5, \alpha_3, \alpha_2, \alpha_1, \alpha_0, \alpha$. It follows that

$$\mathrm{res}(x) \equiv \alpha_5 (2e_2 e_3 - e_1 e_4) + 2\alpha_3 (e_1 e_4) + 2\alpha_2 (e_2 e_3) + 2\alpha_1 (e_1 e_4) + 2\alpha e_5 \pmod 4.$$

On the other hand, $\mathrm{res}(x) = (\mathrm{res} \circ N)(e_5) = 2e_5$ so that $\alpha \equiv 1 \pmod 2$. It follows that the class of c_5 in the quotient ring is a polynomial in e, c_2, c_3, c_4. This excludes the generator c_5 and finishes the proof of (1).

(2): Clearly, the quotient ring is generated by the indicated elements if we allow all $i \geq 0$. We only need to exclude $i \geq 8$. Let $x := N(e_3 e_5) \in \mathrm{CH}^8 X$. The group $\mathrm{CH}^8 X$ is generated by the products

$$e^8,\ e^6 c_2,\ e^5 c_3,\ e^4 c_4,\ e^3 c_2 c_3,\ e^3 c_5,\ e^2 c_2 c_4,\ e c_2 c_5,\ e c_3 c_4.$$

Looking at the coordinates of the images of the generators in $\mathrm{CH}^8 X_L$ (with respect to the basis $\{\prod_{i \in I} e_i\}_{I \subset \{1,\dots,5\}}$), we see that the $e_3 e_5$-coordinate is nonzero modulo 4 for the generator e^8 only (the $e_3 e_5$-coordinate for the image of e^8 is 2). It follows that the class of e^8 in the quotient ring $\mathrm{CH}\, X/N\, \mathrm{CH}\, X_L$ is a linear combination of the other generators, finishing the proof of (2).

(3): Just count the number of the generators in (2). □

Note that the norm map $N : \mathrm{CH}\, X_L \to \mathrm{CH}\, X$ is injective, its image $N\, \mathrm{CH}\, X_L$ is a free subgroup of rank 2^5 in $\mathrm{CH}\, X$.

Corollary 3.4. *The group* $\mathrm{CH}_0 X$ *is torsion-free.*

Proof. It follows by Lemma 3.3 that $\mathrm{CH}_0 X = N\, \mathrm{CH}_0 X_L \simeq \mathbb{Z}$. □

Proof of Theorem 3.1. Besides of the norm homomorphism $N : \mathrm{CH}\, X_L \to \mathrm{CH}\, X$, we have the norm homomorphisms $N : K(X_L) \to K(X)$, inducing the norm homomorphism $N : GK(X_L) \to GK(X)$. The square

$$\begin{array}{ccc} \mathrm{CH}\, X_L & \longrightarrow & GK(X_L) \\ {\scriptstyle N}\big\downarrow & & {\scriptstyle N}\big\downarrow \\ \mathrm{CH}\, X & \longrightarrow & GK(X) \end{array}$$

commutes, inducing an epimorphism

$$f : \mathrm{CH}\, X/N\,\mathrm{CH}\, X_L \twoheadrightarrow GK(X)/NGK(X_L).$$

Since the group $\mathrm{CH}\, X_L$ is torsion-free, the epimorphism $\mathrm{CH}\, X_L \twoheadrightarrow GK(X_L)$ is an isomorphism. Since the group $GK(X_L)$ is torsion-free, the norm homomorphism $GK(X_L) \to GK(X)$ is injective. It follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}\, X_L & \xrightarrow{\text{isomorphism}} & GK(X_L) \\ {\scriptstyle N}\big\downarrow & & {\scriptstyle N}\big\downarrow {\scriptstyle \text{monomorphism}} \\ \mathrm{CH}\, X & \longrightarrow & GK(X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{CH}\, X/N\,\mathrm{CH}\, X_L & \xrightarrow{f} & GK(X)/NGK(X_L) \end{array}$$

that $\mathrm{CH}\, X \to GK(X)$ is an isomorphism provided that the lower map f is an isomorphism.

The map f is an epimorphism of finite-dimensional $\mathbb{F}_2$-vector spaces. We prove that it is an isomorphism comparing the dimensions of the spaces. By Lemma 3.3(3), we already have the upper bound 2^5 on the dimension of the space on the left. A computation similar to [4, Proposition], applied to the monomorphism of filtered groups $N : K(X_L) \to K(X)$, shows that the $\mathbb{F}_2$-vector space $GK(X)/NGK(X_L)$ has the same dimension as $K(X)/NK(X_L)$. (The difference between the two dimensions is the dimension of the kernel of $N : GK(X_L) \to GK(X)$ which is trivial.) By [8], the change of field homomorphism $K(X) \to K(X_L)$ is an isomorphism so that $NK(X_L) = 2K(X)$. The abelian group $K(X) = K(X_L)$ is free of the same rank as $\mathrm{CH}\, X_L$, i.e. of the rank 2^5. Thus $\dim_{\mathbb{F}_2} GK(X)/NGK(X_L) = \dim_{\mathbb{F}_2} K(X)/2K(X) = 2^5$. □

As a byproduct, we made the statement of Lemma 3.3 more precise:

Corollary 3.5. *The $\mathbb{F}_2$-vector space $\mathrm{CH}\, X/N\,\mathrm{CH}\, X_L$ has dimension 2^5 and the products e^i, $e^i c_2$, $e^i c_4$, $e^i c_2 c_4$ with $i = 0, \dots, 7$ form its basis.* □

This corollary is used in the next section for determination of the torsion subgroup.

4. Torsion

For X as in Theorem 3.1, we can now describe the torsion subgroup of $\mathrm{CH}\, X$:

Proposition 4.1. *The torsion subgroup of* $\operatorname{CH} X$ *is an* $\mathbb{F}_2$*-vector space of dimension* 24*; its (homogeneous) basis is given by the elements*

$$e^i(c_2 - 2e^2),\ e^i(c_4 - 2ec_3 + 2e^4),\ e^i(c_2 - 2e^2)(c_4 - 2ec_3 + 2e^4) \ \text{ with } i = 0, \dots, 7.$$

Proof. The torsion subgroup of $\operatorname{CH} X$ vanishes over L and therefore has exponent 2.

The indicated elements, vanishing in $\operatorname{CH} X_L$, are of exponent 2 in $\operatorname{CH} X$ as well. They are linearly independent over $\mathbb{F}_2$ because their images in the quotient $\operatorname{CH} X/N \operatorname{CH} X_L$ are the classes of $e^i c_2$, $e^i c_4$, $e^i c_2 c_4$ $(i = 0, \dots, 7)$ which are linearly independent by Corollary 3.5. Thus we have constructed a torsion subgroup in $\operatorname{CH} X$ of dimension 24. We finish the proof of Proposition 4.1 by showing that 24 is the dimension of the total torsion subgroup of $\operatorname{CH} X$.

The total torsion subgroup of $\operatorname{CH} X$ coincides with the kernel of the change of field homomorphism $\operatorname{CH} X \to \operatorname{CH} X_L$. Using the isomorphism $\operatorname{CH} X \to GK(X)$, we identify it with the kernel of $GK(X) \to GK(X_L)$. By [4, Proposition 2], the order of the kernel of the latter map coincides with the order of its cokernel divided by the order of the cokernel of $K(X) \to K(X_L)$. Since $K(X) \to K(X_L)$ is an isomorphism and replacing $GK(X)$ back to $\operatorname{CH} X$, it remains to compute the order of the cokernel of the change of field homomorphism $\operatorname{res} : \operatorname{CH} X \to \operatorname{CH} X_L$.

To simplify the computation, we first recall that the image of res contains $2 \operatorname{CH} X_L$. The quotient $\operatorname{CH} X_L/2 \operatorname{CH} X_L$ is an $\mathbb{F}_2$-vector space with the basis $\{e_I := \prod_{i \in I} e_i\}_{I \subset \{1,\dots,5\}}$. For any i, the image of c_i in $\operatorname{CH} X_L/2 \operatorname{CH} X_L$ is trivial and the image of e^i is e_I for the set I of the 2-powers with the sum i. It follows that the image of res in $\operatorname{CH} X_L/2 \operatorname{CH} X_L$ is generated by the part of the basis with $I \subset \{1, 2, 4\}$. Therefore the dimension we want to compute is $2^5 - 2^3 = 24$. □

Remark 4.2. The appearance of 2-powers in the end of the above proof can be explained by the main property of the J-invariant, introduced by A. Vishik in [12]. The J-invariant $J(q)$ of the quadratic form q is the collection of i such that the class of e_i in $\operatorname{CH} X_L/2 \operatorname{CH} X_L$ belongs to the image of $\operatorname{CH} X$. By [12, Main Theorem 5.8], the image of $\operatorname{CH} X \to \operatorname{CH} X_L/2 \operatorname{CH} X_L$ is generated by e_i with $i \in J(q)$. In particular, the dimension of the image is $2^{\#J(q)}$.

Remark 4.3. Proposition 4.1 answers by positive a question of N. Yagita, see [13, Remark after Lemma 1.4].

Remark 4.4. Let X be the highest orthogonal grassmannian of an arbitrary non-degenerate 11-dimensional quadratic form q of trivial Clifford invariant with the property that the ring $\operatorname{CH} X$ is generated by $\operatorname{CH}^1 X$ and Chern classes of the tautological vector bundle. Then all results of this and the previous sections hold for X because their proofs do not involve any other condition on X. This observation is exploited in the next (and last) section.

5. Some other generic quadratic forms

We fix an odd integer $n \geq 3$. Let k be a field and let F be the rational function field over k in variables t_{ij}, $1 \leq i \leq j \leq n$. Let $q : F^n \to F$ be the quadratic form on the vector space F^n given by the formula

$$q(x_1,\dots,x_n)=\sum_{1\le i\le j\le n} t_{ij}x_ix_j.$$

Let X be the highest orthogonal grassmannian of q and let Y be the Severi-Brauer variety of the even Clifford algebra of q.

Proposition 5.1. *The ring* $\mathrm{CH}\,X_{F(Y)}$ *is generated by* $\mathrm{CH}^1\,X_{F(Y)}$ *together with the Chern classes of the tautological vector bundle.*

Proof. By [2], the ring $\mathrm{CH}\,X$ is generated by the Chern classes of the tautological vector bundle. The projection $X\times Y\to X$ is a projective bundle so that the $\mathrm{CH}\,X$-algebra $\mathrm{CH}(X\times Y)$ is generated by $\mathrm{CH}^1(X\times Y)$. And the pull-back $\mathrm{CH}(X\times Y)\to \mathrm{CH}\,X_{F(Y)}$ with respect to the generic point of Y is an epimorphism of graded $\mathrm{CH}\,X$-algebras. □

Corollary 5.2. *For* $n\le 11$*, the epimorphism* $\mathrm{CH}\,X_{F(Y)}\twoheadrightarrow GK(X_{F(Y)})$ *is an isomorphism.*

Proof. For $n<11$, the statement is known (see also §1). For $n=11$ it is new and follows by Remark 4.4. □

Remark 5.3. For k of characteristic $\ne 2$, one may replace q by the diagonal quadratic form as in [2, §8]. For k of characteristic 2, one may replace q by the quadratic form as in [2, §9].

Remark 5.4. The quadratic form q is a subform of a unique (up to an isomorphism) non-degenerate $(n+1)$-dimensional quadratic form q' of trivial discriminant. The Clifford invariants of q and q' coincide and the connected component X' of the highest grassmannian of q' is isomorphic to X (see [1, Proposition 85.2]). Therefore Proposition 5.1 and Corollary 5.2 also hold for X' in place of X.

Acknowledgements. I thank Alexander Vishik for productive discussions during our Canmore visit in August 2017, careful reading of the manuscript, and interesting comments.

References

[1] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[2] Karpenko, N. A. On generic quadratic forms. Linear Algebraic Groups and Related Structures (preprint server) 585 (10 June 2017), 10 pages.

[3] Karpenko, N. A. On generically split generic flag varieties. Linear Algebraic Groups and Related Structures (preprint server) 583 (29 April 2017), 17 pages.

[4] Karpenko, N. A. On topological filtration for Severi-Brauer varieties. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, vol. 58 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1995, pp. 275–277.

[5] Karpenko, N. A. Chow groups of some generically twisted flag varieties. *Annals of K-Theory 2*, 2 (2017).

[6] Karpenko, N. A. Chow ring of generic flag varieties. *Math. Nachr.* (2017), Early View.

[7] Karpenko, N. A. Chow ring of generically twisted varieties of complete flags. *Adv. Math. 306* (2017), 789–806.

[8] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.
[9] Petrov, V., Semenov, N., and Zainoulline, K. J-invariant of linear algebraic groups. *Ann. Sci. École Norm. Sup. (4) 41* (2008), 1023–1053.
[10] Pfister, A. Quadratische Formen in beliebigen Körpern. *Invent. Math. 1* (1966), 116–132.
[11] Serre, J.-P. Cohomological invariants, Witt invariants, and trace forms. In *Cohomological invariants in Galois cohomology*, vol. 28 of *Univ. Lecture Ser.* Amer. Math. Soc., Providence, RI, 2003, pp. 1–100. Notes by Skip Garibaldi.
[12] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).
[13] Yagita, N. Chow ring of versal complete flag varieties. arXiv:1609.08721v2 [math.KT] (1 Nov. 2017), 32 pages.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: `karpenko` *at* `ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`