A SHORTENED CONSTRUCTION OF THE ROST MOTIVE

NIKITA A. KARPENKO

Abstract. Using Rost's nilpotence theorem for endomorphisms of a motive of a projective quadric, we give a short and direct construction of the motivic decomposition of a norm quadric obtained by Rost and used in Voevodsky's proof of the Milnor conjecture.

This text is a modified version of [4].

Let F be a field of characteristic different from 2, ϕ a quadratic form over F which is similar to a Pfister form (i.e. ϕ is isomorphic to a tensor product of some non-degenerate binary quadratic forms), and let X be the projective hypersurface $\phi\perp\langle c\rangle = 0$ with some $c \in F^*$ ($\phi\perp\langle c\rangle$ stays for the orthogonal sum of ϕ and of the 1-dimensional quadratic form $\langle c\rangle$). The variety X is called a norm quadric in the literature. Its dimension equals a power of 2 minus 1; we write n for the integer such that $\dim X = 2n+1$.

Let $\mathcal{F}/F$ be a field extension splitting ϕ, $\mathfrak{X}$ the variety $X_{\mathcal{F}}$, and $p \in \mathfrak{X}$ a closed rational point.

Let us consider the Chow group (the group of algebraic cycles modulo rational equivalence, graded by codimension) $\mathrm{CH}^*(\mathfrak{X}\times\mathfrak{X})$. The elements of the group $\mathrm{CH}^{2n+1}(\mathfrak{X}\times\mathfrak{X})$ are also called correspondences on $\mathfrak{X}$. The classical notion of composition for correspondences (see [1, §16.1]) makes this group into a ring, which we also denote by $\mathrm{End}(X)$ (it is the ring of endomorphisms of $\mathfrak{X}$ viewed as an object of the additive category of correspondences).

The classes $[p\times\mathfrak{X}]$ and $[\mathfrak{X}\times p]$ of the cycles $p\times\mathfrak{X}$ and $\mathfrak{X}\times p$ in the Chow group $\mathrm{CH}^{2n+1}(\mathfrak{X}\times\mathfrak{X})$ are easily seen to be orthogonal projectors (that is, orthogonal idempotents in the ring $\mathrm{End}(\mathfrak{X})$). In particular, their sum is a projector. The following theorem states that this projector is defined over F:

Theorem (Rost). *There exists a projector r on X such that*

$$r_{\mathcal{F}} = [p\times\mathfrak{X}] + [\mathfrak{X}\times p]\ .$$

Remarks. The motive (X, r), determined by the projector r (one may think of the classical category of Grothendieck's Chow motives [1, Ex. 16.1.12] as well as of a Voevodsky motivic category [8]) is called the Rost motive. The motivic decomposition $X = (X, r) \oplus (X, \mathrm{id} - r)$ given by r is a part of the motivic decomposition of X established in [6, Th. 17]. It is exactly the part needed in Voevodsky's proof of the Milnor conjecture (see [10, Proof of Th. 4.4] and/or [9, Proof of Th. 4.5]).

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The theorem stated above immediately follows from the following two lemmas.

Lemma 1. *There exists a cycle* $\rho \in \mathrm{CH}^{2n+1}(X \times X)$ *such that*

$$\rho_{\mathcal{F}} = [p \times \mathfrak{X}] + [\mathfrak{X} \times p] \; .$$

Proof. The facts on the Chow groups of split quadrics used below can be found in [3, §2.1] (see also [6, §2.3]).

We need a description of $\mathrm{CH}^*(\mathfrak{X} \times \mathfrak{X})$. Note that $\mathfrak{X}$ is a (completely!) split projective quadric of dimension $2n+1$ and that the description of $\mathrm{CH}^*(\mathfrak{X})$ and $\mathrm{CH}^*(\mathfrak{X} \times \mathfrak{X})$ given below has nothing to do with the fact that X is a norm quadric (the same for Y instead of X appearing below). One has

$$\mathrm{CH}^*(\mathfrak{X} \times \mathfrak{X}) = \mathrm{CH}^*(\mathfrak{X}) \otimes \mathrm{CH}^*(\mathfrak{X})$$

(that is, the ring homomorphism $\mathrm{CH}^*(\mathfrak{X}) \otimes \mathrm{CH}^*(\mathfrak{X}) \to \mathrm{CH}^*(\mathfrak{X} \times \mathfrak{X})$ given by the pull-backs with respect to the first and second projection is bijective) and the group $\mathrm{CH}^*(\mathfrak{X})$ is torsion-free. Write h for the class in $\mathrm{CH}^1(\mathfrak{X})$ of a hyperplane section of $\mathfrak{X}$ (more precisely, h is defined as the pull-back of the hyperplane class with respect to the embedding of the hypersurface $\mathfrak{X}$ in the projective space). Note that h is defined over F.

The group $\mathrm{CH}^i(\mathfrak{X})$ is generated by h^i if $i \leq n$ and by $\frac{1}{2}h^i$ if $i \geq n+1$. The generator $\frac{1}{2}h^i$ coincides with the class of a totally isotropic subspace of the appropriate (co)dimension. In particular, $[p] = \frac{1}{2}h^{2n+1} \in \mathrm{CH}^{2n+1}(\mathfrak{X})$.

It follows that $\mathrm{CH}^{2n+1}(\mathfrak{X} \times \mathfrak{X})$ is a free abelian group on the generators $\frac{1}{2}(h^i \otimes h^{2n+1-i})$, $i = 0, \dots, 2n+1$. Since these generators are orthogonal projectors, the diagonal class Δ (which is the identity of $\mathrm{End}(\mathfrak{X})$) is equal to their sum:

$$\Delta = \frac{1}{2} \sum_{i=0}^{2n+1} h^i \otimes h^{2n+1-i} \in \mathrm{CH}^{2n+1}(\mathfrak{X} \times \mathfrak{X}) \; .$$

Consequently,

$$[p \times \mathfrak{X}] + [\mathfrak{X} \times p] = \frac{1}{2}(h^{2n+1} \otimes 1 + 1 \otimes h^{2n+1}) =$$
$$= \Delta - \frac{1}{2}(h \otimes h^{n+1} + h^{n+1} \otimes h) \cdot \sum_{i=1}^{n} h^{i-1} \otimes h^{n-i} \; .$$

Since Δ is defined over F, it suffices to show that the cycle

$$\frac{1}{2}(h \otimes h^{n+1} + h^{n+1} \otimes h) \in \mathrm{CH}^{n+2}(\mathfrak{X} \times \mathfrak{X})$$

is defined over F as well.

Let Y be the projective quadric $\phi = 0$. Note that Y is a 1-codimensional closed subvariety of X. More precisely, it is a hyperplane section of X. We write $\mathcal{Y}$ for $Y_{\mathcal{F}}$.

Let $\pi \in \mathrm{CH}^{2n}(\mathcal{Y})$ be the class of a maximal totally isotropic subspace of ϕ. The push-forward $\mathrm{in}_*(\pi)$ with respect to the imbedding $\mathrm{in}\colon \mathcal{Y} \hookrightarrow \mathfrak{X}$ equals then $\frac{1}{2}h^{n+1}$. Therefore, the push-forward $(\mathrm{in}\times\mathrm{in})_*(1\otimes\pi+\pi\otimes 1)$ with respect to $\mathrm{in}\times\mathrm{in}\colon \mathcal{Y}\times\mathcal{Y}\hookrightarrow\mathfrak{X}\times\mathfrak{X}$ is our $\frac{1}{2}(h\otimes h^{n+1}+h^{n+1}\otimes h)$. Since the morphism $\mathrm{in}\times\mathrm{in}$ is defined over F, it suffices now to show that the cycle

$$1\otimes\pi+\pi\otimes 1\in\mathrm{CH}^n(\mathcal{Y}\times\mathcal{Y})$$

is defined over F.

We need a description of $\mathrm{CH}^*(\mathcal{Y}\times\mathcal{Y})$. One has

$$\mathrm{CH}^*(\mathcal{Y}\times\mathcal{Y})=\mathrm{CH}^*(\mathcal{Y})\otimes\mathrm{CH}^*(\mathcal{Y})$$

and the group $\mathrm{CH}^*(\mathcal{Y})$ is torsion-free. We write now h for the class in $\mathrm{CH}^1(\mathcal{Y})$ of a hyperplane section of Y.

The group $\mathrm{CH}^i(\mathcal{Y})$ is generated by h^i if $i<n$ and by $\frac{1}{2}h^i$ if $i>n$. The "middle" component $\mathrm{CH}^n(\mathcal{Y})$ has (unlike to the others) two free generators: h^n and π.

It follows that $\mathrm{CH}^n(\mathcal{Y}\times\mathcal{Y})$ is a free abelian group on $h^i\otimes h^{n-i}$ $(i=0,\dots,n)$, $\pi\otimes 1$, and $1\otimes\pi$ (note that all but two last generators are already defined over F, so the situation is now easier to control).

Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^n(\mathcal{Y}\times\mathcal{Y}) & \xrightarrow{(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*} & \mathrm{CH}^n(\mathcal{Y}_{\mathcal{F}(\mathcal{Y})}) \\
{\scriptstyle \mathrm{res}_{\mathcal{F}/F}}\uparrow & & \uparrow{\scriptstyle \mathrm{res}_{\mathcal{F}(\mathcal{Y})/F(Y)}} \\
\mathrm{CH}^n(Y\times Y) & \xrightarrow{(\mathrm{id}_Y\times f)^*} & \mathrm{CH}^n(Y_{F(Y)})
\end{array}$$

where the horizontal arrows are the pull-backs with respect to the flat morphisms $\mathrm{id}_Y\times f: Y_{F(Y)}\to Y\times Y$ and $\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f}\colon\mathcal{Y}_{\mathcal{F}(\mathcal{Y})}\to\mathcal{Y}\times\mathcal{Y}$ and where f (resp. $\mathfrak{f}$) is the generic point morphism of Y (resp. $\mathcal{Y}$).

Since the Pfister form ϕ is isotropic over $F(Y)$, it is hyperbolic over this function field ([5, Cor. 1.6 of Chapter Ten]). Thereby $\pi_{\mathcal{F}(\mathcal{Y})}$ is defined over $F(Y)$. Since $(\mathrm{id}_Y\times f)^*$ is evidently surjective (see for example [2, Prop. 5.1]), it follows that there exists a defined over F cycle $\alpha\in\mathrm{CH}^n(\mathcal{Y}\times\mathcal{Y})$ such that $(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*(\alpha)=\pi_{\mathcal{F}(\mathcal{Y})}$.

It is easy to see how $(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*$ acts on the generators of the group $\mathrm{CH}^n(\mathcal{Y}\times\mathcal{Y})$:

$$(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*(h^i\otimes h^{n-i})=0 \qquad \text{for } i=0,\dots,n-1,$$
$$(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*(h^n\otimes 1)=h^n_{\mathcal{F}(\mathcal{Y})}\ , \qquad \text{and}$$
$$(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*(\pi\otimes 1)=\pi_{\mathcal{F}(\mathcal{Y})}\ .$$

Since $(\mathrm{id}_{\mathcal{Y}}\times\mathfrak{f})^*(\alpha)=\pi_{\mathcal{F}(\mathcal{Y})}$, it follows that

$$\alpha=\pi\otimes 1+a\cdot(1\otimes\pi)+\sum_{i=0}^{n}a_i\cdot(h^i\otimes h^{n-i})$$

with some integers a_i and a (one also has $a_n=0$ but we don't care for this). Since the generators $h^i\otimes h^{n-i}$ are defined over F and since $2(1\otimes\pi)$ is defined over F (because 2π is defined over F by the transfer argument), it follows that either $\pi\otimes 1+1\otimes\pi$ or $\pi\otimes 1$ is defined over F. If we are in the first case, we are done. Since the sum of $\pi \otimes 1$ and its transposition gives the cycle $\pi \otimes 1 + 1 \otimes \pi$, it is defined over F also in the second case (in fact the second case is not possible if the quadratic form ϕ is anisotropic). □

Lemma 2 (see also [7, Lemma 3.12]). *Let X/F be a quadric (not necessarily a norm one) and let ρ be a cycle on $X \times X$. If ρ_E is a projector on X_E for some field extension E/F, then some power of ρ (taken in the ring* End(X) *of correspondences on X) is a projector as well.*

Proof. By Rost's nilpotence theorem [6, Prop. 9] (see also [7, Lemma 3.10]), any element of the kernel of the ring homomorphism

$$\mathrm{res}_{E/F} \colon \mathrm{End}(X) \to \mathrm{End}(X_E)$$

is nilpotent. By the transfer argument, any element of this kernel is of the exponent $2^{\dim X}$ (in the case of a norm quadric it even has the exponent 2).

Since ρ_E is a projector, the difference $\alpha \stackrel{\text{def}}{=} \rho^2 - \rho$ lies in the above kernel (powers of cycles are being taken in End(X) here). Therefore, $\alpha^{2^N} = 0$ and $2^N \alpha = 0$ for certain positive integer N. We claim that ρ^{4^N} is a projector.

To see it, take the 4^N-th power of the equality $\rho^2 = \alpha + \rho$. Since α commutes with ρ, we get the sum of $\binom{4^N}{i} \alpha^i \rho^{4^N - i}$ on the right. Each summand is however zero, because $\alpha^i = 0$ if i is divisible by 2^N, and $\binom{4^N}{i}$ is divisible by 2^N otherwise. Thus $(\rho^{4^N})^2 = \rho^{4^N}$. □

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Westfälische Wilhelms-Universität, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, GERMANY

E-mail address: `karpenk@math.uni-muenster.de`