NOTES ON SPIN GRASSMANNIANS

NIKITA A. KARPENKO

Abstract. After a series of papers [2] , [7], [11] on the indexes of generic spin grassmannians followed by a related series of papers [6], [4], [5] around the classifying spaces of spin groups, we make some additional remarks.

Contents

1. An overview

In §2, we add more precision to the table of indexes of generic spin grassmannians given in [11].

In §3, we show that the indexes of generic spin grassmannians in odd dimensions determine them for even dimensions almost completely. This is useful because in contrast to the even dimensions, there is an algorithm determinating the indexes in odd dimensions.

In §4 we provide an algorithm determinating several last indexes of generic spin grassmannians in even dimensions. In §5, we explain why this algorithm does not extend to the preceding indexes. An interesting invariant of quadratic forms is studied here on the occasion.

2. Completing the table of [11, Appendix B]

In [11, Appendix B], there is a table of the exponent indexes $\mathrm{i}(1),\dots,\mathrm{i}(\lfloor d/2\rfloor)$ of generic spin grassmannians for quadratic forms of dimension $d\leq 24$. Most of the positions in this table are filled with the exact values. However, there are 12 positions filled with lower bounds (which are known to be within 1 from the exact values). For odd d an algorithm

Date: 20 Oct 2022.

Key words and phrases. Quadratic forms over fields; affine algebraic groups; spin groups; torsors; classifying spaces; Chow rings. *Mathematical Subject Classification (2020):* 20G15; 14C25.

Author's work has been supported by a Discovery Grant from the National Science and Engineering Research Council of Canada and accomplished during his stay at the Max-Planck Institute for Mathematics in Bonn.

to determine the exact values is described in [7]. However, in general, this algorithm requires a lot of (computer) computations which are not completed so far for the values in question. For even d, in general, there is no such an algorithm (for $m \geq \lfloor d/2 \rfloor - 5$ there is one described in §4 below).

It turns out that two more exact values can be added to the table without any major effort. For arbitrary d and m, it is easy to see that the value $\mathrm{i}(m) - 1$ cannot be higher than the exponent index number $m - 1$ for the dimension $d - 2$. Therefore $\mathrm{i}(5) = 4$ for $d = 21$ as well as for $d = 22$.

3. Comparing exponent indexes for odd and even dimensions

Since we have an algorithm to determine all exponent indexes in any odd dimension $d = 2n + 1$, it is worthy to explain that they determine all exponent indexes in the even dimension $d' := 2n + 2$ with an exception of two or of one, where the exception of just one actually occurs for most n (see Remark 3.2).

To formulate the precise statement, we first recall that the sequence $\mathrm{i}(1), \dots, \mathrm{i}(n)$ of the exponent indexes for $d = 2n + 1$ satisfies $0 \leq \mathrm{i}(1) \leq \cdots \leq \mathrm{i}(n)$ and is described by two parameters. The first one is the value of $\mathrm{i}(t - 1)$, where $t := \mathrm{i}(n)$ is the exponent of the torsion index 2^t of the algebraic group $\mathrm{Spin}(d)$, determined by Totaro in [15]; note that there are only two possible values for $\mathrm{i}(t - 1)$: $t - 2$ and $t - 1$. (We may assume that $d \geq 15$ to avoid $t - 2 \leq 0$ in this entire section.)

The second parameter is the smallest i with $\mathrm{i}(i) = t$; its possible values are $t, \dots, n - 2$.

The two parameters determine the exponent indexes because $\mathrm{i}(t) \geq t - 1$ by [11, Theorem 3.6] and $\mathrm{i}(m) = m$ for $m < t - 1$ by [11, Corollary A.3] and [2, Theorem 4.2].

Note that 2^t is also the torsion index of the group $\mathrm{Spin}(d')$. Besides, $t = \mathrm{i}(n) = \mathrm{i}(n-1) = \mathrm{i}(n - 2) = \mathrm{i}'(n + 1) = \mathrm{i}'(n) = \mathrm{i}'(n - 1) = \mathrm{i}'(n - 2)$, where $0 \leq \mathrm{i}'(1) \leq \cdots \leq \mathrm{i}'(n + 1)$ are the exponent indexes in dimension $d' = 2n + 2$.

Proposition 3.1. *For any $m \in \{1, \dots, n+1\} \setminus \{t-1, i\}$, one has $\mathrm{i}'(m) = \mathrm{i}(m)$. Besides, $t - 2 \leq \mathrm{i}'(t - 1) \leq \mathrm{i}(t - 1) \leq t - 1$ and $\mathrm{i}'(i) \in \{t - 1, t\}$.*

Proof. For $m < t - 1$, we have $\mathrm{i}'(m) = m = \mathrm{i}(m)$ by [11, Corollary A.3] and [2, Theorems 4.2 and 7.2]. We also have $\mathrm{i}'(t) \geq t - 1$ by [11, Theorem 3.6].

For any $m \in \{1, \dots, n\}$, we have $\mathrm{i}(m - 1) \leq \mathrm{i}'(m) \leq \mathrm{i}(m) \leq m$ by [11, Lemma 2.3], where $\mathrm{i}(0) := 0$. In particular, $t - 2 = \mathrm{i}(t - 2) \leq \mathrm{i}'(t - 1) \leq \mathrm{i}(t - 1) \leq t - 1$. Besides, by definition of i, we have $\mathrm{i}'(m) = t - 1$ for $m \in [t, i - 1]$, $\mathrm{i}'(m) = t$ for $m \in [i + 1, n + 1]$, and $\mathrm{i}'(i) \in \{t - 1, t\}$. $\square$

Remark 3.2. An overwhelming majority of n satisfies the condition of [11, Proposition A.4]. For such n one has $\mathrm{i}'(t - 1) = t - 1$ by [11, Proposition A.4].

4. An algorithm for $\mathrm{i}(\geq n - 5)$ in dimension $2n$

In this section, given any even $d = 2n$, we provide an algorithm for determination of the two highest unknown exponent indexes $\mathrm{i}(n - 4)$ and $\mathrm{i}(n - 5)$. We can assume that $n \geq 6$.

In general, for any $m = 1, \dots, n$, we would get an algorithm to determine $\mathrm{i}(m)$ once we computed the reduced Chow ring $\bar{\mathrm{CH}}(X_m)$ of the orthogonal grassmannian X_m of a generic $2n$-dimensional nondegenerate quadratic form q with trivial discriminant and Clifford invariant. The reduced Chow ring is the Chow ring modulo torsion which we identify with the image of the change of field homomorphism $\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m)$ given by an algebraic closure of the base field.

The following approach to determination of $\bar{\mathrm{CH}}(X_m)$ has been developed in [2], followed by [11]. We assume that $m < n$ below. Let P be the standard parabolic subgroup of the standard split spin group $G := \mathrm{Spin}(d)$ with $\overline{G/P} = \bar{X}_m$. There is a surjective ring homomorphism

$$\mathrm{CH}(BP) \twoheadrightarrow \mathrm{CH}(X_m)$$

out of the Chow ring of the classifying space of P, which yields a surjective ring homomorphism

$$\bar{\mathrm{CH}}(BP) \twoheadrightarrow \bar{\mathrm{CH}}(X_m).$$

We identify $\bar{\mathrm{CH}}(BP)$ with the image of $\mathrm{CH}(BP) \to \mathrm{CH}(BT)$, where $T \subset P$ is the standard split maximal torus. We have an inclusion $\bar{\mathrm{CH}}(BP) \subset \mathrm{CH}(BT)^W$ into the subring of the elements invariant under the action of the Weyl group W of P. We also have a ring homomorphism

$$\varphi \colon \ \mathrm{CH}(BT)^W \to \mathrm{CH}(\bar{X}_m),$$

extending $\bar{\mathrm{CH}}(BP) \twoheadrightarrow \bar{\mathrm{CH}}(X_m)$, and we conclude that $\bar{\mathrm{CH}}(X_m)$ is contained in the image of φ.

It turns out that $\bar{\mathrm{CH}}(X_m)$ is very close to the image of φ and even coincides with it in some cases. One uses a computation of $\mathrm{CH}(BT)^W$ to see it.

To describe $\mathrm{CH}(BT)^W$, we identify $\mathrm{CH}(BT)$ with the ring

$$\mathbb{Z}[z, x_1, \dots, x_m, y_1, \dots, y_l],$$

where $l := n - m$ and the generators are subject to the unique relation

$$2z = x_1 + \dots + x_m + y_1 + \dots + y_l.$$

This is an identification of graded rings, where the degree of every generator of the latter ring is 1. The semisimple part of P is $\mathrm{Spin}(2l)$ so that the Weyl group W of P is the Weyl group of $\mathrm{Spin}(2l)$. It acts on $\mathrm{CH}(BT)$ by arbitrary permutations of $x_1, \dots, x_m$, arbitrary permutations of $y_1, \dots, y_l$, and the sign changes of any even number of the latter. The ring of W-invariants is generated by the following elements: the elementary symmetric polynomials in $x_1, \dots, x_m$; the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$; the *Euler class* $e := y_1 \dots y_l$; certain homogeneous elements $f_0, f_1, \dots, f_{l-2}$ of degrees $2^0, 2^1, \dots, 2^{l-2}$; and certain degree 2^{l-1} homogeneous element $\check{z}$. With the exception of the Euler class, the images under φ of all the generators turn out to be polynomials in Chern classes of some vector bundles on X_m and by this reason lie in $\bar{\mathrm{CH}}(X_m)$. (With the exception of $\check{z}$, they are polynomials in the Chern classes of the tautological vector bundle on X_m.) So, we have the equality $\bar{\mathrm{CH}}(X_m) = \operatorname{Im} \varphi$ if and only if we have the inclusion $\varphi(e) \in \bar{\mathrm{CH}}(X_m)$. We are going to show that for $m \geq n - 5$ we do have the inclusion (and therefore the desired algorithm).

Proposition 4.1. *For $m \geq n - 5$, one has $\varphi(e) \in \bar{\mathrm{CH}}(X_m)$.*

Proof. Let us first consider the case of $m = 1$ (with arbitrary n). The variety $X_m = X_1$ is the projective quadric of q. There are exactly two distinct rational equivalence classes

$$\lambda \neq \lambda' \in \mathrm{CH}^{n-1}(\bar{X}_1)$$

of n-dimensional totally isotropic subspaces in $\bar{q}$. And $\varphi(e)$ is their difference. Their sum $\lambda' + \lambda$ is a power of the hyperplane section of X_m which lies in $\bar{\mathrm{CH}}(X_1)$. Note that the degree of $\check{z}$ is 2^{l-1}, the degree of the Euler class is l and $2^{l-1} > l$ provided that $l \geq 3$. Besides, the image under φ of any of the remaining generators of $\mathrm{CH}(BT)^W$ is a power of the hyperplane section.

Now let us assume that $n \leq 6$ meaning that $\dim q \leq 12$. In this case there exists a finite base field extension of degree dividing 2 such that q is hyperbolic over it. (In dimension 12 this is the famous Pfister theorem.) By the transfer argument, it follows that $2\lambda \in \bar{\mathrm{CH}}(X_1)$. Consequently,

$$\varphi(e) = \lambda' - \lambda = (\lambda' + \lambda) - 2\lambda \in \bar{\mathrm{CH}}(X_1).$$

For $l = 3, 4, 5$, we conclude that $\bar{\mathrm{CH}}(BP)$ contains an element of the form $ae + \alpha$, where α is a polynomial in the remaining (i.e., distinct from e) generators of $\mathrm{CH}(BT)^W$ and a is an odd integer. This is actually a result about the reductive part P_{red} of P because $\mathrm{CH}(BP) = \mathrm{CH}(BP_{\mathrm{red}})$, see [10, Proof of Proposition 6.1], which is the standard split even Clifford group $\Gamma^+(2l)$. Note that this even Clifford group is a spit reductive group with the semisimple part $\mathrm{Spin}(2l)$. For $l \leq 3$, the torsion index of $\Gamma^+(2l)$ (which coincides with the torsion index of its semisimple part) is 1 so that $\mathrm{CH}(B\Gamma^+(2l)) = \mathrm{CH}(BT)^W$. For $l = 4, 5$, the torsion index is 2 so that $2e \in \bar{\mathrm{CH}}(B\Gamma^+(2l))$ and we can replace the appeared above odd integer a by 1. We proved

Lemma 4.2. *For $l \leq 5$, the ring $\bar{\mathrm{CH}}(B\Gamma^+(2l))$ contains a sum $e + \alpha$ of the Euler class and a polynomial in the remaining generators of the ring of Weyl invariants in the Chow ring of the classifying space of the split maximal torus.* □

(We show in Proposition 5.2 that the statement of Lemma 4.2 fails for every $l \geq 6$.)

We now turn attention to arbitrary m, n with $m \geq n - 5$. Since the reductive part of P is $\Gamma^+(2l)$ and $l = n - m \leq 5$, we know by Lemma 4.2 that $\bar{\mathrm{CH}}(BP)$ contains a sum $e + \alpha$. It follows that $\varphi(e + \alpha) \in \bar{\mathrm{CH}}(X_m)$. Since $\varphi(\alpha) \in \bar{\mathrm{CH}}(X_m)$, we get that $\varphi(e) \in \bar{\mathrm{CH}}(X_m)$. □

5. An invariant of quadratic forms

Let q be a nondegenerate quadratic form of even dimension $2n$ and of trivial discriminant. For now, we don't assume that q has trivial Clifford invariant, neither we assume that it is generic in any sense.

Let X_1 be the projective quadric of q and let $\lambda \in \mathrm{CH}^{n-1}(\bar{X}_1)$ be the class of a maximal totally isotropic subspace in $\bar{q}$, where the bar indicates the base field change to an algebraic closure. Since q becomes hyperbolic over a field extension of the base field of degree dividing 2^{n-1}, we have $2^{n-1}\lambda \in \bar{\mathrm{CH}}(X_1)$ by the transfer argument, where, as previously, $\bar{\mathrm{CH}}(X_1)$ is the image of the change of field homomorphism $\mathrm{CH}(X_1) \to \mathrm{CH}(\bar{X}_1)$. Triviality of the discriminant of q ensures that the cokernel of the change of field homomorphism is finite which is needed for the transfer argument to work.

Let $s \geq 0$ be the smallest integer such that $2^s\lambda \in \bar{\mathrm{CH}}^{n-1}(\bar{X}_1)$. The integer $s = s(q)$ seems to be an interesting invariant of q. It is responsible for appearance of some torsion in $\mathrm{CH}(X_1)$, see [8, Theorem 3.10(2)].

The value of this invariant on q coincides with its value on the anisotropic part of q. Also it is easy to check that $s(q) = 0$ if and only if q is hyperbolic (see [3, Corollary 72.6]). So, $s(q)$ is at least 1 for anisotropic q. Another lower bound follows from the computation [14] (see also [12]) of the Grothendieck group $K_0(X_1)$, involving the Clifford algebra $C(q)$: $2^{s(q)}$ is divisible by the index of $C(q)$. In particular, $s(q) = 2^{n-1}$ for a generic d-dimensional quadratic form q of trivial discriminant. This is the maximal possible value of $s(q)$ on $2n$-dimensional q.

However, looking at even-dimensional nondegenerate quadratic forms q of trivial discriminant *and trivial Clifford invariant*, it seems much more difficult to prove that the invariant $s(q)$ ever takes values higher than 1. An upper bound (probably sharp) for such forms is given by the torsion index 2^t of $\mathrm{Spin}(2n)$: any such q becomes hyperbolic over a finite field extension of the base field with the 2-primary component of the degree a divisor of 2^t. In particular, $s(q) \leq 1$ in dimension up to 12.

In dimension 14, the torsion index of $\mathrm{Spin}(14)$, giving an upper bound on $s(q)$, is 2^2. Moreover, if 14-dimensional q (with whatever Clifford invariant) is anisotropic, it cannot become hyperbolic over a field extension of the base field of degree dividing 2: otherwise q is isomorphic to the product of a 7-dimensional form by a binary form and the discriminant of the binary form has to coincide with the discriminant of q implying hyperbolicity of the both. This suggests that $s(q)$ might be 2 for *any* anisotropic 14-dimensional q. And indeed, over a base field of characteristic 0, using the symmetric operations [16] in algebraic cobordism (involving resolution of singularities), Alexander Vishik showed that $s(q) = 2$ for any anisotropic 14-dimensional q.

Using just the Steenrod operations on the modulo 2 Chow groups (available in arbitrary characteristic including 2 – see [13]), we prove

Proposition 5.1. *For even $d \geq 14$, let q be a generic d-dimensional quadratic form of trivial discriminant and Clifford invariant. Then $s(q) \geq 2$.*

Proof. By the arguments of §4, the statement of Proposition 5.1 for every $d \geq 14$ is equivalent to the statement of Proposition 5.2 for $d-2$. Besides, since for every $d > 14$, a generic 14-dimensional quadratic form of trivial discriminant and Clifford invariant is the anisotropic part of a specialization of a generic d-dimensional quadratic form of trivial discriminant and Clifford invariant, the statement of Proposition 5.1 for $d > 14$ is a consequence of the statement for $d = 14$. □

Proposition 5.2 (cf. Lemma 4.2)**.** *For any $l \geq 6$, any element of $\bar{\mathrm{CH}}^l(B\Gamma^+(2l))$ is a polynomial in $2e$ and the remaining (after removing e) standard generators of the ring of Weyl invariants in the Chow ring of the classifying space of the split maximal torus.*

Proof. As explained in the proof of Proposition 5.1, we only need to treat $l = 6$. We are going to work with the spin group $G := \mathrm{Spin}(12)$. Since G is a normal subgroup of $\Gamma^+(12)$ with the quotient $\Gamma^+(12)/G$ isomorphic to $\mathbb{G}_\mathrm{m}$, the ring $\mathrm{CH}(BG)$ is computed in terms of $\mathrm{CH}(B\Gamma^+(12))$, see [9, Proposition 4.1]. In particular, the reduced Chow ring $\bar{\mathrm{CH}}(BG)$ is the quotient of $\bar{\mathrm{CH}}(B\Gamma^+(12))$ by the ideal generated by x_1 in the notation of §4.

Writing T for the standard split maximal torus of G, we have the ring $\bar{\mathrm{CH}}(BG)$ embedded as a subring into $\mathrm{CH}(BT) = \mathbb{Z}[z, y_1, \dots, y_6]$, where $2z = y_1 + \dots + y_6$. More precisely, $\bar{\mathrm{CH}}(BG)$ is contained in the subring generated by: the elementary symmetric polynomials in $y_1^2, \dots, y_6^2$; the *Euler class* $e := y_1 \dots y_6$; certain homogeneous elements $f_0, f_1, \dots, f_4$ of degrees $2^0, 2^1, \dots, 2^4$; and certain degree 2^5 homogeneous element $\check{z}$. Using Steenrod operations in the spirit of what we do below, one can actually show that e can be replaced by $2e$ in the above statement (this gives the positive answer to [6, Question 9])[1]. But here we are only interested in degree 6 for which the arguments are simpler.

We are going to show that the image of $\bar{\mathrm{CH}}(BG)$ in

$$\mathrm{Ch}(BT) := \mathrm{CH}(BT)/2\,\mathrm{CH}(BT) = \mathbb{F}[z, y_1, \dots, y_6],$$

where $\mathbb{F} := \mathbb{F}_2$ is the field of 2 elements, does not contain any (degree 6) homogeneous element of the form $e + \dots$. Note that the unique relation on $z, y_1, \dots, y_6$ we have reads now as $y_1 + \dots + y_6 = 0$. We write $c_1, \dots, c_6 \in \mathbb{F}[y_1, \dots, y_6]$ for the elementary symmetric polynomials in $y_1, \dots, y_6$. We have $c_1 = 0$ and $c_2, \dots, c_6$ are algebraically independent. The elementary symmetric polynomials in the squares of $y_1, \dots, y_6$ are now the squares $c_1^2 = 0, c_2^2, \dots, c_6^2$. Since our aim statement is on degree 6, we only need to care about degrees up to 6. The degree 2^5 of $\check{z}$ is too high. The images of f_0, f_1, f_2 are 0, c_2, and c_4. Therefore any element α in the image of $\bar{\mathrm{CH}}^6(BG)$ is a linear combination of

$$e, c_3^2, c_2^3, \text{ and } c_2c_4. \tag{5.3}$$

The restriction on α that will lead to the success, is the fact (observed in [6]) that for any $i \geq 0$, the value $\mathrm{St}^i(\alpha) \in \mathrm{Ch}^{6+i}(BT)$ of the ith cohomological Steenrod operation at α has also to be in the image of $\bar{\mathrm{CH}}(BG)$. In particular, $\mathrm{St}^i(\alpha)$ has to vanish for any odd i because the image of $\bar{\mathrm{CH}}(BG)$ is generated by elements of even degrees.

It turns out to be enough to apply the above restriction with $i = 1$ and $i = 3$. First we compute St^1 at the elements (5.3):

$$\mathrm{St}^1(e) = 0,\ \mathrm{St}^1(c_3^2) = 0,\ \mathrm{St}^1(c_2^3) = c_2^2c_3,\ \mathrm{St}^1(c_2c_4) = c_3c_4 + c_2c_5. \tag{5.4}$$

To get formulas (5.4), one can use the general formulas for the Steenrod operation on the Chern classes (which are just formulas on symmetric polynomials with coefficients in $\mathbb{F}$) obtained by Wu (announced in [17] and proved in [18]). Another proof is given by Borel in [1, Théorème 7.1]. In particular, $\mathrm{St}^1(c_i) = (i+1)c_{i+1}$ for any $i \geq 1$ (recall that $c_1 := 0$). The formulas for the last Chern class are easy to get directly. In our setting, where the last Chern class is $c_6 = e$, one has $\mathrm{St}^i(e) = c_ie$ for any i, where $c_i := 0$ for $i > 6$.

By (5.4), the condition $\mathrm{St}^1(\alpha) = 0$ implies that α is a linear combination of e and c_3^2 only. Since $\mathrm{St}^3(e) = c_3e = c_3c_6$ and $\mathrm{St}^3(c_3^2) = 0$ (odd Steenrod operations vanish on squares), the coefficient at e has to vanish. □

---

[1]Unfortunately, the positive answer to [6, Question 9] is insufficient for an algorithm of computation for the exponent indexes of spin grassmannians in all (even) dimensions: one needs the positive answer to the modification of [6, Question 9] obtained by replacing the spin group with the corresponding even Clifford group.

References

[1] Borel, A. La cohomologie mod 2 de certains espaces homogènes. *Comment. Math. Helv. 27* (1953), 165–197.

[2] Devyatov, R. A., Karpenko, N. A., and Merkurjev, A. S. Maximal indexes of flag varieties for spin groups. *Forum Math. Sigma 9* (2021), Paper No. e34, 12.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[4] Karpenko, N. A. Envelopes and classifying spaces. Preprint (26 May 2022, 10 pages). Available on author's web page.

[5] Karpenko, N. A. Filtrations on the representation ring of an affine algebraic group. Preprint (19 Jul 2022, 14 pages). Available on author's web page.

[6] Karpenko, N. A. On classifying spaces of spin groups. Preprint (final version of 14 May 2022, 7 pages). To appear in Results Math.

[7] Karpenko, N. A. On generic flag varieties for odd spin groups. Preprint (final version of 23 Nov 2021, 11 pages). To appear in Publ. Mat.

[8] Karpenko, N. A. Algebro-geometric invariants of quadratic forms. *Algebra i Analiz 2*, 1 (1990), 141–162.

[9] Karpenko, N. A. On generic quadratic forms. *Pacific J. Math. 297*, 2 (2018), 367–380.

[10] Karpenko, N. A. On generically split generic flag varieties. *Bull. Lond. Math. Soc. 50* (2018), 496–508.

[11] Karpenko, N. A., and Merkurjev, A. S. Indexes of generic grassmannians for spin groups. Preprint (final version of 7 May 2022, 15 pages). To appear in Proc. Lond. Math. Soc. (3).

[12] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.

[13] Primozic, E. Motivic Steenrod operations in characteristic p. *Forum Math. Sigma 8* (2020), Paper No. e52, 25.

[14] Swan, R. G. K-theory of quadric hypersurfaces. *Ann. of Math. (2) 122*, 1 (1985), 113–153.

[15] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.

[16] Vishik, A. Symmetric operations in algebraic cobordism. *Adv. Math. 213*, 2 (2007), 489–552.

[17] Wu, W.-t. Les i-carrés dans une variété grassmannienne. *C. R. Acad. Sci. Paris 230* (1950), 918–920.

[18] Wu, W.-t. On squares in Grassmannian manifolds. *Acta Sci. Sinica 2* (1953), 91–115.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: `karpenko@ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`