CYCLES ON POWERS OF QUADRICS

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Abstract. These are notes of a part of my lectures on [3] given in the Universität Göttingen during the first week of November 2003. For a projective quadric X, some relative cellular structures on the powers of X are described and, as a consequence, a computation of the Chow groups of powers of X in terms of the Chow groups of powers of the anisotropic part of X, used in [3], is obtained, and this without referring to motives. In contrast to [3], we do not assume that the characteristic of the base field differs from 2.

1. Cellular spaces

Let F be a field. By a *variety* (over F) we mean a reduced (not necessarily irreducible) scheme of finite type over F.

Definition 1.1. A variety X together with a decomposition $X = \cup_{i\in I} C_i$ into a finite disjoint union of its locally closed subvarieties is called a *cellular space*, if every C_i is isomorphic to an affine space and the subvarieties C_i are the successive difference of some filtration of X by closed subvarieties.

Example 1.2. To show that the condition on existence of the filtration in the definition of cellular space is not automatic, let us take as X two projective lines L_1 and L_2 intersecting in two points P_1 and P_2. The locally closed subvarieties $C_i = L_i \setminus P_i$, $i \in I = \{1, 2\}$ of X satisfy all the conditions of Definition 1.1 except the filtration one. The filtration condition is not satisfied because none of C_i is closed in X.

Example 1.3. The standard filtration of a projective space $\mathbb{P}^n$ by its linear subspaces $\mathbb{P}^{n-1}$, $\mathbb{P}^{n-2}$, and so on produces a cellular structure on $\mathbb{P}^n$.

Example 1.4. Let D be a non-negative integer. Let X be a D-dimensional split smooth projective quadric. Up to an isomorphism, X is the hypersurface in the projective space $\mathbb{P}^{D+1}$, given by the equation $x_0x_{d+1} + \cdots + x_dx_{2d+1} = 0$ (in the case of even $D = 2d$) or by the equation $x_0x_{d+1} + \cdots + x_dx_{2d+1} + x^2 = 0$ (in the case of odd $D = 2d + 1$). Consider the descending filtration of X by the closed subvarieties $X^{(i)}$, $i = -1, 0, \ldots, 2d + 2$ with $X^{(-1)} = X$ and with $X^{(i)}$ for $i \geq 0$ defined inside of $X^{(i-1)}$ by the equation $x_i = 0$. This filtration produces a cellular structure on X (every successive difference is easily seen

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to be isomorphic to an affine space). Note that $X^{(d)}$ is $\mathbb{P}^d$ (the equation $x = 0$ in the case of odd D is automatically added because $X^{(d)}$, being a subvariety, should be reduced) and the filtration on $X^{(d)}$ coincides with that of Example 1.3.

Remark 1.5. Every split projective homogeneous variety has a cellular structure (see [4]). And of course every projective homogeneous variety becomes split over an appropriate extension of the base field (for instance, over the algebraic closure).

Example 1.6. Let X and Y be cellular spaces with the cellular decompositions $X = \cup_{i\in I} U_i$ and $Y = \cup_{j\in J} V_j$. Then the direct product $X \times Y$ is cellular as well with the cells $U_i \times V_j$, $(i, j) \in I \times J$. Indeed, the products $U_i \times V_j$ are isomorphic to affine spaces, locally closed in $X \times Y$, disjoint, and cover $X \times Y$. The filtration condition is easy to verify (see, e.g., [2, §7]).

The total integral Chow group $\mathrm{CH}(X)$ of a cellular space X is easy to compute:

Theorem 1.7. *Let X be a cellular space with cells U_i, $i \in I$. Write u_i for the element in $\mathrm{CH}(X)$ given by the closure of the cell U_i. The Chow group $\mathrm{CH}(X)$ is a free $\mathbb{Z}$-module and u_i, $i \in I$ is its basis.*

Proof. Let

$$X = X^{(0)} \supset X^{(1)} \supset \cdots \supset X^{(n)} \supset X^{(n+1)} = \emptyset$$

be a cellular filtration of X. We proof the statement using an induction on its length. Let U be the cell $X^{(0)} \setminus X^{(1)}$ and let u be the corresponding element of $\mathrm{CH}(X)$. By the induction hypothesis, the $\mathbb{Z}$-module $\mathrm{CH}(X^{(1)})$ is free and the elements u_i, given by the cells U_i different from U, form its basis. On the other hand, in the exact sequence of K-homology groups

$$H(X, K_1) \to H(U, K_1) \to \mathrm{CH}(X^{(1)}) \to \mathrm{CH}(X) \to \mathrm{CH}(U) \to 0$$

the left hand side homomorphism is evidently surjective (because the pull-back $H(F, K_1) \to H(U, K_1)$ with respect to the structure morphism of U is an isomorphism by the homotopy invariance of K-cohomology) and the right hand side epimorphism has a splitting $\mathrm{CH}(U) = \mathbb{Z} \cdot [U] \to \mathrm{CH}(X)$ given by $[U] \mapsto u$. □

Remark 1.8. The classes $[X^{(i)}]$, $i = 0, 1, \ldots, n$ of the terms of a cellular filtration

$$X = X^{(0)} \supset X^{(1)} \supset \cdots \supset X^{(n)} \supset X^{(n+1)} = \emptyset$$

of a given cellular space X also form a basis of $\mathrm{CH}(X)$ (one can either deduce it from Theorem 1.7 or repeat the proof of the theorem replacing the splitting $[U] \mapsto u$ by the splitting $[U] \mapsto [X]$).

Remark 1.9. Let $X = \cup_{i\in I} U_i$ and $Y = \cup_{j\in J} V_j$ be cellular spaces with their cellular decompositions, $u_i \in \mathrm{CH}(X)$ and $v_j \in \mathrm{CH}(Y)$ the elements given by the closures of the cells. Note that the class in $\mathrm{CH}(X \times Y)$ of the closure of a cell $U_i \times V_j$ coincides with the class of the product of the closures of the cells (even in the case when this product is not reduced), that is, with the exterior product $u_i \times v_j$. Therefore the exterior products $u_i \times v_j$, $(i,j) \in I \times J$, form a basis of $\mathrm{CH}(X \times Y)$, and the homomorphism $\mathrm{CH}(X) \otimes \mathrm{CH}(Y) \to \mathrm{CH}(X \times Y)$, given by the external product of cycles, is an isomorphism.

Now we get the following description of the Chow ring of a smooth split projective quadric:

Theorem 1.10. *Let X be a split smooth D-dimensional projective quadric, a hypersurface of a projective space $\mathbb{P}$. We set $d = [D/2]$. Let $h \in \mathrm{CH}^1(X)$ be the class of an arbitrary hyperplane section of X. For every $i = 0, 1, \dots, d$, let $l_i \in \mathrm{CH}_i(X)$ be the class of an arbitrary i-dimensional linear subspace of $\mathbb{P}$ lying inside of X. Then the total Chow group $\mathrm{CH}(X)$ is free with the basis h^i, l_i, $i = 0, 1, \dots, d$. The multiplication of the ring $\mathrm{CH}(X)$ is determined by the following rules: $h^{d+1} = 2l_{[(D-1)/2]}$ ($h^{d+1} = 0$ for $D = 0$), $h \cdot l_i = l_{i-1}$ for $i \in [1,\ d]$, and l_d^2 is equal to 0 or l_0 with $l_d^2 = l_0$ if and only if D is divisible by 4.*

Proof. For $D = 0$, the variety X is a disjoint union of two rational points. For $D = 1$, the variety X is isomorphic to a projective line. In both cases the statement of Theorem needs no proof. We assume that $D \geq 2$ in what follows.

The intersection of an arbitrary hyperplane of $\mathbb{P}$ with X is reduced and has codimension 1 in X; therefore the pull-back of the class in $\mathrm{CH}(\mathbb{P})$ of the hyperplane coincides with the class of this intersection; in particular, it does not depend on the choice of the hyperplane. Since the pull-back $\mathrm{CH}(\mathbb{P}) \to \mathrm{CH}(X)$ is a ring homomorphism, any power h^i is the pull-back of the class of an i-codimensional linear subspace of $\mathbb{P}$. It follows that for $i = 0, 1, \dots, d$ the elements h^i are the classes of the first $d+1$ terms of the cellular filtration of X constructed in Example 1.4. The remaining $d+1$ terms are certain linear subspaces of X, one for each dimension $i = d, d-1, \dots, 0$. Since $\mathrm{CH}_i(X) = \mathbb{Z} \cdot [X_i]$ for $i \in [0,\ D/2)$, the push-forward homomorphism $\mathrm{CH}_i(X) \to \mathrm{CH}_i(\mathbb{P})$ is injective (even bijective) for such i; it follows that $[X_i]$ coincides with the class of an arbitrary i-dimensional linear subspace lying on X. At this stage it is already easy to get the relation $h \cdot l_i = l_{i-1}$ using the projection formula with respect to the embedding $X \hookrightarrow \mathbb{P}$.

In contrast to the other graded components of the Chow group of X, the group $\mathrm{CH}_d(X)$ in the case of even D (and only in this case) has a rank different from 1 (namely, 2): its basis is formed by $h^d = [X^{(d-1)}]$ and $l_d = [X^{(d)}]$, where l_d is the class of the special linear subspace $X^{(d)} \subset X$. Let $l'_d \in \mathrm{CH}_d(X)$ be the class of an arbitrary d-dimensional linear subspace of X. Since l_d and l'_d have the same image under the push-forward homomorphism $\mathrm{CH}_d(X) \to \mathrm{CH}_d(\mathbb{P})$ while its kernel is generated by $h^d - 2l_d$, one has $l'_d = l_d + n(h^d - 2l_d)$ with some $n \in \mathbb{Z}$. Since there exists a linear automorphism of X moving l_d to l'_d (while h^d is of course invariant with respect to any linear automorphism), the cycles h^d and l'_d also form a basis of $\mathrm{CH}_d(X)$; consequently, the determinant of the matrix

$$\begin{pmatrix} 1 & n \\ 0 & 1-2n \end{pmatrix}$$

is ± 1, that is, n is 0 or 1 and l'_d is l_d or $h^d - l_d$. So, there are precisely two different rational equivalence classes of d-dimensional linear subspaces of X and their sum is equal to h^d.

Now let l'_d be the class of the linear subspace $x_0 = x_1 = \cdots = x_{d-1} = x_{2d+1} = 0$. Since $X^{(d-1)}$ is the union of $x_0 = x_1 = \cdots = x_{d-1} = x_d = 0$ and $x_0 = x_1 = \cdots = x_{d-1} = x_{2d+1} = 0$, we have $h^d = l_d + l'_d$, that is, the classes l_d and l'_d are different. It follows that the class of $x_{d+1} = x_{d+2} = \cdots = x_{2d+1} = 0$ is l'_d if and only if D is divisible by 4; in this case $l_d \cdot l'_d = 0$ and consequently $l_d^2 = l_d \cdot (h^d - l'_d) = l_d \cdot h^d = l_0$; otherwise $l_d^2 = 0$. □

2. Relative cellular spaces

The notion of a relative cellular space is introduced in [2, §6]. Here we slightly change the definition.

Definition 2.1. A variety X together with a decomposition $X = \cup_{i\in I} C_i$ into a finite disjoint union of its locally closed subvarieties equipped with morphisms $f_i : C_i \to X_i$ to some smooth projective X_i is called a *relative cellular space*, if the subvarieties C_i are the successive difference of some filtration of X by closed subvarieties and for every $i \in I$ the morphism f_i is flat of constant relative dimension and all its fibers are isomorphic to affine spaces. We refer to C_i as to the cells of X and to X_i as to the bases of the cells.

Example 2.2 (The trivial example)**.** Any smooth projective variety can be considered as a relative cellular space in the trivial way: one cell coinciding with its base.

Example 2.3 (The basic example)**.** Let X be a D-dimensional isotropic smooth projective quadric. Up to an isomorphism, X is the hypersurface of $\mathbb{P}^{D+1}$ given by the equation $x_0x_1 + \psi(x_2, \ldots, x_{D+1}) = 0$ with some quadratic form ψ. We define a filtration $X = X^{(-1)} \supset X^{(0)} \supset X^{(1)}$. The subvariety $X^{(0)}$ is defined by the equation $x_0 = 0$. It is a 1-dimensional cone over the smooth projective quadric X_0 given by ψ; we take as $X^{(1)}$ the vertex of the cone (i.e., the point $x_0 = x_2 = x_3 \cdots = x_{D+1} = 0$). This filtration determines a relative cellular structure on X. Indeed, $X^{(-1)} \setminus X^{(0)}$ is an affine space, $X^{(0)} \setminus X^{(1)}$ is a 1-dimensional vector bundle over X_0, and $X^{(1)}$ is a point.

More generally, if X is given by the equation

$$x_0x_1 + \ldots x_{2i-2}x_{2i-1} + \psi(x_{2i}, \ldots, x_{D+1}) \ ,$$

then X has a structure of relative cellular space with the smooth quadric $\psi = 0$ being the base of one cell and points being the bases of the other cells.

Remark 2.4. Every isotropic projective homogeneous variety has a non-trivial structure of cellular space, see [1].

Example 2.5 (Product of cellular spaces)**.** The direct product of two cellular spaces is a cellular space, where the cells are the pairwise products of the cells; their bases are products of the bases of the factors.

The Chow group of a relative cellular space is computed in [2] in terms of the Chow groups of the bases of its cells:

Theorem 2.6 ([2, proof of th. 6.5])**.** *In notation of Definition 2.1, the sum of the homomorphisms* $(\Gamma_{f_i})_*\colon \mathrm{CH}(X_i) \to \mathrm{CH}(X)$*, given by the closures* $\Gamma_{f_i} \subset X_i \times X$ *of the transpositions of the graphs of the morphisms* $f_i\colon C_i \to X_i$*, is an isomorphism of* $\bigoplus_{i\in I} \mathrm{CH}(X_i)$ *onto* $\mathrm{CH}(X)$*.*

Remark 2.7. For an integral closed subvariety $Z \subset X_i$, the element of $\mathrm{CH}(X)$ corresponding to $[Z] \in \mathrm{CH}(X_i)$ under the isomorphism of Theorem 2.6 is given by the closure in X of $f_i^{-1}(Z)$.

References

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