POINCARÉ DUALITY FOR UPPER MOTIVES OF ABSOLUTELY SIMPLE ALGEBRAIC GROUPS

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Abstract. We show that the upper motives of projective homogeneous varieties under absolutely simple algebraic groups of all types other than ${}^6D_4$ satisfy Poincaré duality. New are the cases of ${}^2A_n$ and ${}^3D_4$.

Contents

1. Intro

We start with a brief introduction referring for more details to [6].

Let us fix a prime number p and work with the category of Grothendieck's modulo p Chow motives over an arbitrary field F, defined as in [5, §64] using the coefficient ring $\Lambda := \mathbb{F} := \mathbb{Z}/p\mathbb{Z}$. More specifically, given a reductive group G over F and a projective G-homogeneous F-variety X, we are investigating the structure of its motive $M(X)$. Since the center of G acts trivially on X, we may assume that G is semisimple and adjoint.

By [2, Theorem 34] (see also [9, Corollary 2.6]), $M(X)$ decomposes into a finite direct sum of indecomposable motives, and such a decomposition, called *complete*, is unique in the usual sense. The *upper motive* $U(X)$ is its unique summand with nontrivial $\mathrm{Ch}^0(U(X))$, where Ch is the Chow group with coefficients in $\mathbb{F}$ (for more details see also [9], where the notion of upper motive was introduced originally). Over a separable closure of the base field, $U(X)$ becomes a finite direct sum of Tate motives $\mathbb{F}\{i\}$ with various $i \geq 0$ including one copy of $i = 0$; *dimension* $\dim U(X)$ is the largest i appearing in the decomposition.

We refer to the duality cofunctor $M \mapsto M^*$ of [5, §65]. If for some $i \in \mathbb{Z}$, the motive M is the ith shift of the motive defined by a projector on a connected smooth projective d-dimensional variety, then M^* is the $(-d-i)$th shift of the motive defined by the transposition of the projector.

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We say that the upper motive $U(X)$ satisfies Poincaré duality if its dual $U(X)^*$ admits an isomorphism

$$U(X)^* \simeq U(X)\{-\dim U(X)\}. \tag{1.1}$$

Equivalently, $U(X)^*$ is isomorphic to a shift of $U(X)$: the shifting number is uniquely determined by the isomorphism property applied over a separable closure of the base field.

Property (1.1) has been established in [7, Proposition 5.2] for G of inner type and, more generally, for G becoming of inner type over a finite base field extension of a p-power degree. Since $M(X)$ is a direct sum of shifts of upper motives of (in general, other) projective G-homogeneous varieties (see [9, Theorem 3.5] for the inner case and [8, Theorem 1.1] for the general case), (1.1) provides interesting information on the structure of the entire motive of X, not only on its upper part. Note that the motive of any connected smooth projective variety satisfies Poincaré duality by the very definition of the duality functor $M \mapsto M^*$. The upper motives however are in general not isomorphic to motives of varieties and their duality property is surprising. One may expect that it fails for general reductive groups.

Recall from [9, Corollary 2.15] that the upper motives of projective homogeneous F-varieties X and X' (under possibly different and arbitrary reductive groups over F) are isomorphic if and only if each of the varieties acquires over the function field of the other a closed point of prime to p degree. In particular, if X has a closed point of prime to p degree already over F, then $U(X) \simeq M(\operatorname{Spec} F) =: \mathbb{F}$ (we are following the tradition to use notation of the coefficient ring as notation for the Tate motive $M(\operatorname{Spec} F)$ as well) and (1.1) holds trivially because $\mathbb{F}^* \simeq \mathbb{F}$.

A variant of (1.1), involving a twist by an invertible Artin motive A, has been established in [6] for G acquiring inner type over a finite field extension L/F of prime to p degree provided that the *higher Tits p-indexes* of G (see [3]) are invariant under the *$*$-action* of the absolute Galois group of the base field on the Dynkin diagram of G (see [12, §24.6]): $U(X)^*$ was shown to be isomorphic to a shift of $U(X) \otimes A$ for certain invertible motive A isomorphic to a direct summand in the motive of the spectrum of the F-algebra given by an intermediate field of L/F. In particular, A is an *Artin motive*, i.e., a direct summand in the spectrum of the motive of an étale F-algebra. Over a separable closure of the base field, any Artin motive becomes a finite direct sum of several copies of $\mathbb{F}$; the number of the copies is the *rank* of A. An Artin motive A is *invertible*, if there is a motive M with $A \otimes M \simeq \mathbb{F}$. In fact, M has to be the dual of A; in particular, M is an Artin motive as well. Besides, A is invertible if and only if its rank is 1 (see [6, Lemma 1.5]).

Note that the isomorphism class of the twisting Artin motive A in the discussed above twisted duality of [6] is uniquely determined by X: it can be obtained as the image of $U(X)^*\{\dim U(X)\}$ under the retraction functor $\mathbf{m}$ of [4, §4]. If it happens to be trivial (i.e., if $A \simeq \mathbb{F}$), the twisted duality reduces to (1.1). Some of such cases are already listed in [6, §3], some others occur here below.

Now we turn our attention to absolutely simple algebraic groups and formulate our main result:

Theorem 1.2. *Let p be a prime number and let G be an absolutely simple affine algebraic group. For G of type ${}^6\mathsf{D}_4$, we exclude the prime $p = 2$. Then the upper motive $U(X)$ with* *coefficients in $\mathbb{Z}/p\mathbb{Z}$ of any projective G-homogeneous variety X satisfies Poincaré duality (1.1).*

Let us comment on the possible types of G one by one. As explained above, we do not need to look at inner G and so we start with ${}^2\mathsf{A}_n$, where $n \geq 2$. In this case, Theorem 1.2 is already available for $p = 2$ by [7, Proposition 5.2]. For odd p, a twisted Poincaré duality is obtained in [6], and we show here in §2 that the twist is trivial.

Aside from D_n and E_6, the remaining Dynkin diagrams have no nontrivial automorphisms so that the corresponding groups are inner. The case of ${}^2\mathsf{E}_6$ has already been covered for $p = 2$ by [7], treated for $p = 3$ in [6, §3.4], and is trivial for other p.

For G of type D_n with any n, any projective G-homogeneous variety possesses a closed point of a 2-power degree so that we only need to treat the prime $p = 2$. For G acquiring inner type over a quadratic base field extension, Theorem 1.2 is already proven in [7]. Since the type ${}^6\mathsf{D}_4$ is excluded in the very statement of Theorem 1.2, ${}^3\mathsf{D}_4$, considered here in §3, is the very last case to deal with. To make the case of ${}^3\mathsf{D}_4$, we generalize in Theorem 3.1 the twisted Poincaré duality of [6] to the case of non-Galois-invariant higher Tits p-indexes.

2. ${}^2\mathsf{A}_n$

This section is a proof of Theorem 1.2 for G adjoint absolutely simple of type ${}^2\mathsf{A}_{n-1}$. Any such group G over a field F is isomorphic to the projective unitary group given by a separable quadratic field extension L/F and a degree n central simple L-algebra C with a unitary F-involution, see [10, Theorem 26.9]. The primes p of interest for us are the odd prime divisors of the index of C. Nevertheless, below we are considering an arbitrary odd prime p.

Any projective G-homogeneous variety X is isomorphic to a variety of partial flags of isotropic right ideals in C, cf. [11, (9.12)]: if $n_1, \dots, n_k$ are the ranks (over L) of the ideals in the flags, then X corresponds to the subset of the Dynkin diagram of G given by the union of Galois orbits containing the vertices with the numbers $n_1, \dots, n_k$ (in the sense opposite to [15]). Let us recall that the absolute Galois group acts through the Galois group of the field extension L/F; the nontrivial automorphism of L/F acts on the Dynkin diagram via its cental symmetry.

Let D be a central division L-algebra Brauer-equivalent to the p-primary part of C. Its degree is a p-power p^r with $r \geq 0$. By the mentioned above isomorphism criterion for upper motives, we have $U(X) \simeq U(X_i)$ for some $0 \leq i \leq r$, where X_i is the Weil transfer with respect to L/F of the generalized Severi-Brauer L-variety of rank p^i right ideals in D. Indeed, one takes i such that p^i is the highest p-power common divisor of $n_1, \dots, n_k$. Since the variety X_L is isomorphic to the variety of flags of all rank $n_1, \dots, n_k$ right ideals in the L-algebra D, the variety $X_{L(X_i)}$ possesses a closed point of a prime to p degree d and so the variety $X_{F(X_i)}$ possesses a closed point of degree $2d$. As to the variety $(X_i)_{F(X)}$, it possesses a rational point.

So, it suffices to prove (1.1) for $X = X_i$. We do it using induction on $r - i$. In the starting case of $r = i$, we have $U(X_i) \simeq \mathbb{F}$ and the Poincaré duality holds.

Below we are assuming that $r > i$ (in particular, $r > 0$). We know from [6, Theorem 1.4] that $U(X)$ satisfies a twisted duality with the twist by an invertible motive A: $U(X)^*$

is isomorphic to a shift of $U(X) \otimes A$. More precisely, $A = \mathbb{F}$ or $A = B$, where the isomorphism class of B is defined by the condition that the direct sum $\mathbb{F} \oplus B$ is isomorphic to the motive of the spectrum of the F-algebra L. Note that B becomes isomorphic to $\mathbb{F}$ over L, but is not isomorphic to $\mathbb{F}$ over F (see [4, §3] for details). Besides, both possible values of the Artin motive A are self-inverse: $A \otimes A \simeq \mathbb{F}$, or, in other (equivalent) terms, they are self-dual: $A^* \simeq A$ (cf. [6, Lemma 1.5]). To prove Theorem 1.2 for our current X, we just need to show that $A = \mathbb{F}$.

Let us consider the function fields $F' := F(X_{r-1})$ and $L' := L(X_{r-1})$. By the index reduction formula of [1] (as well as of [14] and of [11]), the tensor product

$$D \otimes_F F' = D \otimes_L L'$$

is Brauer equivalent to a degree p^{r-1} central division L'-algebra D'. By [4, Theorem 6.3], over the function field F', the motive $U(X)$ decomposes into a direct sum of shifts of the tensor products $U(X'_j) \otimes A$ with various $0 \leq j \leq i$, where X'_j is the variety of isotropic rank p^j (over L') right ideals in D' and where A is again an Artin motive (not necessarily the same as above and varying from summand to summand). By [4, Corollary 3.8], A is isomorphic to $\mathbb{F}$ or to B (viewed over F').

By the general isomorphism criterion of [4, Theorem 7.3], two such tensor products $U(X'_{j_1}) \otimes A_1$ and $U(X'_{j_2}) \otimes A_2$ are isomorphic if and only if $j_1 = j_2$ and $A_1 = A_2$. Over L', the factors A_1 and A_2 "disappear"; by [4, Lemma 5.10], the indecomposable motives $U(X'_{j_1})$ and $U(X'_{j_2})$ remain indecomposable over L'; they are isomorphic over L' if and only if $j_1 = j_2$. It follows that the complete decomposition of $U(X)$ over L' is its complete decomposition over F' "up to the Artin factors".

Also note that $U(X'_j)_{L'} \simeq U'_j$, where U'_j is the upper motive of the generalized Severi-Brauer L'-variety of rank p^j right ideals in the central division L'-algebra D'. Similarly, $U(X)_L \simeq U$, where U is the upper motive of the generalized Severi-Brauer L-variety of rank p^i right ideals in the central division L-algebra D. It has been shown in [9, Proof of Theorem 4.1] that the complete decomposition of $U_{L'}$ contains p shifts of U'_i and each of the remaining summands is a shift of U'_j with $j < i$. Note that by the Poincaré duality of upper motives for groups of inner type, for any j, the dual of U'_j is isomorphic to a shift of U'_j.

Let s be the number of shifts of $U(X'_i)$ in the complete decomposition of $U(X)_{F'}$. Then $0 \leq s \leq p$ and $p - s$ is the number of shifts of the tensor product $U(X'_i) \otimes B$ in this decomposition. Note that $B \not\simeq \mathbb{F}$ even though we are viewing the two motives now over the field F' instead of the original field F (see, e.g., [4, Corollary 3.8]). By the induction hypothesis, the dual of $U(X'_i)$ is a shift of $U(X'_i)$. Besides, the motive B is self-dual. It follows that the complete decomposition of $U(X)^*_{F'}$ also contains exactly s shifts of $U(X'_i)$ and exactly $p - s$ shifts of $U(X'_i) \otimes B$.

Now, for the sake of contradiction, assume that the upper motive $U(X)$ satisfies the Poincaré duality involving the twist by B: the dual of $U(X)$ is isomorphic to a shift of $U(X) \otimes B$. This implies that the complete decomposition of $U(X)^*_{F'}$ contains $p - s$ shifts of $U(X'_i)$ and s shifts of $U(X'_i) \otimes B$. Therefore $s = p - s$ contradicting to the assumption that p is odd.

3. ${}^3D_4$

This section provides a proof of Theorem 1.2 for G adjoint absolutely simple of type ${}^3D_4$ over a field F. (As explained in §1, the only prime p of interest for such G is $p = 2$.) The difficulty here is that the higher Tits 2-indexes of G are not Galois-invariant in general: this can be deduced from [13, Appendix by Mathieu Florence] and [15, Table II]. Due to this reason, G is not covered by the twisted Poincaré duality result of [6].

So, we start by extending [6, Theorem 1.4] to reductive groups whose higher Tits p-indexes may not satisfy the Galois invariancy. This extension is based on the extension in a similar direction of [4, Theorem 6.3], made in the latest version of [4].

Theorem 3.1. *Assume that a reductive group G over a field F acquires inner type over a finite field extension L/F of degree prime to p. Then the dual $U(X)^*$ of the upper motive $U(X)$ of any projective G-homogeneous variety X is isomorphic to a shift of the tensor product $U(X) \otimes A$ with some invertible Artin motive A which is a direct summand in the spectrum of the F-algebra given by an intermediate field of L/F.*

Proof. The motive $U(X)^*$ is isomorphic to a shift of an indecomposable summand in $M(X)^*$ – a shift of $M(X)$. Therefore, by [4, Theorem 6.3], it is isomorphic to a shift of a summand in $U(Y)^F$ for some intermediate field K of L/F and some projective G_K-homogeneous K-variety Y, where $U(Y)^F$ is the F-motive given by the K/F-*corestriction* of the K-motive $U(Y)$, see [8, §3]. Moreover, the variety Y can be chosen in such a way that the variety X acquires a rational point over the function field $K(Y)$ (cf. [4, Remark 6.6]).

On the other hand, over the function field $F(X)$ the motive $U(X)$ contains the Tate motive $\mathbb{F}$ as a summand. Therefore a shift of $\mathbb{F}$ is a summand in the motive $U(Y)^F$ which in its turn is a summand in $M(Y)^F = M(Y^F)$, where Y^F is the scheme Y viewed as an F-variety via the composition $Y \to \operatorname{Spec} K \to \operatorname{Spec} F$. By [9, Lemma 2.21], we conclude that the variety $Y_{K(X)}$ possesses a closed point of degree prime to p and therefore $U(Y) \simeq U(X_K)$.

At this point we proved that $U(X)^*$ is isomorphic to a shift of a summand in $U(X_K)^F$. By [4, Remark 5.14], every summand in $U(X_K)^F$ is isomorphic to $U(X) \otimes A$ for some Artin motive A – a direct summand in $M(\operatorname{Spec} K)^F$. So, writing $\sim$ for an isomorphism up to up to a shift, we have $U(X)^* \sim U(X) \otimes A$. Dualizing, we obtain $U(X) \sim U(X)^* \otimes A^*$. Substituting, we come to an actual (not up to a shift) isomorphism $U(X) \simeq U(X) \otimes A \otimes A^*$. Applying the retraction functor $\mathbf{m}$ of [4, §4] to it, we get an isomorphism $\mathbb{F} \simeq A \otimes A^*$, witnessing invertibility of the Artin motive A. □

Proof of Theorem 1.2 for G of type ${}^3D_4$. Recall that to prove Theorem 1.2 for absolutely simple G of type ${}^3D_4$, we only need to treat the case of the prime $p = 2$. By Theorem 3.1, $U(X)$ satisfies a twisted Poincaré duality with a twist by an invertible Artin motive A. By [6, §3.2], the Tate motive $\mathbb{F}$ is the only invertible Artin motive for $p = 2$. □

References

[1] Blanchet, A. Function fields of generalized Brauer-Severi varieties. *Comm. Algebra 19*, 1 (1991), 97–118.

[2] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.

[3] De Clercq, C., and Garibaldi, S. Tits p-indexes of semisimple algebraic groups. *J. Lond. Math. Soc., II. Ser. 95*, 2 (2017), 567–585.

[4] De Clercq, C., Karpenko, N., and Quéguiner-Mathieu, A. A-upper motives of reductive groups. arXiv:2403.11030v3 [math.AG] (5 Mar 2025, 28 pages).

[5] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[6] Karpenko, N. A. Twisted Poincaré duality for upper motives. Preprint (15 Dec 2024, 7 pages). Available on author's web page.

[7] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[8] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.

[9] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[10] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[11] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. II. *K-Theory 14*, 2 (1998), 101–196.

[12] Milne, J. S. *Algebraic groups: The theory of group schemes of finite type over a field*, vol. 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.

[13] Petrov, V., and Semenov, N. Higher Tits indices of algebraic groups. arXiv:0706.2827v3 [math.AG] (16 Jan 2008), 29 pages.

[14] Saltman, D. J. The Schur index and Moody's theorem. *K-Theory 7*, 4 (1993), 309–332.

[15] Tits, J. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*. Amer. Math. Soc., Providence, R.I., 1966, pp. 33–62.

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