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Based on Totaro’s computation of the Chow ring of classifying spaces for orthogonal groups, we compute the Chow rings of all orthogonal Grassmannians associated with a generic quadratic form of any dimension. This closes the gap between the known particular cases of the quadric and the highest orthogonal Grassmannian. We also relate two different notions of generic quadratic forms.

1. Introduction

Let k be a field of characteristic different from 2 and let $F_g = k(t_1, \dots, t_n)$ be the field of rational functions over k in variables t_1, \dots, t_n for some $n \geq 2$. We call *generic* the diagonal quadratic form $q_g := \langle t_1, \dots, t_n \rangle$ over F_g . Thus q_g is the n -dimensional quadratic form $F_g^n \rightarrow F_g$ on the vector space F_g^n given by the formula

$$q_g : (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq n} t_i x_i^2.$$

The Chow ring of the projective quadric defined by q_g has been computed in [Karpenko 1991, Corollary 2.2]. The Chow ring of the highest orthogonal Grassmannian of a generic quadratic form has been computed in [Petrov 2016] (see also [Smirnov and Vishik 2014]), but this was done for a different notion of generic, which we call here *standard generic*. As shown in Section 3, the n -dimensional standard generic quadratic form q lives over the field of rational functions $F = k(t_{ij})_{1 \leq i \leq j \leq n}$ in $n(n+1)/2$ variables t_{ij} and can be defined (in arbitrary characteristic including characteristic 2) by the formula

$$F^n \rightarrow F, (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq j \leq n} t_{ij} x_i x_j.$$

In the present paper we determine the Chow ring $\mathrm{CH} X$ of all orthogonal Grassmannians X associated with the generic and the standard generic quadratic forms.

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(The characteristic $\neq 2$ assumption is removed in the latter case; the characteristic 2 analog for the first case is provided in Section 9.) Namely, our main theorem (Theorem 6.1; see also Corollary 8.2 and Proposition 9.2) affirms that the ring $\text{CH } X$ is generated by the Chern classes of the tautological vector bundle of X . A complete list of relations satisfied by these Chern classes (in general, not only in the generic situation) is provided in Theorem 2.1. All the (well-known) relations that hold over an algebraic closure of the base field actually already hold over the base field itself. This way we obtain a description of the ring $\text{CH } X$ in terms of generators and relations. It also follows that the additive group of $\text{CH } X$ is torsion-free (see Corollary 6.2).

To prove the main theorem, we use the computation of the Chow ring of classifying spaces for orthogonal groups $O(n)$ performed in [Pandharipande 1998], as well as in [Totaro 1999] over the field of complex numbers, and later in [Molina Rojas and Vistoli 2006] over an arbitrary field of characteristic $\neq 2$. We actually need only a piece of this computation which is made in [Totaro 1999] over arbitrary fields (of arbitrary characteristic); see Section 5.

Note that the algebraic group $O(n)$ over a field k is not connected if n is even or $\text{char } k \neq 2$. In the remaining case (when n is odd and $\text{char } k = 2$), the algebraic group $O(n)$ is not smooth. In contrast, the special orthogonal group $O^+(n)$ is always smooth and connected. But since $O(n)$ -torsors correspond to all nondegenerate n -dimensional quadratic forms while $O^+(n)$ -torsors correspond to quadratic forms of trivial discriminant, it is more appropriate to work with $O(n)$ for the question raised in this paper. On the other hand, since orthogonal Grassmannians depend only on the similarity class of the quadratic form in question and any odd-dimensional quadratic form is similar to that of trivial discriminant, $O(n)$ can be replaced by $O^+(n)$ for odd n .

2. Tautological Chern subring

In this section we consider an arbitrary nondegenerate quadratic form $q : V \rightarrow F$ of an arbitrary dimension $n \geq 2$ over an arbitrary field F . (Characteristic 2 is not excluded; nondegenerate quadratic forms are defined as in [Elman et al. 2008, §7.A]. We require $n \geq 2$ everywhere in the paper because the varieties we are interested in, introduced below, do not occur for $n = 1$.) In particular, V is an n -dimensional F -vector space. We fix an integer $1 \leq m \leq n/2$ and write X for the orthogonal Grassmannian of isotropic m -planes (i.e., totally isotropic m -dimensional subspaces) in V . Note that the variety X is smooth projective; it is geometrically connected if and only if $m \neq n/2$.

Let $\mathcal{T} = \mathcal{T}_X$ be the tautological (rank- m) vector bundle on X : the fiber of \mathcal{T} over a point of X , given by an isotropic m -plane, is this very m -plane itself. We define the *tautological Chern subring* $\text{CT } X$ in the Chow ring $\text{CH } X$ as the subring

generated by the Chern classes $c_1(\mathcal{T}), \dots, c_m(\mathcal{T})$. The goal of this section is to determine the ring $\text{CT } X$ by providing a list of defining relations on its generators.

The variety X is a closed subvariety of the usual Grassmannian Γ of all m -planes in V . The Chow ring $\text{CH } \Gamma$ is known to be generated by the Chern classes of the tautological (rank- m) vector bundle on Γ . Therefore the pull-back $\text{CH } \Gamma \rightarrow \text{CH } X$ with respect to the closed imbedding $X \hookrightarrow \Gamma$ provides an epimorphism $\text{CH } \Gamma \twoheadrightarrow \text{CT } X$. Since a description of the ring $\text{CH } \Gamma$ by generators and relations is available (see [Buch et al. 2009, Lemma 1.2] or [Fulton 1984, Example 14.6.6]), we fulfill our goal if we describe the kernel of the epimorphism $\text{CH } \Gamma \twoheadrightarrow \text{CT } X$ in terms of generators of $\text{CH } \Gamma$. For this, it is more convenient to use the generators $c_1, \dots, c_{n-m} \in \text{CH } \Gamma$ given by the Chern classes of $-[\mathcal{T}]$ rather than of $\mathcal{T} = \mathcal{T}_\Gamma$ itself. By $[\mathcal{T}]$ here we mean the class of \mathcal{T} in the Grothendieck ring $K(\Gamma)$. The Chern classes of $-[\mathcal{T}]$ are the Segre classes of \mathcal{T} , i.e., the components of the multiplicative inverse to the total Chern class $c(\mathcal{T})$. The tautological vector bundle \mathcal{T} is a subbundle of the trivial (rank- n) vector bundle V and c_1, \dots, c_{n-m} are the Chern classes of the quotient V/\mathcal{T} . We define $c_i \in \text{CH}^i(\Gamma)$ for every integer i by setting $c_i := c_i(-[\mathcal{T}]) = c_i(V/\mathcal{T})$. Therefore $c_0 = 1$ and $c_i = 0$ for $i < 0$ as well as for $i > n - m$.

Theorem 2.1. *The kernel of the epimorphism $\text{CH } \Gamma \twoheadrightarrow \text{CT } X$ is generated by the elements*

$$(2.2) \quad c_i^2 - 2c_{i-1}c_{i+1} + 2c_{i-2}c_{i+2} - \dots + (-1)^i 2c_0c_{2i}, \quad \text{with } i > n/2 - m,$$

and c_{n-m} . The abelian group $\text{CT } X$ is free with a basis consisting of the images of the products $c_1^{\alpha_1} \dots c_{n-m-1}^{\alpha_{n-m-1}}$ with $\alpha_1 + \dots + \alpha_{n-m-1} \leq m$ and $\alpha_i \leq 1$ for $i > n/2 - m$.

Proof. Let us first check that the elements (2.2) lie in the kernel. The i -th element is mapped to the Chern class $c_{2i}(-[\mathcal{T}] - [\mathcal{T}^\vee]) \in \text{CT } X$, where $\mathcal{T} = \mathcal{T}_X$ and \mathcal{T}^\vee is the dual vector bundle. The isomorphism $V/\mathcal{T}^\perp = \mathcal{T}^\vee$, where \mathcal{T}^\perp is the vector bundle given by the orthogonal complement, shows that $-[\mathcal{T}] - [\mathcal{T}^\vee] = -[\mathcal{T}] + [\mathcal{T}^\perp] = [\mathcal{T}^\perp/\mathcal{T}]$. Since the rank of the quotient $\mathcal{T}^\perp/\mathcal{T}$ is $n - 2m$ (see [Elman et al. 2008, Proposition 1.5]), its Chern classes vanish in degrees $> n - 2m$.

In order to show that c_{n-m} is in the kernel, we proceed similarly to [Vishik 2009, proof of Proposition 2.1]. Notice that the projective bundle $\mathbb{P}(\mathcal{T})$ over X can be identified with the variety of flags of totally isotropic subspaces in V of dimensions 1 and m . In particular, besides the projection $\pi : \mathbb{P}(\mathcal{T}) \rightarrow X$, we have a projection $\pi_1 : \mathbb{P}(\mathcal{T}) \rightarrow X_1$ to the projective quadric X_1 (the orthogonal Grassmannian of 1-planes). Moreover, the tautological line bundle on the projective bundle $\mathbb{P}(\mathcal{T})$ is the pull-back $\pi_1^*(\mathcal{T}_1)$ of the tautological line bundle \mathcal{T}_1 on X_1 . It follows by [Elman et al. 2008, §58] or [Fulton 1984, Chapter 3] that $c_i(-[\mathcal{T}]) = \pi_{*}(\pi_1^*)^*c_{i+m-1}(-[\mathcal{T}_1])$ for any i . Since $\dim X_1 = n - 2$, the Chern class $c_{n-1}(-[\mathcal{T}_1])$ vanishes, implying the vanishing of $c_{n-m}(-[\mathcal{T}])$.

In order to show that the kernel is generated by the elements (2.2) and c_{n-m} , we construct additive generators of the quotient C of the ring $\text{CH } \Gamma$ by the ideal generated by the elements (2.2) and c_{n-m} . We recall that the group $\text{CH } \Gamma$ is free; a basis is given by the products $c_1^{\alpha_1} \cdots c_{n-m}^{\alpha_{n-m}}$ with $\alpha_1 + \cdots + \alpha_{n-m} \leq m$. Using the additional relations in C , we can eliminate squares of c_i for $i > n/2 - m$. Indeed, in the quotient of C by the subgroup generated by the products satisfying the additional condition, any element is divisible by an arbitrary 2-power and therefore is 0 since C is finitely generated.

It follows that the group C is generated by the products $c_1^{\alpha_1} \cdots c_{n-m-1}^{\alpha_{n-m-1}}$ satisfying the additional condition $\alpha_i \leq 1$ for $i > n/2 - m$. It turns out that these are free generators. Moreover, they remain free when we map them to $\text{CT } X$ and this finishes the proof of the theorem.

Our products are free in $\text{CT } X$ because their images in the \mathbb{Q} -vector space $\mathbb{Q} \otimes \text{CH } \bar{X}$ are free, where \bar{X} is X over an algebraic closure of F . For odd n this follows from [Buch et al. 2009, Theorem 2.2(b) and formula (15)] (see Remark 2.3). For even n it follows from [Buch et al. 2009, Theorem 3.2(b) and formula (40)]. \square

Remark 2.3. The paper [Buch et al. 2009], applied in the above proof, actually deals with the singular cohomology ring instead of the Chow ring. The link is explained by the following two well-known facts: the variety \bar{X} is cellular and the ring $\text{CH } \bar{X}$ does not depend on the base field. If the base field is \mathbb{C} , then the cycle map from $\text{CH } \bar{X}$ to the corresponding singular cohomology ring is an isomorphism, [Fulton 1984, Example 19.1.11(b)].

Remark 2.4. In the case of the highest orthogonal Grassmannian, the ring $\text{CH } \bar{X}$ has been described in [Vishik 2005] (see also [Elman et al. 2008, Proposition 86.16 and Theorem 86.12]).

Remark 2.5. Theorem 2.1 shows that the ring $\text{CT } X$ only depends on the integers n and m .

Remark 2.6. For odd n , the ring $\text{CT } X$ can be identified with the full Chow ring $\text{CH } Y$ of the variety of isotropic m -planes in an $n-1$ -dimensional vector space endowed with a nondegenerate alternating bilinear form: there is an isomorphism $\text{CH } Y \rightarrow \text{CT } X$ mapping the Segre classes of the tautological vector bundle on Y to the Segre classes of \mathcal{T}_X . (See [Buch et al. 2009, Theorem 1.2] for a description of the ring $\text{CH } Y$ by generators and relations.) This funny observation in the case of the highest orthogonal Grassmannian turned out to be very useful in [Totaro 2005]. We do not use it here.

Our next and ultimate goal is to show that $\text{CT } X = \text{CH } X$ in the case of generic q . First we need to clarify what is meant by *generic*. We start with the notion of the standard generic quadratic form.

3. The standard generic quadratic form

For a field k (of any characteristic) and an integer $n \geq 2$, the standard generic n -dimensional quadratic form is defined as follows.

We consider the orthogonal group $O(n)$ over k and its tautological imbedding into the general linear group $GL(n)$. The generic fiber of the quotient map

$$GL(n) \rightarrow GL(n)/O(n)$$

is an $O(n)$ -torsor over the function field $F := k(GL(n)/O(n))$. It determines an n -dimensional quadratic form over F (via the identification of [Demazure and Gabriel 1970, Chapitre III, §5, 2.1]; for the case of smooth $O(n)$ see also [Knus et al. 1998, (29.28)]) which we call the *standard generic* one.

In order to describe it explicitly, we first recall the interpretation of the quotient variety $GL(n)/O(n)$ as the variety Q of nondegenerate quadratic forms on the vector space $V := k^n$.¹ The variety of all quadratic forms on V is an affine space (of dimension $n(n+1)/2$) and Q is its open subvariety. The group $GL(n)$ acts on Q in the evident way. The action is such that for any algebraically closed field $K \supset k$, the abstract group $GL(n)(K)$ of K -points of $GL(n)$ acts transitively on the set $Q(K)$ of K -points of Q . Finally, the algebraic group $O(n)$, by its very definition, is the stabilizer of the split quadratic form $q_0 \in Q(k)$, defined by the formulas (7.1) and (7.2).² It follows by [Demazure and Gabriel 1970, Proposition 2.1 of Chapter III §3] that Q is the quotient variety $GL(n)/O(n)$.

For any field extension L/k , an L -point of Q is a nondegenerate quadratic form q on the L -vector space V_L ; the fiber of the quotient map $GL(n) \rightarrow Q$ over this point is an $O(n)$ -torsor E over L , and q is the quadratic form corresponding to E . In particular, the quadratic form given by the generic fiber of $GL(n) \rightarrow Q$ is defined over the field of rational functions $F = k(t_{ij})_{1 \leq i \leq j \leq n}$ (where t_{ij} are indeterminates, and F/k is purely transcendental of the transcendence degree $n(n+1)/2$) by the formula

$$(x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq j \leq n} t_{ij} x_i x_j.$$

4. Chow rings of classifying spaces

Let F be a field (of arbitrary characteristic) and let G be an affine algebraic group over F , not necessarily smooth. The Chow ring CH_G of the classifying space of G , introduced in [Totaro 1999], is the G -equivariant Chow ring $CH_G(\text{Spec } F)$. This is a graded ring; the grading is given by codimension of cycles.

The ring CH_G is cofunctorial in G : a homomorphism $G' \rightarrow G$ of affine algebraic

¹This interpretation is mentioned in [Totaro 1999, §15]; however, due to the context, $k = \mathbb{C}$ there.

²A quadratic form over k is called *split* if it is isomorphic to q_0 .

groups produces a homomorphism of graded rings $\text{CH}_G \rightarrow \text{CH}_{G'}$ (see [Molina Rojas and Vistoli 2006, §2]).

By [Edidin and Graham 1997, Lemma 4] (see also [Karpenko 2012, §3]), if G is a split torus, the homomorphism of graded rings $S(\hat{G}) \rightarrow \text{CH}_G$ is an isomorphism, where \hat{G} is the character lattice of G , $S(\hat{G})$ is the symmetric \mathbb{Z} -algebra, and a character $\chi \in \hat{G} = S^1(\hat{G})$, viewed as a G -equivariant line bundle over $\text{Spec } F$, is mapped to its first equivariant Chern class in CH_G^1 .

We return to the situation where G is an arbitrary affine algebraic group over F :

Proposition 4.1. *Let G' be a closed normal subgroup of G such that the quotient $T := G/G'$ is a split torus. Then the restriction homomorphism $\text{CH}_G \rightarrow \text{CH}_{G'}$ is surjective and its kernel is generated by some elements in CH_G^1 . More precisely, the kernel is generated by the image of the (additive) homomorphism*

$$\hat{T} = S^1(\hat{T}) = \text{CH}_T^1 \rightarrow \text{CH}_G^1$$

induced by the quotient homomorphism $G \rightarrow T$.

Proof. For any integer i , let us consider a generically free G -representation V possessing an open G -equivariant subset $U \subset V$ such that $\text{codim}_V(V \setminus U) \geq i$ and there are a G -torsor $U \rightarrow U/G$ and a G' -torsor $U \rightarrow U/G'$. By definition of CH_G (and similarly for G' in place of G), we have a ring homomorphism $\text{CH}_G \rightarrow \text{CH}(U/G)$ which is bijective in codimensions $< i$. Moreover, the diagram

$$\begin{array}{ccc} \text{CH}_G & \longrightarrow & \text{CH}_{G'} \\ \downarrow & & \downarrow \\ \text{CH}(U/G) & \longrightarrow & \text{CH}(U/G') \end{array}$$

commutes, where the bottom map is the pull-back homomorphism with respect to the T -torsor $U/G' \rightarrow U/G$. Therefore, in order to prove surjectivity of $\text{CH}_G \rightarrow \text{CH}_{G'}$, it suffices to prove surjectivity of $\text{CH}(U/G) \rightarrow \text{CH}(U/G')$. Moreover, to get the description of the kernel for $\text{CH}_G \rightarrow \text{CH}_{G'}$ it suffices to prove the similar description for the kernel of $\text{CH}(U/G) \rightarrow \text{CH}(U/G')$, where the homomorphism $\hat{T} \rightarrow \text{CH}^1(U/G)$ is the composition $\hat{T} \rightarrow \text{CH}_G^1 \rightarrow \text{CH}^1(U/G)$.

Let us first consider the case of $T = \mathbb{G}_m$. Let \mathcal{L} be the line bundle

$$((U/G') \times \mathbb{A}^1)/T$$

over U/G . Then U/G' is an open subvariety in \mathcal{L} and its complement is the zero section. By the homotopy invariance and the localization property of Chow groups ([Ehman et al. 2008, Theorem 57.13 and Proposition 57.9]) we have an exact sequence,

$$\text{CH}(U/G) \rightarrow \text{CH}(U/G) \rightarrow \text{CH}(U/G') \rightarrow 0,$$

where the first map is the multiplication by the first Chern class of \mathcal{L} . This finishes the proof for $T = \mathbb{G}_m$.

In the general case, we induct on the rank of T . We decompose T as $\mathbb{G}_m \times T_1$ and define an intermediate subgroup G_1 with $G' \subset G_1 \subset G$ as the kernel of the composition $G \rightarrow T \rightarrow T_1$. The quotient G/G_1 is then T_1 and the quotient G_1/G' is \mathbb{G}_m . The homomorphism $\text{CH}_G \rightarrow \text{CH}_{G'}$ decomposes into the composition $\text{CH}_G \rightarrow \text{CH}_{G_1} \rightarrow \text{CH}_{G'}$. The surjectivity statement follows because both maps in the composition are surjective by induction. It remains to determine the kernel.

Let $x \in \text{CH}_G$ be an element vanishing in $\text{CH}_{G'}$, then the image of x in CH_{G_1} is the product yx_1 for some $x_1 \in \text{CH}_{G_1}$, where $y \in \text{CH}_{G_1}^1$ is the image of a character of \mathbb{G}_m . Extending the character to T , we get an element $y' \in \text{CH}_G$ lying in the image of $\hat{T} \rightarrow \text{CH}_G^1$ and mapped to y . Using the surjectivity of $\text{CH}_G \rightarrow \text{CH}_{G_1}$, we find an element $x'_1 \in \text{CH}_{G_1}$ mapped to x_1 . The difference $x - y'x'_1$ is then in the kernel of $\text{CH}_G \rightarrow \text{CH}_{G_1}$ and therefore, by induction, lies in the ideal generated by the image of \hat{T} . It follows that x itself lies in the ideal. \square

Corollary 4.2. *In the situation of Proposition 4.1, if the ring $\text{CH}_{G'}$ is generated by Chern classes (in the sense of [Karpenko 2018, §5]), then the ring CH_G is also generated by Chern classes.*

Proof. For any $i \geq 0$ and any $x \in \text{CH}_G^i$, since the image of x in $\text{CH}_{G'}^i$ is a polynomial in Chern classes, there exists an element $x' \in \text{CH}_G^i$, lying in the Chern subring, such that the difference $x - x'$ vanishes in $\text{CH}_{G'}$. By Proposition 4.1, $x - x'$ belongs to the ideal in CH_G generated by CH_G^1 so we can induct on i . \square

Example 4.3. Taking G to be a split connected reductive algebraic group and $G' \subset G$ to be the semisimple group given by the commutator subgroup of G , we are in the situation of Proposition 4.1: G/G' is a split torus. Therefore Proposition 4.1 describes the relation between the Chow ring of the classifying space of a split reductive group G and that of its semisimple part G' . In particular, by Corollary 4.2, if $\text{CH}_{G'}$ is generated by Chern classes, then CH_G is also generated by Chern classes. This has been proved (by a different method) in [Karpenko 2018, Proposition 5.5] in the case of special (split reductive) G , where *special* means that every G -torsor over any field extension of the base field is trivial.

5. Chow rings of classifying spaces for orthogonal groups

The following proposition is a (slightly modified) particular case of [Totaro 1999, Proposition 14.2]. We provide a proof because it is shorter than that of the original statement.

Proposition 5.1. *For any algebraic group G (over any field) and any imbedding of G into a special algebraic group H , the homomorphism $\text{CH}_H \rightarrow \text{CH}_G$ is surjective provided that the Chow groups of the quotient H/G over any field extension of the base field are trivial in positive codimensions.*

Proof. As usual, we replace the homomorphism in question by the pull-back homomorphism $\mathrm{CH}(U/H) \rightarrow \mathrm{CH}(U/G)$ with respect to the morphism $U/G \rightarrow U/H$, where U is an open subvariety in an H -representation, an H -torsor over U/H , and a G -torsor over U/G . Since H is special, every H -torsor is Zariski–locally trivial, [Chevalley et al. 1958]. It follows that the fiber of $U/G \rightarrow U/H$ over any point $x \in U/H$ is isomorphic to the quotient variety H/G with scalars extended to the residue field of x and therefore has trivial Chow groups in positive codimensions. The statement follows from the spectral sequence of [Rost 1996, Corollary 8.2] (see also [Karpenko and Merkurjev 1990, §3]) computing the K -cohomology groups of the total space of the fibration $U/G \rightarrow U/H$ in terms of the K -cohomology groups of the base and of the fibers. \square

We get the following statement for arbitrary base fields of arbitrary characteristic:

Corollary 5.2. *For any $n \geq 2$, the homomorphism $\mathrm{CH}_{\mathrm{GL}(n)} \rightarrow \mathrm{CH}_{O(n)}$, given by the tautological imbedding $O(n) \hookrightarrow \mathrm{GL}(n)$, is surjective.*

Proof. As explained in Section 3, the quotient variety $\mathrm{GL}(n)/O(n)$ is identified with the variety Q of n -dimensional nondegenerate quadratic forms. Since Q is an open subvariety in the affine space of all n -dimensional quadratic forms, we have $\mathrm{CH}^{>0}(Q) = 0$ by the homotopy invariance and the localization property of Chow groups. \square

6. Main theorem and its consequences

In this section, k is a field (of any characteristic), n is an integer ≥ 2 , F is the function field $k(\mathrm{GL}(n)/O(n))$, E is the standard generic $O(n)$ -torsor given by the generic fiber of $\mathrm{GL}(n) \rightarrow \mathrm{GL}(n)/O(n)$, and q is the corresponding standard generic quadratic form.

For m with $1 \leq m \leq n/2$, let X be the m -th orthogonal Grassmannian of q . We would like to determine the ring $\mathrm{CH} X$. The main result is expressed in terms of the tautological (rank- m) vector bundle on X . Its proof will be given in the next section.

Theorem 6.1. *The ring $\mathrm{CH} X$ is generated by the Chern classes of the tautological vector bundle.*

Theorem 6.1 claims that $\mathrm{CH} X = \mathrm{CT} X$, and the ring $\mathrm{CT} X$ has been computed in Section 2.

Before proving Theorem 6.1, let us list some consequences. Let Y be any (partial) flag variety of totally isotropic subspaces in q . Let us consider the standard graded epimorphism $\mathrm{CH} Y \rightarrow GK(Y)$ onto the graded ring associated with the topological filtration (i.e., the filtration by codimension of support) on the Grothendieck ring $K(Y)$.

Corollary 6.2. *The abelian group $CH Y$ is free and, in particular, torsion-free. The ring epimorphism $CH Y \rightarrow GK(Y)$ is an isomorphism. The topological filtration on $K(Y)$ coincides with the gamma filtration.*

Proof. The variety Y is the variety of flags of totally isotropic subspaces in q of some dimensions $m_1 < \dots < m_d$. Let X be the orthogonal Grassmannian of m -planes with $m = m_d$. The projection $Y \rightarrow X$ is a partial flag variety of subspaces in the tautological vector bundle on X . Therefore, it suffices to prove Corollary 6.2 for X instead of Y .

We have that $CH X = CT X$ (Theorem 6.1) and that $CT X$ is a free abelian group (Theorem 2.1).

The kernel of the epimorphism is contained in the torsion subgroup. Since $CH X$ is torsion-free, the epimorphism is an isomorphism.

Since the Chow ring $CH X$ is generated by Chern classes, the topological filtration on $K(X)$ coincides with the gamma filtration; see [Karpenko 1998, Remark 2.17]. □

7. Proof of the main theorem

We continue to work over a field k of arbitrary characteristic. We realize the orthogonal group $O(n)$ as the automorphism group of the following split quadratic form $q_0 : V \rightarrow k$ on the k -vector space $V := k^n$:

$$(7.1) \quad k^n \ni (x_1, \dots, x_{n/2}, y_{n/2}, \dots, y_1) \mapsto x_1y_1 + x_2y_2 + \dots + x_{n/2}y_{n/2}$$

if n is even, and

$$(7.2) \quad k^n \ni (x_1, \dots, x_{(n-1)/2}, z, y_{(n-1)/2}, \dots, y_1) \\ \mapsto x_1y_1 + x_2y_2 + \dots + x_{(n-1)/2}y_{(n-1)/2} + z^2$$

if n is odd.

Instead of the m -th orthogonal Grassmannian X_0 (for some m with $1 \leq m \leq n/2$), we consider the variety Y_0 of flags of totally isotropic subspaces in q_0 of dimensions $1, \dots, m$. The group $O(n)$ acts on Y_0 and for any algebraically closed field $K \supset k$ the action of the group $O(n)(K)$ on the set $Y_0(K)$ is transitive. Therefore, by [Demazure and Gabriel 1970, Proposition 2.1 of Chapter III §3], the variety Y_0 is the quotient $O(n)/P$, where P is the stabilizer of the rational point of Y_0 given by the standard flag $V_1 \subset \dots \subset V_m$ with V_i being the span of the first i vectors in the standard basis of V . (Note that for any m , the variety X_0 is also the quotient of $O(n)$ by the stabilizer of any rational point on X_0 ; this includes $m = n/2$ even though neither X_0 nor Y_0 are connected in this case: recall that the orthogonal group $O(n)$ is also nonconnected for even n .)

Any orthogonal transformation stabilizing this flag also stabilizes the orthogonal complements

$$V_m^\perp = V_{n-m} \subset \cdots \subset V_1^\perp = V_{n-1}.$$

Let \mathcal{F} be the variety of flags of all subspaces in V of dimensions

$$1, \dots, m, n - m, \dots, n - 1.$$

The group $GL(n)$ acts on \mathcal{F} and $\mathcal{F} = GL(n)/S$, where S is the stabilizer of the standard flag $V_1 \subset \cdots \subset V_m \subset V_{n-m} \subset \cdots \subset V_{n-1}$.

Let E be the standard generic $O(n)$ -torsor given by the generic fiber of $GL(n) \rightarrow GL(n)/O(n)$. Let \mathcal{E} be the corresponding $GL(n)$ -torsor obtained via the imbedding $O(n) \hookrightarrow GL(n)$. We have a commutative square,

$$\begin{array}{ccc} CH_S & \longrightarrow & CH(\mathcal{E}/S) \\ \downarrow & & \downarrow \\ CH_P & \longrightarrow & CH(E/P) \end{array}$$

with surjective horizontal mappings (cf. [Karpenko 2017a, Lemma 2.1]).

We claim that the homomorphism $CH_S \rightarrow CH_P$ is surjective. Admitting the claim for the moment, we conclude that the pull-back homomorphism

$$CH(\mathcal{E}/S) \rightarrow CH(E/P) = CH Y$$

from the above commutative square is surjective too. Since the group $GL(n)$ is special, the $GL(n)$ -torsor \mathcal{E} is trivial, implying that $\mathcal{E}/S = \mathcal{F}$. We get a surjection $CH \mathcal{F} \rightarrow CH Y$ implying that the ring $CH Y$ is generated by the Chern classes of the m tautological vector bundles on Y (given by the components of the flags). It follows (see [Karpenko 2017b, Lemma 4.3]) that $CH X = CT X$.

We finish by proving the claim. The subgroup $S' := \mathbb{G}_m^m \times GL(n - 2m) \times \mathbb{G}_m^m \subset S$ is a Levi subgroup of S , its intersection with $P \subset S$ is $P' := \mathbb{G}_m^m \times O(n - 2m)$. The imbedding $P' \hookrightarrow S'$ is the product of the map $\mathbb{G}_m^m \hookrightarrow \mathbb{G}_m^m \times \mathbb{G}_m^m$, $x \mapsto (x, x^{-1})$ and the tautological imbedding $O(n - 2m) \hookrightarrow GL(n - 2m)$.

In the commutative square

$$\begin{array}{ccc} CH_S & \longrightarrow & CH_{S'} \\ \downarrow & & \downarrow \\ CH_P & \longrightarrow & CH_{P'} \end{array}$$

the horizontal maps are isomorphisms (see [Karpenko 2018, proof of Proposition 6.1]). Therefore, in order to prove the claim, it suffices to prove that the homomorphism $CH_{S'} \rightarrow CH_{P'}$ is surjective.

In the commutative square

$$\begin{array}{ccc} \text{CH}_{S'} & \longrightarrow & \text{CH}_{\text{GL}(n-2m)} \\ \downarrow & & \downarrow \\ \text{CH}_{P'} & \longrightarrow & \text{CH}_{O(n-2m)} \end{array}$$

the horizontal maps are epimorphisms by Proposition 4.1. The map on the right is an epimorphism by Corollary 5.2. We can now prove the surjectivity of the map on the left in every codimension $i \geq 0$ using induction on i .

For $i = 0$ there is nothing to prove. For $i = 1$, we have a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathbb{G}_m^m \times \mathbb{G}_m^m} & \longrightarrow & \text{CH}_{S'}^1 & \twoheadrightarrow & \text{CH}_{\text{GL}(n-2m)}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathbb{G}_m^m} & \longrightarrow & \text{CH}_{P'}^1 & \twoheadrightarrow & \text{CH}_{O(n-2m)}^1 \end{array}$$

with a surjection on the left. Since the lower row is exact (by Proposition 4.1),³ the statement for $i = 1$ follows.

For $i \geq 2$, it suffices to show that any element $x \in \text{CH}_{P'}^i$, vanishing in the group $\text{CH}_{O(n-2m)}$, is in the image of $\text{CH}_{S'}$. Since $x = y_1 x_1 + \dots + y_r x_r$ for some $r \geq 0$, some $y_1, \dots, y_r \in \text{CH}_{P'}^1$, and some $x_1, \dots, x_r \in \text{CH}_{P'}^{i-1}$ by Proposition 4.1, we are done.

8. The generic quadratic form in characteristic $\neq 2$

For a field k of characteristic $\neq 2$ and an integer $n \geq 2$, we defined in the introduction the generic n -dimensional quadratic form $q_g := \langle t_1, \dots, t_n \rangle$ over the field of rational functions $F_g := k(t_1, \dots, t_n)$, and the standard generic quadratic form q over the field of rational functions $F := k(t_{ij})_{1 \leq i \leq j \leq n}$. Now we are going to compare q_g with q .

Proposition 8.1. *The field F_g can be k -identified with a subfield in F the way that the field extension F/F_g is purely transcendental and the generic quadratic form q_g with the scalars extended to the field F becomes isomorphic to the standard generic form q .*

Corollary 8.2. *Theorem 6.1 as well as Corollary 6.2 hold for the generic quadratic form in place of the standard generic one.*

Proof. In case of a purely transcendental field extension, the change of field homomorphism for Chow rings is an isomorphism (see [Kahn and Sujatha 2000, Lemma 1.4a]). □

³The upper row is also exact but we don't care.

Proof of Proposition 8.1. Let us apply the standard orthogonalization procedure to the standard basis e_1, \dots, e_n of F^n , where the orthogonality refers to the symmetric bilinear form associated with q . This means that we construct an orthogonal basis e'_1, \dots, e'_n by taking for e'_i the sum of e_i and a linear combination of e_1, \dots, e_{i-1} , where the coefficients of the linear combination are determined by the condition that e'_i is orthogonal to e_1, \dots, e_{i-1} . The procedure works for q because its restriction to the span of e_1, \dots, e_i is nondegenerate for every i .

Then $t_i := q(e'_i)$ equals t_{ii} plus a rational function in $t_{11}, \dots, t_{i-1, i-1}$ and in t_{rs} with $1 \leq r < s \leq n$. It follows that the elements t_{rs} ($1 \leq r < s \leq n$) and t_1, \dots, t_n all together generate the field F over k and therefore—since their number is the transcendence degree—are algebraically independent over k . In particular, t_1, \dots, t_n are algebraically independent so that the field F_g is identified with the subfield $k(t_1, \dots, t_n) \subset F$. This identification has the required properties. \square

9. The generic quadratic form in characteristic 2

In characteristic 2 (actually, in arbitrary characteristic), any nondegenerate quadratic form, depending on the parity of n , is isomorphic to the form

$$[a_1, a_2] \perp \cdots \perp [a_{n-1}, a_n] \quad \text{or} \quad [a_1, a_2] \perp \cdots \perp [a_{n-2}, a_{n-1}] \perp \langle a_n \rangle,$$

where a_1, \dots, a_n are constants from the base field and $a_n \neq 0$ in the case of odd n . The notation $[a_1, a_2]$ stands for the 2-dimensional form

$$(x_1, x_2) \mapsto a_1 x_1^2 + x_1 x_2 + a_2 x_2^2.$$

So, the generic n -dimensional quadratic form q_g will be defined as the form

$$(9.1) \quad [t_1, t_2] \perp \cdots \perp [t_{n-1}, t_n] \quad \text{or} \quad [t_1, t_2] \perp \cdots \perp [t_{n-2}, t_{n-1}] \perp \langle t_n \rangle$$

over the rational function field $F_g := k(t_1, \dots, t_n)$.

Proposition 9.2. *Proposition 8.1 and Corollary 8.2 hold in characteristic 2 as well.*

Proof. We only need to identify the field F_g with a subfield in $F = k(t_{ij})_{1 \leq i \leq j \leq n}$ (over k) the way that the field extension F/F_g is purely transcendental and the generic quadratic form q_g with the scalars extended to the field F becomes isomorphic to the standard generic q .

Starting with the standard basis e_1, \dots, e_n of the vector space F^n , we construct a new basis e'_1, \dots, e'_n as follows. For every odd i , the vector e'_i is e_i plus a linear combination of e_1, \dots, e_{i-1} and if $i < n$ then the vector e'_{i+1} is e_{i+1} plus a linear combination of e_1, \dots, e_{i-1} , where the coefficients of the linear combinations are determined by the condition that the new vectors are orthogonal to each of e_1, \dots, e_{i-1} . Additionally, for every even i , we divide the vector e'_i by the nonzero scalar (e'_{i-1}, e'_i) .

With respect to the new basis, the standard generic quadratic form q has the shape (9.1), where $t_i := q(e'_i)$. For odd i , t_i equals t_{ii} plus a rational function in $t_{11}, \dots, t_{i-1i-1}$ and in t_{rs} with $1 \leq r < s \leq n$. For even i , t_i equals t_{ii}/f_i plus a rational function in $t_{11}, \dots, t_{i-2i-2}$ and in t_{rs} with $1 \leq r < s \leq n$, where f_i is also a rational function in $t_{11}, \dots, t_{i-1i-1}$ and in t_{rs} with $1 \leq r < s \leq n$.

It follows that the elements t_{rs} ($1 \leq r < s \leq n$) and t_1, \dots, t_n all together generate the field F over k and therefore are algebraically independent over k . In particular, t_1, \dots, t_n are algebraically independent so that the field F_g is identified with the subfield $k(t_1, \dots, t_n) \subset F$. This identification has the required properties. \square

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