Canonical $p$-dimension of algebraic groups

Nikita A. Karpenko$^{a,*,1}$, Alexander S. Merkurjev$^{b,2}$

$^a$Laboratoire de Mathématiques de Lens, Faculté des Sciences Jean Perrin, Université d’Artois, Rue Jean Souvraz SP 18, 62307 Lens Cedex, France

$^b$Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA

Received 9 December 2004; accepted 23 July 2005

Communicated by Michel Van Den Bergh

Abstract

We describe a way to compute the $p$-relative version of the Berhuy–Reichstein canonical dimension for an arbitrary split semisimple algebraic group over an arbitrary field of an arbitrary characteristic ($p$ is any prime integer). The canonical $p$-dimension is computed for all split simple groups of classical types.

© 2005 Elsevier Inc. All rights reserved.

MSC: 14L17; 14C25

Keywords: Algebraic groups; Torsors; Splitting fields; Projective homogeneous varieties; Chow groups

Contents

1. Notational conventions and preliminaries .................................................... 3
   1.1. Varieties ............................................................................ 3

*Corresponding author. Current address: Max-Planck Institut für Mathematik, Postfach 7280, 53072 Bonn, Germany.

E-mail addresses: karpenko@euler.univ-artois.fr (N.A. Karpenko), merkurev@math.ucla.edu (A.S. Merkurjev).

1 The James D. Wolfensohn Fund and The Ellentuck Fund support is acknowledged by the first author. The first author was partially supported by the European Community’s Human Potential Programme under contract HPRN-CT-2002-00287 (KTAGS) and by the Max-Planck-Institut für Mathematik in Bonn.

2 The work of the second author has been supported by the NSF grant DMS #0355166.

0001-8708/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
The notion of the canonical dimension of an algebraic structure was introduced by Berhuy and Reichstein in [1]. The canonical dimension measures the size of generic splitting fields of the structure. The formal definition is given in §2. Here we present two basic examples:

- Let $X$ be a scheme over a field $F$. A field extension $L/F$ is called a splitting field of $X$, if $X$ has a point over $L$. A splitting field $L$ is called generic, if for any splitting field $K$ of $X$ there exists an $F$-place $L \rightarrow K$. The canonical dimension of $X$ is the minimum of the transcendence degree (over $F$) of all generic splitting fields of $X$.

- Let $G$ be an algebraic group over $F$. The canonical dimension of $G$ is the maximum of the canonical dimensions of all principal homogeneous varieties ($G$-torsors), defined over field extensions of $F$.

When dealing with a given algebraic structure, we usually have finitely many “significant” prime integers involved. For example, such primes associated with an algebraic group $G$ are the torsion prime integers of $G$ (see Remark 6.7). In order to locate contribution of a prime integer $p$ to the canonical dimension, we define canonical $p$-dimension in a similar fashion.

It turns out that canonical dimension and $p$-dimension of an arbitrary regular complete variety $X$ is closely related to the algebraic cycles on $X$ (see Corollaries 4.7 and 4.12). We express canonical $p$-dimension of a generically cellular variety in terms of its Chow group (see Theorem 5.8).

The main result of the paper is Theorem 6.9, giving a recipe to compute canonical $p$-dimension of an arbitrary split semisimple algebraic group over an arbitrary field (of arbitrary characteristic). The values of the canonical $p$-dimension are given for all split simple groups of classical type (see §8).
1. Notational conventions and preliminaries

1.1. Varieties

We refer as schemes to separated schemes of finite type over a field (there is no restrictions on the field, its characteristic is arbitrary). A variety in the paper is an integral scheme.

For a scheme $X$, the integer $d(X)$ is defined as the g.c.d. of the degrees of all closed points on $X$; for a prime integer $p$, $d_p(X)$ is the $p$-primary part of $d(X)$.

1.2. Chow groups

Let $X$ be a scheme over a field $F$. We write $\text{CH}(X)$ for the integral Chow group of $X$ (see [5]). Fixing a prime $p$, we write $\text{Ch}(X)$ for the modulo $p$ Chow group:

$$\text{Ch}(X) = \text{CH}(X)/p \cdot \text{CH}(X).$$

Furthermore, we write $\overline{\text{Ch}}(X)$ (resp. $\overline{\text{CH}}(X)$) for the colimit of $\text{Ch}(X_L)$ (resp. $\text{CH}(X_L)$) with $L$ running over all field extensions $L/F$, and we write $\overline{\text{CH}}(X)$ (resp. $\overline{\text{CH}}(X)$) for the image of the restriction homomorphism $\text{res} : \text{Ch}(X) \to \text{Ch}(\overline{X})$ (resp. $\text{CH}(X) \to \text{CH}(\overline{X})$). The group $\overline{\text{CH}}(X)$ is called the reduced Chow group of $X$; the group $\overline{\text{Ch}}(X)$ is called the modulo $p$ reduced Chow group of $X$. Note that

$$\overline{\text{Ch}}(X) = \overline{\text{CH}}(X)/(\overline{\text{CH}}(X) \cap p\overline{\text{CH}}(X))$$

is not the same as $\overline{\text{CH}}(X)/p\overline{\text{CH}}(X)$.

1.3. Places

Let $K$ be a field. A valuation ring $R$ of $K$ is a subring $R \subset K$, satisfying $K = R \cup (R \setminus \{0\})^{-1}$. Any valuation ring is local; $R = K$ is a trivial example of a valuation ring.

Given two fields $K$ and $L$, a place $K \to L$ is a local ring homomorphism $\pi : R \to L$ of a valuation ring $R \subset K$ (an embedding of fields is a trivial example of a place).

If $K$ and $L$ are extensions of a field $F$, an $F$-place (or a place over $F$) is a place $K \to L$ with $\pi$ defined and identical on $F$.

Places are composable: if $K \to L$ is a place, given by a ring homomorphism $\pi$, and $L \to E$ a place to a third field $E$, given by a homomorphism $\rho$ of a ring $S \subset L$, then the composition is the place $K \to E$, given the homomorphism $\rho \circ \pi : \pi^{-1}(S) \to E$, defined on the valuation ring $\pi^{-1}(S)$. In particular, any place $L \to E$ can be restricted to any subfield $K \subset L$.

In this paper, an $F$-place $K \to L$ is said to be geometric, if it can be represented as a composition of $F$-places with valuation rings being discrete valuation rings.

1.4. Places and points

Let $X$ be an $F$-variety and let $L$ be a field extension of $F$. If $X$ is complete, then for any valuation ring $R$ of the field $F(X)$ there exists an $F$-morphism $\text{Spec} R \to X$
[7, Chapter II, Theorem 4.7]; therefore an $F$-place $F(X) \to L$ produces an $L$-point of $X$.

Vice versa, if $X$ has an $L$-point and is regular, then there exists a geometric $F$-place $F(X) \to L$. Indeed, since $X$ is regular at the image $x \in X$ of $\text{Spec } L$, there exists a system of local parameters around $x$, which produces a geometric place $F(X) \to F(x)$; composing with the embedding $F(x) \hookrightarrow L$, we get the required place $F(X) \to L$.

2. Canonical dimension of determination functions

Let $F$ be a field, $\text{Fields}_F$ the category of all field extensions of $F$. Let $2^0$ be the category of the subsets of a 1-elemental set 0. A determination function $D$ over $F$ is a continuous functor $\text{Fields}_F \to 2^0$, where by continuity we mean that $D$ commutes with the filtered colimits. In other words, $D$ is a rule assigning to each $E \in \text{Fields}_F$ a value $D(E) \in \{\emptyset, 0\}$ such that

- if $D(E) = 0$ for some $E$, then $D(E') = 0$ for any $E'$ admitting an $F$-embedding $E \to E'$;
- (continuity property) if $D(E) = 0$ for some field $E$ covered by a (possibly infinite) filtered family of subfields $E_i$, then $D(E_i) = 0$ for some $E_i$.

A field $E \in \text{Fields}_F$ is called a splitting field of a determination function $D$, if $D(E) = 0$. A splitting field $E$ of $D$ is called generic, if for any splitting field $L$ there exists an $F$-place $E \to L$. If $D$ has at least one generic splitting field, canonical dimension $\text{cd}(D)$ of $D$ is defined as the minimum of the transcendence degrees (over $F$) of all generic splitting fields of $D$; if $D$ does not admit a generic splitting field, we set $\text{cd}(D) = \infty$.

Lemma 2.1. For a given determination function $D$, any splitting field of $D$, which is a subfield of a generic splitting field, is also generic. Besides, any splitting field contains a finitely generated splitting field and $\text{cd}(D) = \infty$ only if $D$ does not admit generic splitting.

Proof. If $E$ is a generic splitting field and $E'$ a splitting field contained in $E$, then for any splitting field $L$, restricting a place $E \to L$ to $E'$, we get a place $E' \to L$; therefore $E'$ is also generic.

Any splitting field contains a finitely generated splitting field by the continuity of the determination function.

If $D$ has a generic splitting field, then, taking a finitely generated splitting subfield, we get a finitely generated generic splitting field, showing that $\text{cd}(D)$ is finite. □

A determination function $D$ over $F$ is split, if $D(F) = 0$. In this case, $F$ is a generic splitting field of $D$ and $\text{cd}(D) = 0$.

Our basic example of a determination function is the determination function associated with a scheme $X$ over $F$:

$$L \mapsto \begin{cases} \emptyset & \text{if } X(L) = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$
The canonical dimension $\text{cd}(X)$ of an $F$-scheme $X$ is defined as the canonical dimension of the associated determination function (as explained in Remark 4.13, canonical dimension of complete regular $F$-varieties is a birational invariant).

**Example 2.2** (Karpenko and Merkurjev [11, Theorem 4.3]). Let $F$ be a field of characteristic $\neq 2$. Let $X$ be an anisotropic smooth projective quadric over $F$. Then $\text{cd}(X) = \dim X - i_1(X) + 1$, where $i_1(X)$ is the first Witt index of $X$.

Let $\text{PointedSets}$ be the category of the pointed sets and let $k$ be a field. A functor $\mathcal{F} : \text{Fields}_k \to \text{PointedSets}$ is called continuous, if it commutes with filtered colimits. If $\mathcal{F}$ is a continuous functor, then for any $F \in \text{Fields}_k$ and $\alpha \in \mathcal{F}(F)$, we get a determination function $D_\alpha$ over $F$ by setting

$$D_\alpha(L) = \begin{cases} 0 & \text{if } \alpha_L \text{ is the distinguished point of the set } \mathcal{F}(L); \\ \emptyset & \text{otherwise.} \end{cases}$$

Berhuy-Reichstein canonical dimension of a continuous functor $\mathcal{F}$ [1, §10] is the supremum of $\text{cd}(D_\alpha)$ for all $F$ and $\alpha \in \mathcal{F}(F)$. If $G$ is an algebraic group over the field $k$, canonical dimension of $G$ is defined as canonical dimension of the (continuous) functor $\text{Tors}_G$, taking a field $F$ to the set of isomorphism classes $\text{Tors}_G(F)$ of $G$-torsors over $F$. We note that canonical dimension of an algebraic group $G/k$ is not the same as canonical dimension of the underlying variety of $G$ (which is always 0 because $G(k) \neq \emptyset$).

### 3. Canonical $p$-dimension

Let us fix an arbitrary prime $p$ and refer to a splitting field $E$ of a determination function $D$ as $p$-generic, if for any splitting field $L$ of $D$ there exists a finite field extension $L'/L$ of degree prime to $p$ admitting a place $E \to L'$. Replacing generic splitting fields by the $p$-generic ones in the definitions of section 2, we get a modified notion of canonical dimension which we call canonical $p$-dimension and denote $\text{cd}_p$.

We refer to a finite field extension as $p$-coprime, if its degree is not divisible by $p$.

The following two lemmas are useful when working with $\text{cd}_p$.

**Lemma 3.1** (cf. [11, Lemma 3.3]). Let $K$ be an arbitrary field, $p$ a prime, $K'/K$ a $p$-coprime field extension, and $L/K$ an arbitrary field extension. Then there exists a field $L'$, containing $K'$ and $L$, such that the extension $L'/L$ is also $p$-coprime.

**Proof.** We argue as in the proof of [11, Lemma 3.3], where the case of $p = 2$ was treated. We may assume that $K'$ is generated over $K$ by one element; let $f(t) \in F[t]$ be its minimal polynomial. Since the degree of $f$ is coprime with $p$, there exists an

---

\(^3\)Our notion of $p$-generic splitting is based on the notion of $p$-generic splitting varieties of symbols in a modulo $p$ Milnor’s $K$-group of a field, introduced in [14].
irreducible divisor $g \in L[t]$ of $f$ over $L$ such that $\deg(g)$ is coprime with $p$ as well. We set $L' = L[t]/(g)$. □

Replacing the field embedding $K \hookrightarrow L$ by a place, one generalizes Lemma 3.1 as follows:

**Lemma 3.2.** Let $K$ be a field extension of $F$ of finite transcendence degree over $F$; let $K \rightarrow L$ be a geometric $F$-place and let $K'$ be a $p$-coprime field extension of $K$. Then there exists a $p$-coprime field extension $L'/L$ such that the place $K \rightarrow L$ extends to a place $K' \rightarrow L'$.

**Proof.** By Lemma 3.1 it suffices to prove Lemma 3.2 in the case where the place $K \rightarrow L$ is surjective and its valuation ring $R$ is a discrete valuation ring. Also it is clear, that is suffices to consider only two cases: (1) $K'/K$ is purely inseparable and (2) $K'/K$ is separable.

In the first case, the degree $[K':K]$ is a power of a prime $q \neq p$. We take an arbitrary valuation ring $R'$ of $K'$, lying over $R$, i.e., such that $R' \cap K = R$ and the embedding $R \rightarrow R'$ is local (such $R'$ exists in the case of an arbitrary field extension $K'/K$, [15, Chapter VI, Theorem 5]). Let $L'$ be the residue field of $R'$ so that we have a surjective place $K' \rightarrow L'$. We show that $L'$ is also purely inseparable over $L$ (and therefore $[L':L]$, being a power of the same $q$, is coprime to $p$). For this, we take an element $l \in L'$ and show that $l^q^n \in L$ for some $n$: let $k \in R'$ be a preimage of $l$; then $k^{q^n} \in K$ for some $n$ and consequently $l^{q^n} \in L$ for the same $n$.

In the second case we consider all valuation rings $R_1, \ldots, R_r$ of $K'$, lying over $R$ (the number of such valuation rings is finite by [15, Chapter VI, Theorem 12, Corollary 4]). The residue field of each $R_i$ is a finite extension of $L$. Moreover, $\sum_{i=1}^{r} e_i n_i = [K':K]$ [15, Chapter VI, Theorem 20, and p. 63] (the discrete valuation ring assumption and the separability assumption are needed for this equality), where $n_i$ is the degree over $L$ of the residue field of $R_i$, and $e_i$ is the reduced ramification index of $R_i$ over $R$, [15, Definition on pp. 52–53]. It follows that at least one of $n_i$ is not divisible by $p$. □

Let us make some first general observations on $cd_p$. Clearly, a generic splitting field of a determination function is also $p$-generic; therefore we always have $cd \geq cd_p$.

Also it is clear, that $cd_p$ is not interesting, if the determination function in question has a $p$-coprime splitting field. More precisely, one has a simple

**Lemma 3.3.** Assume that a determination function $D$ is split by a $p$-coprime extension $E/F$. Then $cd_p(D) = 0$.

**Proof.** It follows by Lemma 3.1, that the splitting field $E$ of $D$ is $p$-generic. □

**Example 3.4.** The computation of $cd(X)$ for an anisotropic smooth projective quadric $X$ (over a field of characteristic $\neq 2$), given in [11] (see Example 2.2), shows in fact also that $cd_2(X) = cd(X)$. 
Example 3.5. Let \( X \) be a Severi-Brauer variety. If \( d(X) = d_p(X) \) (that is, \( d(X) \) is a power of the prime \( p \)), then \( \text{cd}(X) = \text{cd}_p(X) = d_p(X) - 1 \), [1, Theorem 11.4]. Now if \( d(X) \) is not a power of a prime, the value of \( \text{cd}(X) \) is not known, while \( \text{cd}_p(X) \) is still \( d_p(X) - 1 \) (see Example 5.10).

Example 3.6. The computation of \( \text{cd}(\text{SO}_n) \), given in [10], also shows that \( \text{cd}_2(\text{SO}_n) = \text{cd}(\text{SO}_n) \) (see also Example 5.11 as well as (8.2) and (8.4)).

Remark 3.7. Let \( F \) and \( F' \) be continuous functors \( \text{Fields}_k \to \text{PointedSets} \) with a morphism \( f : F \to F' \). If the kernel of \( f \) is trivial, then for any \( F \in \text{Fields}_k \) and any \( x \in F(F) \) the determination function of \( x \) coincides with the determination function of \( f(x) \) (cf. [1, Lemma 10.2(a)]); therefore \( \text{cd}(F) \leq \text{cd}(F') \) (cf. [1, Lemma 10.2(b)]) and \( \text{cd}_p(F) \leq \text{cd}_p(F') \) (for any \( p \)). If moreover \( f \) is surjective (but not necessarily injective), then \( \text{cd}(F) = \text{cd}(F') \) (cf. [1, Lemma 10.2(c)]) and \( \text{cd}_p(F) = \text{cd}_p(F') \).

4. Canonical \((p-)\)dimension of regular complete varieties

Lemma 4.1. The function field of a regular variety \( X \) is a generic splitting field of \( X \); in particular, \( \text{cd}(X) \leq \text{dim } X \) for regular \( X \).

Proof. The function field \( F(X) \) is a splitting field of \( X \) (even in the non-regular case). If \( L \) is an arbitrary splitting field of regular \( X \), then by §1.4 there exists an \( F \)-place \( F(X) \to L \); this shows that the splitting field \( F(X) \) is generic. \( \square \)

Remark 4.2. The \( F \)-place \( F(X) \to L \) we get in the proof of Lemma 4.1 is geometric (as defined in §1.3).

We have the following generalization of Lemma 4.1:

Lemma 4.3. If \( Y \) is a closed subvariety of a regular variety \( X \), admitting a dominant rational morphism \( X \to Y \), then the function field of \( Y \) is a generic splitting field of \( X \). In particular, \( \text{cd}(X) \leq \text{dim } Y \).

Proof. Clearly, \( F(Y) \) is a splitting field of \( X \). A dominant rational morphism \( X \to Y \) produces an \( F \)-embedding of \( F(Y) \) into the field \( F(X) \), which by Lemma 4.1 is a generic splitting field of \( X \). It follows by Lemma 2.1 that \( F(Y) \) is a generic splitting field too. \( \square \)

Lemma 4.4. Let \( Y \) be a scheme over a field \( F \), \( X \) a variety over \( F \).

1. If \( Y \) admits a dominant rational morphism \( X \to Y \), then the \( F(X) \)-scheme \( Y_{F(X)} \) has a closed rational point.

2. If the \( F(X) \)-scheme \( Y_{F(X)} \) has a closed rational point, then there exists a closed subvariety \( Y' \subset Y \), admitting a dominant rational morphism \( X \to Y' \).
Proof. Existence of a rational morphism $X \to Y$ is equivalent to existence of a closed rational point on $Y_{F(X)}$. To prove the second statement, we take as $Y'$ the closure of the image of the rational morphism $X \to Y$. □

Proposition 4.5. Any regular complete variety $X$ has a closed subvariety $Y \subset X$ of dimension $\dim Y = \text{cd}(X)$, admitting a dominant rational morphism $X \to Y$.

Proof. Let us take a generic splitting field $E$ of $X$, having the transcendence degree $\text{cd}(X)$ over $F$. Since $E$ is a splitting field of $X$, there exists a morphism $\text{Spec } E \to X$; let $T \subset X$ be the closure of its image. Since the splitting field $E$ is generic, there exists an $F$-place $E \to F(X)$; composing it with the embedding of the function field $F(T)$ into $E$, we get an $F$-place $F(T) \to F(X)$, producing by completeness of $T$ a morphism $\text{Spec } F(X) \to T$; we define $Y$ as the closure of its image. Clearly, $Y$ admits a dominant rational morphism $X \to Y$ and $\dim Y \leq \dim T \leq \text{tr.deg } E = \text{cd}(X)$. On the other hand, by Lemma 4.3, $\dim Y \geq \text{cd}(X)$. Therefore, $\dim Y = \text{cd}(X)$. □

Combining Lemma 4.3 and Proposition 4.5, we get

Corollary 4.6. Canonical dimension of a regular complete variety $X$ is the minimal dimension of the closed subvarieties $Y \subset X$, admitting a dominant rational morphism $X \to Y$.

Taking into account Lemma 4.4, we get the following variant of Corollary 4.6:

Corollary 4.7. Canonical dimension of a regular complete variety $X$ is the minimal dimension of the closed subvarieties $Y \subset X$, satisfying $Y(F(X)) \neq \emptyset$.

Now we establish variants of Lemma 4.3, Proposition 4.5, and Corollaries 4.6 and 4.7, related to the canonical $p$-dimension.

We say that a closed subvariety $Y$ of an $F$-variety $X$ satisfies condition $(\ast)$, if the function field $F(Y)$ embeds (over $F$) into a $p$-coprime field extension of $F(X)$.

Lemma 4.8. If $Y$ is a closed subvariety of a regular variety $X$, satisfying condition $(\ast)$, then the function field of $Y$ is a $p$-generic splitting field of $X$. In particular,

$$\text{cd}_p(X) \leq \dim Y.$$

Proof. Clearly, $F(Y)$ is a splitting field of $X$. Let $K'/F(X)$ be a $p$-coprime field extension with an $F$-embedding $F(Y) \hookrightarrow K'$. For an arbitrary splitting field $L$ of $X$ we can find a geometric $F$-place $F(X) \to L$ (see Lemma 4.1 with Remark 4.2). Applying Lemma 3.2 to this place and the field extension $K'/F(X)$, we get a place $K' \to L'$ to some $p$-coprime field extension $L'/L$. Restricting the latter place to the subfield $F(Y) \subset K'$, we get a place $F(Y) \to L'$; therefore, the splitting field $F(Y)$ is $p$-generic. □
Lemma 4.9. Let $Y$ be a scheme over a field $F$, $X$ a variety over $F$.

(1) If $Y$ satisfies condition (*), then $d_p(Y_{F(X)}) = 1$ (see Section 1.1 for definition of $d_p$).

(2) If $d_p(Y_{F(X)}) = 1$, then there exists a closed subvariety $Y' \subset Y \subset X$, satisfying condition (*).

Proposition 4.10. Any regular complete variety $X$ has a closed subvariety $Y \subset X$ of dimension $\dim Y = \text{cd}_p(X)$, satisfying condition (*).

Proof. Let us take a $p$-generic splitting field $E$ of $X$, having the transcendence degree $\text{cd}_p(X)$ over $F$. Since $E$ is a splitting field of $X$, there exists a morphism $\text{Spec } E \to X$; let $T \subset X$ be the closure of its image. Since the splitting field $E$ is $p$-generic, there exists an $F$-place $E \to K'$, where $K'/F(X)$ is a $p$-coprime field extension. Restricting to $F(T) \subset E$, we get an $F$-place $F(T) \to K'$. By completeness of $T$, the place $F(T) \to K'$ produces a morphism $\text{Spec } K' \to T$; we define $Y$ as the closure of its image. Clearly, $Y$ satisfied condition (*) and $\dim Y \leq \dim T \leq \text{tr.deg} E = \text{cd}_p(X)$. On the other hand, by Lemma 4.8, $\dim Y \geq \text{cd}_p(X)$. Therefore, $\dim Y = \text{cd}_p(X)$. □

Lemma 4.8 and Proposition 4.10 together produce

Corollary 4.11. The canonical $p$-dimension of a regular complete variety $X$ is the minimal dimension of the closed subvarieties $Y \subset X$, satisfying (*).

By Lemma 4.9, the following variant of Corollary 4.11 also holds:

Corollary 4.12. The canonical $p$-dimension of a regular complete variety $X$ is the minimal dimension of the closed subvarieties $Y \subset X$ with $d_p(Y_{F(X)}) = 1$.

Remark 4.13. We would like to notice that the canonical ($p$-)dimension of a complete regular $F$-variety $X$ is a birational invariant of $X$. Indeed, $\text{cd}(X)$ for such $X$ can be determined in terms of $F(X)$ as the minimal transcendence degree of the field extensions $L/F$ possessing $F$-places to and from $F(X)$; similarly, $\text{cd}_p(X)$ is the minimal transcendence degree of the field extensions $L/F$ possessing an $F$-place from $F(X)$ and an $F$-place to a $p$-coprime extension of $F(X)$.

5. Generically $p$-split varieties

In this section $X$ stands for a smooth complete absolutely irreducible variety over a field $F$.

Lemma 5.1. The degree homomorphism $\deg : \text{Ch}_0(X) \to \mathbb{F}_p$ is an isomorphism if and only if $\dim_{\mathbb{F}_p} \text{Ch}_0(X) = 1$.

Proof. The degree homomorphism is non-zero and therefore surjective. □
Lemma 5.2. Assume that $\dim_{\mathbb{F}_p} \text{Ch}_0(X) = 1$. Let $T$ be an arbitrary $F$-scheme, let $E_1$ and $E_2$ be field extensions of $F$, and let $f_1 : \text{Spec } E_1 \to X$ and $f_2 : \text{Spec } E_2 \to X$ be $F$-morphisms. Then the diagram

$$
\begin{array}{ccc}
\text{Ch}(T \times X) & \xleftarrow{(\text{id}_T \times f_1)^*} & \text{Ch}(T_{E_1}) \\
\downarrow & & \downarrow \\
\text{Ch}(T_{E_2}) & \xrightarrow{(\text{id}_T \times f_2)^*} & \text{Ch}(T) \\
\end{array}
$$

is commutative.

Proof. Let $E$ be a field extension of $F$, containing $E_1$ and $E_2$. Replacing $T$ and $X$ by $T_E$ and $X_E$, we come to the following situation: $E = E_1 = E_2 = F$ and for some closed rational points $x_1, x_2 \in X$, $f_i$ is the embedding $T = T \times \{x_i\} \hookrightarrow T \times X$. We want to show that $f_1^* = f_2^* : \text{Ch}(T \times X) \to \text{Ch}(T)$. Since $pr_i \circ f_i^* = \text{id}$ for $i = 1, 2$, where $pr$ is the projection $T \times X \to T$, it suffices to show that $f_1^* \circ f_1^* = f_2^* \circ f_2^* : \text{Ch}(T \times X) \to \text{Ch}(T \times X)$.

The composition $f_1^* \circ f_1^*$ coincides with the multiplication by $[T \times x_1]$. Since by the assumption on $\dim_{\mathbb{F}_p} \text{Ch}_0(X)$ and Lemma 5.1, the degree homomorphism $\deg : \text{Ch}(X) \to \mathbb{F}_p$ is an isomorphism, we have $[x_1] = [x_2] \in \text{Ch}(X)$, and therefore $[T \times x_1] = [T \times x_2]$ in $\text{Ch}(T \times X)$. The required assertion follows. \[\square\]

Let $in : Y \hookrightarrow X$ be a closed subvariety of $X$. The closed embedding

$$(\text{id}_Y, in) \times \text{id}_X : Y \times X \to Y \times X \times X$$

is regular, and we define a paring

$$\text{Ch}(Y) \otimes \text{Ch}(X \times X) \to \text{Ch}(Y \times X)$$

by the formula $\alpha \otimes \beta \mapsto ((\text{id}_Y, in) \times \text{id}_X)^*(\alpha \times \beta)$.

Proposition 5.3. Let $Y$ be a closed subvariety of $X$. Assume that $\dim_{\mathbb{F}_p} \text{Ch}_0(X) = 1$ and that for any field $E \supset F(X)$ the restriction homomorphism $\text{Ch}(X_{F(Y)}) \to \text{Ch}(X_{E})$ is an isomorphism. Then the above paring is surjective.

Proof. We proceed by induction on $\dim Y$. We have a commutative diagram

$$
\begin{array}{ccccccc}
\bigoplus_{Y'} \text{Ch}(Y') \otimes \text{Ch}(X \times X) & \xrightarrow{\text{Ch}(Y) \otimes \text{Ch}(X \times X)} & \text{Ch}(Y) \otimes \text{Ch}(X \times X) & \xrightarrow{\text{Ch}(X \times X)} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{Y'} \text{Ch}(Y' \times X) & \xrightarrow{\text{Ch}(Y \times X)} & \text{Ch}(Y \times X) & \xrightarrow{\text{Ch}(X_{F(Y)})} & 0
\end{array}
$$
where \( Y' \) runs over closed subvarieties of codimension 1 in \( Y \). The rows are exact. Indeed, the upper row is the obvious exact sequence \( \bigoplus_{Y'} \overline{\text{Ch}}(Y') \to \overline{\text{Ch}}(Y) \to \mathbb{Z} \to 0 \), tensored by \( \overline{\text{Ch}}(X \times X) \) over \( \mathbb{F}_p \). To see that the lower row is exact, one notices that the row with \( \text{Ch} \) in place of \( \overline{\text{Ch}} \) is exact and that the restriction homomorphism \( \text{Ch}(X_F(Y)) \to \overline{\text{Ch}}(X) \) is injective as the composite of the homomorphism \( \text{Ch}(X_F(Y)) \to \text{Ch}(X_F(Y)(X)) \), which is injective due to the specialization of [5, §20.3], and the isomorphism \( \text{Ch}(X_F(Y)(X)) \to \overline{\text{Ch}}(X) \) (see the assumption on \( \text{Ch}(X_F(X)) \)). Furthermore, the left vertical map of the diagram is surjective by the induction hypothesis. The right vertical map is surjective because the rhombus

\[
\begin{array}{c}
\text{Ch}(X \times X) \\
\text{Ch}(X_F(Y)) \\
\text{Ch}(X_F(X)) \\
\text{Ch}(X)
\end{array}
\]

is commutative (as guaranteed by the assumption on \( \text{Ch}_0(X) \) and Lemma 5.2 applied to \( T = X \)). \( \square \)

**Corollary 5.4.** Under the assumptions of Proposition 5.3, if the push-forward

\[
 \left( \text{in} \times \text{id}_X \right)_* : \overline{\text{Ch}}(Y \times X) \to \overline{\text{Ch}}(X \times X)
\]

is non-zero, then the push-forward \( \text{in}_* : \overline{\text{Ch}}(Y) \to \overline{\text{Ch}}(X) \) is also non-zero and, in particular, \( \overline{\text{Ch}}_i(X) \neq 0 \) for at least one \( i \leq \dim Y \).

**Proof.** The square

\[
\begin{array}{ccc}
\overline{\text{Ch}}(Y) \otimes \overline{\text{Ch}}(X \times X) & \longrightarrow & \overline{\text{Ch}}(Y \times X) \\
\downarrow \text{in}_* \otimes \text{id} & & \downarrow \text{(in id)}_* \\
\overline{\text{Ch}}(X) \otimes \overline{\text{Ch}}(X \times X) & \longrightarrow & \overline{\text{Ch}}(X \times X)
\end{array}
\]

is commutative. \( \square \)

**Definition 5.5.** We say that a (complete smooth absolutely irreducible) variety \( X \) over \( F \) is \( p \)-balanced, if the symmetric bilinear form

\[
\text{Ch}(X) \times \text{Ch}(X) \to \mathbb{F}_p, \quad (\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)
\]

is non-degenerate (in the sense that its radical is trivial; note that \( \dim_{\mathbb{F}_p} \text{Ch}(X) \) can be infinite).

A variety \( X \) over \( F \) is called cellular, if there is a filtration

\[
\emptyset = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X
\]
by closed subschemes such that for every $i = 0, 1, \ldots, n - 1$ the scheme $X_{i+1} \setminus X_i$ is isomorphic to an affine space over $F$.

**Remark 5.6.** Let $X$ be a geometrically cellular variety, that is, $X_E$ is cellular for some field extension $E/F$. We claim that $X$ is $p$-balanced (for any $p$). Indeed, the Chow-motive of the cellular variety $X_E$ decomposes into a finite direct sum of twists of the motive of the point (see, e.g., [9, Theorem 6.5]). Therefore $CH(X_E) = CH(X)$. Moreover, the mutually inverse isomorphisms of the motive of $X_E$ with the above direct sum are given by certain sequences $e_0, \ldots, e_n$ and $e'_0, \ldots, e'_n$ of homogeneous elements in $CH(X_E)$, which are bases of $CH(X_E)$ mutually dual with respect to the $\mathbb{Z}$-bilinear form $(\alpha, \beta) \mapsto \deg(\alpha \cdot \beta)$ (simply because they define mutually inverse isomorphisms of motives).

Note that for any $p$-balanced $X$ and any integer $i$, one has $\dim_{\mathbb{F}_p} Ch_i(X) = \dim_{\mathbb{F}_p} Ch_i(X)$, if at least one of these two dimensions is finite. Since $\dim_{\mathbb{F}_p} Ch_0(X) = 1$, the above equality with $i = 0$ implies that $\dim_{\mathbb{F}_p} Ch_0(X) = 1$ for a $p$-balanced $X$.

**Definition 5.7.** A $p$-balanced variety $X$ over $F$ is called $p$-split, if for any field $E \supset F$ the restriction homomorphism $Ch(X) \to Ch(X_E)$ is an isomorphism (in particular, one has $\overline{Ch}(X) = Ch(X)$ for a $p$-split $X$).

A cellular variety is $p$-split.

We say that a variety $X$ has a property generically, if $X$ over its own function field has this property. This way we get a notion of generically $p$-split variety. According to above remarks, a generically cellular variety is generically $p$-split.

We are ready to prove the main result of the first half of the paper, interpreting the canonical $p$-dimension of a generically $p$-split variety in terms of its modulo $p$ reduced Chow group:

**Theorem 5.8.** If $X$ is a generically $p$-split variety (see Definitions 5.7 and 5.5), then

$$\text{cd}_p(X) = \min\{i \mid \overline{Ch}_i(X) \neq 0\}.$$

In particular, the formula holds for a generically cellular $X$.

**Proof.** Two inequalities are proved separately.

Let $i$ be an integer such that the group $\overline{Ch}_i(X)$ is non-zero. Then $[Y] \neq 0$ for a closed $i$-dimensional subvariety $Y \subset X$. We are going to show that $d_p(Y_{F(X)}) = 1$ for such $Y$ (this suffices for our purposes by Corollary 4.12).

Since the variety $X_{F(X)}$ is $p$-split, there exists a prime cycle $Z \subset X_{F(X)}$ such that $\deg([Y_{F(X)}] \cdot [Z]) \neq 0$. Since the product $[Y_{F(X)}] \cdot [Z]$ can be represented by a cycle on the intersection $Y_{F(X)} \cap Z$ (see [5, §8.1]), the scheme $Y_{F(X)}$ has a closed $p$-coprime point, meaning that $d_p(Y_{F(X)}) = 1$. 


Let now \( \text{in} : Y \hookrightarrow X \) be a closed subvariety of \( X \), satisfying condition \((*)\), that is, \( F(Y) \hookrightarrow K \) for some \( p \)-coprime extension \( K/F(X) \). We will show that \( \overline{\text{Ch}}_i(X) \neq 0 \) for some \( i \leq \dim Y \). The desired inequality will then follow by Proposition 4.10.

Let \( Z \) be the closure of the image of the morphism \( \text{Spec} K \to Y \times X \). The cycle \((\text{in} \times \text{id}_X)_s([Z]) \in \overline{\text{Ch}}(X^2)\) is non-zero, because for the second projection \( pr : X^2 \to X \), we have

\[
pr_s((\text{in} \times \text{id}_X)_s[Z]) = [K : F(X)] \cdot [X] \neq 0 \in \overline{\text{Ch}}(X).
\]

It follows by Corollary 5.4 that \( \overline{\text{Ch}}_i(X) \neq 0 \) for some \( i \leq \dim Y \). \( \square \)

**Remark 5.9.** If we take \( Y \) with \( \dim Y = \text{cd}_p(X) \) in the beginning of the \((\geq)\)-part of the proof of Theorem 5.8, then, since we have already proved the \((\leq)\)-part of the theorem, we come to the conclusion that the \((\dim Y)\)-dimensional component of the homomorphism \( \text{in}_s : \overline{\text{Ch}}(Y) \to \overline{\text{Ch}}(X) \) is non-zero. Since the \((\dim Y)\)-dimensional component of the image of \( \text{in}_s \) is generated by \([Y] \in \overline{\text{Ch}}(X)\), we see that in fact the class in \( \overline{\text{Ch}}(X) \) of \( Y \) itself is non-zero.

**Example 5.10.** Let \( X \) be the Severi-Brauer variety of a central simple \( F \)-algebra \( A \). Since by Theorem 5.8, \( \text{cd}_p(X) = \text{cd}_p(X_L) \) for any \( p \)-coprime field extension \( L/F \), \( \text{cd}_p(X) = \text{cd}_p(Y) \), where \( Y \) is the Severi-Brauer variety of a division algebra, Brauer-equivalent to the \( p \)-primary part of \( A \). Furthermore, \( \overline{\text{Ch}}(Y) = \overline{\text{Ch}}^0(Y) \) by [8, Proposition 2.1.1]. Therefore, by Theorem 5.8, \( \text{cd}_p(Y) = \dim Y \), so that we get

\[
\text{cd}_p(X) = \text{cd}_p(Y) = \dim Y = d_p(X) - 1.
\]

**Example 5.11.** In this example \( p = 2 \). Let \( X/F \) be the orthogonal grassmannian of \( n \)-dimensional totally isotropic subspaces of a \((2n + 1)\)-dimensional non-degenerate quadratic form. If \( d_2(X) = 2^n \), then \( \overline{\text{Ch}}(X) = \overline{\text{Ch}}^0(X) \) by [10, Proposition 1.4] and therefore

\[
\text{cd}_2(X) = \dim(X) = n(n + 1)/2.
\]

Without any restriction on \( d_2(X) \), canonical 2-dimension of \( X \) can be expressed as the sum of all \( i \) such that the \( i \)th special Schubert class \( e_i \in \text{Ch}^i(X) \) is *non-rational*, i.e., does not lie in \( \overline{\text{Ch}}(X) \): indeed, by [16, Main Theorem 5.7], the product of all *rational* \( e_i \) is a non-zero element of \( \overline{\text{Ch}}(X) \) of the smallest possible dimension.

6. Canonical \( p \)-dimension of algebraic groups

If \( P \) is an algebraic group over a field \( F \), we write \( \text{CH}(BP) \) for the \( P \)-equivariant Chow ring \( \text{CH}_P(\text{Spec} F) \) of the point \( \text{Spec} F \) (see [4]).

Let \( G \) be a connected algebraic group over \( F \) and let \( P \) be a subgroup of \( G \). Consider the homomorphism

\[
\varphi_G = \varphi_{G,P} : \text{CH}(BP) = \text{CH}_P(\text{Spec} F) \xrightarrow{q^*} \text{CH}_P(G) = \text{CH}(G/P),
\]

where \( q : G \to \text{Spec} F \) is the structure morphism.
Remark 6.1. If $G$ is a subgroup of a group $G'$, then $\varphi_G = i^* \circ \varphi_{G'}$, where $i: G/P \to G'/P$ is the morphism, induced by the embedding of $G$ into $G'$.

Proposition 6.2. Let $G = \text{GL}_n$. Then the map $\varphi_G$ is surjective and the left $G$-action on $G/P$ induces the trivial action on $\text{CH}(G/P)$.

Proof. The group $G$ is embedded into the affine space of $\text{End}(F^n)$ as a $G$-equivariant open subset. The map $q^*$ factors as the composite

$$
\text{CH}_P(\text{Spec } F) \to \text{CH}_P(\text{End}(F^n)) \to \text{CH}_P(G),
$$

where the first pull-back map is an isomorphism by the homotopy invariance property and the second restriction map is surjective by the localization. Hence $\varphi_G$ is surjective.

For a rational point $g$ of $G$, let $\lambda_g: G \to G$ be the morphism of the left multiplication by $g$. It follows from $q \circ \lambda_g = q$ that $\lambda_g^* \circ q^* = q^*$. Since $q^*$ is surjective, $\lambda_g^*$ is the identity, i.e., $G$ acts trivially on $\text{CH}(G/P)$. □

Recall that we write $\text{CH}(G/P)$ for the colimit of $\text{CH}(GL/P_L)$ over all field extensions $L/F$. We define a homomorphism $\varphi_G$ as the composite

$$
\varphi_G : \text{CH}(BP) \xrightarrow{\varphi_G} \text{CH}(G/P) \xrightarrow{\text{res}} \text{CH}(G/P).
$$

Let $E$ be a (right) $G$-torsor over a field extension $F'$ of $F$. Set $K = F'(E)$. Let $\psi_E : \text{CH}(E/P) \to \text{CH}(G_K/P_K)$ be the pull-back map with respect to the morphism $G_K/P_K \to E/P$, induced by the $G$-equivariant morphism $G_K \to E$, taking the identity of $G$ to the generic point of $E$. We define a homomorphism $\psi_E$ as the composite

$$
\psi_E : \text{CH}(E/P) \xrightarrow{\psi_E} \text{CH}(G_K/P_K) \xrightarrow{\text{res}} \text{CH}(G/P).
$$

We identify $G$ with a subgroup of $S = \text{GL}_n$ for some $n$.

Lemma 6.3. Suppose that there is a $G$-equivariant morphism $E \to S$ over $F$ and let $h: E/P \to S/P$ be the induced morphism. Then $\varphi_G = \psi_E \circ h^* \circ \varphi_S$.

Proof. The composition $G_K \to E_K \to S_K$ of the morphisms induced by $G_K \to E$ and $E \to S$, differs from the inclusion $G_K \hookrightarrow S_K$ by a left multiplication by an element of $S(K)$. By Proposition 6.2, the induced pull-back homomorphisms $\text{CH}(S_K/P_K) \to \text{CH}(G_K/P_K)$ coincide. Composing with the restriction homomorphism $\text{CH}(S/P) \to \text{CH}(S_K/P_K)$, we get $\psi_E \circ h^* = \text{res}_{K/F} \circ i^*$, where $i: G/P \to S/P$ is the morphisms, induced by the embedding $G \hookrightarrow S$. We have:

$$
\varphi_G = \text{res} \circ \varphi_G = \text{res}_K \circ \text{res}_{K/F} \circ i^* \circ \varphi_S = \text{res}_K \circ \psi_E \circ h^* \circ \varphi_S = \psi_E \circ h^* \circ \varphi_S
$$

(for the second equality, see Remark 6.1). □

Theorem 6.4. (1) For any $G$-torsor $E$ (over any field extension of $F$) we have

$$
\text{Im}(\varphi_G) \subset \text{Im}(\psi_E).
$$
(2) There exists a $G$-torsor $E$ (over a field extension of $F$) such that $\text{Im}(\overline{\varphi}_G) = \text{Im}(\overline{\psi}_E)$.

Proof. (1) We may assume that $E$ is a $G$-torsor over $F$. By the Hilbert theorem 90, the $S$-torsor $(E \times S)/G$ is trivial (where $(E \times S)/G$ stands for the quotient of $E \times S$ by the action $(e, s) \cdot g = (eg, g^{-1}s)$ of $G$; the action of $G$ on this quotient is defined by the formula $(e, s) \cdot g = (e, sg)$, so that the embedding $E = E \times 1 \hookrightarrow E \times S$ induces a $G$-equivariant morphism $E \to (E \times S)/G$). In particular, there is a $G$-equivariant morphism $E \to S$. By Lemma 6.3, $\text{Im}(\overline{\varphi}_G) \subset \text{Im}(\overline{\psi}_E)$.

(2) Let $X = S/G$ and $K = F(X)$. Denote by $E \to \text{Spec} K$ the generic fiber of the projection $S \to X$. Clearly, $E$ is a $G$-torsor over $K$. Denote by $h : E/P_K \to S/P$ the morphism induced by the canonical $G$-equivariant morphism $E \to S$. Since $E/P_K$ is a localization of $S/P$, the pull-back homomorphism $h^*$ is surjective. By Proposition 6.2, $\varphi_S$ is also surjective. It follows from Lemma 6.3 that $\text{Im}(\overline{\varphi}_G) = \text{Im}(\overline{\psi}_E)$. □

Let $G$ be an algebraic group over a field $F$ and let $\text{Tors}_G$ be the functor $\text{Fields}_F \to \text{PointedSets}$, taking a field $K$ to the set of isomorphism classes $\text{Tors}_G(K)$ of $G$-torsors over $K$. For a $G$-torsor $E/K$, the determination function of $E \in \text{Tors}_G(K)$ coincides with the determination function of the $K$-variety $E$.

Let $P \subset G$ be a subgroup. We assume that $P$ is a special group, that is, the functor $\text{Tors}_P$ is trivial.

Lemma 6.5. The determination functions of the varieties $E$ and $E/P$ coincide.

Proof. Suppose $E/P$ has a point over $K$. We need to show that $E(K) \neq \emptyset$. The fiber of the natural morphism $E \to E/P$ over the point is a $P$-torsor. Since $P$ is special, this torsor is trivial, i.e., the fiber has a point over $K$. □

Remark 6.6. Let $G$ be a split semisimple algebraic group and let $P$ be a parabolic subgroup. The variety $G/P$ is cellular (see, e.g., [2]), therefore $\text{CH}(G/P) = \text{CH}(G/P)$.

Remark 6.7. Suppose further that $P$ is a Borel subgroup of $G$. The image of the composite

$$
\text{CH}(BP) \xrightarrow{\varphi_G} \text{CH}(G/P) \xrightarrow{\deg} \mathbb{Z}
$$

is a subgroup $t_G \mathbb{Z}$ with a positive integer $t_G$ known as the torsion index of $G$ (see [6]). It follows from Theorem 6.4 and Lemma 6.5 that $t_G$ is the l.c.m. of the numbers $d(E)$ over all $G$-torsors over all field extension. This statement is known as Grothendieck’s theorem [6, Theorem 2]. The prime divisors of the torsion index $t_G$ are called the torsion primes of $G$.

Let $G$ be a split semisimple group and let $P \subset G$ be a parabolic subgroup. Suppose $P$ is a special group (for example, $P$ is a Borel subgroup of $G$). By Lemma 6.5, it follows that the canonical dimension of $G$ (resp. $\text{cd}_P(G)$) is the supremum of the canonical dimension of $E/P$ (resp. $\text{cd}_P(E/P)$) over all $G$-torsors over all field extensions of
F. Let $E$ be a $G$-torsor. Note that the variety $E/P$ is projective. In order to apply Theorem 5.8 to the variety $E/P$, we need the following:

**Corollary 6.8.** The variety $E/P$ is generically cellular.

**Proof.** By Lemma 6.5, the torsor $E$ is split over the function field $L = F(E/P)$, hence $E_L \cong G_L$ and therefore $(E/P)_L \cong (G/P)_L$. The latter variety is cellular. □

Theorems 5.8 and 6.4 yield

**Theorem 6.9.** Let $G$ be a split semisimple group and let $P \subset G$ be a special parabolic subgroup (for example, a Borel subgroup). Denote by $\tilde{\text{CH}}(G/P)$ the image of the graded ring homomorphism $\varphi_G : \text{CH}(BP) \to \text{CH}(G/P)$. Then $\text{cd}_p(G)$ for a prime $p$, is equal to the smallest integer $i$ such that $\tilde{\text{CH}}_i(G/P)$ is not contained in $p\text{CH}_i(G/P)$.

**Remark 6.10.** The canonical dimension $\text{cd}_p(G)$ is positive if and only if $p$ is a torsion prime of $G$ (see Remark 6.7). Indeed, by Theorem 6.9, $\text{cd}_p(G) = 0$ if and only if $\text{CH}_0(G/P)$ is not divisible by $p$ in $\text{CH}_0(G/P)$, where $P$ is a Borel subgroup of $G$. Since $\text{CH}_0(G/P)$ is an infinite cyclic group generated by the class of a rational point, the latter is equivalent to the condition that $p$ does not divide $t_G$, i.e., $p$ is not a torsion prime of $G$.

7. Remarks on $\tilde{\text{CH}}(G/P)$

Let $P$ be an arbitrary subgroup of an algebraic group $G$. Let $P \to \text{GL}(V)$ be a finite-dimensional representation. The group $P$ acts (on the right) on the product $G \times V$ by $(g, v) \cdot p = (g \cdot p, p^{-1} \cdot v)$. The factor variety $(G \times V)/P$ is a vector bundle over $G/P$, we denote it by $\text{Bun}(V)$.

We can view $V$ as a $P$-equivariant vector bundle over the point $\text{Spec} F$. For any $n \geq 0$, the $n$th $P$-equivariant Chern class $c_n(P)$ is an element of $\text{CH}^n(BP)$ (see [4]).

Let $T$ be a split torus. There is a canonical isomorphism

$$S(\hat{T}) \xrightarrow{\sim} \text{CH}(BT),$$

(where $\hat{T}$ is the character group of $T$, $S$ stands for the symmetric algebra) defined by the property that the image of a character $\chi$ is the first Chern class $c_1(\chi)$ where $\chi$ is considered as a 1-dimensional representation of $T$ [4, 3.2].

Let $P$ be a special parabolic subgroup of a split semisimple algebraic group $G$. Let $T$ be a maximal split torus contained in $P$ and let $W_P$ be the Weyl group of $P$. Since $P$ is special, the canonical homomorphism

$$\text{CH}(BP) \to \text{CH}(BT)^{W_P} = S(\hat{T})^{W_P}$$

is an isomorphism [4, Proposition 6]. Identifying $\text{CH}(BP)$ with $S(\hat{T})^{W_P}$, we get a homomorphism

$$\varphi_G : S(\hat{T})^{W_P} \to \text{CH}(G/P)$$

with the image the subring $\tilde{\text{CH}}(G/P)$. 

Lemma 7.1. Let \( \chi_1, \chi_2, \ldots, \chi_m \in \hat{T} \) be the characters (with multiplicities) of a representation \( P \to GL(V) \). Let \( s_n \in S^n(\hat{T})^W \) be the elementary symmetric polynomials in the characters \( \chi_i \). Then \( \varphi_G(s_n) = c_n(Bun(V)) \).

Proof. By naturality of the Chern classes, we have \( \varphi_G(c_n(V)) = c_n(Bun(V)) \). On the other hand, \( c_n(V) \) is the \( n \)th elementary symmetric polynomial in the characters of \( V \). □

Remark 7.2. Let \( G \) be a split semisimple group over an arbitrary field (of an arbitrary characteristic), \( B \subset G \) a Borel subgroup, \( T \subset B \) a split maximal torus, \( W \) the Weyl group of \( G \). The closures \( X_w \) of the cells \( BwB \) of the cellular variety \( G/B \) are indexed by the elements \( w \in W \) and called generalized Schubert varieties of \( G/B \); moreover, \( \dim X_w = l(w) \), where \( l: W \to \mathbb{Z}_{\geq 0} \) is the length function. Taking the unique maximal length element \( w_0 \in W \) and setting \( X^w = X_{w_0w} \), we get a different (preferable for us) indexation of the same varieties, for which \( \operatorname{codim} X^w = l(w) \). The group \( \operatorname{CH}(G/B) \) is free and the classes \([X^w]\), called generalized Schubert classes, form its basis.

The following formula for the product of a 1-codimension Schubert class with an arbitrary Schubert class is given in [3, §4.4 Corollary 2]:

\[
[X^{s_x}] \cdot [X^w] = \sum_{\beta} (\beta^\vee, \omega_x) \cdot [X^{w^{-1}s_\beta}],
\]

where \( x \) is a simple root, \( \omega_x \) its fundamental weight, \( s_x \in W \) the reflection with respect to \( x \); \( \beta \) runs over the set of positive roots such that \( l(w \cdot s_\beta) = l(w) + 1 \), and \( \beta^\vee \) is the dual to \( \beta \) root. Note that the coefficients of this formula depend only on the root system; in particular, they do not depend on the base field and its characteristic. Moreover, this formula completely determines the multiplication table of the basis \([X^w], w \in W\), because the \( \mathbb{Q} \)-algebra \( \operatorname{CH}(G/B) \otimes \mathbb{Q} \) is generated by \( \operatorname{CH}^1(G/B) \) [3].

Remark 7.3. Let \( P = B \) be a Borel subgroup of \( G \). We have \( W_B = 1 \) and therefore the subring \( \hat{\operatorname{CH}}(G/B) \) is generated by \( \varphi_G(\hat{T}) \). In the case of simply connected \( G \), for the weight \( \omega_x \) of a simple root \( x \), one has the formula

\[
\varphi_G(\omega_x) = -[X^{s_x}], \quad [3, \S 4 \text{ formula (7)}],
\]

which also determines \( \varphi_G \) in the non simply connected case. This formula also shows that if the group \( G \) is simply connected, then \( \varphi_G(\hat{T}) = \operatorname{CH}^1(G/B) \), and therefore \( \hat{\operatorname{CH}}(G/B) \) is the subring of \( \operatorname{CH}(G/B) \), generated by \( \operatorname{CH}^1(G/B) \).

Remark 7.4. From Theorem 6.9 and Remark 7.3, we see that

(1) if \( G_1 \) and \( G_2 \) are split semisimple groups, then \( \operatorname{cd}_p(G_1 \times G_2) = \operatorname{cd}_p(G_1) + \operatorname{cd}_p(G_2) \); (2) if \( G' \to G \) is a central isogeny of split semisimple groups, then \( \operatorname{cd}_p(G') \leq \operatorname{cd}_p(G) \).

Remark 7.5. Let us consider pairs \((\Phi, A)\), consisting of a root system \( \Phi \) and a subgroup of the quotient of the weight lattice of \( \Phi \) by its root lattice. An isomorphism of pairs \((\Phi, A) \to (\Phi', A')\) is an isomorphism of the root systems \( \Phi \to \Phi' \) such that the
induced isomorphism of the lattice quotients maps $A$ to $A'$. To any split semisimple algebraic group $G$ one attaches an isomorphism class of above pairs, to which we refer as extended type of $G$. Theorem 6.9 with Remarks 7.2 and 7.3 shows that $\text{cd}_p(G)$ (for any $p$) depends only on the extended type of $G$. It does not depend on the base field $F$ and, in particular, on the characteristic of $F$ (so that computing $\text{cd}_p(G)$ one may always assume that $G$ is defined over $\mathbb{C}$).

8. Canonical $p$-dimension of split simple groups of classical types

In this section we compute canonical $p$-dimension of all split simple groups of classical types. We will need the following:

**Lemma 8.1.** Let $R$ be a commutative ring, $r \in R$, and let $A$ be the factor ring of the polynomial ring $R[x_1, x_2, \ldots, x_n]$ modulo the ideal generated by the polynomial $x_1 + x_2 + \cdots + x_n - r$. The symmetric group $W = S_n$ acts on $A$ by permuting the $x_i$. If $R$ has trivial $\mathbb{Z}$-torsion, then $A^W = R[s_2, s_3, \ldots, s_n]$, where $s_i$ are the elementary symmetric polynomials.

**Proof.** Consider the natural $W$-action on the ring $R[x] = R[x_1, x_2, \ldots, x_n]$. We have the exact sequence $0 \rightarrow R[x] \xrightarrow{f} R[x] \rightarrow A \rightarrow 0$, where the first map is the multiplication by $f = x_1 + x_2 + \cdots + x_n - r$. Passing to $W$-invariants we get an exact sequence \[0 \rightarrow R[x]^W \xrightarrow{f} R[x]^W \rightarrow A^W \rightarrow H^1(W, R[x]).\]

The ring $R[x]^W$ coincides with $R[s] = R[s_1, s_2, \ldots, s_n]$. The monomials in the variables $x_i$ form a permutation basis of the $R$-module $R[x]$. By the Faddeev-Shapiro lemma, the group $H^1(W, R[x])$ is a direct sum of the groups $H^1(W', R) = \text{Hom}(W', R)$ for certain subgroups $W' \subset W$. Since $R$ has trivial $\mathbb{Z}$-torsion, the latter group is trivial. Therefore, $A^W = R[s]/(f) = R[s_2, s_3, \ldots, s_n]$. □

8.1. Type $A_{n-1}$

A split simple group of type $A_{n-1}$ is isomorphic to $G = \text{SL}(n)/\mu_l$, where $l$ is a divisor of $n$. Let $P \subset \text{SL}(n)$ be the stabilizer of the line $U = [1 : 0 : \ldots : 0] \in \mathbb{P}^{n-1}$ with respect to the natural action of $\text{SL}(n)$ on $\mathbb{P}^{n-1}$. The semisimple part of $P$ is $\text{SL}(n-1)$ and it intersects $\mu_l$ trivially. Hence the parabolic subgroup $P_l = P/\mu_l$ of $G$ is special. We have $G/P_l = \mathbb{P}^{n-1}$.

The intersection $T$ of the group of diagonal matrices $D(n)$ of $\text{GL}(n)$ with $\text{SL}(n)$ is a maximal torus of $\text{SL}(n)$. The character group $\widehat{T}$ is identified with the factor group of $\mathbb{Z}^n = \widehat{D(n)}$ with the standard basis $x_1, x_2, \ldots, x_n$ by the subgroup generated by $x_1 + x_2 + \cdots + x_n$. The character group of the maximal torus $T_l = T/\mu_l$ of $G$ is the subgroup of $\widehat{T}$ consisting of all sums $\sum a_i x_i$ such that $\sum a_i$ is divisible by $l$. 

8.2. Type $B_n$
Hence, $\widehat{T}_l$ is generated by $lx_1$ and $x_i - x_1$ for all $i = 2, \ldots, n$ with the relation $\sum_{i \geq 2} (x_i - x_1) = -nx_1$.

The Weyl group $W = W_{P_l}$ is the symmetric group $S_{n-1}$, permuting $x_2, \ldots, x_n$. Applying Lemma 8.1 to the ring $R = \mathbb{Z}[l x_1]$, the element $r = -nx_1$, the variables $x_i - x_1$ and the group $W$, we get $\mathbb{S}(\widehat{T}_l)^W = \mathbb{Z}[l x_1, s_2, s_3, \ldots, s_{n-1}]$, where the $s_i$ are the elementary symmetric polynomials in the $x_i - x_1$, $i \geq 2$.

The group $P$ acts naturally on the space $V = F^n$. The characters of this representation are $x_1, x_2, \ldots, x_n$. The corresponding vector bundle $\text{Bun}(V)$ over $\mathbb{P}^{n-1} = \text{SL}(n)/P = G/P_l$ is the trivial vector bundle of rank $n$. The line $U$ can be viewed as a 1-dimensional representation of $P$ given by the character $x_1$. We have $\text{Bun}(U) = L^\vee$, where $L$ is the canonical line bundle on $\mathbb{P}^{n-1}$ (with the sheaf of sections $\mathcal{O}(1)$). Consider the representation $M = (V/U) \otimes U^\vee$ of the group $P$ with the characters $x_i - x_1$ for all $i = 2, \ldots, n$. Note that the group $\mu_l$ is contained in the kernel of the representation, hence $M$ is a representation of $P_l$.

By Lemma 7.1, we have $\varphi_G(x_1) = lc_1(L^\vee) = -lh$, where $h \in \text{CH}_1(\mathbb{P}^{n-1})$ is the class of a hyperplane, and also $\varphi_G(s_i) = c_i(\text{Bun}(M))$ for all $i$. Hence the subring, $\mathbb{C}(\mathbb{P}^{n-1})$ of $\text{CH}(\mathbb{P}^{n-1}) = \mathbb{Z}[h]/(h^n)$ is generated by $lh$ and the Chern classes $c_i(\text{Bun}(M))$. Since

$$\text{Bun}(M) = (\text{Bun}(V)/\text{Bun}(U)) \otimes \text{Bun}(U^\vee) = (\text{Bun}(V)/L^\vee) \otimes L,$$

the class $[\text{Bun}(M)]$ is equal to $n[L] - 1$ in $K_0(\mathbb{P}^{n-1})$. Hence, $c_\bullet(\text{Bun}(M)) = c_\bullet(L)^n = (1 + h)^n$. Thus the subring $\mathbb{C}(\mathbb{P}^{n-1})$ is generated by $lh$ and $\left(\begin{array}{c} n \\ i \end{array}\right) h^i$ for $i = 2, \ldots, n - 1$.

Let $p$ be a prime integer and let $p^k$ be the largest power of $p$ dividing $n$. Note that the binomial coefficient $\left(\begin{array}{c} n \\ i \end{array}\right)$ is divisible by $p$ unless $i$ is divisible by $p^k$. The largest value of $i < n$ such that $\left(\begin{array}{c} n \\ i \end{array}\right)$ is not divisible by $p$ is $n - p^k$. By Theorem 6.9,

$$\text{cd}_p(\text{SL}(n)/\mu_l) = \begin{cases} p^k - 1 & \text{if } p \text{ divides } l, \\ 0 & \text{otherwise}. \end{cases}$$

Denote by $\text{CSA}_{n,l}(K)$ the set of isomorphism classes of central simple $K$-algebras of degree $n$ and exponent dividing $l$. The exact sequence $1 \to \mu_l \to \text{SL}(n) \to \text{SL}(n)/\mu_l \to 1$ yields a surjective map $\text{Tors}_{\text{SL}(n)/\mu_l}(K) \to \text{CSA}_{n,l}(K)$ with trivial kernel. By Remark 3.7,

$$\text{cd}_p(\text{CSA}_{n,l}) = \begin{cases} p^k - 1 & \text{if } p \text{ divides } l, \\ 0 & \text{otherwise}. \end{cases}$$

8.2. Type $B_n$

The only torsion prime is $p = 2$. 
Taking a \((2n+1)\)-dimensional vector space, endowed with a completely split quadratic form, let a vector \(g\) together with vectors \(e_i, f_i, i = 1, 2, \ldots, n\) form a basis such that \(\{e_i, f_i\}\) are pairwise orthogonal hyperbolic pairs, while \(g\) is orthogonal to all \(e_i, f_i\). Let \(G = \text{SO}(2n+1)\) be the corresponding special orthogonal group. The inclusion of \(D(n)\) into \(\text{SO}(2n+1)\) given by \(t(e_i) = t_i e_i, t(f_i) = t_i^{-1} f_i\) and \(t(g) = g\), where \(t = \text{diag}(t_1, \ldots, t_n)\), identifies \(D(n)\) with a maximal torus \(T\) of \(\text{SO}(2n+1)\). In particular, the group \(\widehat{T}\) is identified with \(\mathbb{Z}^n = \mathbb{D}(n)\). We write \(x_1, x_2, \ldots, x_n\) for the standard basis of \(\mathbb{Z}^n\).

Let \(V\) be the totally isotropic subspace of dimension \(n\) generated by all the \(e_i\). Denote by \(P\) the stabilizer of \(V\) in \(G\), so that \(X = G/P\) is the variety of all dimension \(n\) totally isotropic subspaces. The characters of the natural representation \(P \to \text{GL}(V)\) are \(x_1, x_2, \ldots, x_n\). The vector bundle \(\text{Bun}(V)\) over \(X\) is the tautological vector bundle.

The group \(W = W_P\) is the symmetric group \(S_n\) permuting the \(x_i\). The semisimple part of \(P\) is \(\text{SL}(n)\), so that \(P\) is special.

We have \(S(\widehat{T})^W = \mathbb{Z}[s_1, s_2, \ldots, s_n]\), where \(s_k\) are the elementary symmetric polynomials in the \(x_i\). By Lemma 7.1, the subring \(\text{CH}(X)\) of \(\text{CH}(X)\) is generated by the Chern classes of \(\text{Bun}(V)\). These Chern classes are divisible by 2 in \(\text{CH}(X)\) [13, Chapter III, Theorem 6.11]. Thus, \(\text{CH}^j(X) = 0\) if \(j > 0\). We conclude by Theorem 6.9 that

\[
\text{cd}_2 \text{SO}(2n+1) = \frac{n(n+1)}{2}
\]

(see also Examples 3.6 and 5.11). The set \(\text{Tors}_{\text{SO}(2n+1)}(K)\) is identified with the set of similarity classes \(Q_{2n+1}(K)\) of non-degenerate quadratic forms of dimension \(2n+1\) over \(K\). Thus,

\[
\text{cd}_2 Q_{2n+1} = \frac{n(n+1)}{2}.
\]

Let \(G = \text{Spin}(2n+1)\) be the spinor group. There is an exact sequence

\[1 \to \mu_2 \to T' \to T \to 1,
\]

where \(T'\) is a maximal torus of \(\text{Spin}(2n+1)\). We have \(\widehat{T}' = \widehat{T} + \mathbb{Z} y = \mathbb{Z}^n + \mathbb{Z} y\), where \(y = (x_1 + \cdots + x_n)/2\). By Lemma 8.1 applied to the ring \(R = \mathbb{Z}[y]\), the element \(r = 2 y\) and the group \(W\), the ring \(S(\widehat{T})^W\) is the polynomial ring \(\mathbb{Z}[y, s_2, s_3, \ldots, s_n]\).

By Lemma 7.1, \(\varphi_G(s_1) = c_1(\text{Bun}(V))\). The latter class coincides with \(2 e\) where \(e\) is a generator of \(\text{CH}^1(X)\) [13, Chapter III, Theorem 6.11]. Since \(s_1 = 2 y\) and \(\text{CH}^1(X)\) is torsion free, we have \(\varphi_G(y) = e\).

As noted above, the images of the \(s_i\) in \(\text{CH}(X)\) are divisible by 2. Hence the image of \(\widehat{\text{CH}}(X)\) in \(\text{Ch}(X) = \text{CH}(X)/2\) is the subring generated by \(e\) mod 2. Let \(m\) be the smallest integer such that \(2^m > n\). Then \(e^{2^m} = 0\) and \(e^{2^m-1} \neq 0\) in \(\text{CH}(X)\) [13, Chapter III, Theorem 6.11]. Thus,

\[
\text{cd}_2 \text{Spin}(2n+1) = \frac{n(n+1)}{2} - 2^m + 1.
\]

Let \(\overline{Q}_{2n+1}(K)\) be the subset of \(Q_{2n+1}(K)\) consisting of all classes of forms with trivial even Clifford invariant. The exact sequence

\[1 \to \mu_2 \to \text{Spin}(2n+1) \to \text{SO}(2n+1)
\]
→ 1 yields a surjective map $\text{Tors}_{\text{Spin}}(2n+1)(K) \to \overline{Q}_{2n+1}(K)$ with trivial kernel. In particular,

$$\text{cd}_2 \overline{Q}_{2n+1} = \frac{n(n+1)}{2} - 2^m + 1.$$ 

8.3. Type $C_n$

The group $\text{Sp}(2n)$ is special, so that $\text{cd}_p \text{Sp}(2n) = 0$ for all $p$.

Let $G = \text{PGSp}(2n)$ be the projective symplectic group. The number $p = 2$ is the only torsion prime of $G$. Instead of applying the general method, we proceed as follows.

The set $\text{Tors}_{\text{PGSp}}(2n)(K)$ is identified with the set of isomorphism classes $\text{ASI}_{2n}(K)$ of central simple $K$-algebras $A$ of degree $2n$ with a symplectic involution [12, §29.22]. The forgetful functor $\text{ASI}_{2n} \to \text{CSA}_{2n,2}$ has trivial kernel and is surjective. Therefore, by Remark 3.7 and (8.1),

$$\text{cd}_2 \text{PGSp}(2n) = \text{cd}_2 \text{ASI}_{2n} = \text{cd}_2 \text{CSA}_{2n,2} = 2^k - 1,$$

where $2^k$ is the largest power of 2 dividing $2n$.

8.4. Type $D_n$

Let $\{e_i, f_i\}, i = 1, 2, \ldots, n$ be pairwise orthogonal hyperbolic pairs of a hyperbolic quadratic form of dimension $2n$. The inclusion of $D(n)$ into $\text{SO}(2n)$ given by $t(e_i) = t_i e_i$ and $t(f_i) = t_i^{-1} f_i$, where $t = \text{diag}(t_1, \ldots, t_n)$, identifies $D(n)$ with a maximal torus $T'$ of $\text{SO}(2n)$. In particular, the group $\hat{T}'$ is identified with $\mathbb{Z}^n = \hat{D}(n)$. We write $x_1, x_2, \ldots, x_n$ for the standard basis of $\mathbb{Z}^n$.

Let $V$ be the totally isotropic subspace of dimension $n$ generated by all the $e_i$ and let $U$ be the line $Fe_1$. Denote by $P$ the stabilizer of the flag $U \subset V$ in $G = \text{Spin}(2n)$ and set $X = G/P$. The semisimple part of $P$ is isomorphic to $\text{SL}(n-1)$ and intersects trivially the center of $G$. Hence the image of $P$ in any simple group of type $D_n$ (under a central isogeny of $G$) is a special group.

Let $Y$ be the connected component of the scheme of maximal $(n$-dimensional) totally isotropic subspaces such that $V$ is a point of $Y$. The natural morphism $f : X \to Y$ is the projective bundle associated with the tautological vector bundle $E$ over $Y$ of rank $n$. In particular,

$$\dim X = \dim Y + (n - 1) = \frac{n(n-1)}{2} + (n - 1).$$

Note that $Y$ is isomorphic to the projective homogeneous variety of the group $\text{Spin}(2n - 1)$ considered in the type $B_{n-1}$. The Chern classes of $E$ in $\text{CH}(Y)$ are divisible by 2 (see the type $B_n$), hence $\text{Ch}(X) = \text{Ch}(Y)[h]/(h^n)$, where $h = c_1(L)$ for the canonical line bundle $L$ over $X$.

Similar to the case $B_n$, the character group of the maximal torus $T$ of $\text{Spin}(2n)$ is equal to $\mathbb{Z}^n + \mathbb{Z} y$, where $y = (x_1 + x_2 + \cdots + x_n)/2$. Set $x'_i = x_i - x_1$ for $i = 2, \ldots, n$, so
that \( x'_2 + \cdots + x'_n = 2y - nx_1 \). The symmetric group \( W = W_P \) permutes the \( x'_i \) and acts trivially on \( y \) and \( x_1 \). Applying Lemma 8.1 to the variables \( x'_i \), the ring \( R = \mathbb{Z}[y, x_1] \) and the element \( r = 2y - nx_1 \) we see that \( S(\hat{T})^W = \mathbb{Z}[y, x_1, x_2, \ldots, x_{n-1}] \), where the \( s_i \) are the elementary symmetric polynomials in the \( x'_i \).

Consider the homomorphism (reduced modulo 2)

\[
\varphi_G : \mathbb{Z}[y, x_1, x_2, \ldots, x_{n-1}] \to \text{Ch}(X) = \text{Ch}(Y)[h]/(h^n).
\]

As in the case \( A_{n-1} \), we have \( \text{Bun}(U) = L^\vee \) and therefore \( \varphi_G(x_1) = c_1(L^\vee) = -h \).

Similar to the case \( B_n \), we have \( e = \varphi_G(y) \) is a generator of \( \text{Ch}^1(Y) \). Recall that \( e^{2m-1} \neq 0 \) and \( e^{2m} = 0 \) where \( m \) is the smallest integer such that \( 2^m \geq n \).

Similar to the case \( A_{n-1} \), we observe by Lemma 7.1 that the images of the \( s_i \) in \( \text{Ch}(X) \) are the Chern classes of the vector bundle \( (f^*(E)/L^\vee) \otimes L \). The class of this bundle in \( K_0(X) \) is equal to \( [f^*(E) \otimes L]-1 \). Since the Chern classes of \( E \) are divisible by 2, we can replace \( E \) by the trivial bundle of rank \( n \) and replace \( [f^*(E) \otimes L] \) by \( n[L] \). As in the case \( A_{n-1} \), we see that \( \varphi_G(s_i) = \binom{n}{i} h^i \).

The subring \( \tilde{\text{Ch}}(X) = \text{Im}(\varphi_G) \) is generated by \( h \) and \( e \). The largest degree nontrivial monomial in \( h \) and \( e \) is \( h^{n-1}e^{2n-1} \). By Theorem 6.9,

\[
\text{cd}_2 \text{Spin}(2n) = \dim X - (n - 1) - (2^m - 1) = \frac{n(n - 1)}{2} - 2^m + 1.
\]

Let \( \overline{Q}_{2n}(K) \) be the subset of the set \( Q_{2n}(K) \) of isomorphism classes of non-degenerate quadratic forms of dimension \( 2n \) consisting of all classes of forms with trivial discriminant and Clifford invariant. The exact sequence \( 1 \to \mu_2 \to \text{Spin}(2n) \to \text{SO}(2n) \to 1 \) yields a surjective map \( \text{Tors}_{\text{Spin}(2n)}(K) \to \overline{Q}_{2n}(K) \) with trivial kernel. In particular,

\[
\text{cd}_2 \overline{Q}_{2n} = \frac{n(n - 1)}{2} - 2^m + 1.
\]

Now let \( G = \text{SO}(2n) \). Recall that the character group \( \hat{T}' \) of the maximal torus \( T' \) of \( G \) is the subgroup of \( \hat{T} \) generated by all the \( x_i \). Thus we have \( S(\hat{T}') = \mathbb{Z}[x_1, x_2, \ldots, x_n] \) and therefore, \( S(\hat{T})^W = \mathbb{Z}[x_1, s_1, \ldots, s_{n-1}] \). The subring \( \tilde{\text{Ch}}(X) \) is then generated by \( h \). The largest degree nontrivial monomial in \( h \) is \( h^{n-1} \). By Theorem 6.9,

\[
\text{cd}_2 \text{SO}(2n) = \dim X - (n - 1) = \frac{n(n - 1)}{2}.
\]

Let \( Q'_{2n}(K) \) be the subset of the set \( Q_{2n}(K) \) consisting of all classes of forms with trivial discriminant. There is a canonical bijection \( \text{Tors}_{\text{SO}(2n)}(K) \sim Q'_{2n}(K) \). Therefore,

\[
\text{cd}_2 Q'_{2n} = \frac{n(n - 1)}{2}.
\]

Let \( G = \text{PGO}^+(2n) \) be the projective orthogonal group. Let \( \overline{T} \) be the image of the maximal torus \( T \) under the canonical isogeny \( \text{Spin}(2n) \to G \). The character group \( \overline{T} \) is the subgroup of \( \hat{T} \) generated by all the simple roots. Thus we have \( S(\overline{T}) = \)}
\[ Z[2x_1, x'_2, \ldots, x'_n] \] and therefore, \( S(\hat{T})^W = Z[2x_1, s_1, \ldots, s_{n-1}] \). The subring \( \widehat{Ch}(X) \) is then generated by \( \left( \begin{array}{c} n \\ i \end{array} \right) h^i \). Let \( 2^k \) be the largest power of 2 dividing \( n \). Note that the binomial coefficient \( \left( \begin{array}{c} n \\ i \end{array} \right) \) is even unless \( i \) is divisible by \( 2^k \). The largest value of \( i < n \) such that \( \left( \begin{array}{c} n \\ i \end{array} \right) \) is odd is \( n - 2^k \). The largest degree nontrivial monomial in \( h \) is \( \left( \begin{array}{c} n \\ n - 2^k \end{array} \right) h^{n - 2^k} \). By Theorem 6.9,

\[
\text{cd}_2 \ PGO^+(2n) = \dim X - (n - 2^k) = \frac{n(n - 1)}{2} + 2^k - 1.
\]

Let \( AQP_{2n}(K) \) be the set of isomorphism classes of central simple algebras of degree \( 2n \) with a quadratic pair with trivial discriminant \([12, \S 29.F]\). The exact sequence \( 1 \to PGO^+(2n) \to PGO(2n) \to \mathbb{Z}/2\mathbb{Z} \to 1 \) yields a surjective map \( \text{Tors}^{PGO^+(2n)}(K) \to AQP_{2n}(K) \) with trivial kernel. In particular,

\[
\text{cd}_2 \ AQP_{2n} = \frac{n(n - 1)}{2} + 2^k - 1.
\]

Suppose now that \( n \) is even. There are two isomorphic semispinor groups. We set \( \Spin^\sim(2n) = \Spin(2n)/H \), where \( H \) is the intersection of \( \text{Ker}(y) \) with the center of \( \Spin(2n) \). Let \( T'' \) be the image of the maximal torus \( T \) under the canonical isogeny \( \Spin(2n) \to G \). The character group of \( T'' \) is the subgroup of \( \hat{T} \) generated by all the simple roots and \( y \). Thus we have \( S(\hat{T}'') = Z[y, 2x_1, x'_2, \ldots, x'_n] \).

Applying Lemma 8.1 to the elements \( x'_1 \), the ring \( R = Z[y, 2x_1] \), and the element \( r = 2y - nx_1 \), we see that \( S(\hat{T}'')^W = Z[y, 2x_1, s_2, \ldots, s_{n-1}] \).

The subring \( \widehat{Ch}(X) \) is then generated by \( e \) and \( \left( \begin{array}{c} n \\ i \end{array} \right) h^i \). The largest degree nontrivial monomial in \( h \) and \( e \) is \( \left( \begin{array}{c} n \\ n - 2^k \end{array} \right) h^{n-2^k} e^{2^m-1} \). By Theorem 6.9,

\[
\text{cd}_2 \ Spin^\sim(2n) = \dim X - (n - 2^k) - (2^m - 1) = \frac{n(n - 1)}{2} + 2^k - 2^m.
\]

Let \( AQP'_{2n}(K) \) be the set of isomorphism classes of central simple algebras of degree \( 2n \) with a quadratic pair with trivial discriminant and trivial component of the Clifford algebra. The exact sequence \( 1 \to \mu_2 \to Spin^\sim(2n) \to PGO^+(2n) \to 1 \) yields a surjective map \( \text{Tors}^{Spin^\sim(2n)}(K) \to AQP'_{2n}(K) \) with trivial kernel. In particular,

\[
\text{cd}_2 \ AQP'_{2n} = \frac{n(n - 1)}{2} + 2^k - 2^m.
\]
Acknowledgments

Authors thank V. Chernousov, I. Panin, and M. Rost for helpful remarks. The first author thanks the Institute for Advanced Study in Princeton, New Jersey: the results were obtained and the initial version of the paper was written during his stay at the Institute; he was also inspired by the lecture course of A. Suslin on the Rost norm varieties.

Appendix. Type $G_2$

The only torsion prime is 2. Since $\text{Tors}_G \simeq \overline{O}_8$ for a split simple $G$ of type $G_2$, we have $\text{cd}_2(G) = \text{cd}_2(\overline{O}_8) = 3$ (see §8.4).

References