INCOMPRESSIBILITY OF PRODUCTS BY GRASSMANNIANS OF ISOTROPIC SUBSPACES

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Abstract. We prove that the product of an arbitrary projective homogeneous variety Y by an orthogonal, symplectic, or unitary Grassmannian X is 2-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are so. Some new properties of incompressible Grassmannians are established on the way.

1. Introduction

Let F be a field and let p be a prime number. We refer to [6] and [16] for definitions and general discussion of canonical p-dimension and p-incompressibility. We only recall that canonical p-dimension $\operatorname{cdim}_p X$ of a smooth complete variety X is the least dimension of the image of a self-correspondence $X \rightsquigarrow X$ of multiplicity prime to p; X is called p-incompressible, if $\operatorname{cdim}_p X = \dim X$, that is, if every self-correspondence of multiplicity prime to p is dominant.

We work with *projective homogeneous varieties*, i.e., twisted flag varieties under semisimple affine algebraic groups. A necessary condition for p-incompressibility of the product $X \times Y$ of projective homogeneous F-varieties X and Y is p-incompressibility of the varieties $X_{F(Y)}$ and $Y_{F(X)}$. This necessary condition is known by [10] to be sufficient in the case where X is a *generalized Severi-Brauer variety*, that is, the Grassmannian of right ideals of a fixed dimension in a central simple F-algebra. Such a variety is a twisted form of a usual Grassmannian of subspaces of a fixed dimension in a finite-dimensional vector space over F and is projective homogeneous for an algebraic group of the Dynkin type ${}^1\mathcal{A}$.

In the present paper we show that the above necessary condition is also sufficient in the case where X is a Grassmannian of totally isotropic spaces of a fixed dimension for:

- a non-degenerate quadratic form (orthogonal case, algebraic groups of types $\mathcal{B}$ and $\mathcal{D}$), or
- a hermitian form over a separable quadratic extension field of F (unitary case, type ${}^2\mathcal{A}$), or
- a hermitian form over a quaternion division F-algebra (symplectic case, type $\mathcal{C}$), where in the characteristic 2 case the hermitian form is supposed to be *alternating*, [14, §4.A].

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Note that in the symplectic case, only the Grassmannians of subspaces of *integral* dimension over the quaternion algebra are considered because the others are not interesting from the viewpoint of the canonical dimension. Also note that $p = 2$ is the only interesting prime for the varieties treated here.

Proving the result, we establish some properties of incompressible Grassmannians which seem to be of interest on their own, even in the most classical orthogonal case. Note however, that we prove the main result – a criterion of incompressibility for $X \times Y$ in terms of incompressibility of a Grassmannian X and an arbitrary projective homogeneous variety Y – without possessing a criterion of incompressibility for the factors themselves, even for the more specific X. So, the situation here is different from the situation of [10], dealing with the usual twisted Grassmannians, where a criterion of their incompressibility was already available.

Fortunately, we at least have [8], [9], and [11], telling, roughly, that there are many incompressible orthogonal, unitary, and symplectic Grassmannians.

The main results of this paper (Theorems 4.1, 5.2, and 6.3) are covered by a more recent [3]. However, the proofs given here are different and less ad-hoc. Not only the method of the proofs but also their byproducts (Propositions 3.1, 5.1, and 6.2) seem to be still of interest.

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2. Canonical p-dimension of a fibration

As shown in [10] (see also Corollary 2.5 here), canonical p-dimension of the product $X \times Y$ of projective homogeneous F-varieties X and Y has $\operatorname{cdim}_p X + \operatorname{cdim}_p Y_{F(X)}$ as an upper bound. We may view X as the base of the projection (a "trivial fibration") $X \times Y \to X$, and $Y_{F(X)}$ is its generic fiber. In this section, we generalize this upper bound relation to the case of a more general fibration and also we sharpen the upper bound, replacing $\operatorname{cdim}_p X$ by an, in general, smaller integer $\operatorname{cdim}'_p X$ defined in terms of X and the function field of the total variety of the fibration.

Here is the type of the fibrations we are interested in. Let G be a quasi-split semisimple affine algebraic group over F becoming split over a finite extension field of a p-power degree, T a G-torsor over F, P a parabolic subgroup of G and P' a parabolic subgroup of G contained in P. We consider the fibration

$$\pi\colon Z := T/P' \to T/P =: X$$

of projective homogeneous varieties, and we write Y for its generic fiber. We are using the Chow group Ch with coefficients in $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. In particular, the degree homomorphism deg on Ch_0 of a complete variety takes its values in $\mathbb{F}_p$.

Let us first recall

Proposition 2.1 ([8, Corollary 6.2])**.** *Canonical p-dimension* $\operatorname{cdim}_p X$ *of* X *is the minimal integer d such that there exist a cycle class* $\alpha \in \operatorname{Ch}^d X_{F(X)}$ *and a cycle class* $\beta \in \operatorname{Ch}_d X$ *with* $\deg(\beta_{F(X)} \cdot \alpha) = 1 \in \mathbb{F}_p$.

Lemma 2.2. *In the above settings, we have*

$$\operatorname{cdim}_p Z \le \operatorname{cdim}'_p X + \operatorname{cdim}_p Y,$$

where $\operatorname{cdim}'_p X$ *is the minimal integer d such that there exist elements* $\alpha \in \operatorname{Ch}^d(X_{F(Z)})$ *and* $\beta \in \operatorname{Ch}_d(X)$ *with* $\deg(\alpha \cdot \beta_{F(Z)}) = 1$.

Remark 2.3. Replacing $F(Z)$ by $F(X)$ in the definition of $\operatorname{cdim}'_p X$, we get $\operatorname{cdim}_p(X)$, see Proposition 2.1. Since $F(Z) \supset F(X)$, we have $\operatorname{cdim}'_p X \le \operatorname{cdim}_p X$.

Remark 2.4. In the case where the parabolic subgroup P' is special, Lemma 2.2 has been proved in [5, Lemma 5.3]. The proof was more complicated (and the statement – more specific) because Proposition 2.1 was not available at the time.

Proof of Lemma 2.2. By the definition of $\operatorname{cdim}'_p X$, for $x := \operatorname{cdim}'_p X$ we can find

$$\alpha_X \in \operatorname{Ch}^x(X_{F(Z)}) \;\text{ and }\; \beta_X \in \operatorname{Ch}_x(X)$$

with $\deg(\alpha_X \cdot (\beta_X)_{F(Z)}) = 1$.

Note that the function field $F(X)$ is the field of definition of the variety Y. By Proposition 2.1, for $y := \operatorname{cdim}_p Y$ we can find

$$\alpha_Y \in \operatorname{Ch}^y(Y_{F(X)(Y)}) \;\text{ and }\; \beta_Y \in \operatorname{Ch}_y(Y)$$

with $\deg(\alpha_Y \cdot (\beta_Y)_{F(X)(Y)}) = 1$. We are going to produce certain

$$\alpha'_Y \in \operatorname{Ch}^y(Z_{F(Z)}) \quad\text{and}\quad \beta'_Y \in \operatorname{Ch}_{y+\dim X}(Z)$$

out of α_Y and β_Y.

By [1, Proposition 57.10], the flat pull-back homomorphisms

$$\operatorname{Ch}_{y+\dim X}(Z) \to \operatorname{Ch}_y(Y) \;\text{ and }\; \operatorname{Ch}^y(Z_{F(Z)}) \to \operatorname{Ch}^y(Y \times_{\operatorname{Spec} F} \operatorname{Spec} F(Z))$$

are surjective. We define β'_Y simply as a preimage of β_Y.

In order to define α'_Y, we notice that $F(Z) = F(X)(Y)$ so that $Y \times_{\operatorname{Spec} F} \operatorname{Spec} F(Z) = Y_{F(X)(Y)(X)}$. We define α'_Y as a preimage of $(\alpha_Y)_{F(X)(Y)(X)}$.

Now we set

$$\alpha := \pi^*(\alpha_X) \cdot \alpha'_Y \in \operatorname{Ch}^{x+y}(Z_{F(Z)}) \quad\text{and}\quad \beta := \pi^*(\beta_X) \cdot \beta'_Y \in \operatorname{Ch}_{x+y}(Z).$$

We claim that $\deg(\alpha \cdot \beta_{F(Z)}) = 1$. According to Proposition 2.1, the claim implies that

$$\operatorname{cdim}_p Z \le x + y = \operatorname{cdim}'_p X + \operatorname{cdim}_p Y.$$

It remains to prove the claim. The claim is about cycles over $F(Z)$ so that we replace the base field F by $F(Z)$.

The product in $\operatorname{Ch}(Z)$, whose degree we are interested in, can be rewritten as the product of $\pi^*(\delta)$, where δ is a 0-cycle class on X of degree 1, by an element $\gamma \in \operatorname{Ch}(Z)$ whose image under $\operatorname{Ch}(Z) \twoheadrightarrow \operatorname{Ch}(Y_{F(X)})$ is a 0-cycle class Y of degree 1. We have

$$\deg(\pi^*(\delta) \cdot \gamma) = \deg \pi_*(\pi^*(\delta) \cdot \gamma) = \deg(\delta \cdot \pi_*(\gamma)).$$

To prove the claim it suffices to check that $\pi_*(\gamma) = [X]$. For this, it suffices to check that the pull-back in $\mathrm{Ch}(\mathrm{Spec}\, F(X)) = \mathbb{F}_p$ of $\pi_*(\gamma)$ along the generic point morphism $\mathrm{Spec}\, F(X) \to X$ is equal to 1. By [2, Proposition 1.7 of Chapter 1] applied to the square

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Spec}\, F(X) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

the pull-back of $\pi_*(\gamma)$ coincides with the degree of the image of γ in $\mathrm{Ch}(Y)$ which is indeed 1. The claim and therefore Lemma 2.2 as well are proved. □

Corollary 2.5 ([10])**.** *Let X and Y be projective homogeneous varieties. Then*

$$\mathrm{cdim}_p(X \times Y) \le \mathrm{cdim}_p X + \mathrm{cdim}_p Y_{F(X)}.$$

In particular, $X \times Y$ is p-incompressible only if $X_{F(Y)}$ and $Y_{F(X)}$ are so.

Proof. Since canonical p-dimension is not changed under base field extensions of degree prime to p (see [15, Proposition 1.5(2)]), we may assume that the semisimple affine algebraic groups acting on the projective homogeneous varieties X and Y become of inner type over a finite p-primary extension of F. Therefore we may apply Lemma 2.2 to the projection $X \times Y \to X$. Taking into account Remark 2.3, we get Corollary 2.5. □

3. Incompressible orthogonal Grassmannians

In this section, n is a positive integer, φ is a non-degenerate n-dimensional quadratic form over F, and X_i with $0 \le i < n/2$ is the ith orthogonal Grassmannian of φ. Note that we do not consider the case of $i = n/2$ (with even n), see Remark 4.2.

We recall that $\dim X_i = i(i-1)/2 + i(n-2i)$. (This formula is valid for $i = n/2$ as well.)

Now assume that $n \ge 3$. The variety X_1 is the projective quadric of φ and the quadratic form $\varphi_{F(X_1)}$ is isotropic. Let φ' be an $(n-2)$-dimensional quadratic form Witt-equivalent to $\varphi_{F(X_1)}$. For i with $0 \le i < (n-2)/2$, we write X'_i for the ith orthogonal Grassmannian of φ'.

Proposition 3.1. *For some $n \ge 3$ and some $i \ge 1$, assume that the variety X_i is 2-incompressible. Then φ is anisotropic, all the first i higher Witt indexes $\mathfrak{i}_1(\varphi), \dots, \mathfrak{i}_i(\varphi)$ of φ are equal to 1, and the variety X'_{i-1} is also 2-incompressible.*

Remark 3.2. In [8], the following sufficient condition for 2-incompressibility of X_i has been given: $\mathfrak{i}_i(\varphi) = 1$ and the degree of every closed point on X_i is divisible by 2^i. The degree condition implies that φ is anisotropic and $\mathfrak{i}_1(\varphi) = \dots = \mathfrak{i}_{i-1}(\varphi) = 1$. So, Proposition 3.1 shows that a part of the sufficient condition for 2-incompressibility of X_i is actually necessary.

Remark 3.3. For $i = 1$, Proposition 3.1 says that the 2-incompressibility of the projective quadric X_1 implies that φ is anisotropic and its first Witt index is 1 (the variety X'_{i-1} is just a point and as such is 2-incompressible automatically). This statement is easy to check (as in the proof below), Lemma 2.2 is not used. A known old result due to A. Vishik actually says that X_1 is 2-incompressible if and only if φ is anisotropic of first Witt index 1. For a proof (based on ideas of [18]) see [1, Theorem 90.2].

Remark 3.4. For odd n, the statement of Proposition 3.1 on maximal $i = (n-1)/2$ can be easily deduced from the properties of the *J-invariant* of φ. The original paper introducing J-invariant and establishing its main properties is [19]. The monograph [1, §88], where the J-invariant is replaced by its "opposite", can also be consulted. In particular, see [1, Theorem 90.3] for the relation between canonical 2-dimension and the J-invariant.

Here is the deduction. For $i = (n-1)/2$, let $X := X_i$ and let

$$e_1 \in \mathrm{Ch}^1(X_{F(X)}), \ \ldots, \ e_i \in \mathrm{Ch}^i(X_{F(X)})$$

be the ring generators of $\mathrm{Ch}(X_{F(X)})$ defined as in [1, §86]. (The generators e_i are originally introduced in [19], see also [21], and called *elementary classes* there.)

We call an element of $\mathrm{Ch}(X_{F(X)})$ *rational*, if it lies in the image of the change of field homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(X_{F(X)})$. Assuming that X is 2-incompressible, none of $e_1, \ldots, e_i$ is rational by [1, Theorem 90.3]. It follows by [1, Proposition 88.8] that the quadratic form φ in question is anisotropic and has $\mathfrak{i}_1(\varphi) = \cdots = \mathfrak{i}_{i-1}(\varphi) = 1$. Therefore $\mathfrak{i}_i(\varphi) = 1$ as well. Finally, replacing the base field F by the function field $F(X_1)$ of the quadric X_1, the element e_i becomes rational, but none of $e_1, \ldots, e_{i-1}$, see [1, Corollary 88.6].[1] This shows that

$$\mathrm{cdim}_2 X = 1 + \cdots + (i-1) = \dim X'_{i-1}$$

for the now *equivalent* (in the sense of existence of rational maps in both directions) varieties $X = X_i$ and X'_{i-1}, meaning that X'_{i-1} is 2-incompressible.

Proof of Proposition 3.1. If φ is isotropic, the variety X_i is equivalent (in the above sense of existence of rational maps in both directions) to the variety X'_{i-1}. Therefore $\mathrm{cdim}_2 X_i = \mathrm{cdim}_2 X'_{i-1}$ (see, e.g., [12, Lemma 3.3]). Since $\dim X_i > \dim X'_{i-1}$, we get a contradiction with the 2-incompressibility of X_i.

We have shown that φ is anisotropic. Next, assuming that $i \geq 2$, we are going to check the 2-incompressibility of X'_{i-1}. For this, we apply Lemma 2.2 to the fibration $\pi : X_{1\subset i} \to X_1$, where $X_{1\subset i}$ is the 2-flag variety projecting onto X_1 and X_i. Note that the variety X'_{i-1} is the generic fiber of π. The $(i-1)$th power $h^{i-1} \in \mathrm{Ch}^{i-1}(X_1)$ of the hyperplane section class $h \in \mathrm{Ch}^1(X_1)$ and the class $l_{i-1} \in \mathrm{Ch}_{i-1}(X_1)_{F(X_{1\subset i})}$ of an i-dimensional totally isotropic subspace satisfy the relation

$$\deg(l_{i-1} \cdot h^{i-1}_{F(X_{1\subset i})}) = 1.$$

Therefore $\mathrm{cdim}'_2 X_1 \leq \dim X_1 - (i-1) = (n-2) - (i-1) = n - i - 1$. It follows by Lemma 2.2 that

$$\mathrm{cdim}_2 X_{1\subset i} \leq (n-i-1) + \mathrm{cdim}_2 X'_{i-1}.$$

Since the flag variety $X_{1\subset i}$ is equivalent to X_i, we may replace $\mathrm{cdim}_2 X_{1\subset i}$ by $\mathrm{cdim}_2 X_i$ in this inequality. Since $\dim X_i = (n-i-1) + \dim X'_{i-1}$, the upper bound on $\mathrm{cdim}_2 X_i$, we obtained, shows that the 2-incompressibility of X_i implies that of X'_{i-1}.

[1] In characteristic $\neq 2$, the statement also follows from A. Vishik's Main Tool Lemma [20, Theorem 3.1], see also [4, Theorem 1.1].

We have shown that X'_{i-1} is 2-incompressible. It follows by the preceding part that the quadratic form φ' is anisotropic. This means that $\mathfrak{i}_1(\varphi) = 1$. Continuing this induction procedure, we get that $\mathfrak{i}_1(\varphi) = \cdots = \mathfrak{i}_{i-1}(\varphi) = 1$.

To finish the proof of Proposition 3.1, it remains to show that $\mathfrak{i}_i(\varphi) = 1$. Assume that $\mathfrak{i}_i(\varphi) \geq 2$. In particular, $i < [n/2]$. Let ψ be a $(n-1)$-dimensional non-degenerate subform of φ. (Note that the notion of non-degeneracy for quadratic forms in characteristic 2 we are using is that of [1] so that non-degenerate quadratic forms exist in any dimension.) Let Y_i be the ith orthogonal Grassmannian of ψ. The condition $\mathfrak{i}_i(\varphi) \geq 2$ ensures that the varieties X_i and Y_i are equivalent. Indeed, we always have a rational map $Y_i \dashrightarrow X_i$; on the other hand, the intersection of an $(i+1)$-dimensional totally isotropic subspace of $\varphi_{F(X_i)}$ with the hyperplane of definition of $\psi_{F(X_i)}$ has dimension at least i, showing that $Y_i(F(X_i)) \neq \emptyset$, so that a rational map $X_i \dashrightarrow Y_i$ exists as well. But $\dim Y_i < \dim X_i$ contradicting 2-incompressibility of X_i. $\square$

4. Products by orthogonal Grassmannians

Theorem 4.1. *Let X be an orthogonal Grassmannian of a non-degenerate quadratic form over F and let Y be a projective homogeneous variety under a semisimple affine algebraic group over F. The product $X \times Y$ is 2-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are so.*

Remark 4.2. One may include in the statement of Theorem 4.1 the case where the quadratic form (let us call it φ) is of even dimension n and X is a component of $X_{n/2}$. However the proof of this case easily reduces to the case of $(n-2)/2$th orthogonal Grassmannian of a $(n-1)$-dimensional quadratic form (namely, any $(n-1)$dimensional non-degenerate subform in φ_L, where L/F is the discriminant field extension of φ).

Proof of Theorem 4.1. By Corollary 2.5 we only need to prove the "if" part.

Since canonical 2-dimension is not changed under base field extensions of odd degree (see [15, Proposition 1.5(2)]), we may assume that there exists a finite 2-primary Galois field extension E/F such that G_E is of inner type, where G is the semisimple algebraic group over F acting on Y. This assumption allows us to apply results of [7] and [8, §6].

The variety X is the ith orthogonal Grassmannian X_i of a non-degenerate n-dimensional quadratic form φ for some i, n with $0 \leq i < n/2$. We induct on $n \geq 2$. For $n = 2$ the statement we are proving is trivial. We assume that $n \geq 3$ below. We also may assume that $i \geq 1$ (otherwise, the variety X is just the point $\operatorname{Spec} F$).

Let F' be the function field of the variety X_1 and let φ' be a $(n-2)$-dimensional quadratic form over F' Witt-equivalent to $\varphi_{F'}$. We set $X' := X'_{i-1}$, the $(i-1)$st orthogonal Grassmannian of φ'.

We are using Chow motives with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$, [1, §64]. For a motive M and an integer j, we write $M\{j\}$ (and not $M(j)$ as in [1]) for the jth shift of M.

The proof below is a modification of [10, Proof of Theorem 3.1].

The motive $M(X_{F'})$ of the variety $X_{F'}$ decomposes into a direct sum

$$M(X_{F'}) \simeq M(X') \oplus M(X')\{\dim X - \dim X'\} \oplus N, \tag{4.3}$$

where N is a shift of $M(X'_i)$, cf. [8, (2.6)]. (The shifting number is $(\dim X - \dim X'_i)/2$ but the knowledge of it is not needed in the sequel.) Multiplying by the motive of $Y_{F'}$, we get that

$$M(X\times Y)_{F'}\simeq M(X'\times Y_{F'})\oplus M(X'\times Y_{F'})\{\dim X-\dim X'\}\oplus N,$$

with N being a shift of $M(X'_i\times Y_{F'})$ now. We claim that the variety $X'\times Y_{F'}$ is 2-incompressible by the induction hypothesis. To check the claim, we check that the varieties $Y_{F'(X')}$ and $X'_{F'(Y)}$ are 2-incompressible. To check that $Y_{F'(X')}$ is 2-incompressible, we check that Y over a larger field $F'(X'\times X_{F'})$ is so. Since the projective homogeneous varieties X_1 and X' possess $F(X)$-points, the field

$$F'(X'\times X_{F'})=F(X)(X_1)(X')$$

is purely transcendental over $F(X)$. And the variety $Y_{F(X)}$ is 2-incompressible (this is the place in the proof of the "if" part of Theorem 4.1, where the assumption on $Y_{F(X)}$ is used). Since canonical 2-dimension of projective homogeneous varieties does not change under purely transcendental field extensions, Y over $F'(X'\times X_{F'})$ is 2-incompressible.

To check that $X'_{F'(Y)}$ is 2-incompressible, we apply Proposition 3.1 to the base field $F(Y)$. The variety $X_{F(Y)}$ over the field $F(Y)$ is assumed to be 2-incompressible. Therefore the variety $X'_{F'(Y)}$ is 2-incompressible by Proposition 3.1.

The claim being proved, writing $U(Z)$ for the upper motive of a projective homogeneous variety Z, we obtain the following decomposition:

$$M(X\times Y)_{F'}\simeq U(X'\times Y_{F'})\oplus U(X'\times Y_{F'})\{\dim X-\dim X'\}\oplus N, \tag{4.4}$$

with N having the property that no summand of its complete decomposition is isomorphic to a shift of $U(X'\times Y_{F'})$. Indeed, by 2-incompressibility of $X'\times Y_{F'}$, no summand of the complete decomposition of the complement of $U(X'\times Y_{F'})$ in the total motive of the variety is isomorphic to a shift of the upper motive. Besides, since the higher Witt index $\mathfrak{i}_{i-1}(\varphi'_{F'(Y)})=\mathfrak{i}_i(\varphi_{F(Y)})$ is equal to 1 by Proposition 3.1, the degree of any closed point on the variety $(X'_i\times Y_{F'})_{F'(X'\times Y_{F'})}$ is odd so that the motive of this variety does not contain any Tate motive as a summand. Since $U(X'\times Y_{F'})_{F'(X'\times Y_{F'})}$ does contain a Tate motive, it follows that no indecomposable summand of $M(X'_i\times Y_{F'})$ is isomorphic to a shift of $U(X'\times Y_{F'})$.

Now, to show that the variety $X\times Y$ is 2-incompressible, it suffices by [6, Theorem 5.1] to show that the complete decomposition of $U(X\times Y)$ contains the second summand of (4.4). So, we assume the contrary and we look for a contradiction.

By [8, Proposition 2.4], the complete decomposition of $U(X\times Y)_{F'}$ contains as a summand the motive $U(X'\times Y_{F'})$ shifted by the difference

$$\operatorname{cdim}_2(X\times Y)-\operatorname{cdim}_2(X'\times Y_{F'}).$$

Therefore, our assumption implies that the difference is 0, and we come to

$$\operatorname{cdim}_2(X\times Y)=\operatorname{cdim}_2(X'\times Y_{F'})=d,$$

where $d:=\dim(X'\times Y_{F'})=\dim X'+\dim Y$.

By [8, Proposition 6.1], there exist $\alpha\in\operatorname{Ch}^d(X\times Y)_{F(X\times Y)}$ and $\beta\in\operatorname{Ch}_d(X\times Y)$ with $\deg(\alpha\cdot\beta)\neq 0\in\mathbb{F}_2$. In the last formula, we consider both cycles over a common field extension of their fields of definition, before we multiply them. We use this convention below in the proof (in similar formulas on degree of products) as well.

Since $\operatorname{cdim}_2(X_{F'}) = \dim X' =: d'$, we can find $\alpha' \in \operatorname{Ch}^{d'}(X_{F'(X)})$ and $\beta' \in \operatorname{Ch}_{d'}(X_{F'})$ with $\deg(\alpha' \cdot \beta') \neq 0$. Using these α' and β' and a rational point $\mathbf{pt} \in Y_{F(Y)}$, we get the cycles

$$\alpha' \times [\mathbf{pt}] \in \operatorname{Ch}^d(X \times Y)_{F'(X\times Y)} \quad \text{and} \quad \beta' \times [Y] \in \operatorname{Ch}_d(X \times Y)_{F'},$$

having the same property as α and β:

$$\deg\Big((\alpha' \times [\mathbf{pt}]) \cdot (\beta' \times [Z])\Big) \neq 0.$$

It follows by [8, Lemma 6.5] that one can "mix up" the old cycles with the new ones and get the relation

$$\deg\Big((\alpha' \times [\mathbf{pt}]) \cdot \beta\Big) \neq 0.$$

Since $\alpha' \times [\mathbf{pt}] = (\alpha' \times [Y]) \cdot ([X] \times [\mathbf{pt}])$, the last degree relation can be rewritten as $\deg(\alpha' \cdot \beta'') \neq 0$, where $\beta'' \in \operatorname{Ch}_{d'}(X_{F(Y)})$ is the push-forward of the product $([X] \times [\mathbf{pt}]) \cdot \beta$ along the projection $(X \times Y)_{F(Y)} \to X_{F(Y)}$. Since the field extension $F'(X)/F(X)$ is purely transcendental, there exists $\alpha'' \in \operatorname{Ch}^{d'}(X_{F(X)})$ mapped to α' under the change of field homomorphism. Changing notation, we write α'' for the image of α'' in $\operatorname{Ch}^{d'}(X_{F(Y)(X)})$.

The cycles $\alpha'' \in \operatorname{Ch}^{d'}(X_{F(Y)(X)})$ and $\beta'' \in \operatorname{Ch}_{d'}(X_{F(Y)})$ thus constructed have the property $\deg(\alpha'' \cdot \beta'') \neq 0$. It follows by Proposition 2.1 that $\operatorname{cdim}_2(X_{F(Y)}) \leq d'$. Since

$$d' = \dim X' = (i-1)(i-2)/2 + (i-1)(n-2i) < i(i-1)/2 + i(n-2i) = \dim X,$$

the relation $\operatorname{cdim}_2(X_{F(Y)}) \leq d'$ we obtained contradicts the assumption on 2-incompressibility of the variety $X_{F(Y)}$. □

5. Unitary Grassmannians

Let K/F be a separable quadratic field extension, $n \geq 0$ an integer, V an n-dimensional vector space over K, and φ a K/F-hermitian form on V. For any integer i with $0 \leq i \leq n/2$, we write X_i for the unitary Grassmannian of totally isotropic i-dimensional K-subspaces in V.

We recall that $\dim X_i = i(2n - 3i)$.

Assume that $n \geq 2$. Then the variety X_1 is defined and the hermitian form $\varphi_{F(X_1)}$ is isotropic. Let φ' be an $(n-2)$-dimensional $K(X_1)/F(X_1)$-hermitian form Witt-equivalent to φ. For i with $0 \leq i \leq (n-2)/2$, we write X'_i for the corresponding unitary Grassmannian of totally isotropic i-dimensional subspaces.

Proposition 5.1. *For some integers n and i with $1 \leq i \leq n/2$, assume that the variety X_i is 2-incompressible. Then φ is anisotropic, all the first i higher Witt indexes $\mathfrak{i}_1(\varphi), \dots, \mathfrak{i}_i(\varphi)$ of φ are equal to 1, and the variety X'_{i-1} is also 2-incompressible.*

Proof. We start by repeating word by word the proof of Proposition 3.1. The first change that occurs in the unitary case compared to the orthogonal one is in the formula for $\dim X_i$. But we still have $\dim X_i > \dim X'_{i-1}$ in the setting, so that anisotropy of φ is proved.

As the next step, we apply Lemma 2.2 to the projection $\pi : X_{1\subset i} \to X_1$ in order to prove the 2-incompressibility of X'_{i-1}, which is again the generic fiber of π. The base X_1 of the projection, which was a usual quadric in the orthogonal case, is now a sort of *unitary quadric*, and we need information on its Chow group analogous to the information we have and have used for usual quadrics. Such an information is provided in [13, §3]. It is shown there that there are elements $h \in \mathrm{Ch}^2(X_1)$ and $l_{i-1} \in \mathrm{Ch}_{2(i-1)}(X_1)_{F(X_i)}$ with $\deg(l_{i-1} \cdot h^{i-1}) = 1$. (The notation for the elements we use is similar to the quadric case and differs from [13].) Therefore

$$\mathrm{cdim}'_2 X_1 \leq \dim X_1 - 2(i-1) = (2n-3) - 2(i-1) = 2n - 2i - 1.$$

It follows by Lemma 2.2 that

$$\mathrm{cdim}_2 X_{1\subset i} \leq (2n - 2i - 1) + \mathrm{cdim}_2 X'_{i-1}.$$

Since the flag variety $X_{1\subset i}$ is equivalent to X_i, we may replace $\mathrm{cdim}_2 X_{1\subset i}$ by $\mathrm{cdim}_2 X_i$ in this inequality. Since $\dim X_i = (2n-2i-1) + \dim X'_{i-1}$, the upper bound on $\mathrm{cdim}_2 X_i$, we obtained, shows that 2-incompressibility of X_i implies that of X'_{i-1}.

We have shown that X'_{i-1} is 2-incompressible. It follows by the preceding part that the quadratic form φ' is anisotropic. This means that $\mathfrak{i}_1(\varphi) = 1$. Continuing this induction procedure, we get that $\mathfrak{i}_1(\varphi) = \cdots = \mathfrak{i}_{i-1}(\varphi) = 1$.

To finish the proof of Proposition 5.1, it remains to show that $\mathfrak{i}_i(\varphi) = 1$. Assume that $\mathfrak{i}_i(\varphi) \geq 2$. In particular, $i < [n/2]$. Let ψ be a $(n-1)$-dimensional non-degenerate subform of φ. The form ψ exists because φ can be diagonalized, [17, Theorem 6.3 of Chapter 7]. Let Y_i be the ith unitary Grassmannian of ψ. By precisely the same argument as in the orthogonal case, the condition $\mathfrak{i}_i(\varphi) \geq 2$ ensures that the varieties X_i and Y_i are equivalent. But again $\dim Y_i < \dim X_i$, contradicting 2-incompressibility of X_i. □

Theorem 5.2. *Let X be a unitary Grassmannian of a K/F-hermitian form over F (where K/F is a separable quadratic field extension) and let Y be a projective homogeneous variety under a semisimple affine algebraic group over F. The product $X \times Y$ is 2-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are so.*

Proof. The proof goes through as that of Theorem 4.1 (with Proposition 5.1 replacing Proposition 3.1). The motivic decomposition (4.3) is given by [9, Lemma 7.1]. Note that N is now given by a direct sum of a shift of $M(X'_i)$ and several shifted copies of $\operatorname{Spec} K$. But this addition of shifted copies of $\operatorname{Spec} K$ does not make any difference for the sequel. □

6. Symplectic Grassmannians

Let Q be a quaternion division F-algebra, $n \geq 0$ an integer, V a right n-dimensional vector space over Q, φ a hermitian (with respect to the canonical symplectic involution on Q) form on V. In the case of $\operatorname{char} F = 2$, we require φ to be alternating.

For any integer i with $0 \leq i \leq n/2$, we write X_i for the symplectic Grassmannian of totally isotropic i-dimensional Q-subspaces in V. We refer to [11] for a general discussion on these varieties and recall that

$$\dim X_i = i(4n - 6i + 1).$$

Note that besides the symplectic Grassmannians of φ introduced right above, for any odd integer j with $1 \leq j \leq n$ there is the variety of totally isotropic subspaces in V of "dimension" $j/2$ over Q. Any such variety however is equivalent to the conic of Q and therefore is not interesting regarding the questions we consider in the paper.

Lemma 6.1. *Assume that $n \geq 2$. There exists an element $h \in \mathrm{Ch}^4(X_1)$ and for any integer i with $1 \leq i \leq n/2$ there exists an element $l_{i-1} \in \mathrm{Ch}^{4(i-1)}(X_1)_{F(X_i)}$ such that* $\deg(l_{i-1} \cdot h^{i-1}_{F(X_i)}) = 1$.

Proof. The variety X_1 is a closed hypersurface (a sort of "symplectic quadric") in the "projective space" $Q\mathbb{P}(V)$ – the variety of 1-dimensional Q-subspaces of V, cf. [11]. Note that $\dim Q\mathbb{P}(V) = 4(n-1)$, where $n = \dim V$. Picking up a hyperplane W in V, we consider the class

$$H := [Q\mathbb{P}(W)] \in \mathrm{Ch}^4(Q\mathbb{P}(V))$$

and define $h \in \mathrm{Ch}^4(X_1)$ as the pull-back of H.

Now we replace the base field F by the function field $F(X_i)$, pick up a totally isotropic i-dimensional Q-subspace in V, and let l_{i-1} be its class in $\mathrm{Ch}_{4(i-1)}(X_1)$. For the closed imbedding $in : X_1 \hookrightarrow Q\mathbb{P}(V)$ we have $in_*(l_{i-1}) = H^{n-i}$. Since $\deg(H^{n-1}) = 1$, we get, applying the projection formula for in:

$$\deg(l_{i-1} \cdot h^{i-1}) = \deg in_*(l_{i-1} \cdot h^{i-1}) = \deg(in_*(l_{i-1}) \cdot H^{i-1}) = \deg(H^{n-1}) = 1. \qquad \square$$

As in Lemma 6.1, assume that $n \geq 2$. Then the variety X_1 is defined and the hermitian form $\varphi_{F(X_1)}$ is isotropic. Let φ' be an $(n-2)$-dimensional hermitian form Witt-equivalent to φ. For i with $0 \leq i \leq (n-2)/2$, we write X'_i for the corresponding Grassmannian of totally isotropic i-dimensional subspaces.

Proposition 6.2. *For some $n \geq 2$ and some $i \geq 1$, assume that the variety X_i is 2-incompressible. Then φ is anisotropic, all the first i higher Witt indexes $\mathfrak{i}_1(\varphi), \dots, \mathfrak{i}_i(\varphi)$ of φ are equal to 1, and the variety X'_{i-1} is also 2-incompressible.*

Proof. Again everything goes through as in the proof of Proposition 3.1 (or also Proposition 5.1), although the symplectic Grassmannians we are working with now satisfy different dimension formulas compared to the varieties we had in the previous sections. Nevertheless we still have $\dim X_i > \dim X'_{i-1}$ in the current setting, so that we obtain the anisotropy of φ.

The next step is, as before, the proof of 2-incompressibility of X'_{i-1}, involving Lemma 2.2 and the projection $\pi : X_{1\subset i} \to X_1$. The information on the Chow group of X_1 that we need here now is provided by Lemma 6.1. It shows that

$$\mathrm{cdim}'_2 X_1 \leq \dim X_1 - 4(i-1) = (4n-5) - 4(i-1) = 4n - 4i - 1.$$

It follows by Lemma 2.2 that

$$\mathrm{cdim}_2 X_{1\subset i} \leq (4n - 4i - 1) + \mathrm{cdim}_2 X'_{i-1}.$$

Since again the flag variety $X_{1\subset i}$ is equivalent to X_i, we may replace $\mathrm{cdim}_2 X_{1\subset i}$ by $\mathrm{cdim}_2 X_i$ in this inequality. Since $\dim X_i = (4n-4i-1) + \dim X'_{i-1}$, the upper bound on $\mathrm{cdim}_2 X_i$, we obtained, shows that 2-incompressibility of X_i implies that of X'_{i-1}.

We have shown that X'_{i-1} is 2-incompressible. It follows as in the orthogonal case that $\mathfrak{i}_1(\varphi) = \cdots = \mathfrak{i}_{i-1}(\varphi) = 1$.

To finish the proof of Proposition 6.2, it remains to show that $\mathfrak{i}_i(\varphi) = 1$. Assume that $\mathfrak{i}_i(\varphi) \geq 2$. In particular, $i < [n/2]$. Let ψ be a $(n-1)$-dimensional non-degenerate subform of φ. Let Y_i be the ith unitary Grassmannian of ψ. The condition $\mathfrak{i}_i(\varphi) \geq 2$ ensures that the varieties X_i and Y_i are equivalent. But now again $\dim Y_i < \dim X_i$ contradicting the 2-incompressibility of X_i. □

Theorem 6.3. *Let X be a symplectic Grassmannian of a hermitian form over a quaternion F-algebra and let Y be a projective homogeneous variety under a semisimple affine algebraic group over F. The product $X \times Y$ is 2-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are so.*

Proof. The proof goes through as that of Theorem 4.1 (with Proposition 6.2 replacing Proposition 3.1). The analogue of the motivic decomposition (4.3) is provided by [11, Lemma 9.1]. Note that N is now given by a direct sum of a shift of $M(X'_i)$ and several shifted copies of the motive of the conic of Q. □

Theorems 4.1, 5.2, and 6.3 give all together

Corollary 6.4 (cf. [10, Corollary 3.6])**.** *A product $X_1 \times \cdots \times X_r \times Y$, where for each $i = 1, \ldots, r$ the ith factor X_i is a generalized Severi-Brauer variety or a (orthogonal, symplectic, or unitary) Grassmannian of isotropic subspaces and Y is an arbitrary projective homogenous variety, is 2-incompressible if and only each factor of the product considered over the function field of the product of the remaining factors is 2-incompressible.* □

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