ERRATUM TO [2]

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The proof of the implication (3) $\Rightarrow$ (2) of [2, Proposition 3.1] makes use of a wrong formula for the endomorphism ring of a corestriction of a motive. Example 0.1 here below demonstrates that the formula is indeed wrong.

Proposition 0.2 further below is a weaker version of [2, Corollary 3.2] satisfactory for the purposes of [2]. Proposition 0.2 also proves a modified version of [2, Proposition 3.1] with its condition (2) referring to a more specific motive M.

Example 0.1. Let E/F be a cyclic degree 4 field extension and let L/F be its quadratic subextension. For $M := M_L(\operatorname{Spec} E)$ – the motive of the L-variety $\operatorname{Spec} E$, the formula from [2, Proof of Proposition 3.1] claims that

$$\operatorname{End}(\operatorname{cor}_{L/F} M) = (\operatorname{End} M) \otimes (\mathbb{F}[x](x^2 - 1)).$$

For the coefficients $\mathbb{F} = \mathbb{F}_2$ this yields the polynomial ring $\mathbb{F}[x, y]$ modulo the relations $(x - 1)^2 = 0 = (y - 1)^2$. In this ring, the square of every nilpotent vanishes.

On the other hand, $\operatorname{cor}_{L/F} M = M_F(E)$ and

$$\operatorname{End}(M_F(E)) = \mathbb{F}[z]/(z^4 - 1) = \mathbb{F}[z]/((z - 1)^4).$$

The class of $z - 1$ in this ring is nilpotent, but its square does not vanish.

Now let p be a prime number and let L/F be a subextension of a finite Galois field extension E/F of a p-power degree. Let G be a reductive group over F and let X be a projective G_L-homogeneous L-variety.

Proposition 0.2. *Let $M \in \operatorname{CM}(L, \mathbb{F}_p)$ be an upper indecomposable motivic summand of X. Then $\operatorname{cor}_{L/F} M$ is an upper indecomposable summand of the F-variety X.*

Proof. We only need to show that the motive $\operatorname{cor}_{L/F} M$ is indecomposable. We induct on the degree $[L : F]$. The starting case $[L : F] = 1$ is trivial.

Assuming that $[L : F] > 1$, we find a degree p Galois subextension K/F of L/F. (It exists because $\operatorname{Gal}(E/L)$ is a proper subgroup in the p-group $\operatorname{Gal}(E/F)$, and any proper subgroup of a p-group is contained, normal, and of index p in some larger subgroup of the p-group.) By the induction hypothesis, the K-motive $N := \operatorname{cor}_{L/K} M$ is indecomposable. We just need to check that the motive $\operatorname{cor}_{K/F} N = \operatorname{cor}_{L/F} M$ is also indecomposable.

The K-motive $(\operatorname{cor}_{K/F} N)_K$ is the direct sum of the indecomposable motives N_σ with σ running over the Galois group of K/F, where N_σ is the base change of N via $\sigma\colon K \to K$. Since

$$\dim \operatorname{Ch}^0(N_\sigma)_E = \dim \operatorname{Ch}^0(N_E) = [L : K],$$

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it suffices to show that $\dim \mathrm{Ch}^0(N'_E) = [L : F]$ for any nonzero summand N' of $\operatorname{cor}_{K/F} N$. Over the function field $F(Y)$ of the variety Y of Borel subgroups in G, the motive $\operatorname{cor}_{K/F} N$ becomes a sum of shifts of copies of the motive

$$M_F(\operatorname{Spec} L)_{F(Y)} = M_{F(Y)}(\operatorname{Spec} L(Y))$$

with (exactly) one non-shifted (i.e., shifted by 0) copy. Note that the field extension $L(Y)/F(Y)$ is separable of degree $[L : F]$. By Corollary 0.4, the motive $M_{F(Y)}(\operatorname{Spec} L(Y))$ is indecomposable. Therefore

$$\dim \mathrm{Ch}^0(N'_E) = \dim \mathrm{Ch}^0(N'_{E(Y)}) = \mathrm{Ch}^0\left(M_F(\operatorname{Spec} L)_{E(Y)}\right) = [L : F]. \qquad \square$$

Theorem 0.3 (A.S. Merkurjev). *Let p be prime number, G a finite group, and X a G-set. If G acts transitively on X and $|X|$ is a p-power, the G-module $\mathbb{F}_p[X]$ is indecomposable.*

Proof. Let $H \subset G$ be the stabilizer of a point in X, so we have $X = G/H$. Choose a Sylow p-subgroup $P \subset G$. The integer $[G : H]$ divides

$$[G : (H \cap P)] = [G : P] \cdot [P : (H \cap P)].$$

Since $[G : H]$ is a p-power and $[G : P]$ is prime to p, the integer $[G : H]$ divides the index of $H \cap P$ in P. In particular, $[G : H] \leq [P : (H \cap P)]$.

On the other hand, the natural map $P/(H \cap P) \to G/H$ is injective. Therefore, this map is a bijection, hence P acts transitively of $X = G/H$. We can then replace G by P and therefore assume that G is a p-group.

By [3, Theorem 1.11.1], the augmentation ideal I of the group ring $\mathbb{F}[G]$ is nilpotent. Since the $\mathbb{F}[G]$-module $\mathbb{F}[G/H]$ satisfies $\mathbb{F}[G/H]/I\mathbb{F}[G/H] = \mathbb{F}$, in any its decomposition in a direct sum of two, one summand – let's call it M – will satisfy $M/IM = 0$, i.e., $M = IM$. Nakayama's lemma then tells that $M = 0$. Indeed, for any system of generators $m_1, \ldots, m_n$ of M one has $m_n = i_1 m_1 + \cdots + i_n m_n$ with some nilpotents $i_1, \ldots, i_n \in I$. Since the difference $1 - i_n$ is invertible, the generator m_n can be eliminated. $\square$

Let L/F be a finite separable field extension, G the Galois groups of its Galois closure, $H \subset G$ the subgroup corresponding to L. According to [1, §7], the motive $M_F(L) \in \mathrm{CM}(F, \mathbb{F}_p)$ is indecomposable if and only if the $\mathbb{F}_p[G]$-module $\mathbb{F}_p[G/H]$ is indecomposable. Thus Theorem 0.3 yields

Corollary 0.4. *For any finite separable field extension L/F of a p-power degree, the motive $M_F(\operatorname{Spec} L)$ is indecomposable.* $\square$

References

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