ISOTROPY OF ORTHOGONAL INVOLUTIONS

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WITH AN APPENDIX BY JEAN-PIERRE TIGNOL

Abstract. An orthogonal involution on a central simple algebra becoming isotropic over any splitting field of the algebra, becomes isotropic over a finite odd degree extension of the base field (provided that the characteristic of the base field is not 2). The proof makes use of a structure theorem for Chow motives with finite coefficients of projective homogeneous varieties, of incompressibility of certain generalized Severi-Brauer varieties, and of Steenrod operations.

The main result of this paper is as follows:

Theorem 1. *Let F be a field of characteristic not 2, A a central simple F-algebra, σ an orthogonal involution on A. The following two conditions are equivalent:*

(1) *σ becomes isotropic over any splitting field of A;*
(2) *σ becomes isotropic over some finite odd degree extension of the base field.*

The proof of Theorem 1 is given in the very end of the paper; it makes use of Chow motives with finite coefficients, of incompressibility of certain projective homogeneous varieties, and of Steenrod operations. A sketch of the proof is given shortly below.

For F with no finite field extensions of odd degree, Theorem 1 proves [7, Conjecture 5.2]. (I learned this conjecture in 1994 from A. Wadsworth during a Luminy conference.) For general F, the question whether condition (2) implies isotropy of σ over F remains open. Note that any orthogonal involution becomes isotropic over some 2-primary field extension, so that the mentioned open question is about existence of a rational point on a variety possessing a 0-cycle of degree 1. Such a question can hardly be attacked by the methods of the paper, so that Theorem 1 seems to be the best possible result in this direction which can be achieved by such methods.

The general reference on central simple algebras and involutions is [11].

The implication (2) $\Rightarrow$ (1) is a consequence of the Springer theorem on quadratic forms. We only prove the implication (1) $\Rightarrow$ (2). Condition (1) is equivalent to the condition that σ becomes isotropic over some (and therefore any) generic splitting field of the algebra, such as the function field of the Severi-Brauer variety of any central simple algebra Brauer-equivalent to A.

We prove this theorem over all fields simultaneously using an induction on the index $\operatorname{ind} A$ of A. The case of $\operatorname{ind} A = 1$ is trivial. The case of $\operatorname{ind} A = 2$ is done in [15] (with

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The statement of Proposition 15 has been tested and Example 16 has been detected using the Maple *Chow Ring Package* by S. Nikolenko, V. Petrov, N. Semenov, and K. Zainoulline.

"σ is isotropic (over F)" in place of condition (2)). From now on we are assuming that $\operatorname{ind} A > 2$. Therefore $\operatorname{ind} A = 2^r$ for some integer $r \geq 2$.

Let us list our basic notation: F is a field of characteristic different from 2; r is an integer ≥ 2; A is a central simple F-algebra of the index 2^r; σ is an orthogonal involution on A; D is a central division F-algebra (of degree 2^r) Brauer-equivalent to A; V is a right D-module with an isomorphism $\operatorname{End}_D(V) \simeq A$; v is the D-dimension of V (therefore $\operatorname{rdim} V = \deg A = 2^r \cdot v$, where $\operatorname{rdim} V := \dim_F V/\deg D$ is the reduced dimension of V); we fix an orthogonal involution τ on D; h is a hermitian (with respect to τ) form on V such that the involution σ is adjoint to h; $\mathfrak{X} = X(2^r; (V, h))$ is the variety of totally isotropic submodules in V of the reduced dimension 2^r which is isomorphic (via Morita equivalence) to the variety $X(2^r; (A, \sigma))$ of right totally isotropic ideals in A of the same reduced dimension; $\mathcal{Y} = X(2^{r-1}; D)$ is the variety of right ideals in D of reduced dimension 2^{r-1}.

We assume that the hermitian form h (and therefore, the involution σ) becomes isotropic over the function field of the Severi-Brauer variety $X(1; D)$ of D, and we want to show that h (and σ) becomes isotropic over a finite odd degree extension of F. By [8], the Witt index of h (which coincides with the Witt index of σ) over this function field is at least $2^r = \operatorname{ind} A$. In particular, $v \geq 2$. If the Witt index is bigger than 2^r, we replace V by a submodule in V of D-codimension 1 (that is, of the reduced dimension $2^r(v-1)$) and we replace h by its restriction on this new V. The Witt index of $h_{F(X(1;D))}$ drops by at most 2^r or stays unchanged. We repeat the procedure until the Witt index becomes equal to 2^r (we come down eventually to the Witt index 2^r because the Witt index is at most 2^r for V with $\dim_D V = 2$).

If $\dim_D V = 2$, then h becomes hyperbolic over $F\big(X(1; D)\big)$. Therefore, by the main result of [9], h is hyperbolic over F and we are done. By this reason, we assume that $\dim_D V \geq 3$, that is, $v \geq 3$. In particular, the variety $\mathfrak{X}$ is projective *homogeneous* (in the case of $v = 2$, the variety $\mathfrak{X}$ has two connected components each of which is homogeneous).

Remark 2. Note that the hyperbolicity theorem (HT) of [9] is a formal consequence of Theorem 1. Keeping the case $v = 2$ (therefore avoiding the only point where we use HT) and slightly modifying the sequel, one can get a proof of Theorem 1 which does not rely on HT (see Remark 14). This will give a new proof of HT which (although having much in common) is (at several points) essentially different from the original one.

The variety $\mathfrak{X}$ has an $F\big(X(1; D)\big)$-point and $\operatorname{ind} D_{F(\mathcal{Y})} = 2^{r-1}$. Consequently, by the induction hypothesis, the variety $\mathfrak{X}_{F(\mathcal{Y})}$ has an odd degree closed point. We prove Theorem 1 by showing that the variety $\mathfrak{X}$ has an odd degree closed point. Here is a sketch of the proof:

Sketch of Proof of Theorem 1. We assume that the variety $\mathfrak{X}$ (and therefore also $\mathfrak{X} \times \mathfrak{X}$) has no odd degree closed point and we are looking for a contradiction. First we show that the Chow motive with coefficients in $\mathbb{F}_2$ of $\mathfrak{X}$ contains a summand isomorphic to a shift of the *upper* indecomposable summand $M_{\mathcal{Y}}$ of the motive of $\mathcal{Y}$ (Corollary 8), where *upper* means that the 0-codimensional Chow group of $M_{\mathcal{Y}}$ is non-zero. (At this point we use the 2-incompressibility of $\mathcal{Y}$ which is due to [12].) Moreover, the corresponding projector on $\mathfrak{X}$ can be *symmetrized* (Proposition 9). This makes it possible to compute the degree modulo

4 of any integral representative of the 0-cycle class on $\mathfrak{X} \times \mathfrak{X}$, given by the value of the appropriate Steenrod operation on this projector. Namely (see Corollary 11), this degree is identified with the *rank* of $M_{\mathcal{Y}}$ and therefore is 2 modulo 4 by a result of [5] (which is a consequence of the 2-incompressibility of $\mathcal{Y}$ and a structure theorem for motives with finite coefficients of projective homogeneous varieties established in [5] and generalized in [10]; this structure theorem says that any indecomposable summand of the motive of a projective G-homogeneous variety X, where G is semisimple affine algebraic group, is isomorphic to the *upper* indecomposable summand of another projective G-homogeneous variety X' such that the Tits index of $G_{F(X')}$ contains the Tits index of $G_{F(X)}$). On the other hand, a computation of Steenrod operations on the split orthogonal grassmannian $\bar{\mathfrak{X}}$ (Proposition 12) allows one to show that the above degree is 0 modulo 4. This is the required contradiction. □

We need an enhanced version of [9, Proposition 4.6]. This is a statement about the *Grothendieck Chow motives* (see [4, Chapter XII]) with coefficients in a prime field $\mathbb{F}_p$ (which we shall apply to $p = 2$). We write Ch for Chow groups with coefficients in $\mathbb{F}_p$ and we write $M(X)$ for the motive of a complete smooth F-variety X. Saying "sum of motives", we always mean the direct sum. We call X *split*, if $M(X)$ is isomorphic to a sum of Tate motives (which are defined as shifts of the motive of a point), and we call X *geometrically split*, it it splits over an extension of the base field. We say that X satisfies *nilpotence principle*, if for any field extension E/F the kernel of the change of field homomorphism $\operatorname{End} M(X) \to \operatorname{End} M(X)_E$ consists of nilpotents. Finally, X is *p-incompressible*, if it is connected and for any proper closed subvariety $Y \subset X$, the degree of any closed point on $Y_{F(X)}$ is divisible by p.

The base field F may have arbitrary characteristic in this statement:

Proposition 3. *Let Y be a geometrically split, geometrically irreducible F-variety satisfying the nilpotence principle and let X be a smooth complete F-variety. Assume that there exists a field extension E/F such that*

1. *for some field extension $\overline{E(Y)}/E(Y)$, the image of the change of field homomorphism $\operatorname{Ch}(X_{E(Y)}) \to \operatorname{Ch}(X_{\overline{E(Y)}})$ coincides with the image of the change of field homomorphism $\operatorname{Ch}(X_{F(Y)}) \to \operatorname{Ch}(X_{\overline{E(Y)}})$;*
2. *the E-variety Y_E is p-incompressible;*
3. *a shift of the upper indecomposable summand of $M(Y)_E$ is a summand of $M(X)_E$.*

Then the same shift of the upper indecomposable summand of $M(Y)$ is a summand of $M(X)$.

Proof. We recall that this Proposition is an enhanced version of [9, Proposition 4.6]. The only difference with the original version is in the condition (1): the field extension $E(Y)/F(Y)$ is assumed to be purely transcendental in the original version. However, only the new condition (1), a consequence of the pure transcendentality, is used in the original proof. □

Everywhere below, the prime p is 2. We are going to apply Proposition 3 (with $p = 2$) to $Y = \mathcal{Y}$, $X = \mathfrak{X}$, and $E = F(\mathfrak{X})$. We do not know if the field extension $E(\mathcal{Y})/F(\mathcal{Y})$ is purely transcendental because we do not know whether the variety $\mathfrak{X}_{F(\mathcal{Y})}$ has a rational point (we only know that this variety has an odd degree closed point).

Next we are going to check that conditions (1)–(3) of Proposition 3 are satisfied for these Y, X, E. We start with condition (3) for which we need a motivic decomposition of $X_E = \mathfrak{X}_{F(\mathfrak{X})}$. We have the decomposition of [2] arising from the fact that $\mathfrak{X}(F(\mathfrak{X})) \neq \emptyset$. More generally, the "same" decomposition holds for $F(\mathfrak{X})$ replaced by any field K/F with $\mathfrak{X}(K) \neq \emptyset$. Over such K, the hermitian form h decomposes in the orthogonal sum of a hyperbolic D_K-plane and a hermitian form h' on a right D_K-module V' with $\operatorname{rdim} V' = 2^r(v-2)$.

It requires some work to derive the decomposition from the general theorem of [2]. We use a ready answer from [6], where the projective homogeneous varieties under the *classical* semisimple affine algebraic groups have been treated:

Lemma 4 ([6, Corollary 15.14])**.** $M(\mathfrak{X}_K) \simeq$

$$\bigoplus_{i,j} M\big(X(i, i+j; D_K) \times X(j; (V', h'))\big)\big(i(i-1)/2 + j(i+j) + i(\operatorname{rdim} V' - j)\big),$$

where $X(i, i+j; D_K)$ is the variety of flags given by a right ideal in the K-algebra D_K of the reduced dimension i contained in a right ideal of the reduced dimension $i+j$ (this is a non-empty variety if and only if $0 \le i \le i+j \le \deg D$).

Proof. Unfortunately, [6, Corollary 15.14] is not the above statement on motives, but its consequence. However, the needed statement on motives is actually proved in the proof of [6, Corollary 15.14]. □

In particular, a shift of the motive of the variety $\mathcal{Y}_{F(\mathfrak{X})}$ is a motivic summand of $\mathfrak{X}_{F(\mathfrak{X})}$: namely, the summand of Lemma 4 given by $i = 2^{r-1}$ and $j = 0$ (with $K = F(\mathfrak{X})$). This summand has as the shifting number the integer

$$n := 2^{r-2}(2^{r-1} - 1) + 2^{2r-1}(v-2). \tag{5}$$

We note that $\dim \mathfrak{X} = 2^{r-1}(2^r - 1) + 2^{2r}(v-2)$, $\dim \mathcal{Y} = 2^{2r-2}$, and therefore

$$n = (\dim \mathfrak{X} - \dim \mathcal{Y})/2.$$

We have checked condition (3) of Proposition 3 and we start checking condition (2). By [12] (see [5] for a different proof and generalizations), the variety $\mathcal{Y}_{F(\mathfrak{X})}$ is 2-incompressible if (and only if) the division algebra D remains division over the field $F(\mathfrak{X})$. This is indeed the case:

Lemma 6. *The algebra $D_{F(X)}$ is division, that is,* $\operatorname{ind} D_{F(\mathfrak{X})} = \operatorname{ind} D$.

Proof. Of course, the statement can be checked using the index reduction formulas of [13] (in the inner case, that is, in the case when the discriminant of h is trivial) and of [14] (in the outer case). However, we prefer to do it in a different way which is more internal with respect to the methods of this paper.

Assume that $\operatorname{ind} D_{F(\mathfrak{X})} < \operatorname{ind} D$. Then $\mathcal{Y}(F(\mathfrak{X})) \neq \emptyset$. Since in the same time the variety $\mathfrak{X}_{F(\mathcal{Y})}$ has an odd degree closed point, it follows (by the main property of the upper motives established in [5, Corollary 2.15]) that the upper indecomposable motivic summand of $\mathcal{Y}$ is a motivic summand of $\mathfrak{X}$. This implies (because the variety $\mathcal{Y}$ is 2-incompressible) that the complete motivic decomposition of the variety $\mathfrak{X}_{F(\mathcal{Y})}$ contains the Tate summand $\mathbb{F}_2(\dim\mathcal{Y}) = \mathbb{F}_2(2^{2r-2})$. On the other hand, all the summands of the motivic decomposition of Lemma 4 (applied to the field $K = F(\mathcal{Y})$) are shifts of the motives of anisotropic varieties besides the following three: $\mathbb{F}_2$ (given by $i = j = 0$), $\mathbb{F}_2(\dim\mathfrak{X}) = \mathbb{F}_2\big(2^{r-1}(2^r-1) + 2^{2r}(v-2)\big)$ (given by $i = 2^r$ and $j = 0$), and $M(\mathcal{Y}_{F(\mathcal{Y})})(n)$ (given by $i = 2^{r-1}$ and $j = 0$) with n defined in (5). Here a variety is called anisotropic, if all its closed points are of even degree. The motive of an anisotropic variety does not contain Tate summands by [5, Lemma 2.21]. Taking into account the Krull-Schmidt principle of [3] (see also [10, §2]), we get a contradiction because $0 < 2^{2r-2} < n$ (the assumption $v \geq 3$ is used here). □

We have checked condition (2) of Proposition 3. It remains to check condition (1).

Lemma 7. *Let L/K be a finite odd degree field extension of a field K containing F. Let $\bar{L}$ be an algebraically closed field containing L. Then*

$$\operatorname{Im}\big(\operatorname{res}_{\bar{L}/L} : \operatorname{CH}(\mathfrak{X}_L) \to \operatorname{CH}(\mathfrak{X}_{\bar{L}})\big) = \operatorname{Im}\big(\operatorname{res}_{\bar{L}/K} : \operatorname{CH}(\mathfrak{X}_K) \to \operatorname{CH}(\mathfrak{X}_{\bar{L}})\big).$$

Proof. We write I_L and I_K for these images and we evidently have $I_K \subset I_L$.

Inside of $\bar{L}$, the variety $\mathfrak{X}_K$ has a finite 2-primary splitting field K'/K.

If the discriminant $\operatorname{disc} h_K$ is trivial, then $[L:K]\cdot I_L \subset I_K$. Since moreover $[K':K]\cdot\operatorname{CH}(\mathfrak{X}_{\bar{L}}) \subset I_K$ and $[K':K]$ is coprime with $[L:K]$, it follows that $I_L \subset I_K$.

If $\operatorname{disc} h_K$ is non-trivial, then also $\operatorname{disc} h_L \neq 1$ and the group $G := \operatorname{Aut}(\bar{L}/K)$, acting on $\operatorname{CH}(\mathfrak{X}_{\bar{L}})$, acts trivially on I_L. Therefore we still have $[L:K]\cdot I_L \subset I_K$. Besides, $[K':K]\cdot\operatorname{CH}(\mathfrak{X}_{\bar{L}})^G \subset I_K$, and it follows that $I_L \subset I_K$. □

We write $M_{\mathcal{Y}}$ for the upper indecomposable motivic summand of $\mathcal{Y}$.

Corollary 8. *$M_{\mathcal{Y}}(n)$ is a motivic summand of $\mathfrak{X}$.*

Proof. As planned, we apply Proposition 3 to $p = 2$, $Y = \mathcal{Y}$, $X = \mathfrak{X}$, and $E = F(\mathfrak{X})$. There exists a finite odd degree extension $L/F(Y)$ such that $X(L) \neq \emptyset$. The field extension $L(X)/L$ is purely transcendental. Since $E(Y) \subset L(X)$, condition (1) is satisfied (with $\overline{E(Y)}$ being an algebraically closed field containing $L(X)$) by Lemma 7.

As already pointed out, condition (2) is satisfied by Lemma 6, and condition (3) is satisfied by Lemma 4. □

We need the following enhancement of Corollary 8:

Proposition 9. *There exists a* symmetric *projector $\pi_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the motive $(\mathfrak{X}, \pi_{\mathfrak{X}})$ is isomorphic to $M_{\mathcal{Y}}(n)$.*

Remark 10. In fact, for *any* projector $\pi_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the motive $(\mathfrak{X}, \pi_{\mathfrak{X}})$ is isomorphic to $M_{\mathcal{Y}}(n)$, the motive $(\mathfrak{X}, \pi^t_{\mathfrak{X}})$ given by the transposition $\pi^t_{\mathfrak{X}}$ of $\pi_{\mathfrak{X}}$ is *isomorphic* to $(\mathfrak{X}, \pi_{\mathfrak{X}})$. However, $\pi_{\mathfrak{X}}$ is not necessarily symmetric, that is, the equality $\pi^t_{\mathfrak{X}} = \pi_{\mathfrak{X}}$ may fail.

Proof of Proposition 9. Let us start by checking that the motive $M_{\mathcal{Y}}$ can be given by a symmetric projector $\pi_{\mathcal{Y}}$ on $\mathcal{Y}$. The proof we give is valid for any projective homogeneous 2-incompressible variety in place of the variety $\mathcal{Y}$. Let π be a projector on $\mathcal{Y}$ such that

$(\mathcal{Y},\pi)\simeq M_{\mathcal{Y}}$. Since our Chow groups are with finite coefficients, there exists an integer $l\geq 1$ such that $\pi_{\mathcal{Y}}:=(\pi^t\circ\pi)^{\circ l}$ is a (symmetric) projector, where π^t is the transposition of π. Since the variety $\mathcal{Y}$ is 2-incompressible, $\operatorname{mult}\pi^t=1$ by [5, §2], where mult is the multiplicity (sometimes also called *degree* in the literature) of a correspondence. It follows that $\operatorname{mult}\pi_{\mathcal{Y}}=1$ and therefore the motive $(\mathcal{Y},\pi_{\mathcal{Y}})$ is non-zero. In the same time, it is a direct summand of the indecomposable motive $(\mathcal{Y},\pi)$ (the morphisms to and from $(\mathcal{Y},\pi)$ having the identical composition are given, for instance, by $\pi\circ\pi_{\mathcal{Y}}$ and simply $\pi_{\mathcal{Y}}$). Therefore $M_{\mathcal{Y}}\simeq(\mathcal{Y},\pi_{\mathcal{Y}})$ by indecomposability of $(\mathcal{Y},\pi)$.

Now let $\alpha:(\mathcal{Y},\pi_{\mathcal{Y}})(n)\to M(\mathfrak{X})$ and $\beta:M(\mathfrak{X})\to(\mathcal{Y},\pi_{\mathcal{Y}})(n)$ be morphisms with $\beta\circ\alpha=\pi_{\mathcal{Y}}=\operatorname{id}_{(\mathcal{Y},\pi_{\mathcal{Y}})}$ (existing because $(\mathcal{Y},\pi_{\mathcal{Y}})(n)$ is a motivic summand of $\mathfrak{X}$). Note that α^t is a morphism

$$M(\mathfrak{X})\to(\mathcal{Y},\pi^t_{\mathcal{Y}})(\dim\mathfrak{X}-\dim\mathcal{Y}-n)=(\mathcal{Y},\pi_{\mathcal{Y}})(n)$$

because $\pi^t_{\mathcal{Y}}=\pi_{\mathcal{Y}}$ and $2n=\dim\mathfrak{X}-\dim\mathcal{Y}$. There exists an integer $l\geq 1$ such that $(\alpha^t\circ\alpha)^{\circ l}$ is a projector. If $\operatorname{mult}(\alpha^t\circ\alpha)\neq 0$, then $(\alpha^t\circ\alpha)^{\circ l}=\pi_{\mathcal{Y}}$. Therefore $(\alpha\circ\alpha^t)^{\circ l}$ is a (symmetric) projector on $\mathfrak{X}$ and $\alpha:(\mathcal{Y},\pi_{\mathcal{Y}})(n)\to\big(\mathfrak{X},(\alpha\circ\alpha^t)^{\circ l}\big)$ is an isomorphism of motives, so that we are done in this case.

Similarly, if $\operatorname{mult}(\beta\circ\beta^t)\neq 0$, then $\beta^t:(\mathcal{Y},\pi_{\mathcal{Y}})(n)\to\big(\mathfrak{X},(\beta^t\circ\beta)^{\circ l}\big)$ for some (other) l is an isomorphism, and we are done in this case also.

In the remaining case we have $\operatorname{mult}(\alpha^t\circ\alpha)=0=\operatorname{mult}(\beta\circ\beta^t)$. Let $\mathbf{pt}\in\operatorname{Ch}_0(\mathcal{Y}_{F(\mathcal{Y})})$ be the class of a rational point. The compositions $\alpha\circ([\mathcal{Y}_{F(\mathcal{Y})}]\times\mathbf{pt})\circ\beta$ and $\beta^t\circ([\mathcal{Y}_{F(\mathcal{Y})}]\times\mathbf{pt})\circ\alpha^t$ are orthogonal projectors on $\mathfrak{X}_{F(\mathcal{Y})}$, and each of two corresponding motives is isomorphic to $\mathbb{F}_2(n)$. It follows that the complete motivic decomposition of $\mathfrak{X}_{F(\mathcal{Y})}$ contains two exemplars of $\mathbb{F}_2(n)$. However, as shown in the end of the proof of Lemma 6, the complete motivic decomposition of $\mathfrak{X}_{F(\mathcal{Y})}$ contains only one exemplar of $\mathbb{F}_2(n)$ (because the motive of $\mathcal{Y}_{F(\mathcal{Y})}$ contains only one exemplar of $\mathbb{F}_2$). □

From now on we are assuming that all closed points on the variety $\mathfrak{X}$ have even degrees. Then all closed points on the product $\mathfrak{X}\times\mathfrak{X}$ also have even degrees. Therefore the homomorphism $\deg/2:\operatorname{Ch}_0(\mathfrak{X}\times\mathfrak{X})\to\mathbb{F}_2$ is defined (as in [9, §5]).

Corollary 11. *Let $\pi_{\mathfrak{X}}$ be as in Proposition 9. Then $\pi^2_{\mathfrak{X}}$ is a 0-cycle class on $\mathfrak{X}\times\mathfrak{X}$ for which we have $(\deg/2)(\pi^2_{\mathfrak{X}})=1\in\mathbb{F}_2$.*

Proof. For any symmetric projector π on $\mathfrak{X}$, we have $(\deg/2)(\pi^2)=\operatorname{rk}(\mathfrak{X},\pi)/2\pmod 2$, where rk is the rank of the motive (the number of the Tate summands in the complete decomposition over a splitting field). Indeed, taking a complete motivic decomposition of $\bar{\mathfrak{X}}$ (here and below $\bar{\mathfrak{X}}$ is $\mathfrak{X}$ over a splitting field of $\mathfrak{X}$) which is a refinement of the decomposition $M(\mathfrak{X})\simeq(\mathfrak{X},\pi)\oplus(\mathfrak{X},\Delta_{\mathfrak{X}}-\pi)$, we get a homogeneous basis B of $\operatorname{Ch}(\bar{\mathfrak{X}})$ such that $\bar\pi=\sum_{b\in B_\pi}b\times b^*$, where B_π is a subset of B and $\{b^*\}_{b\in B}$ is the dual basis. Note that $\operatorname{rk}(\mathfrak{X},\pi)=\#B_\pi$. For every $b\in B$, let us fix an integral representative $\mathfrak{b}\in\operatorname{CH}(\bar{\mathfrak{X}})$ of b and an integral representative $\mathfrak{b}^*\in\operatorname{CH}(\bar{\mathfrak{X}})$ of b^*. Then the sum $\sum_{b\in B_\pi}\mathfrak{b}\times\mathfrak{b}^*$, as well as the sum $\sum_{b\in B_\pi}\mathfrak{b}^*\times\mathfrak{b}$, is an integral representative of $\bar\pi$, and for the integral degree homomorphism $\deg : \mathrm{CH}_0(\bar{\mathfrak{X}}) \to \mathbb{Z}$ we have:

$$\deg\left(\Big(\sum_{b\in B_\pi} \mathfrak{b}\times\mathfrak{b}^*\Big)\Big(\sum_{b\in B_\pi} \mathfrak{b}^*\times\mathfrak{b}\Big)\right) \equiv \#B_\pi \pmod 4.$$

By definition of deg/2, the element $(\deg/2)(\pi^2) \in \mathbb{F}_2$ is represented by the half of the degree of an arbitrary integral representative of π^2 (over F !). So, let $\Pi \in \mathrm{CH}(\mathfrak{X}\times\mathfrak{X})$ be an integral representative of π (Π does not need to be a projector). Then $\Pi\cdot\Pi^t$ is a representative of π^2, so that we have $(\deg/2)(\pi^2) = (\deg(\Pi\cdot\Pi^t))/2 \pmod 2$. On the other hand, there exists an element $\alpha \in \mathrm{CH}(\bar{\mathfrak{X}}\times\bar{\mathfrak{X}})$ such that

$$\sum_{b\in B_\pi} \mathfrak{b}\times\mathfrak{b}^* = \bar{\Pi} + 2\alpha,$$

and we get the following congruences modulo 4:

$$\#B_\pi \equiv \deg\big((\bar{\Pi}+2\alpha)\cdot(\bar{\Pi}^t + 2\alpha^t)\big) \equiv \deg(\Pi\cdot\Pi^t)$$

because $\deg(\alpha\cdot\bar{\Pi}^t) = \deg(\bar{\Pi}\cdot\alpha^t)$.

We have shown that $(\deg/2)(\pi^2_{\mathfrak{X}}) = \mathrm{rk}(\mathfrak{X},\pi_{\mathfrak{X}})/2 \pmod 2$. Now the rank of the motive $(\mathfrak{X},\pi_{\mathfrak{X}})$ coincides with the rank of the motive $(\mathcal{Y},\pi_{\mathcal{Y}}) \simeq M_{\mathcal{Y}}$ which is shown to be 2 modulo 4 in [5, Theorem 4.1]. □

The following Proposition is a general statement on the action of the cohomological Steenrod operation $\mathrm{Sq}^\bullet$ (see [4, Chapter XI]) on the Chow groups modulo 2 of a split orthogonal grassmannian G (which we shall apply to $G = \bar{\mathfrak{X}}$, where, as above, $\bar{\mathfrak{X}}$ is $\mathfrak{X}$ over a splitting field of $\mathfrak{X}$):

Proposition 12. *Let d be an integer ≥ 1, m an integer satisfying $0 \leq m \leq d-1$, G the variety of the totally isotropic $(m+1)$-dimensional subspaces of a hyperbolic $(2d+2)$-dimensional quadratic form q (over a field of characteristic $\neq 2$). Then for any integer $i > (d-m)(m+1)$ we have $\mathrm{Sq}^i\,\mathrm{Ch}_i(G) = 0$.*

Proof. Let Q be the projective quadric of q, Φ the variety of flags consisting of a line contained in a totally isotropic $(m+1)$-dimensional subspace of q, and $pr_G : \Phi \to G$, $pr_Q : \Phi \to Q$ the projections. We write $h \in \mathrm{CH}^1(Q)$ for the (integral) hyperplane section class and we write $l_i \in \mathrm{CH}_i(Q)$, where $i = 0,\dots,d$, for the (integral) class of an i-dimensional linear subspace in Q (for $i = d$ we *choose* one of the two classes, call it l_d, and write l'_d for the other). As in [17, §2], we define the integral classes

$$W_i \in \mathrm{CH}^i(G) \quad \text{for} \quad i = 1,\dots,d-m \quad \text{by} \quad W_i := (pr_G)_*\,pr_Q^*(h^{m+i})$$

and we define the integral classes

$$Z_i \in \mathrm{CH}^i(G) \quad \text{for} \quad i = d-m,\dots,2d-m \quad \text{by} \quad Z_i = (pr_G)_*\,pr_Q^*(l_{2d-m-i}).$$

The elements $W_1,\dots,W_{d-m}, Z_{d-m},\dots,Z_{2d-m}$ generate the ring $\mathrm{CH}(G)$ by [17, Proposition 2.9]. We call them *the generators of* $\mathrm{CH}(G)$. We refer to $W_1,\dots,W_{d-m}$ as W-generators, and we refer to $Z_{d-m},\dots,Z_{2d-m}$ as Z-generators.

Note that $Z_{d-m} = (pr_G)_*\,pr_Q^*(l_d)$. We also set $Z'_{d-m} = (pr_G)_*\,pr_Q^*(l'_d)$. Since $l_d + l'_d = h^d$, we have $Z_{d-m} + Z'_{d-m} = W_{d-m}$.

Note that any element of $\mathrm{O}_{2d+2}(F) \setminus \mathrm{SO}_{2d+2}(F)$ gives an automorphism of G such that the corresponding automorphism of the ring $\mathrm{CH}(G)$ acts trivially on all the generators but Z_{d-m} which is interchanged with Z'_{d-m}.

For any $i \geq 0$, let $c_i \in \mathrm{CH}^i(G)$ be the ith Chern class of the quotient bundle on G. According to [17, Proposition 2.1], $c_i = W_i$ for any i for which W_i is defined, and $c_i = 2Z_i$ for all i satisfying $d - m < i \leq 2d - m$.

A computation similar to [4, (86.15)] (see also [1, (44) and (45) in Theorem 3.2]) shows that for any $i = d-m, \dots, 2d-m$, the generators of $\mathrm{CH}(G)$ satisfy the following relation

$$Z_i^2 - Z_i c_i + Z_{i+1} c_{i-1} - Z_{i+2} c_{i-2} + \dots .$$

(This is not and we do not need a complete list of relations.)

We denote the images of the generators of $\mathrm{CH}(G)$ under the epimorphism $\mathrm{CH}(G) \to \mathrm{Ch}(G)$ to the modulo 2 Chow group using the small letters w and z (with the same indices), and call them *the generators of* $\mathrm{Ch}(G)$. We say that an element of $\mathrm{Ch}(G)$ is of level l, if it can be written as a sum of products of generators such that the number of the z-factors in each product is at most l (so, any level l element is also of level $l+1$). A z-generator raised to power k is counted k times here, that is, we are looking at the total degree assigning weight 1 to each z-generator (and weight 0 to each w-generator). For instance, the monomial z_d^2 is of level 2 (but because of the relation $z_d^2 = z_d c_d - z_{d+1} c_{d-1} + \dots$, the element z_d^2 is also of level 1).

By [17, Proposition 2.8], the value of the total cohomological Steenrod operation $\mathrm{Sq}^\bullet : \mathrm{Ch}(G) \to \mathrm{Ch}(G)$ on any single z-generator is of level 1. Similar computation shows that the value of $\mathrm{Sq}^\bullet$ on any w-generator is of level 0. Since $\mathrm{Sq}^\bullet$ is a ring homomorphism, it follows that for any $l \geq 0$, the image under $\mathrm{Sq}^\bullet$ of a level l element is also of level l.

The above relations on the generators show that any element of $\mathrm{Ch}(G)$ is a polynomial of the generators such that the exponent of any z-generator in any monomial of the polynomial is at most 1. Since the dimension of such (biggest-dimensional level $m+1$) monomial $z_{d-m} \dots z_d$ is equal to

$$\dim G - \Big((d-m) + \dots + d\Big) = \dim G - \Big((d-m)(m+1) + m(m+1)/2\Big) = (d-m)(m+1),$$

any homogeneous element $\alpha \in \mathrm{Ch}(G)$ of dimension $i > (d-m)(m+1)$ is of level m. Therefore $\mathrm{Sq}^i(\alpha) \in \mathrm{Ch}_0(G)$ if also of level m.

We finish by showing that any level m element in $\mathrm{Ch}_0(G)$ is 0. For this we turn back to the integral Chow group $\mathrm{CH}(G)$ and show that any odd degree element $\beta \in \mathrm{CH}_0(G)$ is not of level m. The integral version of the notion of level used here is defined in the same way as the above modulo 2 version (using the generators of $\mathrm{CH}(G)$ instead of the generators of $\mathrm{Ch}(G)$).

Since the description of the ring $\mathrm{CH}(G)$ does not depend on the base field F, we may assume that $G = G'_F$, where G' is the grassmannian of a *generic* quadratic form defined over a subfield $F' \subset F$. We say that an element of $\mathrm{CH}(G)$ is *rational*, if it is in the image of the change of field homomorphism $\mathrm{res}_{F/F'} : \mathrm{CH}(G') \to \mathrm{CH}(G)$.

For any $i \geq 0$, the element c_i is rational. Therefore, for any $l \geq 0$, the 2^l-multiple of any level l element in $\mathrm{CH}_0(G)$ is rational. Indeed, this statement is a consequence of the formulas $W_i = c_i$ for any i such that W_i is defined, and the formulas $Z_i + \sigma Z_i = c_i$ for any i such that Z_i is defined, where σ is the ring automorphism of $\mathrm{CH}(G)$ given by an element of $\mathrm{O}_{2d+2}(F) \setminus \mathrm{SO}_{2d+2}(F)$ (note that σ is the identity on $\mathrm{CH}_0(G)$). The degree of any closed point on G' is divisible by 2^{m+1}. Therefore the element $2^m\beta$ is not rational, and it follows that β is not of level m. □

Remark 13. It might look strange that we are using the trick with the generic quadratic form proving a statement about a split quadratic form. Indeed, the ring $\mathrm{Ch}(G)$ is completely described in terms of generators and relations (moreover, the Pieri rule [1, Theorem 3.1] is obtained) and the action of the Steenrod operations is computed in terms of the generators. However a direct proof based only on this information seems to be very complicated.

Remark 14. The statement of Proposition 12 also holds in the case of $m = d$, that is, in the case of a split *maximal* orthogonal grassmannian. The proof is even simpler and also the given proof of Proposition 12 can be easily modified to cover this case. Using this, one can cover the case of $v = 2$, excluded in the very beginning, and obtain this way a new proof for the hyperbolicity result of [9].

Corollary 15. *For any integer $i \geq n$ (where n is as in (5)) we have* $\mathrm{Sq}^i \mathrm{Ch}_i(\bar{\mathfrak{X}}) = 0$.

Proof. We apply Proposition 12 to $G = \bar{\mathfrak{X}}$. We have $d = 2^{r-1}v - 1$ and $m = 2^r - 1 \leq d - 1$ (because $v \geq 3$). Therefore $(d-m)(m+1) = 2^{2r-1}(v-2)$ and

$$n := 2^{r-2}(2^{r-1} - 1) + 2^{2r-1}(v-2) > (d-m)(m+1)$$

(because $r \geq 2$). □

Example 16. Corollary 15 fails for $r = 1$. For instance, if $v = 6$ (and therefore $d = 5$), we have: $n = 8$, $z_4 z_5 \in \mathrm{Ch}^9(\bar{\mathfrak{X}}) = \mathrm{Ch}_8(\bar{\mathfrak{X}})$, and $\mathrm{Sq}^8(z_4 z_5) \neq 0$. Therefore, an additional argument is needed to prove the quaternion case (fortunately already proved in [15]) by the method of this paper.

Proof of Theorem 1. We are going to show that $(\deg/2)(\pi_{\mathfrak{X}}^2) = 0$. This will contradict to Corollary 11 thus proving Theorem 1.

Since $\pi_{\mathfrak{X}}^2 = \mathrm{Sq}^{\dim \mathfrak{X}} \pi_{\mathfrak{X}}$, we have $(\deg/2)(\pi_{\mathfrak{X}}^2) = (\deg/2)(\mathrm{Sq}^\bullet \pi_{\mathfrak{X}})$. Let $\alpha : M(\mathcal{Y})(n) \to M(\mathfrak{X})$ and $\beta : M(\mathfrak{X}) \to M(\mathcal{Y})(n)$ be morphisms of motives with $\alpha \circ \beta = \pi_{\mathfrak{X}}$ and let

$$pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}} : \mathfrak{X} \times \mathcal{Y} \times \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$$

be the projection. Since $\alpha \circ \beta = (pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}})_*\big(([\mathfrak{X}] \times \alpha) \cdot (\beta \times [\mathfrak{X}])\big)$, we have

$$\mathrm{Sq}^\bullet \pi_{\mathfrak{X}} = (pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}})_*\Big(\big([\mathfrak{X}] \times \mathrm{Sq}^\bullet(\alpha)\big) \cdot \big(\mathrm{Sq}^\bullet(\beta) \times [\mathfrak{X}]\big) \cdot \big([\mathfrak{X}] \times c_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)\Big),$$

where $T_{\mathcal{Y}}$ is the tangent bundle of $\mathcal{Y}$ and $c_\bullet$ is the total Chern class modulo 2. Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral representatives of $\mathrm{Sq}^\bullet(\alpha)$ and $\mathrm{Sq}^\bullet(\beta)$. It suffices to show that the degree of the integral cycle class

$$\mathfrak{d} := (pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathfrak{X}\mathfrak{X}})_*\Big(\big([\mathfrak{X}] \times \mathfrak{a}\big) \cdot \big(\mathfrak{b} \times [\mathfrak{X}]\big) \cdot \big([\mathfrak{X}] \times \mathfrak{c}_\bullet(-T_{\mathcal{Y}}) \times [\mathfrak{X}]\big)\Big)$$

is divisible by 4, where $\mathfrak{c}_\bullet$ stands for the *integral* total Chern class.

We have

$$(pr^{\mathfrak{X}\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}\mathfrak{X}})_*\Big(\big([\mathfrak{X}]\times\mathfrak{a}\big)\cdot\big(\mathfrak{b}\times[\mathfrak{X}]\big)\cdot\big([\mathfrak{X}]\times\mathfrak{c}_\bullet(-T_\mathcal{Y})\times[\mathfrak{X}]\big)\Big)= \mathfrak{a}\cdot\big((pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b})\times[\mathfrak{X}]\big)\cdot\big(\mathfrak{c}_\bullet(-T_\mathcal{Y})\times[\mathfrak{X}]\big)$$

and

$$(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*\Big(\mathfrak{a}\cdot\big((pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b})\times[\mathfrak{X}]\big)\cdot\big(\mathfrak{c}_\bullet(-T_\mathcal{Y})\times[\mathfrak{X}]\big)\Big)= (pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\mathfrak{a})\cdot(pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b})\cdot\mathfrak{c}_\bullet(-T_\mathcal{Y}).$$

Therefore

$$\deg(\mathfrak{d})=\deg\Big((pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\mathfrak{a})\cdot(pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\mathfrak{b})\cdot\mathfrak{c}_\bullet(-T_\mathcal{Y})\Big)$$

and it suffices to show that the cycle classes $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\bar{\mathfrak{a}})$ and $(pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\bar{\mathfrak{b}})$ are divisible by 2.

The (modulo 2) cycle class $\bar\alpha$ is a sum of $a'\times a$ with some $a'\in \mathrm{Ch}(\bar{\mathcal{Y}})$ and some homogeneous $a\in\mathrm{Ch}(\bar{\mathfrak{X}})$ of dimension $\geq n$. By Corollary 15, $\deg\mathrm{Sq}^\bullet(a)=0\in\mathbb{F}_2$ for such a. Therefore $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*\big(\mathrm{Sq}^\bullet(\bar\alpha)\big)=0$ and the integral cycle class $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*(\bar{\mathfrak{a}})$, which represents the modulo 2 cycle class $(pr^{\mathcal{Y}\mathfrak{X}}_{\mathcal{Y}})_*\big(\mathrm{Sq}^\bullet(\bar\alpha)\big)$, is divisible by 2. Similarly, the cycle class $\bar\beta$ is a sum of $b\times b'$ with some $b'\in\mathrm{Ch}(\bar{\mathcal{Y}})$ and some homogeneous $b\in\mathrm{Ch}(\bar{\mathfrak{X}})$ of dimension $\geq n$, and it follows that the cycle class $(pr^{\mathfrak{X}\mathcal{Y}}_{\mathcal{Y}})_*(\bar{\mathfrak{b}})$ is also divisible by 2. ☐

Appendix: Isotropy of symplectic and unitary involutions

by Jean-Pierre Tignol[1]

Using a technique from [16], we derive from Theorem 1 the following analogues for symplectic and unitary involutions:

Theorem A. Let A be a central simple algebra over a field F of characteristic different from 2 and let σ be a symplectic involution on A. The following conditions are equivalent:

(1) σ becomes isotropic over every field extension E of F such that $\operatorname{ind} A_E=2$;
(2) σ becomes isotropic over some odd-degree field extension of F.

If A is split, then (1) is void and (2) always holds since symplectic involutions on split algebras are adjoint to alternating forms.

Theorem B. Let B be a central simple algebra of exponent 2 over a field K of characteristic different from 2, let τ be a unitary involution on B, and let $F\subset K$ be the subfield of K fixed under τ. The following conditions are equivalent:

(1) τ becomes isotropic over every field extension E of F such that $B\otimes_F E$ is split;
(2) τ becomes isotropic over some odd-degree field extension of F.

In (1) it suffices to consider field extensions E that are linearly disjoint from K, since τ becomes isotropic over every field extension containing K.

In each case, (2) ⇒ (1) readily follows from Springer's theorem on the anisotropy of quadratic forms under odd-degree extensions: symplectic involutions on central simple

[1]This appendix was written while the author was a Senior Fellow of the Zukunftskolleg, Universität Konstanz, whose hospitality is gratefully acknowledged.

algebras of index 2 are adjoint to hermitian forms h over quaternion algebras, which are isotropic if and only if the associated quadratic form $h(x,x)$ is isotropic. Likewise, unitary involutions on split central simple algebras are adjoint to hermitian forms over quadratic extensions, and the same observation applies. Therefore, we just prove (1) $\Rightarrow$ (2).

Proof of Theorem A. Since (2) holds when A is split, we may assume A is not split. Adjoining to F two Laurent series indeterminates x, y, we let $\widehat{F} = F((x))((y))$ and consider the quaternion algebra $(x,y)_{\widehat{F}}$ with its conjugation involution γ. Let $\widetilde{A} = A \otimes_F (x,y)_{\widehat{F}}$ with the involution $\widetilde{\sigma} = \sigma \otimes \gamma$, which is orthogonal since σ is symplectic. Let E be the function field of the Severi–Brauer variety of $\widetilde{A}$. The Brauer group kernel of the scalar extension map from $\widehat{F}$ to E is generated by $\widetilde{A}$, hence A is not split over E; but A_E is Brauer-equivalent to $(x,y)_E$, hence $\operatorname{ind} A_E = 2$. Assuming (1), we have σ isotropic over E, hence $\widetilde{\sigma}$ also becomes isotropic over E and therefore, by Theorem 1, there is an odd-degree extension L of $\widehat{F}$ over which $\widetilde{\sigma}$ becomes isotropic. The x,y-adic valuation on $\widehat{F}$ (with value group $\mathbb{Z}^2$ ordered lexicographically from right to left) is Henselian, hence it extends to a valuation v on L. To prove (2), we show that σ becomes isotropic over the residue field $\overline{L}$, which is an odd-degree extension of F.

Let $\Gamma = v(L^\times) \subset \mathbb{Q}^2$. Since L is an odd-degree extension of $\widehat{F}$, the quaternion algebra $(x,y)_{\widehat{F}}$ remains a division algebra over L, hence v extends to a valuation on $(x,y)_L$ defined by

$$v(q) = \tfrac{1}{2} v\bigl(\operatorname{Nrd}(q)\bigr) \in \tfrac{1}{2}\Gamma \cup \{\infty\} \qquad \text{for } q \in (x,y)_L,$$

where Nrd is the reduced norm. Since $(\Gamma{:}\mathbb{Z}^2)$ is odd, we have $v(x)$, $v(y) \notin 2\Gamma$, and the residue division algebra $\overline{(x,y)_L}$ is therefore easily checked to be $\overline{L}$. We further extend v to a map

$$w\colon \widetilde{A}_L \to \tfrac{1}{2}\Gamma \cup \{\infty\}$$

as follows: let $(a_i)_{i=1}^n$ be a base of A, so every element in $\widetilde{A}_L$ has a unique representation of the form $\sum_i a_i \otimes q_i$ for some $q_i \in (x,y)_L$; we set

$$w(\textstyle\sum_i a_i \otimes q_i) = \min\{v(q_i) \mid 1 \le i \le n\}.$$

The map w is not a valuation[2] ($\widetilde{A}_L$ is not a division algebra) but it satisfies $w(a+b) \ge \min\{w(a), w(b)\}$ and $w(ab) \ge w(a) + w(b)$ for a, $b \in \widetilde{A}_L$. (It is also easy to see that it does not depend on the choice of the base $(a_i)_{i=1}^n$.) We may therefore consider a residue algebra $(\widetilde{A}_L)_0$, which consists of residue classes of elements a such that $w(a) \ge 0$ modulo elements a such that $w(a) > 0$. Since $\overline{(x,y)_L} = \overline{L}$, we have $(\widetilde{A}_L)_0 = A_{\overline{L}}$. Moreover, $w\bigl(\widetilde{\sigma}(a)\bigr) = w(a)$ for all $a \in \widetilde{A}_L$, and if $w(a) = 0$ we have

$$\overline{\widetilde{\sigma}(a)} = \sigma_{\overline{L}}(\overline{a}).$$

Now, since $\widetilde{\sigma}_L$ is isotropic we may find a nonzero $a \in \widetilde{A}_L$ such that $\widetilde{\sigma}_L(a) \cdot a = 0$. Multiplying a on the right by a suitable quaternion in $(x,y)_L$, we may assume $w(a) = 0$,

[2]The map w is a *gauge* in the terminology of [J.-P. Tignol, A.R. Wadsworth, Value functions and associated graded rings for semisimple algebras, *Trans. Amer. Math. Soc.* **362** (2010) 687–726].

hence $\overline{a} \in A_{\overline{L}}$ is defined and nonzero. We have

$$\sigma_{\overline{L}}(\overline{a}) \cdot \overline{a} = \overline{\widetilde{\sigma}_L(a) \cdot a} = 0,$$

hence $\sigma_{\overline{L}}$ is isotropic, as claimed. □

Proof of Theorem B. As in [16, A.2], we choose an orthogonal involution ν on B and set $g = \nu \circ \tau$, which is an outer automorphism of B. Consider the algebra of twisted Laurent series $\widetilde{B} = B((\xi; g))$ in one indeterminate ξ. It carries an orthogonal involution $\widetilde{\tau}$ extending τ such that $\widetilde{\tau}(\xi) = \xi$. Let $u \in B^\times$ be such that $\nu(u) = \tau(u) = u$ and $g^2(b) = ubu^{-1}$ for all $b \in B$. The center of $\widetilde{B}$ is $F((x))$ where $x = u^{-1}\xi^2$. Let E be the function field of the Severi–Brauer variety of $\widetilde{B}$. Extension of scalars to E splits B, since $B \otimes_F F((x))$ is the centralizer of K in $\widetilde{B}$. Therefore, assuming (1), τ becomes isotropic over E, hence $\widetilde{\tau}$ also becomes isotropic over E. By Theorem 1, it follows that there is an odd-degree extension L of $F((x))$ over which $\widetilde{\tau}$ becomes isotropic. The x-adic valuation on $F((x))$ (with value group $\mathbb{Z}$) extends to a valuation v on L with value group $\Gamma \subset \mathbb{Q}$, and further to a map

$$w\colon \widetilde{B}_L \to \tfrac{1}{2}\Gamma \cup \{\infty\}$$

defined as follows: let $(b_i)_{i=1}^n$ be an F-base of B, so every element in $\widetilde{B}_L$ has a unique representation of the form $\sum_i b_i \otimes \ell_i + (\sum_j b_j \otimes \ell'_j)\xi$ for some ℓ_i, $\ell'_j \in L$; set

$$w(\textstyle\sum_i b_i \otimes \ell_i + (\sum_j b_j \otimes \ell'_j)\xi) = \min\{v(\ell_i), v(\ell'_j) + \frac{1}{2} \mid 1 \leq i, j \leq n\}.$$

The corresponding residue ring $(\widetilde{B}_L)_0$ is $B \otimes_F \overline{L}$, and for $b \in \widetilde{B}_L$ with $w(b) = 0$ we have $w\big(\widetilde{\tau}_L(b)\big) = 0$ and $\overline{\widetilde{\tau}_L(b)} = \tau_{\overline{L}}(\overline{b})$. Since $\widetilde{\tau}_L$ is isotropic, we may find a nonzero $b \in \widetilde{B}_L$ such that $\widetilde{\tau}_L(b) \cdot b = 0$. Multiplying b on the right by a suitable power of ξ, we may assume $w(b) = 0$, hence $\overline{b} \in B \otimes_F \overline{L}$ is defined and nonzero. We have

$$\tau_{\overline{L}}(\overline{b}) \cdot \overline{b} = \overline{\widetilde{\tau}_L(b) \cdot b} = 0,$$

hence $\tau_{\overline{L}}$ is isotropic. Note that $\overline{L}$ is an odd-degree extension of F since L is an odd-degree extension of $F((x))$. □

In [15, §4], Parimala–Sridharan–Suresh give an example of a central simple algebra B with an anisotropic unitary involution that becomes isotropic over an odd-degree extension L of the field of symmetric central elements. The algebra B in this example has odd exponent (and is split by L).

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