MOTIVES AND CHOW GROUPS OF QUADRICS WITH APPLICATION TO THE U-INVARIANT (AFTER OLEG IZHBOLDIN)

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These are the notes of my lectures delivered during the mini-course "Méthodes géometriques en théorie des formes quadratiques" at the Université d'Artois, Lens, 26–28 June 2000. Part I is based on [11], Part II on [10].

In this text we consider only non-degenerate quadratic forms over fields of characteristic different from 2.

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Part 1. Virtual Pfister neighbors and first Witt index

1. Introduction

Let ϕ be a quadratic form over a field F. The *splitting pattern* of ϕ (cf. [6]) is defined as the set of integers $\{i_{\mathrm{W}}(\phi_E)\}$ where E runs over all field extensions of F and $i_{\mathrm{W}}(\phi_E)$ stays for the Witt index of the quadratic form ϕ_E.

In [5], the list of all splitting patterns of anisotropic quadratic forms of dimensions up to 10 is given. For example, in dimension 9 the only possible splitting patterns are $\{0, 1, 4\}$ and $\{0, 1, 2, 3, 4\}$ (moreover, a 9-dimensional form ϕ has the splitting pattern $\{0, 1, 4\}$ if and only if its even Clifford algebra $C_0(\phi)$ is split).

One difficulty appears in dimension 11 and remains unsolved in [5]: it is not clear whether the set $\{0, 2, 3, 4, 5\}$ is the splitting pattern of an 11-dimensional form. In characteristic 0 these question was answered by negative in [28] (see also [12]) where it was shown that the difference $i_1 - i_0$ can be strictly bigger as every other difference $i_2 - i_1$, $i_3 - i_2$,..., $i_n - i_{n-1}$ for a splitting pattern $\{i_0, i_1, \ldots, i_n\}$ of a form ϕ only in the case where $\dim \phi - i_0$ is a power of 2. The proof made use of methods developed in [30] and in particular of the existence and certain properties of Voevodsky's cohomological operations in the motivic cohomology.

In contrast to that, the proof of the following theorem, also answering the question raised, works in any characteristic and makes use of much simpler and more classical tools. Recall that the first Witt index of an anisotropic quadratic form ϕ is defined as the smallest number in the splitting pattern of ϕ:

$$i_1(\phi) = \min\{i_{\mathrm{W}}(\phi_E) > 0 \mid E/F \text{ a field extension}\}\ .$$

Theorem 1.1 (Izhboldin, cf. [11, Theorem 8.10])**.** *Let ϕ be an anisotropic quadratic form of dimension $2^n + 3$ with some n. Then $i_1(\phi) \neq 2$.*

Since $11 = 2^3 + 3$, this implies

Corollary 1.2. *The splitting pattern $\{0, 2, 3, 4, 5\}$ is not possible for an 11-dimensional quadratic form.* □

Note that Theorem 1.1 also provides a restriction on the first Witt index for quadratic forms of dimensions different from $2^n + 3$:

Corollary 1.3. *Let ϕ be an anisotropic quadratic form of dimension $2^n + k$ with $3 \leq k \leq 2^n$. Then $i_1(\phi) \neq k - 1$ (we put $k \leq 2^n$ in order to have a non-trivial statement).*

Proof. Assume that $i_1(\phi) = k - 1$ and let ψ be a $(2^n + 3)$-dimensional subform of ϕ. The forms $\phi_{F(\psi)}$ and $\psi_{F(\phi)}$ are isotropic: [1] the latter is isotropic as a $k-3$-codimensional subform in the form $\phi_{F(\phi)}$ of a Witt index $> k - 3$. Therefore, by a theorem of A. Vishik [29, Corollary A. 18] (see also [18, Theorem 8.1])

$$\dim \phi - i_1(\phi) = \dim \psi - i_1(\psi)\ .$$

It follows that $i_1(\psi) = 2$ which is in contradiction with Theorem 1.1. □

Besides, we would like to remark that Theorem 1.1 proves a particular case of the following general conjecture, due to D. Hoffmann, on the possible values of the first Witt index of quadratic forms:

Conjecture 1.4. *For any anisotropic quadratic form ϕ, the number $i_1(\phi) - 1$ is the remainder of $\dim(\phi) - 1$ modulo an appropriate 2 power.*

[1] this type of relation between two quadratic forms is called *stably birational equivalence* of the forms and means in fact that the corresponding projective quadrics are stably birationally equivalent algebraic varieties

2. Proof of Theorem 1.1

We fix an anisotropic quadratic form ϕ of dimension $2^n + 3$ with $n \geq 2$.

Case 1: ϕ is a Pfister neighbor. Then $i_1(\phi)$ equals 3 which differs from 2.

Case 2: ϕ is a *virtual Pfister neighbor*, that is, ϕ becomes an anisotropic Pfister neighbor over some field extension of F. Here we need a couple of simple observations concerning imbeddings of quadratic forms into Pfister forms.

Lemma 2.1 (cf. [7, Lemma 2.1])**.** *Let π and τ be anisotropic quadratic forms over F which are similar to some n-fold Pfister forms. There exists a field extension of F over which the forms are isomorphic while still being anisotropic.*

Proof. Consider the generic splitting tower of the form $\pi \perp -\tau$. Over the top of the tower the forms π and τ become isomorphic, and we only need to check that they are still anisotropic over the top.

Since $\pi \perp -\tau \in I^n$, where $I \subset W(F)$ is the fundamental ideal in the Witt ring, it follows from the Arason-Pfister Hauptsatz ([24, Theorem 5.6 of Chapter 4]) that every step of the tower is the function field of a quadratic form of some dimension $\geq 2^n$. By the Cassels-Pfister subform theorem ([24, Theorem 5.4(ii) of Chapter 4]), any of π and τ can not become isotropic over the function field of dimension strictly bigger that 2^n (recall that a form similar to a Pfister form is either anisotropic or hyperbolic). So we only need to see what can be done in the case where the anisotropic part of the difference $\pi \perp -\tau$ is an 2^n-dimensional form ρ such that the forms $\pi_{F(\rho)}$ and $\tau_{F(\rho)}$ are hyperbolic. This case is not possible however: again by the Cassels-Pfister subform theorem, π and τ should be now both similar to ρ whence similar to each other; therefore the difference $\pi \perp -\tau$ is in I^{n+1} and (again by the Arason-Pfister Hauptsatz) can not have an anisotropic part of dimension smaller that 2^{n+1}. □

Corollary 2.2. *Let ϕ be an anisotropic quadratic form over a field F, let $K = F(t_1, \dots, t_n)$ be the field of rational functions in n variables, and let $\pi = \langle\langle t_1, \dots, t_n \rangle\rangle$ be the "generic n-fold Pfister form" (π is a quadratic form over the field K). If there exists a field extension $\tilde{F}/F$ over which $\phi_{\tilde{F}}$ is similar to a subform of an anisotropic n-fold Pfister form τ, then there exists a field extension E/K such that π_E is anisotropic and contains a subform isomorphic to ϕ_E.*

Proof. We assume that $\phi_{\tilde{F}} \subset k\tau$ for some $\tilde{F}$ and $k \in \tilde{F}^*$. Put $\tilde{K} = \tilde{F}(t_1, \dots, t_n)$. The forms $\pi_{\tilde{K}}$ and $k\tau_{\tilde{K}}$ are clearly anisotropic ($\pi_{\tilde{K}}$ is still a generic n-fold Pfister form; $\tau_{\tilde{K}}$ is anisotropic because the extension $\tilde{K}/\tilde{F}$ is purely transcendental). We take as E an extension of $\tilde{K}$ over which they become isomorphic while still being anisotropic. Such an extension exists according to Lemma 2.1 □

Lemma 2.3 ([4, Proof of Theorem 2])**.** *If a 1-codimensional subform ψ of an anisotropic form ϕ is contained in a Pfister form π, then there exists a field extension E/F such that π_E contains the whole ϕ_E while still being anisotropic.*

Proof. We have $\pi = \psi \perp \psi'$ for some quadratic form ψ' and $\phi = \psi \perp \langle a \rangle$ for some $a \in F^*$. We define E as the function field of the quadratic form $\psi' \perp \langle -a \rangle$. Over E the form ψ'_E represents a, therefore $\phi_E \subset \pi_E$ and the only thing to check is the anisotropy of π_E.

Assume that π becomes isotropic over $E = F(\psi' \perp \langle -a \rangle)$. By the Cassels-Pfister subform theorem we then have $\psi' \perp \langle -a \rangle \subset k\pi$ for any $k \in F^*$ being the product of a value of the form $\psi' \perp \langle -a \rangle$ by a value of the form π. Since $\psi' \subset \pi$, one may take $k = 1$. So, $\psi' \perp \langle -a \rangle \subset \pi = \psi \perp \psi'$. Applying the Witt cancellation, we get the inclusion $\langle -a \rangle \subset \psi$ which means that the form $\phi = \psi \perp \langle a \rangle$ is isotropic, a contradiction. □

We continue the proof of Theorem 1.1. We are considering the case where ϕ is a virtual Pfister neighbor. We set

$$K = F(t_1, \dots, t_{n+1}) \quad \text{and} \quad \pi = \langle\langle t_1, \dots, t_{n+1} \rangle\rangle / K \ .$$

Let us consider the generic splitting tower of the quadratic form $\phi_K \perp -\pi$. Let L be the smallest field in the tower having the property $i_{\mathrm{W}}(\phi_K \perp -\pi)_L \geq 2^n + 2$ (i.e., the dimension of the anisotropic part $(\phi_L \perp -\pi_L)_{\mathrm{an}}$ of the form $\phi_L \perp -\pi_L)$ is at most $2^n - 1$).

Let us show that the form ϕ_L is anisotropic. Since ϕ is a virtual Pfister neighbor and according to Corollary 2.2, we can find an extension E/K such that ϕ_E is anisotropic and contained in π_E. The inclusion $\phi_E \subset \pi_E$ provides us with the inequality $i_{\mathrm{W}}(\phi_E \perp -\pi_E) \geq \dim \phi_E = 2^n + 3 \geq 2^n + 2$ which implies that the free composite $E \cdot_K L$ (defined as the field of fractions of the ring $E \otimes_K L$; this ring is an integral domain because the extension L/K is a tower of the function fields of quadrics which are absolutely integral varieties) is a purely transcendental field extension of E. Therefore the anisotropy of ϕ_E implies the anisotropy of ϕ_{EL}. In particular, the quadratic form ϕ_L is anisotropic.

Since our finite goal is to show that $i_1(\phi) \neq 2$, we may assume that $i_1(\phi) \geq 2$. First of all we are going to show that $i_1(\phi) = i_1(\phi_L)$ in this case. Since $\dim \phi = 2^n + 3$, the condition $i_1(\phi) \geq 2$ means that $\dim \left(\phi_{F(\phi)}\right)_{\mathrm{an}} \leq 2^n - 1$. The statement we are going to check means that the form $\left(\phi_{F(\phi)}\right)_{\mathrm{an}}/F(\phi)$ remains anisotropic over the field $L(\phi)$. Recall that the field extension L/F is a tower with the first step being purely transcendental and the other steps given by the function fields of quadratic forms of dimensions at least 2^n+1. The same can be said about the extension $L(\phi)/F(\phi)$. Therefore, by Hoffmann's theorem [4, Theorem 1], every anisotropic quadratic form over $F(\phi)$ of any dimension $< 2^n + 1$ remains anisotropic over the field $L(\phi)$. In particular, the form $\left(\phi_{F(\phi)}\right)_{\mathrm{an}}$ remains anisotropic indeed over the field $L(\phi)$.

The condition $i_{\mathrm{W}}(\phi_L \perp -\pi_L) \geq 2^n + 2$ (appeared in the choice of L) means that the forms ϕ_L and π_L have a common subform of the dimension $2^n + 2$. In other words, ϕ_L contains a 1-codimensional subform which is a Pfister neighbor (of π_L). But since $i_{\mathrm{W}}(\phi_L) \geq 2$, the form ϕ_L is stably birationally equivalent with any its 1-codimensional subform. It follows that ϕ_L is a Pfister neighbor (more precisely, it is a neighbor of the form π_L) and by that reason $i_1(\phi_L)$ is 3. Having $i_1(\phi) = i_1(\phi_L)$, we get $i_1(\phi) = 3$. So, $i_1(\phi) \neq 2$ for any virtual (2^n+3)-dimensional Pfister neighbor ϕ.

Case 3: the general case. Here we also start by considering the generic splitting tower of the quadratic form $\phi_K \perp -\pi$ with K and π/K as in the proof of the previous case. Let now L be the smallest field in the tower satisfying the property $i_{\mathrm{W}}(\phi_K \perp -\pi)_L \geq 2^n$ (or, equivalently, $\dim(\phi_L \perp -\pi_L)_{\mathrm{an}} \leq 2^n+3$).

Let us check that the form π_L is anisotropic. Let $\psi \subset \phi$ be a subform of dimension 2^n. By Hoffmann's [4, Main Lemma], there exists an extension of F over which ψ is imbeddable into an anisotropic $(n+1)$-Pfister form. Therefore, by Corollary 2.2, we can find a field extension E/K such that the form π_E is anisotropic and contains ψ_E. The inequality

$$i_{\mathrm{W}}(\psi_E \perp -\pi_E) \geq \dim \psi_E = 2^n$$

shows that $i_{\mathrm{W}}(\phi_E \perp -\pi_E) \geq 2^n$ as well, whence the field extension $L \cdot_K E/E$ is purely transcendental. Therefore π, being anisotropic over E, remains anisotropic over the composite $L \cdot_K E$; in particular, π is anisotropic over the smaller field $L \subset L \cdot_K E$.

Let us check that the field extension $L(\pi)/F$ is unirational. The function field $F(t)(\langle\langle t \rangle\rangle)$ of the 2-dimensional quadratic form $\langle\langle t \rangle\rangle = \langle 1, -t \rangle$ (t is transcendental over F) is easily seen to be purely transcendental over F. Therefore the extension $K' = K(\langle\langle t_1 \rangle\rangle)/F$ is also purely transcendental. However the form $\pi_{K'}$ is hyperbolic whence inequality $i_{\mathrm{W}}(\phi_{K'} \perp -\pi_{K'}) \geq \frac{1}{2}\dim \pi_{K'} = 2^n$ by the reason of which the extension $L \cdot_K K'/K'$ is purely transcendental. Since the extension $(L \cdot_K K')(\pi)/L_K K'$ is purely transcendental as well (because of the hyperbolicity of $\pi_{K'}$, it follows that the extension $(L \cdot_K K')(\pi)/F$ is made of three purely transcendental steps and so is itself purely transcendental. Thus the subextension $L(\pi)/F$ is unirational.

Note that the smaller extension L/F is therefore now also known to be unirational. In particular, the form ϕ_L is anisotropic and $i_1(\phi) = i_1(\phi_L)$.

Now, assuming that $i_1(\phi) = 2$, let us check that $\dim(\phi_L \perp -\pi_L)_{\mathrm{an}} = 2^n+3$. By the choice of L we have the inequality $\dim(\phi_L \perp -\pi_L)_{\mathrm{an}} \leq 2^n+3$. If the inequality is strict, then $i_{\mathrm{W}}(\phi_L \perp -\pi_L) \geq 2^n+1$, i.e., the forms ϕ_L and π_L have a common (2^n+1)-dimensional subform. So, ϕ_L contains a (2^n+1)-dimensional Pfister neighbor. By Lemma 2.3, it follows that ϕ_L contains a (2^n+2)-dimensional virtual Pfister neighbor (to get it, one takes just any (2^n+2)-dimensional subform of ϕ_L containing the (2^n+1)-dimensional Pfister neighbor). Since $i_1(\phi_L) = 2 > 1$, the form ϕ_L is stably birationally equivalent with any its 1-codimensional (i.e., (2^n+2)-dimensional) subform, thus ϕ_L is a virtual Pfister neighbor as well and so ϕ over F is already a virtual Pfister neighbor. Thereafter $i_1(\phi) \neq 2$ by the case which is already done, a contradiction.

So, for the form $\psi = (\phi_L \perp -\pi_L)_{\mathrm{an}}$, we have $\dim \psi = 2^n+3$. Going one step further in the generic splitting tower of $\phi_K \perp -\pi$, we see that if the form

$\phi_{L(\psi)}$ would be anisotropic, the form ϕ/F would be a virtual Pfister neighbor. Therefore the anisotropic form ϕ_L becomes isotropic over the function field of the form ψ/L. We claim that the form ψ also becomes isotropic over the function field $L(\phi)$. This claim will be checked in a moment, but before this we show how it ends the proof of Theorem 1.1.

The equality $\pi_L = \phi_L - \psi$ taking place in the Witt group $W(L)$ leads to the equality

$$\pi_{L(\phi)} = (\phi_{L(\phi)})_{\mathrm{an}} - (\psi_{L(\phi)})_{\mathrm{an}} \in W(L(\phi)) \ .$$

We have: $\dim(\phi_{L(\phi)})_{\mathrm{an}} \leq 2^n - 1$ and $\dim(\psi_{L(\phi)})_{\mathrm{an}} \leq 2^n - 1$ (to get the second relation we use the equality $\dim(\phi_L) - i_1(\phi_L) = \dim(\psi) - i_1(\psi)$ for the stably birationally equivalent forms ϕ_L and ψ). Thus the form $\pi_{L(\phi)}$ should be isotropic as being represented in the Witt group by a form of dimension $(2^n-1)+(2^n-1) < \dim \pi = 2^{n+1}$. Hence it is hyperbolic which implies that ϕ_L is a Pfister neighbor (of π_L) and ϕ is a virtual Pfister neighbor, a contradiction (recall our assumption $i_1(\phi) = 2$ which is already known to be impossible for a virtual Pfister neighbor of dimension $2^n + 3$).

The claim that ψ becomes isotropic over $L(\phi)$ which we did not prove so far, follows from the following general conjecture worthy to be mentioned anyway:

Conjecture 2.4. *Let ϕ and ψ be anisotropic quadratic forms over a field F.*

1. *If the form $\phi_{F(\psi)}$ is isotropic, then* $\dim \phi - i_1(\phi) \geq \dim \psi - i_1(\phi)$*;*
2. *if the form $\phi_{F(\psi)}$ is isotropic and if moreover* $\dim \phi - i_1(\phi) = \dim \psi - i_1(\phi)$*, then the form $\psi_{F(\phi)}$ is isotropic as well.*

Remark 2.5. To prove Conjecture 2.4 in general it suffices to handle the case where $i_1(\phi) = 1 = i_1(\psi)$.

Although it will not help us to finish the proof of Theorem 1.1 in a correct way, we first show how to deduce from Conjecture 2.4 the claim we need. We have $\dim \phi_L = \dim \psi$ and $i_1(\phi_L) = 2$. The first part of Conjecture 2.4 shows then that $i_1(\psi) \geq 2$. However over the field $L(\pi)$ the forms ϕ and ψ are anisotropic and isomorphic (because $0 = \pi_{L(\pi)} = \phi_{L(\pi)} - \psi_{L(\pi)} \in W(L(\pi))$). The extension $L(\pi)/F$, being unirational, does not change the first Witt index of a form, therefore $i_1(\psi_{L(\pi)}) = i_1(\phi_{L(\pi)}) = i_1(\phi) = 2$; thus $i_1(\psi) = 2$ as well, and the isotropy of the form $\psi_{L(\phi)}$ follows now from the second part of Conjecture 2.4.

To prove the claim in an honest way we need the following result which is in the heart of the whole business:

Proposition 2.6 (Izhboldin, cf. [11, Theorem 6.6])**.** *Let ϕ and ψ be some quadratic forms over a field F such that $\phi_{F(\psi)}$ is isotropic. We assume that* $\dim \phi, \dim \psi \geq 3$*. If the forms ϕ and ψ are anisotropic and stably birationally equivalent over some field extension E/F not affecting the first Witt index of the form ϕ, then they are stably birationally equivalent already over F.*

The proof of Proposition 2.6 will be given in the next section. Now we use Proposition 2.6 in order to finish the proof of Theorem 1.1.

We apply Proposition 2.6 to the quadratic forms ϕ_L and ψ over the field L. The function field $E = L(\pi)$ is an extension of L with the properties required in Proposition 2.6: it does not affect the first index of ϕ_L by the unirationality over F; by the same reason the form ϕ_E is anisotropic; since $\phi_E \simeq \psi_E$, the form ψ_E is anisotropic too; the forms ϕ_E and ψ_E are stably birationally equivalent simply because they are isomorphic. Therefore ψ is isotropic over $L(\phi)$.

The proof of Theorem 1.1 is done.

3. Proof of Proposition 2.6

For ϕ and ψ satisfying the conditions of Proposition 2.6, let us choose some subforms $\phi_0 \subset \phi$ and $\psi_0 \subset \psi$ of the dimension $\dim \phi - i_1(\phi) + 1$. Then ϕ_0 becomes isotropic over $F(\phi)$, ϕ over $F(\psi)$, and ψ over $F(\psi_0)$. Therefore, by the transitivity, the form $(\phi_0)_{F(\psi_0)}$ is isotropic. Note that $i_1(\phi_0) = 1$ because of the relation $\dim \phi - i_1(\phi) = \dim \phi_0 - i_1(\phi_0)$ for the stably birationally equivalent forms ϕ and ϕ_0.

Thus, replacing ϕ and ψ by the subforms ϕ_0 and ψ_0, we reduce the proof of Proposition 2.6 to the following particular case:

Lemma 3.1. *Let ϕ and ψ be some quadratic forms over a field F having one and the same dimension ≥ 3, and assume that the form $\phi_{F(\psi)}$ is isotropic. If ϕ and ψ are anisotropic and stably birationally equivalent over some field extension E/F such that $i_1(\phi_E) = 1$ (therefore $i_1(\phi) = 1$), then ϕ and ψ are stably birationally equivalent already as forms over F.*

We will deduce Lemma 3.1 from the following statement about the integral Chow correspondences on a projective quadric of the first Witt index 1:

Lemma 3.2 ([18, Theorem 6.4])**.** *Let ϕ be an anisotropic quadratic form of dimension ≥ 3 with $i_1(\phi) = 1$. Let X be the projective quadric $\phi = 0$ and $n = \dim X$ ($= \dim \phi - 2$). For any element $\alpha \in \mathrm{CH}^n(X \times X)$ of the Chow group of n-codimensional cycles on the variety $X \times X$, one then has: $\deg_1(\alpha) \equiv \deg_2(\alpha) \pmod 2$, where $\deg_i$ stays for the degree of α over the i-th factor of the product $X \times X$.*

For reader's convenience we recall the definition of $\deg_i(\alpha)$ (cf. [2, Example 16.1.4]): $\deg_i(\alpha)$ is the integer such that $(pr_i)_*(\alpha) = \deg_i(\alpha) \cdot [X] \in \mathrm{CH}^0(X)$ for the push-forward $(pr_i)_*$ with respect to the i-th projection $pr\colon X \times X \to X$.

Proof of Lemma 3.1. We denote by Y the projective quadric $\psi = 0$. The fact that the form $\phi_{F(\psi)}$ is isotropic means that the variety $X_{F(Y)}$ has a rational point, i.e. there exists a rational morphism $f\colon Y \to X$. Let $\alpha \in \mathrm{CH}^n(Y \times X)$ be the correspondence given by the closure of the graph of f. We have $\deg_1(\alpha) = 1$ ([18, Example 1.2]). By the Springer theorem, in order to show that the form $\psi_{F(\phi)}$ is isotropic, it suffices to show that the variety $Y_{F(X)}$ possesses a 0-dimensional cycle of an odd degree. Since the pull-back of α to $Y_{F(X)}$ is a 0-dimensional cycle of the degree $\deg_2(\alpha)$, it suffices to show that $\deg_2(\alpha)$ is odd.

Since a base change does not affect $\deg_i(\alpha)$, it suffices to show that $\deg_2(\alpha_E)$ is odd. But the variety $Y_{E(X)}$ has a rational point. So there exists a correspondence $\beta \in \mathrm{CH}^n(X_E \times Y_E)$ with $\deg_1(\beta) = 1$. For $\gamma = \alpha_E \circ \beta \in \mathrm{CH}^n(X_E \times X_E)$ (γ is defined as the composition of the correspondences α_E and β, see [2, §16.1] for the notion of composition for correspondences) one has:

$$\deg_1(\gamma) = \deg_1(\beta) \cdot \deg_1(\alpha_E) = 1 \cdot 1 = 1$$

and $\deg_2(\gamma) = \deg_2(\beta) \cdot \deg_2(\alpha_E)$. Since $\deg_1(\gamma) \equiv \deg_2(\gamma) \pmod 2$, the integer $\deg_2(\gamma)$ is odd. Therefore $\deg_2(\alpha_E)$ is odd, too. □

4. A characterization of virtual Pfister neighbors

Note that an anisotropic $(2^n + 1)$-dimensional quadratic form is always a virtual Pfister neighbor ([4, Theorem 2]). By methods similar to those of above, one can obtain the following characterization of $(2^n + 2)$-dimensional virtual Pfister neighbors:

Theorem 4.1 (Izhboldin). *An anisotropic quadratic (2^n+2)-dimensional form is a virtual Pfister neighbor if and only if its splitting pattern contains 2.*

Proof. The first Witt index of an honest anisotropic Pfister neighbor of dimension $2^n + 2$ is equal to 2. Therefore the "only if" part of the theorem is trivial. Let us prove the "if" part.

We take an anisotropic $(2^n + 2)$-dimensional virtual Pfister neighbor ϕ over a field F, put $K = F(t_1, \dots, t_{n+1})$ and consider over K the $(n+1)$-fold Pfister form $\pi = \langle\langle t_1, \dots, t_{n+1}\rangle\rangle$. In the generic splitting tower of the form $\phi_K \perp -\pi$ we take the smallest field L satisfying the condition $i_{\mathrm{W}}(\phi_L \perp -\pi_L) \geq 2^n$, i.e., $\dim(\phi_L \perp -\pi_L)_{\mathrm{an}} \leq 2^n + 2$. By the same reason as in the proof of the general case of Theorem 1.1, the form π remains anisotropic over L.

If the inequality $i_{\mathrm{W}}(\phi_L \perp -\pi_L) \geq 2^n$ is strict, the form ϕ_L contains a (2^n+1)-dimensional Pfister neighbor. Then it follows from Lemma 2.3 that ϕ_L is a virtual Pfister neighbors. Whence ϕ over F is a virtual Pfister neighbor.

It remains to consider the case with $i_{\mathrm{W}}(\phi_L \perp -\pi_L) = 2^n$. In this case $\dim \psi = 2^n + 2$ for $\psi = (\phi_L \perp -\pi_L)_{\mathrm{an}}$. If $\phi_{L(\psi)}$ is anisotropic, then ϕ is a virtual Pfister neighbor; hence we may assume that $\phi_{L(\psi)}$ is isotropic. Over the function field $L(\pi)$ the quadratic form ϕ is anisotropic and isomorphic to ψ; moreover, $i_1(\phi) = i_1(\phi_{L(\pi)})$, because the field extension $L(\pi)/F$ is unirational (by the same argument as in the proof of the general case of Theorem 1.1). Applying Proposition 2.6, we get the stably birational equivalence for the forms ϕ_L and ψ.

Let now F'/F be a field extension such that $i_{\mathrm{W}}(\phi_{F'}) = 2$. In the Witt group $W(F' \cdot_F L)$ of the free composite $F' \cdot_F L$ we have the equality $\pi_{F'L} = \phi_{F'L} - \psi_{F'L}$. Since $\dim(\phi_{F'L})_{\mathrm{an}} \leq 2^n - 2$ and $\dim(\psi_{F'L})_{\mathrm{an}} \leq 2^n$ ($\psi_{F'L}$ is isotropic as $\phi_{F'L}$ is so), we see that $\pi_{F'L}$ should be isotropic. On the other hand, one can check that $\pi_{F'L}$ is anisotropic by constructing a field extension E of $F'(t_1, \dots, t_{n+1})$ such that π_E is anisotropic and $i_{\mathrm{W}}(\phi_E \perp -\pi_E) \geq 2^n$: for the $(2^n - 2)$-dimensional anisotropic part ϕ' of the form $\phi_{F'}$ we take an extension $E/F'(t_1, \dots, t_{n+1})$ over which π is anisotropic and contains ϕ'. Such an extension exists by Hoffmann's [4, Main Lemma] together with Corollary 2.2. Since $i_W(\phi'_E \perp -\pi_E) \geq 2^n - 2$ for that extension and since $\phi_E \simeq \phi'_E \perp 2\mathbb{H}$ (where $\mathbb{H}$ stays for the hyperbolic plane), we get $i_W(\phi_E \perp -\pi_E) \geq 2^n$. □

Part 2. U-invariant 9

5. Introduction

We recall the definition of the u-invariant $u(F)$ of a field F: $u(F) = \sup\{\dim \phi\}$ where ϕ runs over anisotropic quadratic forms over F. The classical question in the theory of quadratic form asks about the possible finite values of the u-invariant.

Since 1991 we know by [21] that every even positive integer is possible (before this result one was able to realize the powers of 2 only).

The u-invariant of a quadratically (e.g., separably or algebraically) closed field is 1. Is an odd value > 1 possible? The answer is classically known to be negative for the first three odd integers: 3, 5, and 7. Here we will prove the following

Theorem 5.1 ([10]). *There exists a field E with $u(E) = 9$.*

Proof. The construction of E is not a problem: for any n, if one knows that n is a value of the u-invariant, then n is the u-invariant of the field E constructed by the following procedure.

We start with an arbitrary field F and consider the field $K = F(t_1, \ldots, t_n)$ of rational functions in n variables $t_1, \ldots, t_n$ over F. Let ϕ be the generic n-dimensional quadratic form $\langle t_1, \ldots, t_n \rangle$ $/K$. We construct an infinite tower of fields $K = K_0 \subset K_1 \subset K_2 \subset \ldots$ as follows: for every $i \geq 0$ the field K_{i+1} is the free composite of the function fields $K_i(\psi)$ where ψ runs over all $(n+1)$-dimensional anisotropic forms over K_i (more precisely, one takes one ψ in every isomorphism class of such quadratic forms; the infinite free composite is defined as the directed direct limit of all finite subcomposites). This tower evidently has the following property: any anisotropic quadratic form of any dimension $> n$ over a field K_i becomes isotropic over the field K_{i+1}. Thus the union $E = \cup_i K_i$ is a field with $u(E) \leq n$. By the genericity of the construction, we have $u(E) = n$ (an anisotropic n-dimensional form over E is the form ϕ_E).

We do not prove the statement just announced, because we do not need it. But looking at the construction, we see what can be done in order to realize a number n: it is enough to find a list of properties of n-dimensional quadratic forms over fields such that the generic forms satisfy them and if a form ϕ satisfies them over a field F, then ϕ is anisotropic and still satisfies them over the function field $F(\psi)$ of any $(n+1)$-dimensional anisotropic quadratic form ψ/F.

If we have such a list, then $u(E) = n$ because the n-dimensional form ϕ_E is anisotropic. Of course in this case we are not obliged to take $K = F(t_1, \ldots, t_n)$ with $\phi/K = \langle t_1, \ldots, t_n \rangle$ anymore: we may start the construction of the tower giving E by any field K and a form ϕ/K satisfying the conditions of the list: the form ϕ_E will be anisotropic.

The problem of the choice of a list of properties needed is quite delicate. Of course, we can not take the list of the only one property "the form is generic", because we can not guaranty that a generic form over F will be still generic over $F(\psi)$.

Let us recall the property used in [21] working for any even n: *the even Clifford algebra $C_0(\phi)$ is a division algebra.* This property guaranties that ϕ is anisotropic and this property is conserved when climbing over the function field of an $(n+1)$-dimensional quadratic form according to the index reduction formula for quadrics [21, thm. 1] (which is in fact the basic point of the even n business).

For n odd this property does not work (see **(1)** in the proof of Theorem 5.3). An appropriate list of properties for $n = 9$ is as follows:

1. $\operatorname{ind} C_0(\phi) \geq 4$ where $\operatorname{ind} C_0(\phi)$ is the Schur index of $C_0(\phi)$ which is a central simple F-algebra (a stronger condition $\operatorname{ind} C_0(\phi) \geq 8$ can be also taken);
2. ϕ is anisotropic;
3. ϕ is not a Pfister neighbor.

We remark that these properties (with 4 in the first one) are also necessary in order that a field extension E/F with $u(E) = 9$ and ϕ_E anisotropic would exist: clearly, if (3) is not satisfied, then ϕ_E is a neighbor of a 4-fold Pfister form and hence is isotropic because the dimension of a 4-fold Pfister form is $16 > 9$; besides of that, since the 10-dimensional form $\phi_E \perp \langle -\det\phi\rangle_E$ is isotropic, the form ϕ_E represents its determinant and therefore contains an 8-dimensional subform q of determinant 1. The Clifford algebra $C(q)$ of q is isomorphic to the even Clifford algebra $C_0(\phi_E)$. If the condition (1) is not satisfied, then $\operatorname{ind} C(q) \leq 2$ whence $q \simeq \langle\langle a\rangle\rangle \otimes \langle b_1, b_2, b_3, b_4\rangle$ for some $a, b_1, b_2, b_3, b_4 \in E^*$ ([20, example 9.12]). Therefore ϕ_E is isomorphic to a subform of the 10-dimensional quadratic form $\langle\langle a\rangle\rangle \otimes \langle b_1, b_2, b_3, b_4, \det\phi\rangle$. This form is isotropic. Since its Witt index is divisible by 2, it is at least 2. Hence the 1-codimensional subform ϕ_E is isotropic.

Definition 5.2. A 9-dimensional quadratic form ϕ satisfying the properties (1)–(3) is called *essential*.

Theorem 5.3. *For an essential quadratic form ϕ and a 10-dimensional quadratic form ψ over a field F, the form $\phi_{F(\psi)}$ is also essential.*

Proof. For the form $\phi_{F(\psi)}$, let us check the conditions of the essentiality (1)–(3) one by one:

(1) According to the index reduction formula for quadrics, the Schur index $\operatorname{ind} C_0(\phi_{F(\psi)})$ of the central simple $F(\psi)$ algebra $C_0(\phi_{F(\psi)}) = C_0(\phi)_{F(\psi)}$ is either the same as that of $C_0(\phi)$ or $\big(\operatorname{ind} C_0(\phi)\big)/2$, depending on whether the even Clifford algebra $C_0(\psi)$ maps homomorphically into the underlying division algebra of $C_0(\phi)$ (this is the simplified formulation of Merkurjev's index reduction [21] due to J.-P. Tignol [27]).

We only have to take care about the situation where $\operatorname{ind} C_0(\phi)$ is 4, that is, $\dim_F D = 4^2 = 2^4$. Although the algebra $C_0(\psi)$ is not always simple, its subalgebra $C_0(\psi')$ is simple for any 9-dimensional subform $\psi' \subset \psi$. Thus an algebra homomorphism $C_0(\psi) \to D$ would give an imbedding $C_0(\psi') \hookrightarrow D$ which is far from being possible by the simple dimension reason: $\dim_F C_0(\psi') = 2^{\dim \psi' - 1} = 2^8 > \dim_F D$ (as we see, the equality $\operatorname{ind} C_0(\phi_{F(\psi)}) = \operatorname{ind} C_0(\phi)$ also holds for any ϕ with $\operatorname{ind} C_0(\phi) = 8$; however, if the Schur index is 16 – the maximal possible value for a 9-dimensional quadratic form – it *can* go down over the function field of ψ; thus we would not come through if only looking at the Schur indexes which was enough for constructing the even u-invariants).

(2) The proof of the fact that the form $\phi_{F(\psi)}$ is still anisotropic is based on the following criterion of isotropy of an essential form ϕ over the function field of a 9-dimensional form (in the place of a 10-dimensional) form ψ ([19, thm. 1.13]): $\phi_{F(\psi)}$ is isotropic if and only if the forms ϕ and ψ are similar. This criterion is obtained as a consequence of the characterization of the 9-dimensional Pfister neighbors obtained in [19]: an anisotropic 9-dimensional quadratic form is a Pfister neighbor if and only if the projective quadric given by the form has a Rost correspondence. The details will be given in Section 6.

(3) The proof makes use of certain results on the unramified cohomology of projective quadrics due to B. Kahn, M. Rost and Sujatha. It also involves computation of the Chow group CH^3 for certain projective quadrics. The details will be explained in Section 7. The computation of CH^3 needed will be done in Section 8. □

Theorem 5.1 is proved (modulo **(2)** and **(3)** in the proof of Theorem 5.3). □

6. Checking (2)

In this section we check that an essential quadratic form ϕ/F remains anisotropic over the function field of any 10-dimensional quadratic form ψ/F. We shall indicate four different ways to do this (due respectively to myself, O. Izhboldin, D. Hoffmann, and B. Kahn).

First of all, this can be done by the same method as in the proof of the anisotropy of $\phi_{F(\psi)}$ for a 9-dimensional ψ non-similar to ϕ (Theorem 6.1). However the proof for a 10-dimensional ψ turns out to be a little bit more complicated as that for a 9-dimensional ψ because of some special effects in the intermediate Chow group of an even-dimensional quadric. Since in the same time it turns out that the 10-dimensional case is a formal consequence of the 9-dimensional one (see the three other ways which follow), it does not seem not to be reasonable to argue this way anymore.

Now we assume that we already know the criterion of the isotropy of the essential forms over the function fields of 9-dimensional forms. We indicate three ways to deduce the statement on the 10-dimensional forms from it (the proof of the criterion itself will be explained right after). All of them are based on the following observation. If an essential quadratic form ϕ would be isotropic over the function field of some 10-dimensional form ψ, then it would be also isotropic over the function field of any 9-dimensional subform $\psi_0 \subset \psi$. Therefore, to show that $\phi_{F(\psi)}$ is anisotropic, it suffices to find insight of ψ a 9-dimensional form ψ_0 non similar with ϕ. It can be always done, at least over a purely transcendental field extension of F (which is also enough for our purposes). Here are the three different ways to construct the subform ψ_0.

In [10], a sort of generic 9-dimensional subform of ψ is taken for ψ_0 (see [10, Lemma 7.9]). To be precise, the form $\tilde{\psi} = \psi_{F(t)} \perp \langle t \rangle$ over the field $\tilde{F}$ of rational functions in one variable t is considered, and ψ_0 is defined to be the anisotropic part of $\tilde{\psi}_{\tilde{F}(\tilde{\psi})}$. Note that the extension $\tilde{F}(\tilde{\psi})/F$ is purely transcendental. It is then shown in [10, Lemma 7.9] that for any 9-dimensional quadratic form q/F and any $k \in \tilde{F}(\tilde{\psi})^*$, the difference $\psi_0 - k \cdot q_{\tilde{F}(\tilde{\psi})}$ in the Witt ring $W(\tilde{F}(\tilde{\psi}))$ is not in $I^4(\tilde{F}(\tilde{\psi}))$, that is, ψ_0 is not similar to $q_{\tilde{F}(\tilde{\psi})}$ modulo I^4 (and, in particular, ψ_0 is not similar to $q_{\tilde{F}(\tilde{\psi})}$ in the usual sense).

This statement is interesting on its own. But of course it is much more stronger as our simple needs.

As suggested by Detlev Hoffmann during the course, for $\psi = \langle a_1, \dots, a_{10} \rangle$ one may take the subform $\psi_0 = \langle a_1 + a_2 t^2, a_3, \dots, a_{10} \rangle \subset \psi_{\tilde{F}}$. This is a 1-codimensional subform of $\psi_{\tilde{F}}$ which is far from being generic. However, using exactly the same arguments as in [3, page 224], one may show that ψ_0 is not similar to $q_{\tilde{F}}$ for any q/F with $\operatorname{ind} C_0(q) > 2$ (in particular, for any essential q).

Finally, a third method has been suggested by Bruno Kahn during the course. Let ψ/F be an anisotropic quadratic form of an even dimension $2n$. Assume that ψ represents 1 and that all 1-codimensional subforms of ψ are similar. Then it is easy to check that $D(\psi) \subset G(\psi)$ with $G(\psi) \subset F^*$ staying for the group of the similarity factors of ψ and $D(\psi) \subset F^*$ the set of non zero elements represented by ψ: we have $\psi = \langle 1 \rangle \perp \psi'$ with some 1-codimensional subform $\psi' \subset \psi$; for $a \in D(\psi)$, we also can write $\psi = \langle a \rangle \perp \psi''$ with some ψ''; we know that the forms ψ' and ψ'' are similar; comparing their determinants, we get $a\psi' \simeq \psi''$, whence $a\psi = \langle a \rangle \perp a\psi' \simeq \langle a \rangle \perp \psi'' = \psi$. This is not yet enough to get a contradiction, but if we assume additionally that for any purely transcendental extension $\tilde{F}/F$ (it suffices to assume this for $\tilde{F}$ being the function field of the affine space given by the vector space $/F$ of definition of ψ) all 1-codimensional subforms of $\psi_{\tilde{F}}$ are still similar, the inclusion $D(\psi_{\tilde{F}}) \subset G(\psi_{\tilde{F}})$ we get implies by [24, thm. 4.4(v) of Ch. 4] that ψ is a Pfister form and thus can not be 10-dimensional.

Now we explain the proof of the criterion of isotropy of the essential forms over the function fields of the 9-dimensional forms, namely

Theorem 6.1 ([19, thm. 1.13])**.** *Let ϕ be an essential quadratic form over F and let ψ be any 9-dimensional quadratic form over F. Then $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to ϕ.*

Proof. Of course, a proof is needed only the "only if" part. We shall give two proofs. The first one makes use of motives and is more conceptual. However the motives are not really needed: the second proof is much more elementary (although it seems to be more tricky) and is in fact a translation of the "motivic" proof into an elementary language. All the details will be given in the second proof; as to the first one, we shall give only a sketch.

The first proof. The "motivic" proof starts with the following observation. Let ϕ be a 9-dimensional anisotropic quadratic form (essential or not) such that $\operatorname{ind} C_0(\phi) \geq 4$. Let X be the projective quadric $\phi = 0$. One observes that any non-trivial decomposition of the Chow-motive $M(X)$ in a direct sum contains a summand R which is a *Rost motive*, that is $R_{\bar{F}} \simeq \mathbf{pt}_{\bar{F}} \oplus \mathbf{pt}_{\bar{F}}(d)$, where $\bar{F}$ is an algebraic closure of F, $\mathbf{pt}$ is the motive of $\operatorname{Spec} F$, $d = \dim X$, and $\mathbf{pt}(d)$ is the d-fold twist of $\mathbf{pt}$. In particular, if the motive of X decomposes, then there exists a Rost correspondence on X, that is, in the Chow group $\mathrm{CH}^d(X \times X)$ there exists an element ρ such that $\rho_{\bar{F}} = [\bar{X} \times x] + [x \times \bar{X}] \in \mathrm{CH}^d(\bar{X} \times \bar{X})$, where $\bar{X} = X_{\bar{F}}$ and $x \in \bar{X}$ is a rational point. If we now assume that the form ϕ is essential (in other words, we additionally assume that ϕ is not a Pfister neighbor), then by [19, thm. 1.7] we know that there are no Rost correspondences on X. Thus the motive $M(X)$ of an essential quadric X is indecomposable.

Let ϕ and ψ be anisotropic quadratic forms over F such that $\dim \phi = \dim \psi = 2^n + 1$ for some n. A theorem of Izhboldin [9, thm. 0.2] states that if $\phi_{F(\psi)}$ is isotropic, then $\psi_{F(\phi)}$ is also isotropic. This theorem can be considered as a complement to [4, thm. 1]. Hoffmann's proof of [4, thm. 1] as well as Izhboldin's proof of [9, thm. 0.2] are tricky and do not give a feeling to explain why do the things happen this way in the nature. Such explanation (and new proofs) are given by the Rost degree formula ([22, §5]).

Applying Izhboldin's theorem to our particular situation, where ϕ is an essential form which becomes isotropic over the function field of some other 9-dimensional form ψ, we see that ψ also becomes isotropic over $F(\phi)$. In other words, there are rational morphisms in both direction: $X \to Y$ and $Y \to X$, where X and Y are the projective quadrics given by ϕ and ψ. An observation due to A. Vishik ([28]) says that every time we have rational morphisms in both directions for two projective quadrics X and Y, there is a non-trivial direct summand of $M(X)$ isomorphic to some direct summand of $M(Y)$. Since the motive of X is indecomposable in our setup, it follows that $M(X)$ as whole is isomorphic to a direct summand of $M(Y)$. Finally, since $\dim X = \dim Y$, we obtain a motivic isomorphism $M(X) \simeq M(Y)$ for X and Y.

Now we apply a theorem of Izhboldin [8, cor. 2.9] stating that two projective quadrics of an odd dimension can be motivically isomorphic only if they are isomorphic as algebraic varieties, what means that the quadratic forms defining them are similar. Thus the quadratic forms ϕ and ψ are similar.

The second proof. In this proof all the details will be given. The word “motive” will be not pronounced in the proof. It will only appear in the comments indicating the motivic meaning of an intermediate result achieved.

Let X be the projective quadric given by a 9-dimensional quadratic form ϕ. We first assume that ϕ is completely split, i.e., the Witt index of ϕ is 4, $\phi \sim \mathbb{H} \perp \mathbb{H} \perp \mathbb{H} \perp \mathbb{H} \perp \langle 1 \rangle$. So, our X is the hypersurface in the projective space $\mathbb{P}^8$ given by the equation $x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + t^2 = 0$. The variety X is known to be cellular: all successive differences of the filtration $X = X^0 \supset X^1 \supset X^2 \supset X^3 \supset \mathbb{P}^3 \supset \mathbb{P}^2 \supset \mathbb{P}^1 \supset \mathbb{P}^0$ are affine spaces, where X^i for $i = 1, 2, 3$ is the closed (singular !) subvariety of X given by the equations $x_0 = 0, \dots, x_i = 0$, while $\mathbb{P}^i$ is an i-dimensional projective subspace of the 3-dimensional projective subspace $\mathbb{P}^3 \subset \mathbb{P}^8$ contained in X and determined by the equations $x_0 = 0, \dots, x_4 = 0$ and $t = 0$. Therefore (see [2, example 1.9.1]) the whole Chow group $\mathrm{CH}^*(X)$ of X is the free abelian group on $[X^i] \in \mathrm{CH}^i(X)$ and $[\mathbb{P}^i] \in \mathrm{CH}_i(X) = \mathrm{CH}^{7-i}(X)$, $i = 0, 1, 2, 3$. We write h^i for $[X^i]$, and l_i for $[\mathbb{P}^i]$. So, for every $i = 0, 1, 2, 3$, the groups $\mathrm{CH}^i(X)$ and $\mathrm{CH}_i(X)$ are infinite cyclic with the generators h^i and l_i respectively.

We are more interested in the Chow group $\mathrm{CH}^7(X \times X)$ however. To understand the Chow group of the product $X \times X$, note that the cellular structure on X induces a cellular structure on $X \times X$ (see, e.g., [17, §7]). In particular, it follows that $\mathrm{CH}^*(X \times X)$ is the free abelian group on $h^i \times l_j$ and $l_j \times h^i$, $i, j = 0, 1, 2, 3$. Since $h^i \times l_j$ and $l_j \times h^i$ are in $\mathrm{CH}^{i+7-j}(X \times X)$, the generators of the group $\mathrm{CH}^7(X \times X)$ are $h^i \times l_i$ and $l_i \times h^i$, $i = 0, 1, 2, 3$.

Now we do not assume anymore that the quadratic form ϕ giving the quadric X is completely split. Nevertheless, it is completely split over an algebraic closure $\bar{F}$ of F, and for any $\alpha \in \mathrm{CH}^7(X \times X)$ we may define the *type* of α as the sequence of integers

$$\mathrm{type}(\alpha) = (a_0, a_1, a_2, a_3, a'_3, a'_2, a'_1, a'_0)\ , \quad a_i, a'_i \in \mathbb{Z}$$

such that $\alpha_{\bar{F}} = \sum_{i=0}^{3} a_i(h^i \times l_i) + a'_i(l_i \times h^i)$.

Here is a couple of examples: $\mathrm{type}(\alpha) = (1, 0, \dots, 0, 1)$ means that α is a Rost correspondence; the type of the diagonal class is $(1, 1, \dots, 1)$.

In the case where $\alpha \in \mathrm{CH}^7(X \times X)$ is a projector (i.e., an idempotent with respect to the composition of the correspondences), the type of α is a sequence of 0 and 1 having the following meaning: over $\bar{F}$, the motive (X, α) becomes isomorphic to the direct sum $\bigoplus_{i=0}^{r} \mathbf{pt}(j_i)$, where $j_1, \dots, j_r$ are the numbers of places of the non-zero entries in the type of α (the places are numbered starting from 0).

Of course, one also may define the type for an $\alpha \in \mathrm{CH}^7(X \times Y)$ where Y is another projective quadric of the same dimension as X. We note that the first and the last entries of type α are the *degrees* (or *indices*, see [2, example 16.1.4]) of α over the first and over the second factor of the product $X \times Y$ respectively (see [18, example 1.2]).

If $\alpha \in \mathrm{CH}^7(X \times Y)$ and $\beta \in \mathrm{CH}^7(Y \times Z)$ with one more 7-dimensional projective quadric Z, the type of the composition $\beta \circ \alpha$ of the correspondences α and β is the componentwise product of type(α) and type(β).

Starting from this point, we shall consider the types *modulo 2*. The types $(1, 1, \ldots, 1)$ and $(0, 0, \ldots, 0)$ will be called *trivial*. It is not difficult to check (see [19, §9]) that in the case of an anisotropic ϕ with $\operatorname{ind} C_0(\phi) \geq 4$, the only possible non-trivial types are $(1, 0, \ldots, 0, 1)$ and its complement $(0, 1, \ldots, 1, 0)$. Thus for an essential ϕ, by [19, 1.7] (see also [19, lemma 9.3]), there are no non-trivial types (this is a reflection of the fact that the motive of X is indecomposable for an essential ϕ).

Now we assume that our essential form ϕ becomes isotropic over the function field of some 9-dimensional form ψ. By Izhboldin's theorem the form $\psi_{F(\phi)}$ is then isotropic as well, and we have two rational morphisms $f: X \to Y$ and $g: Y \to X$, where Y is the quadric $\psi = 0$. Let $\alpha \in \mathrm{CH}^7(X \times Y)$ be given by the closure of the graph of f while $\beta \in \mathrm{CH}^7(Y \times X)$ be given by the closure of the graph of g. Recall that one may define the types of α and β in the same way as in the case $X = Y$. Moreover, the first entry of such a type is the degree of the correspondence over the first factor. Since α and β are given by the closures of the graphs of rational morphisms, these degrees are 1 (see [18]). Therefore, the first entry in the type of $\gamma = \beta \circ \alpha \in \mathrm{CH}^7(X \times X)$ is also 1. In particular, type(γ) $\neq 0$. Since the only possible types for X are the trivial ones, we therefore have type(γ) $= (1, 1, \ldots, 1)$ whence type(α) $=$ type(β) $= (1, 1, \ldots, 1)$ (at this stage we almost have constructed a motivic isomorphism between X and Y; this "almost" however turns out to be enough for our purposes).

In the first proof we applied Izhboldin's theorem [8, cor. 2.9] to get $X \simeq Y$ from $M(X) \simeq M(Y)$. However the theorem [8, cor. 2.9] has nothing to do with the motives: in its proof, the isomorphism $X \simeq Y$ is obtained as a consequence of the equalities $i_{\mathrm{W}}(\phi_E) = i_{\mathrm{W}}(\psi_E)$ for any field extension E/F. Now we are able to get these equalities directly, without passing through the motives.

For any i the inequality $i_{\mathrm{W}}(\phi_E) > i$ is equivalent to the statement that the element $l_i \in \mathrm{CH}_i(X_{\bar{E}})$ is defined over E (i.e., is in the image of the restriction $\mathrm{CH}_i(X_E) \to \mathrm{CH}_i(X_{\bar{E}})$). The image of l_i with respect to the push-forward $(\alpha_{\bar{E}})_*: \mathrm{CH}_i(X_{\bar{E}}) \to \mathrm{CH}_i(Y_{\bar{E}})$ is l_i again. The same holds for $(\beta_{\bar{E}})_*$. Therefore, for any i, one has $i_{\mathrm{W}}(\phi_E) > i$ if and only if $i_{\mathrm{W}}(\psi_E) > i$. Thus $i_{\mathrm{W}}(\phi_E) = i_{\mathrm{W}}(\psi_E)$ for any E/F.

we have finished the second proof of Theorem 6.1. □

For the reader convenience we formulate and prove Izhboldin's theorem used in the end of the proof of Theorem 6.1:

Theorem 6.2 (Izhboldin, [8])**.** *Let ϕ and ψ be some quadratic forms over F. Assume that the dimension of ϕ coincides with the dimension of ψ and is odd. If $i_{\mathrm{W}}(\phi_E) = i_{\mathrm{W}}(\psi_E)$ for any field extension E/F, then $\phi \sim \psi$.*

Proof (cf. [8]*).* Replacing ψ by $\det(\phi) \cdot \det(\psi) \cdot \psi$, we come to the situation where $\det(\phi) = \det(\psi)$. We shall prove that $\phi \simeq \psi$ in this situation.

Replacing ϕ and ψ by their anisotropic parts, we come to the situation where both ϕ and ψ are anisotropic. We prove that $\phi \simeq \psi$ by induction on $\dim \phi$.

We put $\pi = \phi \perp -\psi$ and need to show that the quadratic form π is hyperbolic. Suppose that it is not. The form $\pi_{F(\phi)}$ is hyperbolic by the induction hypothesis. Since the anisotropic part π_{an} of π clearly has a common value with ϕ, we get that $\phi \subset \pi_{\mathrm{an}}$. Now if π_{an} would be different from π, the form ψ would be isotropic. So, the form π is anisotropic.

Over the function field $F(\pi)$ of π the forms ϕ and ψ are anisotropic by Hoffmann's theorem [4, thm. 1]. Since the form $\pi_{F(\pi)}$ is no more anisotropic, it should be hyperbolic by the above arguments. It follows from [24, thm. 5.4(i)] that π is similar to a Pfister form. In particular, the dimension of π is a 2 power which contradicts to the assumption that the dimension of the forms ϕ and ψ is odd (we do not consider the trivial case where $\dim \phi = \dim \psi = 1$). □

7. Checking (3)

The link to the unramified staff comes with the following, as simple as crutial, observation:

Lemma 7.1 (c.f. [10, lemma 6.2]). *Let ϕ be a quadratic form over F and let L/F be a field extension such that ϕ_L is a neighbor of an n-fold Pfister form π/L. Then the class of π in the Witt group $W(L)$ is unramified over F.*

Proof. We recall that an element $x \in W(L)$ is called unramified over F if $\partial_v(x) = 0$ for any discrete valuation v of L trivial on F, where ∂_v stays for the second residue homomorphism.

Let v be a discrete valuation of L trivial on F with a prime $p \in L^*$. We write k_v for the residue field of v. Recall that the second residue homomorphism $\partial_v \colon W(L) \to W(k_v)$ is the group homomorphism (depending on the choice of the prime p) such that

$$\partial_v(\langle l \rangle) = \begin{cases} 0, & \text{if } v(l) \text{ is even;} \\ \left\langle \text{the class of } lp^{-v(l)} \in L \text{ in } k_v \right\rangle, & \text{if } v(l) \text{ is odd} \end{cases}$$

(note that even though ∂_v depends on the choice of p, its kernel does not).

We are going to prove that $\partial_v(\pi) = 0$. We may assume that ϕ represents 1 over F (because we may replace ϕ/F by a similar form). Then ϕ_L is a subform of π so that we can write π as $\phi_L \perp \phi'$. Since $\partial_v(\pi) = \partial_v(\phi_L) + \partial_v(\phi')$ and $\partial_v(\phi_L) = 0$, the Witt class $\partial_v(\pi) \in W(k_v)$ is represented by a form of dimension $\leq \dim \phi' < \frac{1}{2} \dim \pi = 2^{n-1}$.

On the other hand, since π is an n-fold Pfister form, the Witt class $\partial_v(\pi) \in W(k_v)$ is represented by a form similar to an $(n-1)$-fold Pfister form. Comparing with the previous paragraph, we obtain that the form representing $\partial_v(\pi)$ is isotropic. Hence it is hyperbolic, that is, $\partial_v(\pi) = 0$. □

We need some notation concerning the Galois cohomology. We write $H^n(F)$ for the Galois cohomology group $H^n(F, \mathbb{Z}/2\mathbb{Z})$. We write $GP_n(F)$ for the set of (isomorphism classes of) quadratic forms over F which are similar to n-fold

Pfister forms. We write $e^n\colon \mathrm{GP}_n(F) \to H^n(F)$ for the degree n cohomological invariant of such quadratic forms defined as

$$e^n(a\,\langle\langle a_1,,\dots,a_n\rangle\rangle) = (a_1,\dots,a_n)\ ,$$

where $(a_1,\dots,a_n) = (a_1)\cup\dots\cup(a_n)$. For a field extension L/F we write $H^n(L/F)$ for the relative Galois cohomology group $\mathrm{Ker}(H^n(F)\to H^n(L))$, and we write $H^n_{\mathrm{ur}}(L/F)$ for the group of the cohomology classes in $H^n(L)$ unramified over F. Note that $H^n(L/F)\subset H^n(F)$ while $H^n_{\mathrm{ur}}(L/F)\subset H^n(L)$. Recall that the unramified cohomology group $H^n_{\mathrm{ur}}(L/F)$ is defined in the similar way as $W_{\mathrm{ur}}(L/F)$:

$$H^n_{\mathrm{ur}}(L/F) = \bigcap \mathrm{Ker}(\partial_v)\ ,$$

where the intersection runs over all discrete valuations of L trivial on F and $\partial_v\colon H^n(L)\to H^{n-1}(k_v)$ is the residue homomorphism.

One more convention: we shall write $H^n_{\mathrm{ur}}(L/F)/H^n(F)$ for the cokernel of the restriction homomorphism $H^n(F)\to H^n_{\mathrm{ur}}(L/F)$ even in the case where the restriction homomorphism is not injective.

Corollary 7.2. *In the condition of Lemma 7.1, the cohomological invariant $e^n(\pi)\in H^n_{\mathrm{nr}}(L)$ is unramified over F.*

Proof. It follows from the formula $\partial_v(e^n(\pi)) = e^{n-1}(\partial_v(\pi))$. Note that we do note use the fact that the cohomological invariant $e^n\colon I^n(L)\to H^n(L)$ is well defined on the whole $I^n(L)$: we only apply it to quadratic forms from GP_n. □

The unramified cohomology group $H^4_{\mathrm{ur}}(F(\psi)/F)$ of the function field of a quadratic form ψ/F, as well as the relative cohomology group $H^4(F(\psi)/F)$ was investigated in [14]. We shall use only the following list of results obtained their:

Theorem 7.3 ([14])**.** *We consider quadratic forms ψ/F with $\dim\psi\ge 9$.*

(i) *For any ψ being a 4-fold Pfister form there is a monomorphism*

$$H^4_{\mathrm{ur}}(F(\psi)/F)/H^4(F)\hookrightarrow H^4(F)$$

natural in F.

(ii) *For any ψ which is not a 4-fold Pfister neighbor, there is a monomorphism $H^4_{\mathrm{ur}}(F(\psi)/F)/H^4(F)\hookrightarrow \mathrm{Tors}\,\mathrm{CH}^3(X_\psi)$, where $\mathrm{Tors}\,\mathrm{CH}^3(X_\psi)$ is the torsion subgroup of the Chow group $\mathrm{CH}^3(X_\psi)$ of the projective quadric given by ψ.*

(iii) *For any ψ which is not a 4-fold Pfister neighbor, the relative cohomology group $H^4(F(\psi)/F)$ is trivial.*

Now we recall that the goal of this section is the proof of the following statement: if ϕ/F is an essential form and ψ/F is an arbitrary quadratic form of dimension 10, then the form $\phi_{F(\psi)}$ is essential. To prove this, it suffices to find a field extension E/F such that the form ϕ_E is still essential while the form ψ_E is isotropic.

To begin we show that one can always climb over the function field of a 4-fold Pfister form (which will allow us later on to kill the Galois cohomology of the base field in degree 4).

Proposition 7.4. *Let ϕ/F be an essential quadratic form and let q/F be a 4-fold Pfister form. Then the form $\phi_{F(q)}$ is still essential.*

Proof. We know already that $\operatorname{ind} C_0(\phi_{F(q)}) \geq 4$ and that the form $\phi_{F(q)}$ is anisotropic. The only thing to check is that $\phi_{F(q)}$ does not become a Pfister neighbor.

Let us assume the contrary: $\phi_{F(q)}$ is a neighbor of some 4-fold Pfister form $\pi/F(q)$. The element $e^4(\pi) \in H^4_{\mathrm{ur}}(F(q)/F)$ is different from 0 (since the form $\phi_{F(q)}$ is anisotropic, the form π is anisotropic too, therefore $e^4(\pi) \neq 0$ simply by the classical "injectivity on symbols" known for e^n with any n).

Applying Theorem 7.3 (i) to the field extension $F(\phi)/F$, we get a commutative diagram

$$\begin{array}{ccc} H^4_{\mathrm{ur}}(F(q)/F)/H^4(F) & \longrightarrow & H^4_{\mathrm{ur}}(F(\phi,q)/F(\phi))/H^4(F(\phi)) \\ \downarrow & & \downarrow \\ H^4(F) & \longrightarrow & H^4(F(\phi)) \ , \end{array}$$

where $F(\phi, q)$ is the function field of the direct product of the projective quadrics $\phi = 0$ and $q = 0$. Note that the vertical arrows of the diagram are monomorphisms (Theorem 7.3(i)). Moreover, the lower horizontal arrow is a monomorphism as well (Theorem 7.3 (iii)). Hence the upper horizontal arrow is a monomorphism, too. By this reason, the class of $e^4(\pi)$ in the quotient $H^4_{\mathrm{ur}}(F(q)/F)/H^4(F)$, evidently vanishing in the quotient

$$H^4_{\mathrm{ur}}(F(\phi,q)/F(\phi))/H^4(F(\phi)) \ ,$$

is 0, that is, $e^4(\pi)$ is in the image of the restriction homomorphism $H^4(F) \to H^4_{\mathrm{ur}}(F(q)/F)$, say $e^4(\pi) = \lambda_{F(q)}$ for some $\lambda \in H^4(F)$.

For this λ, we have $\lambda_{F(\phi,q)} = e^4(\pi)_{F(\phi)} = 0$, whence

$$\lambda_{F(\phi)} \in H^4(F(\phi,q)/F(\phi)) \ .$$

Since $q_{F(\phi)}$ is a 4-fold Pfister form, we have ([13] and [26])

$$H^4(F(\phi,q)/F(\phi)) = \{0, \ e^4(q)_{F(\phi)}\} \ ,$$

whence $\lambda_{F(\phi)} = 0$ or $\lambda_{F(\phi)} = e^4(q)_{F(\phi)}$. By the injectivity of $H^4(F) \to H^4(F(\phi))$ (Theorem 7.3 (iii)) we get that $\lambda = 0$ or $\lambda = e^4(q)$ already over F. Therefore $\lambda_{F(q)} = 0$ which is a contradiction with $\lambda_{F(q)} = e^4(\pi) \neq 0$. □

Corollary 7.5. *For any F and any essential ϕ/F there exists a field extension $\tilde{F}/F$ such that $H^4(\tilde{F}) = 0$ while $\phi_{\tilde{F}}$ is still essential.*

Proof. The extension $\tilde{F}/F$ we construct is common for all essential ϕ/F. Let $F_0 = F$ and for every $i \geq 0$ let F_{i+1} be the free composite of the function fields of all 4-fold Pfister forms over F_i. The union $\tilde{F} = \cup F_i$ is a field extension of F

with trivial $I^4(\tilde{F})$ and the form $\phi_{\tilde{F}}$ is still essential. Of course, we may conclude that $H^4(\tilde{F}) = 0$ by using the fact that $H^4(\tilde{F})$ is generated by $e^4(\mathrm{GP}_4(\tilde{F}))$. The things are much more simple however. If for every F_i we consider a maximal odd extension E_i/F_i and put $\tilde{F} = \cup E_i$, then this new $\tilde{F}$ is a field with trivial $I^4(\tilde{F})$ and without odd extensions. Therefore $H^4(\tilde{F}) = 0$ already by [1]. To show that $\phi_{\tilde{F}}$ is essential for this choice of $\tilde{F}$ one uses [19, cor. 1.12]. □

Definition 7.6. We say that an anisotropic quadratic form q/F is *special*, if

1. $\dim q = 9$ or 10;
2. for a 9-dimensional q, we require that $\operatorname{ind} C_0(q) \leq 2$;
3. $\operatorname{Tors} \mathrm{CH}^3 X_q = 0$.

Remark 7.7. The 2nd condition ensures that a special form is never similar to an essential form.

Proposition 7.8. *Assume that F is a field with $H^4(F) = 0$, ϕ/F is an essential and q/F a special quadratic forms. Then the form $\phi_{F(q)}$ is also essential.*

Proof. Since $q \not\sim \phi$ (Remark 7.7), it follows by Theorem 6.1 that the form $\phi_{F(q)}$ is anisotropic. Therefore, if $\phi_{F(q)}$ is a neighbor of a 4-fold Pfister form $\pi/F(q)$, the cohomology class $e^4(\pi) \in H^4_{\mathrm{ur}}(F(q)/F)$ is non-trivial.

On the other hand, since q is special, the restriction $H^4(F) \to H^4_{\mathrm{ur}}(F(q)/F)$ is an epimorphism by Theorem 7.3 (ii), while $H^4(F) = 0$. We get a contradiction. □

Now we recall that for given essential form ϕ and 10-dimensional form ψ over a field F, we are looking for a field extension E/F such that ψ_E is isotropic while ϕ_E is still essential. For this we need a list of special forms which is "large enough". Note that one can not take all 10-dimensional forms in such a list because not all of them are special (there are 10-dimensional forms Q with non-trivial torsion in $\operatorname{Tors} \mathrm{CH}^3 X_q$, see [10, thm. 0.5]); also we can not simply take all 10-dimensional quadratic forms q with no torsion in $\operatorname{Tors} \mathrm{CH}^3 X_q$: it is not clear whether such a list is large enough. One possible choice of the list is given in the following definition. We use some 9-dimensional quadratic forms as well. This choice is particularly nice because the absence of torsion in $\operatorname{Tors} \mathrm{CH}^3 X_q$ is particularly easy to check for the forms q of this list (we note that the Chow group $\mathrm{CH}^3 X_q$ is computed for all quadratic forms q of all dimensions ≥ 9 in [10, thm. 0.5]).

Definition 7.9. An anisotropic quadratic form q is called *particular* if it is of one of the following 4 types:

(i) q with $\dim q = 10$ and $\operatorname{ind} C_0(q) \geq 4$;
(ii) q with $\dim q = 10$, $\operatorname{ind} C_0(q) = 2$, such that q contains a subform $q' \subset q$ with $\dim q' = 8$ and $\operatorname{disc} q' = 1$;
(iii) q with $\dim q = 9$, $\operatorname{ind} C_0(q) = 2$, such that q contains a subform $q' \subset q$ with $\dim q' = 8$ and $\operatorname{disc} q' = 1$;

(iv) q with $\dim q = 9$, $\operatorname{ind} C_0(q) = 2$, such that q contains a 7-dimensional Pfister neighbor $q' \subset q$.

Proposition 7.10. *A particular quadratic form is special.*

The proof of the proposition will be given in the next section. Now we only check that such a list of special forms is really big enough. First of all we notice that the particular forms are particularly nice because of the following additional property:

Lemma 7.11. *Let q/F be particular and let $\tilde{F}/F$ be the extension constructed in Corollary 7.5. Then $q_{\tilde{F}}$ is also particular.*

Proof. By the construction of $\tilde{F}/F$ it suffices to check that $q_{F(\pi)}$ is particular for any 4-fold Pfister form π/F.

By Hoffmann's theorem $q_{F(\pi)}$ is anisotropic.

Since $C_0(\pi) \simeq M_{2^7}(F) \times M_{2^7}(F)$, where $M_n(F)$ is the algebra of the $n \times n$-matrices over F, for any central division algebra D there is no homomorphisms $C_0(\pi) \to D$. It follows by the index reduction formula that $\operatorname{ind} C_0(q_{F(\pi)}) = \operatorname{ind} C_0(q)$ for any q/F. □

Corollary 7.12. *Let F be an arbitrary field, ϕ/F essential, and q/F particular. Then $\phi_{F(q)}$ is also essential.*

Proof. The form $\phi_{\tilde{F}(q)}$ is essential. □

The following statement shows that the list of special forms given by the particular ones is "large enough":

Lemma 7.13. *Let ψ be a 10-dimensional quadratic form over a field F. There exists a finite chain of field extensions $F = F_0 \subset F_1 \subset \cdots \subset F_n$ such that ψ_{F_n} is isotropic and every step F_{i+1}/F_i is the function field either of a particular form or of a 4-fold Pfister form.*

Proof. We assume that ψ/F is anisotropic (otherwise we take $n = 0$).

If $\operatorname{ind} C_0(\psi) \geq 4$, then ψ is particular of type (i). So, we may simply take $n = 1$ with $F_1 = F(\psi)$.

Now we assume that $\operatorname{disc} \psi = 1$. If $\operatorname{ind} C_0(\psi) = 1$, the form ψ is isotropic ([23]), so that we assume $\operatorname{ind} C_0(\psi) = 2$. Such a form ψ contains a 7-dimensional Pfister neighbor q' ([5, thm. 5.1]). Let q be an "intermediate" 9-dimensional form: $q' \subset q \subset \psi$. Since $\operatorname{ind} C_0(q) = \operatorname{ind} C_0(\psi) = 2$, q is a form of type (iv), and ψ is isotropic over $F(q)$.

At this stage we have already shown that we can make isotropic any 10-dimensional quadratic form over F with trivial discriminant. Hence for a given 9-dimensional form over F one may assume that it contains an 8-dimensional subform of trivial discriminant. It remains us to show that every ψ with $\operatorname{ind} C_0(\psi) \leq 2$ is isotropic in this situation.

If $\operatorname{ind} C_0(\psi) = 2$, then ψ is of the type (ii), hence there is no problem with such ψ.

Finally, we assume that $\operatorname{ind} C_0(\psi) = 1$. We choose a 9-dimensional subform $q \subset \psi$. We have $C_0(\psi) \simeq C_0(q) \otimes_F F(\sqrt{d})$ with $d = \operatorname{disc}(\psi)$. Therefore $\operatorname{ind} C_0(q) = 1$ or 2. In the second case, q is of the type (iii), while in the first case q is a neighbor of a 4-fold Pfister form. □

We have finished the part **(3)** of the proof of Theorem 5.3 modulo computation of CH^3 for the particular forms needed for Proposition 7.10. This computation will be done in the next section.

8. Computing CH^3

In this section we prove Proposition 7.10. More precisely, we prove that $\operatorname{Tors}\mathrm{CH}^3 X_q = 0$ for any *particular* (see Definition 7.9) quadratic form q.

Lemma 8.1. *Every 9-dimensional quadratic form q with $\operatorname{ind} C_0(q) = 4$ is a subform of some 13-dimensional quadratic form ρ with $\operatorname{ind} C_0(\rho) = 1$.*

Proof. Let $q\perp \langle a\rangle$ be the 10-dimensional quadratic form of discriminant 1 containing q. The Clifford invariant $[C(q\perp \langle a\rangle)] = [C_0(q)] \in \mathrm{Br}(F)$ of this form is represented by a biquaternion algebra. Let $\langle -a\rangle \perp q'$ be an Albert form corresponding to this biquaternion algebra (the quadratic form q' here is 5-dimensional with $\det q' = a$ and $C_0(q')$ Brauer-equivalent to $C_0(q)$). Since the Clifford invariant of the Witt class

$$[q\perp \langle a\rangle] + [\langle -a\rangle \perp q'] = [q\perp q'] \in W(F)$$

is trivial, one can take $\rho = q\perp q''$ where q'' is a 4-dimensional subform of q' (in this case ρ is a 13-dimensional subform of the 14-dimensional form $q\perp q'$ with trivial $\operatorname{disc}(q\perp q')$, and therefore $[C_0(\rho)] = [C(q\perp q')] = 0 \in \mathrm{Br}(F)$). □

Corollary 8.2. *For any 9-dimensional quadratic form q with $\operatorname{ind} C_0(q) = 4$, one has $\operatorname{Tors}\mathrm{CH}^3 X_q = 0$.*

Proof. We write $K(X)$ for the Grothendieck group $K_0'(X)$ of a variety X. We consider the topological filtration on $K(X)$ given by the codimension of support and write $K^{(i)}(X)$ $(i \geq 0)$ for its i-th term. Since the canonical epimorphism $CH^i(X) \twoheadrightarrow K^{(i)}(X)/K^{(i+1)}(X)$ is an isomorphism for $i \leq 3$ in the case where X is a projective quadric (see [15, cor. 4.5] for $i = 3$), it suffices to show that the successive quotient group $K^{(3)}(X_q)/K^{(4)}(X_q)$ is torsion-free for q as in the statement under proof. According to [15, thm. 3.8], this is equivalent to the fact that $l_1 \in K^{(4)}(X_q)$ where $l_1 \in K(\bar{X}_q)$ is the class of a line on $\bar{X}_q$ (given by some totally isotropic 2-dimensional subspace of $q_{\bar{F}}$). Note that according to Swan's computation [25] of the K-theory of projective quadrics, $K(X_q)$ is a subgroup of $K(\bar{X}_q)$ containing l_1.

Let ρ be a 13-dimensional quadratic form as in Lemma 8.1. Since $\operatorname{ind} C_0(\rho) = 1$, we have $l_5 \in K(X_\rho)$ ([25]) for the class l_5 of a 5-dimensional projective subspace on $\bar{X}_\rho$. Since $\dim\rho$ is bigger than 12, the group $\mathrm{CH}^3(X_\rho)$ is torsion-free by [16]. Note that the groups $CH^i(X_\rho)$ for $i < 3$ are torsion-free as well (see [15, thm. 6.1] for $i = 2$). It follows that the groups $K^{(i)}(X_\rho)/K^{(i+1)}(X_\rho)$

are torsion-free for $i \leq 3$ by which reason $l_5 \in K^{(4)}(X_\rho)$. Applying to this l_5 the pull-back $K^{(4)}(X_\rho) \to K^{(4)}(X_q)$ with respect to the imbedding $X_q \hookrightarrow X_\rho$, we get l_1 (because $\operatorname{codim}_{X_\rho} X_q = 4$). Thus $l_1 \in K^{(4)}(X_q)$. □

Corollary 8.3. *For any 9-dimensional quadratic form q with $\operatorname{ind} C_0(q) \geq 4$, one has $\operatorname{Tors} \operatorname{CH}^3 X_q = 0$ as well.*

Proof. The possible values of $\operatorname{ind} C_0(q)$ (q is 9-dimensional) greater than 4 are 8 and 16. In the case of the maximal index, there is no torsion in the successive quotients of the topological filtration on $K(X_q)$ at all ([15, thm. 3.8]).

Let $\operatorname{ind} C_0(q) = 8$. To see that there is no torsion in $\operatorname{CH}^3(X_q)$ is is enough to show that $l_0 \in K^{(4)}(X_q)$ where $l_0 \in K(X_q) \subset K(\bar{X}_q)$ is the class of a rational point.

We may assume that the base field F has no extensions of odd degree. Then there exists a quadratic field extension E/F such that $\operatorname{ind} C_0(q_E) = 4$. It follows by Corollary 8.2 that $l_1 \in K^{(4)}(X_{q_E})$ over E. Taking the transfer we get that $2l_1 \in K^{(4)}(X_q)$ over F. Since $2l_1 = h^6 + l_0 \in K(X_q)$ where $h^6 \in K^{(6)}(X_q)$ is the 6-th power of the hyperplane section class $h \in K^{(1)}(X_q)$ (cf. [15, proof of lemma 3.9]), the desired relation $l_0 \in K^{(4)}(X_q)$ follows. □

Corollary 8.4. *Let q be an 8-dimensional quadratic form, $a \in F^*$, and let $U_{q,a}$ be the affine quadric $q + a = 0$. If $\operatorname{ind} C_0(q \perp \langle a \rangle) \geq 4$ then $\operatorname{CH}^3 U_{q,a} = 0$.*

Proof. Since $U_{q,a}$ is the complement of X_q in $X_{q\perp\langle a\rangle}$, we have the exact sequence

$$\operatorname{CH}^2 X_q \to \operatorname{CH}^3 X_{q\perp\langle a\rangle} \to \operatorname{CH}^3 U_{q,a} \to 0 .$$

The middle term is torsion-free by Corollary 8.3, therefore it is generated by the 3-d power h^3 of the hyperplane section $h \in \operatorname{CH}^1 X_{q\perp\langle a\rangle}$. Since this h^3 is the image of $h^2 \in \operatorname{CH}^2 X_q$, the first arrow of the exact sequence is surjective. □

Lemma 8.5. *Let q be an 8-dimensional quadratic form over F and let $a \in F$. If either $a \neq 0$ or q is not similar to a 3-fold Pfister form, then $\operatorname{Tors} \operatorname{CH}^2 U_{q,a} = 0$.*

Proof. We first consider the case where $a \neq 0$. Here the group $\operatorname{Tors} \operatorname{CH}^2 X_{q\perp\langle a\rangle}$ is torsion-free by [15, 6.1], and the exact sequence

$$\operatorname{CH}^1 X_q \to \operatorname{CH}^2 X_{q\perp\langle a\rangle} \to \operatorname{CH}^2 U_{q,a} \to 0$$

gives the statement desired.

For $a = 0$ the following sequence is exact:

$$\operatorname{CH}^1 X_q \to \operatorname{CH}^2 X_q \to \operatorname{CH}^2 U_{q,a} \to 0$$

with the first arrow given by multiplication by h. Since q is not similar to a 3-fold Pfister form, the middle term is generated by h^2 ([15, 6.1]) which is the image of $h \in \operatorname{CH}^1 X_q$. □

Now we are able to prove that $\operatorname{Tors} \operatorname{CH}^3 X_q = 0$ for a particular form q of type (i). Let us write q as $q = q' \perp \langle a \rangle$. The exact sequence

$$\operatorname{CH}^2 X_{q'} \to \operatorname{CH}^3 X_q \to \operatorname{CH}^3 U_{q',a} \to 0$$

gives an isomorphism of $\operatorname{Tors}\operatorname{CH}^3 X_q$ with $\operatorname{CH}^3 U_{q',a}$. For q' written down as $q' = q'' \perp \langle b\rangle$, we have an exact sequence as follows (cf. [15, §1.3.2]):

$$\coprod_p \operatorname{CH}^2 U_{q''_{F(p)},bt^2+a} \to \operatorname{CH}^3 U_{q',a} \to \operatorname{CH}^3 U_{q''_{F(t)},bt^2+a} \to 0$$

where the direct sum is taken over all closed points p of the affine line $\mathbb{A}^1 = \operatorname{Spec} F[t]$, t a variable (here $bt^2 + a$ is considered as an element of the residue field $F(p)$). We claim that the both side terms of the exact sequence are 0 (this gives the triviality of the middle term and finishes the proof of Proposition 7.10 for the particular forms of type (i)).

The even Clifford algebra of an even-dimensional quadratic form is isomorphic to the even Clifford algebra of any 1-codimensional subform tensored by the etale quadratic F-algebra given by the square root of the discriminant of the even-dimensional form. Applying this to $q''_{F(t)} \perp \langle bt^2 + a\rangle \subset q_{F(t)}$ we get:

$$C_0(q_{F(t)}) \simeq C_0(q''_{F(t)} \perp \left\langle bt^2 + a\right\rangle) \otimes_{F(t)} F(t)(\sqrt{\operatorname{disc} q})\ .$$

In particular,

$$\operatorname{ind} C_0(q''_{F(t)} \perp \left\langle bt^2 + a\right\rangle) \geq \operatorname{ind} C_0(q_{F(t)}) = \operatorname{ind} C_0(q) \geq 4\ .$$

By Corollary 8.4 it follows that $\operatorname{CH}^3 U_{q''_{F(t)},bt^2+a} = 0$.

Now let us consider a summand $\operatorname{CH}^2 U_{q''_{F(p)},bt^2+a}$ from the left hand side term of the exact sequence. If $bt^2+a \neq 0 \in F(p)$, this summand is 0 by the first part of Lemma 8.5. Let us assume that $bt^2 + a = 0 \in F(p)$. This may happen only for a unique closed point $p \in \mathbb{A}^1$, namely, for the point given by the principle prime ideal of the polynomial ring $F(t)$ generated by $bt^2 + a$. In particular, $F(p) \simeq F(\sqrt{-a/b})$. If the form $q''_{F(p)}$ is not similar to a 3-fold Pfister form, $\operatorname{CH}^2 U_{q''_{F(p)},0} = 0$ according to the second part of Lemma 8.5. In the opposite case, $q''_{F(p)}$ has the trivial discriminant and Clifford invariant. Since $[q''_{F(p)}] = [q_{F(p)}] \in W(F(p))$, the quadratic form $q_{F(p)}$ also has the trivial discriminant and Clifford invariant. In particular, $\operatorname{ind} C_0(q_{F(p)}) = \operatorname{ind} C(q_{F(p)}) = 1$ (here we use that the even Clifford algebra of an even-dimensional quadratic form with trivial discriminant is isomorphic to $A \times A$, where A is a central simple algebra such that the algebra of 2 by 2 matrices over A is isomorphic to the whole Clifford algebra of the quadratic form). On the other hand, $\operatorname{ind} C_0(q_{F(p)})$ is at least 2, because $\operatorname{ind} C_0(q) \geq 4$ and $[F(p) : F] = 2$. Thus every particular form of type (i) is special.

Now let us check that a particular form q of type (iv) is special. In order to show that $\operatorname{Tors}\operatorname{CH}^3 X_q = 0$, it suffices to show that $l_2 \in K^{(4)}(X_q)$. Let q' be a 7-dimensional Pfister neighbor sitting insight of q. According to Swan's computation of $K(X_{q'})$, the element $l_2 \in K(\bar{X}_{q'})$ lies in $K(X_{q'}) \subset K(\bar{X}_{q'})$. Since the quotients $K^{(0)}(X_{q'})/K^{(1)}(X_{q'})$ and $K^{(1)}(X_{q'})/K^{(2)}(X_{q'})$ have no torsion, the element l_2 is in $K^{(2)}(X_{q'})$. Now taking the push-forward of this l_2 with respect to the 2-codimensional imbedding $X_{q'} \hookrightarrow X_q$, we get $l_2 \in K^{(4)}(X_q)$. Thus every particular form of type (iv) is special as well.

For a particular form q of type (iii) we will use the 1-codimensional imbedding $X_{q'} \hookrightarrow X_q$, where $q' \subset q$ is an 8-dimensional subform of trivial discriminant. The Clifford invariant of q' is represented by the even Clifford algebra of q which has index 2 and therefore is non-trivial. Hence q' is not similar to a 3-fold Pfister form and according to [15, thm. 6.1] the group $\mathrm{CH}^2 X_{q'}$ is torsion-free. We obtain that $l_2 \in K^{(3)}(X_{q'})$ and, taking the push-forward, $l_2 \in K^{(4)}(X_q)$. Thus every particular form of type (iii) is special.

Finally, consider a particular quadratic form q of type (ii). Let E be the quadratic field extension of F given by the square root of the discriminant of q. The form q_E has trivial discriminant and $\operatorname{ind} C_0(q_E) = 2$. According to [15, prop. 3.5], $2l_4 \in K(X_{q_E})$ where $l_4 \in K(\bar{X}_q)$ is the class of a 4-dimensional projective subspace on $\bar{X}$. Note that $4 = (\dim X)/2$ by which reason it is not true that all the 4-dimensional subspaces on $\bar{X}$ have the same class in the Chow group: there are precisely two different classes of such subspaces. We have denoted one of them as l_4 and we write l'_4 for the second one.

For the subform $q' \subset q$ as in the definition of this type of particular forms, we have $q_E \simeq q'_E \perp \mathbb{H}$. Therefore, for $i = 1, 2, 3$ there are isomorphisms $\mathrm{CH}^i X_{q_E} \simeq \mathrm{CH}^{i-1} X_{q'_E}$ ([15, §2.2]). It follows that the isomorphic groups are torsion-free ($\mathrm{CH}^2 X_{q'_E}$ is so because $\operatorname{ind} C(q'_E) = 2$ and so q'_E is not similar to a 3-fold Pfister form) and therefore $2l_4 \in K^{(4)}(X_{q_E})$. Applying the transfer homomorphism $K^{(4)}(X_{q_E}) \to K^{(4)}(X_q)$ to the element $2l_4$, we get $2(l_4 + l'_4)$. Using the relation $l_4 + l'_4 = h^4 + l_3 \in K(\bar{X}_q)$, we get $2l_3 = 2(l_4 + l'_4) - 2h^4 \in K^{(4)}(X_q)$. Finally, since $2l_3 = l_2 + h^5$, it follows that $l_2 \in K^{(4)}(X_q)$. Hence the group $K^{(3)}(X_q)/K^{(4)}(X_q) \simeq \mathrm{CH}^3 X_q$ has no torsion, i.e., q is special.

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