MINIMAL CANONICAL DIMENSIONS
OF QUADRATIC FORMS

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Abstract. Canonical dimension of a smooth complete connected variety is the minimal dimension of image of its rational endomorphism. The i-th canonical dimension of a non-degenerate quadratic form is the canonical dimension of its i-th orthogonal grassmannian. The maximum of a canonical dimension for quadratic forms of a fixed dimension is known to be equal to the dimension of the corresponding grassmannian. This article is about the minima of the canonical dimensions of an anisotropic quadratic form. We conjecture that they equal the canonical dimensions of an excellent anisotropic quadratic form of the same dimension and we prove it in a wide range of cases.

Contents

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1. Introduction

The *canonical dimension* $\mathrm{cd}(X)$ of a smooth complete connected algebraic variety X over a field F is the minimum of dimension of the image of a rational map $X \dashrightarrow X$. This integer depends only on the class of field extensions L/F with $X(L) \neq \emptyset$. We refer to [9] and [16] for interpretations and basic properties of $\mathrm{cd}(X)$. We will also use a 2-local version $\mathrm{cd}_2(X)$ of $\mathrm{cd}(X)$ called *canonical* 2*-dimension*.

All fields here are of characteristic $\neq 2$. (The questions we are discussing can be raised in characteristic 2 as well, but all results we get are for characteristic $\neq 2$ mainly because their proofs need the Steenrod operations on Chow groups modulo 2 which are not available in characteristic 2.)

Let φ be a non-degenerate quadratic form over a field F. (Our general reference for quadratic forms is [3].) For any integer i lying in the interval $[1,\ (\dim\varphi)/2]$, the i-th canonical dimension $\mathrm{cd}[i](\varphi)$ is defined as the canonical dimension of the orthogonal grassmannian of i-dimensional totally isotropic subspaces of φ (*i-grassmannian of φ* for short). A little care should be given to the case of $i = (\dim\varphi)/2$ because the corresponding i-grassmannian is not connected if the discriminant of φ is trivial. However, the (two) connected components it has are isomorphic to each other so that we can define the canonical dimension by taking any of them.

For arbitrary i and a given field extension L/F, the i-grassmannian of φ has an L-point if and only if the Witt index $\mathfrak{i}_0(\varphi_L)$ is at least i. Therefore, $\mathrm{cd}[i](\varphi)$ is an invariant of the class of field extensions L/F satisfying $\mathfrak{i}_0(\varphi_L) \geq i$.

Similarly, the i-th canonical 2-dimension $\mathrm{cd}_2[i](\varphi)$ is the canonical 2-dimension of the i-grassmannian. Since in general, canonical 2-dimension is a lower bound for canonical dimension, we have $\mathrm{cd}[i](\varphi) \geq \mathrm{cd}_2[i](\varphi)$ for any i. This is known to be equality for $i = 1$ (see Section 5) and no example when this inequality is not an equality (for some $i > 1$) is known.

The study of canonical dimensions of quadratic forms naturally commences with the question about the range of their possible values for anisotropic quadratic forms of a fixed dimension (over all fields or over all field extensions of a given field). It has been shown in [12] (see also [13]) that the evident upper bound on $\mathrm{cd}[i](\varphi)$ and $\mathrm{cd}_2[i](\varphi)$, given by the dimension of the i-grassmannian, is sharp. Here is a formula for this dimension:

$$i(i-1)/2 + i(\dim\varphi - 2i).$$

The question on the sharp upper bound being therefore closed, the present paper addresses the question about the sharp lower bound. Natural candidates are canonical dimensions of excellent quadratic forms. We do not really have a strong evidence supporting this, but we may, for instance, recall [3, Theorem 84.1] where the excellent forms appear in the answer to the question about the minimal height of quadratic forms.

For any $n \geq 1$ and any $i \in [1,\ n/2]$, we write $\mathrm{cd}[i](n)$ (resp., $\mathrm{cd}_2[i](n)$) for the i-th canonical (2-)dimension of an anisotropic excellent n-dimensional quadratic form over some field. Note that $\mathrm{cd}[i](n)$ depends only on i, n and coincides with $\mathrm{cd}_2[i](n)$ (see Section 2).

The following conjecture therefore gives a complete answer to the question about the sharp lower bound on canonical dimension and canonical 2-dimension of anisotropic quadratic forms:

Conjecture 1.1. *Let φ be an anisotropic quadratic form over a field F satisfying* $\dim\varphi > 2i$ *for some* $i \geq 1$*. Then* $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$.

The reason of excluding the case $2i = \dim\varphi$ in the statement is that in this case $\mathrm{cd}_2[i](\varphi) = \mathrm{cd}_2[i-1](\varphi_E)$ and $\mathrm{cd}[i](\varphi) = \mathrm{cd}[i-1](\varphi_E)$, where E/F is the discriminant field extension of φ ($E = F$ if the discriminant of φ is trivial) and $i \geq 2$. So, understanding of $\mathrm{cd}_2[i](\varphi)$ and $\mathrm{cd}[i](\varphi)$ for $i < (\dim\varphi)/2$ would provide their understanding for $i = (\dim\varphi)/2$ and, on the other hand, using these relations it is easy to get counter-examples to the formula of Conjecture 1.1 with $i = (\dim\varphi)/2$ (see Section 9).

In this paper we prove Conjecture 1.1 for "small" values of i, namely, for i not exceeding the 2-nd absolute Witt index of φ (see Theorem 6.1) as well as for $i \leq 5$ (see Theorems 7.1, 10.1 and 11.1). Finally, we prove Conjecture 1.1 with arbitrary i for all quadratic forms of height ≤ 3 (see Theorem 8.2).

The proofs make use of a wide spectrum of modern results on quadratic forms and Chow motives (the question seems to be a good testing ground for them). However most of the results under use already became "classical" at least in the sense that they have been exposed in a book (in [3] in most of the cases). For instance, we are using only a part of Excellent Connections Theorem [20, Theorem 1.3], called Outer, which was available already before the whole result and is exposed in [3, Corollary 80.13].

The most recent (and certainly yet non-classical) tool is a kind of going down principle for Chow motives due to Charles De Clercq [2], used in the proofs of Theorem 3.2 and (in a slightly different situation) Theorem 8.2. Applications of some particular cases of this principle exist already in the literature (see, e.g., [4]). We are using it here (in the proof of Theorem 3.2) in a new situation (still not in its full generality but in the biggest generality which may occur in the case of projective homogeneous varieties). This principle generalizes [10, Proposition 4.6], this older result is not sufficient for our purposes here.

Those methods can certainly be used to prove a bit more of Conjecture 1.1, but it seems that something is missing for a complete solution.

One could expect that the case of maximal i should be more accessible because maximal orthogonal grassmannians are so well-understood (mainly due to results of [19] also exposed in [3, Chapter XVI]). Though in our approach we have to go through all values of i in order to get to the maximal one.

This paper is an extended version of [6].

For more introduction see §12.

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2. Excellent forms

Here we recall some standard facts about excellent forms needed to complete the statement of Conjecture 1.1. Proofs (along with a definition) can be found, e.g., in [3, §28].

Every positive integer n is uniquely representable in the form of an alternating sum of 2-powers:

$$n = 2^{p_0} - 2^{p_1} + 2^{p_2} - \cdots + (-1)^{r-1}2^{p_{r-1}} + (-1)^r 2^{p_r}$$

for some integers $r \geq 0$ and $p_0, p_1, \dots, p_r$ satisfying $p_0 > p_1 > \dots > p_{r-1} > p_r + 1 > 0$.

For any integer $i \in [1,\ n/2]$, we define an integer $\operatorname{cd}[i](n)$ as

$$\operatorname{cd}[i](n) := 2^{p_{s-1}-1} - 1,$$

where s is the minimal positive integer with

$$n - 2i \geq 2^{p_s} - 2^{p_{s+1}} + \dots + (-1)^{r-s} 2^{p_r}.$$

Note that $\operatorname{cd}[i](n) \geq \operatorname{cd}[i+1](n)$ (for any i, n such that both sides are defined).

Lemma 2.1. *For any field k and any positive integer n, there exists an n-dimensional anisotropic quadratic form φ over an appropriate extension field F/k such that*

$$\operatorname{cd}[i](\varphi) = \operatorname{cd}_2[i](\varphi) = \operatorname{cd}[i](n)$$

for any $i \in [1,\ n/2]$.

Proof. One may take as F a field extension of k generated by p_0 algebraically independent elements. (For $k \subset \mathbb{R}$ one may simply take $F = \mathbb{R}$.) Then there exists an anisotropic p_0-fold Pfister form over F and therefore an anisotropic excellent quadratic form φ of dimension n. (For $F = \mathbb{R}$, the unique up to isomorphism anisotropic n-dimensional quadratic form is excellent.) We claim that canonical dimensions of such φ are as required. Indeed, for $i \in [1,\ n/2]$ let s be the defined above integer. Then by [3, Theorem 28.3], there exists a p_{s-1}-fold Pfister form ρ over F such that for any field extension L/F the condition $\mathfrak{i}_0(\varphi_L) \geq i$ is equivalent to isotropy of ρ_L. It follows that $\operatorname{cd}_2[i](\varphi) = \operatorname{cd}[i](\varphi) = 2^{p_{s-1}-1} - 1$. □

3. Upper motives

By *motives* we always mean the Chow motives with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$; we use related terminology and notation as in [3, Chapter XII]. In particular, $M(X)$ is the motive of a variety X; the motive $M(\operatorname{Spec} F)$ and all its shifts $M(\operatorname{Spec} F)(i)$, $i \in \mathbb{Z}$, are called Tate motives. If M is a motive over F, $\bar{M}$ is the corresponding motive over an algebraic closure of F.

Let φ be a non-degenerate quadratic form over a field F. For an integer i with $0 \leq i < \dim \varphi/2$, let $X_i = X_i(\varphi)$ be the i-grassmannian of φ. In particular, X_0 is the point and $X := X_1$ is the projective quadric of φ.

According to the general notion of upper motive, introduced in [14] and [11], the upper motive $U(X_i)$ of the variety X_i is the unique summand in the complete motivic decomposition of X with the property that $\bar{U}(X_i)$ contains a Tate summand with no shift (i.e., with the shift 0). According to the general criterion of isomorphism for upper motives, $U(X_i) \simeq U(X_j)$ if and only if

$$\mathfrak{i}_0(\varphi_L) \geq i \iff \mathfrak{i}_0(\varphi_L) \geq j$$

for any extension field L/F. This means that i and j are in the same semi-open interval $(\mathfrak{j}_{r-1},\ \mathfrak{j}_r]$ for some $r \geq 0$, where $\mathfrak{j}_r$ is the r-th absolute Witt index of φ and $\mathfrak{j}_{-1} := -\infty$.

According to the general [11, Theorem 1.1], applied to quadrics, any summand of the complete motivic decomposition of X is a shift of $U(X_i)$ for some i or – in the case of even-dimensional φ with non-trivial discriminant – $U(\operatorname{Spec} E)$, where E/F is the quadratic discriminant field extension. Shifts of $U((X_i)_E)$, which may a priori appear by [11, Theorem 1.1], aren't possible because for any $j \neq (\dim X)/2$ the motive $\bar{M}(X)$ contains at most one Tate summand with the shift j while $\bar{U}((X_i)_E)$ contains two Tate summands without shift and two Tate summands with the shift $\dim U((X_i)_E)$.

A more precise information can be derived from [18, §4] (see also [3, §73]): if a shift of $U(X_i)$ for some $i \in (\mathfrak{j}_{r-1},\ \mathfrak{j}_r]$ with $r \geq 1$ really appears in the decomposition (note that this is always the case for $r = 1$), then it appears precisely $\mathfrak{i}_r := \mathfrak{j}_r - \mathfrak{j}_{r-1}$ times and the shifting numbers are $\mathfrak{j}_{r-1}, \mathfrak{j}_{r-1} + 1, \dots, \mathfrak{j}_r - 1$. A shift of $U(\operatorname{Spec} E)$ appears if and only if φ_E is hyperbolic in which case it appears only once and with the shifting number $(\dim X)/2$. Note that $U(X_i)$ for $i \leq \mathfrak{j}_0$ is just the motive of a point (= the Tate summand with no shift), it appears precisely $2\mathfrak{j}_0$ times and the shifting numbers are $0, \dots, \mathfrak{j}_0 - 1$ and $\dim X, \dots, \dim X - (\mathfrak{j}_0 - 1)$.

Given any i and setting $Y := X_i$, one can answer the question, whether a shift of $U(Y)$ does appear, in terms of canonical dimension. First of all we have

Theorem 3.1 ([9, Theorem 5.1])**.** $\operatorname{cd}_2(Y) = \dim U(Y)$.

The following result is new. It provides a criterion of appearance of $U(Y)$ and is proved with a help of the going down principle of [2].

Theorem 3.2. *Assume that $i \in (\mathfrak{j}_{r-1},\ \mathfrak{j}_r]$ for some $r \geq 1$ and set $T := X_{\mathfrak{j}_{r-1}}$, $Y := X_i$. A shift of $U(Y)$ appears in the complete motivic decomposition of X if and only if*

$$\operatorname{cd}_2(Y) = \operatorname{cd}_2(Y_{F(T)}).$$

Remark 3.3 (cf. §5)**.** $\operatorname{cd}_2(Y_{F(T)}) = \dim \varphi - 2\mathfrak{j}_{r-1} - \mathfrak{i}_r - 1$.

Remark 3.4. Note that $\operatorname{cd}_2(Y) \geq \operatorname{cd}_2(Y_{F(T)})$ in general, [16].

Remark 3.5. As already mentioned, for $i = \mathfrak{j}_1$, the $\mathfrak{i}_1$ shifts of $U(X_i)$ appear always.

Remark 3.6. Sufficient criteria of appearance given in [18, Theorems 4.15 and 4.17] are easily derived from Theorem 3.2.

Proof of Theorem 3.2. By Theorem 3.1, we may replace $\operatorname{cd}_2(Y)$ with $\dim U(Y)$ as well as $\operatorname{cd}_2(Y_{F(T)})$ with $\dim U(Y_{F(T)})$ in the statement.

If a shift of $U(Y)$ does appear, then $\dim U(Y) = \dim U(Y_{F(T)})$ by [18, §4] (see also [3, §73]). This proves one ("easy") direction of Theorem 3.2. Let us concentrate on the opposite direction.

Note that a shift of $U(Y_{F(T)})$ is a summand in $M(X_{F(T)})$ (see Remark 3.5). If $\dim U(Y) = \dim U(Y_{F(T)})$, then we conclude by [2, Theorem 1.1] that the same shift of $U(Y)$ is a summand in $M(X)$. □

4. Some tools

In this section we recall some results which appear most frequently in the proofs below.

4a. **Outer excellent connections.** The following statement is a part of [20, Theorem 1.3]. It is also proved in [3, Corollary 80.13].

Theorem 4.1 (Outer Excellent Connections)**.** *Let X be the quadric of an anisotropic quadratic form of dimension $2^n + m$ with $n \geq 1$ and $m \in [1,\ 2^n]$. Let M be a summand* *of the complete motivic decomposition of X. If $\bar{M}$ contains a Tate summand with a shift $i < m$, then it also contains a Tate summand with the shift $2^n - 1 + i = \dim X - (m-1) + i$.*

Using Theorem 4.1, we will be able to see that no shift of $U(Y)$ is a summand of $M(X)$ for certain concrete X and Y as in Theorem 3.2. The latter theorem will then tell us that $\mathrm{cd}_2(Y) > \mathrm{cd}_2(Y_{F(T)})$ (see Remark 3.4). Afterwards, we usually get even a sharper lower bound on $\mathrm{cd}_2(Y)$ using the motivic decomposition described right below.

4b. **A motivic decomposition.** Let φ be a non-degenerate quadratic form over F of dimension n and let Y be the $\mathfrak{i}_0$-grassmannian of φ. A variety is called *anisotropic* if all its closed points are of even degree.

Lemma 4.2 ([7, Theorem 15.8 and Corollary 15.14] or [1])**.** *The motive of Y decomposes in a sum of shifts of motives of some anisotropic varieties plus*

$$\bigoplus_{i=0}^{\mathfrak{i}_0} M(\Gamma_i)\Big(i(i-1)/2 + i(n-2\mathfrak{i}_0)\Big),$$

where Γ_i is the i-grassmannian of an $\mathfrak{i}_0$-dimensional vector space (Γ_0 and $\Gamma_{\mathfrak{i}_0}$ are points, Γ_1 and $\Gamma_{\mathfrak{i}_0-1}$ – projective spaces).

Corollary 4.3. *The motive of Y does not contain any Tate summand with a positive shift strictly below $n - 2\mathfrak{i}_0$.*

Proof. By preceding Lemma, the motive of Y decomposes in a sum of shifts of motives of certain varieties. Those summands of this motivic decomposition which are motives of *isotropic* varieties[1] (and therefore can contain Tate summands while the motives of anisotropic varieties cannot, see, e.g., [14, Lemma 2.21]) come with shifts $i(i-1)/2 + i(n-2\mathfrak{i}_0)$, $i \geq 0$. For $i = 0$ the shifting number is 0 and the corresponding variety is just the point. For $i \geq 1$ the shifting numbers are at least $n - 2\mathfrak{i}_0$. □

4c. **Maximal orthogonal grassmannian.** Let φ be a non-degenerate quadratic form of dimension $2n+1$ and let $Y = X_n(\varphi)$ be the maximal orthogonal grassmannian of φ. Let $e_i \in \mathrm{Ch}^i(\bar{Y})$, $i = 0, 1, \dots, e_{2^{n-1}+1}$, be the standard generators of the modulo 2 Chow ring $\mathrm{Ch}(\bar{Y})$ defined as in [3, §86]. We say that e_i is *rational* if it is in the image of the change of field homomorphism $\mathrm{Ch}^i(Y) \to \mathrm{Ch}^i(\bar{Y})$; otherwise is *irrational.* We recall [3, Theorem 90.3] stating that $\mathrm{cd}_2(Y)$ is equal to the sum of all j such that e_j is irrational.

4d. **Values of first Witt index.** By [3, Proposition 79.4 and Remark 79.5], the first Witt index $\mathfrak{i}_1$ of an anisotropic quadratic form of dimension $d \geq 2$ satisfies the relations

$$\mathfrak{i}_1 \equiv d \pmod{2^r} \quad \text{and} \quad 1 \leq \mathfrak{i}_1 \leq 2^r$$

for some integer $r \geq 0$ with $2^r < d$.

[1] A variety is *isotropic* here if it has a closed point of odd degree.

4e. **Dimensions of forms in I^n.** By [3, Proposition 82.1], dimension d of an anisotropic quadratic form in I^n (the n-th power of the fundamental ideal in the Witt ring of the base field), where $n \geq 1$, is either $\geq 2^{n+1}$ or equals $2^{n+1} - 2^i$ with $1 \leq i \leq n+1$. Actually, apart from the old Arason-Pfister Hauptsatz (saying that $d \notin (0,\ 2^n)$), we are only using the statement about the "first hole", saying that d is outside of the open interval $(2^n,\ 2^n + 2^{n-1})$ and proved earlier ([18, Theorem 6.4]).

5. Level 1

We explain here that Conjecture 1.1 is actually already known in "level 1", that is, for i not exceeding the first Witt index of φ.

It is well-known that $\mathrm{cd}[1](\varphi) = \mathrm{cd}_2[1](\varphi) \geq \mathrm{cd}[1](\dim\varphi)$ for any anisotropic φ. This is a consequence of the formula $\mathrm{cd}[1](\varphi) = \mathrm{cd}_2[1](\varphi) = \dim\varphi - \mathfrak{i}_1(\varphi) - 1$ ([3, Theorem 90.2]) and the fact that the first Witt index of an excellent form is maximal among the first Witt indexes of quadratic forms of a given dimension ([5, Corollary 1]).

As an immediate consequence, we get the following, formally more general statement – (a bit more than) the "level 1" case of Conjecture 1.1:

Proposition 5.1. *Let φ be an anisotropic quadratic form over F of height ≥ 1. For any $i \leq \mathfrak{i}_1(\varphi)$ one has* $\mathrm{cd}[i](\varphi) = \mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$.

Proof. $\mathrm{cd}[i](\varphi) = \mathrm{cd}_2[i](\varphi) = \mathrm{cd}_2[1](\varphi) \geq \mathrm{cd}[1](\dim\varphi) \geq \mathrm{cd}[i](\dim\varphi)$. □

6. Level 2

In this Section we prove (a bit more than) the "level 2" case of Conjecture 1.1:

Theorem 6.1. *Let φ be an anisotropic quadratic form over F of height ≥ 2. For any positive integer $i \leq \mathfrak{i}_1(\varphi) + \mathfrak{i}_2(\varphi)$ one has* $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$.

Corollary 6.2. *Let φ be an anisotropic quadratic form over F of dimension* $\dim\varphi \geq 4$. *Then* $\mathrm{cd}_2[2](\varphi) \geq \mathrm{cd}[2](\dim\varphi)$. □

Corollary 6.3. *Let φ be an anisotropic quadratic form over F of height ≤ 2. Then* $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$ *for any* $i \in [1, (\dim\varphi)/2]$. □

Proof of Theorem 6.1. We write $\mathfrak{i}_1$ for $\mathfrak{i}_1(\varphi)$ and $\mathfrak{i}_2$ for $\mathfrak{i}_2(\varphi)$. By Proposition 5.1, we may assume that $i \in (\mathfrak{i}_1,\ \mathfrak{i}_1 + \mathfrak{i}_2]$.

Let us write $\dim\varphi = 2^n + m$ with $n \geq 1$ and $m \in [1,\ 2^n]$. In the case of $\mathfrak{i}_1 = m$ we have

$$\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}_2[i-m](\varphi_1) \geq \mathrm{cd}[i-m](\dim\varphi_1) = \mathrm{cd}[i](\dim\varphi),$$

where φ_1 is the 1-st anisotropic kernel of φ, [3, §25]. The first inequality here is a particular case of the general principle saying that $\mathrm{cd}_2(T_L) \leq \mathrm{cd}_2(T)$ for a variety T over F and a field extension L/F, [16]. The second inequality holds by Proposition 5.1.

Below we are assuming that $\mathfrak{i}_1 < m$ and we have to show that $\mathrm{cd}_2[i](\varphi) \geq 2^n - 1$.

In the case of $\mathfrak{i}_1 < m/2$ we have

$$\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}_2[1](\varphi_1) \geq \mathrm{cd}[1](2^n + m - 2\mathfrak{i}_1(\varphi)) = 2^n - 1.$$

Below we are assuming that $m/2 \leq \mathfrak{i}_1 < m$. It follows by §4d that $\mathfrak{i}_1 = m/2$ (in particular, m is ≥ 2 and even). This implies that $\mathfrak{i}_2 \leq 2^{n-1}$.

If $\mathfrak{i}_1 + \mathfrak{i}_2 < m$, then $\mathfrak{i}_1 + \mathfrak{i}_2 \leq m - \mathfrak{i}_1$ by [17, Theorem 1.2] which is impossible with $\mathfrak{i}_1 = m/2$. Therefore $\mathfrak{i}_1+\mathfrak{i}_2 \geq m$ and it follows by Theorem 4.1 that $U(Y)(\mathfrak{i}_1)$ is not a direct summand of the motive of X, where X is the quadric of φ and Y is the ($\mathfrak{j}_2 = \mathfrak{i}_1 + \mathfrak{i}_2$)-th grassmannian of φ.

Since $\mathrm{cd}_2[i](\varphi) = \mathrm{cd}_2(Y)$, all we need to show is $\mathrm{cd}_2(Y) \geq 2^n - 1$.

First of all we have $\mathrm{cd}_2(Y) > \mathrm{cd}_2(Y_{F(X)})$ by Theorem 3.2 and Remark 3.4.

Now we claim that the complete decomposition of $M(Y_{F(X)})$ does not contain a summand $U(Y_{F(X)})(j)$ with j inside of the open interval

$$(0,\ 2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)).$$

Indeed, if $U(Y_{F(X)})(j)$ with some j is there, then $M(Y_{F(Y)})$ contains a Tate summand with the shift j. By Corollary 4.3 we necessarily have $j = 0$ or $j \geq 2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)$, and the claim is proved.

By [9, Proposition 5.2], the complete decomposition of $U(Y)_{F(X)}$ ends with a summand $U(Y_{F(X)})(j)$ with some $j \geq 0$. (We say "ends" meaning that $\dim U(Y)_{F(X)} = \dim U(Y_{F(X)}) + j$.) By the first claim, $j \neq 0$. It follows by the second claim that $j \geq 2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)$. Thus

$$\begin{aligned}\mathrm{cd}_2(Y) = \dim U(Y) = \dim U(Y)_{F(X)} = \dim U(Y_{F(X)}) + j = \mathrm{cd}_2[1](\varphi_1) + j = \\ (2^n + m - \mathfrak{i}_1 - 1) + j \geq (2^n + m - \mathfrak{i}_1 - 1) + (2^n + m - 2(\mathfrak{i}_1 + \mathfrak{i}_2)) = \\ 2^{n+1} + 2m - 3\mathfrak{i}_1 - 2\mathfrak{i}_2 - 1 = 2^{n+1} + m/2 - 2\mathfrak{i}_2 - 1 \geq 2^n.\end{aligned}$$

The last inequality here holds because $\mathfrak{i}_2 \leq 2^{n-1}$ and $m \geq 2$ (see above). The very first equality holds by Theorem 3.1. □

7. Third canonical dimension

Theorem 7.1. *For any positive integer $i \leq 3$ and any anisotropic quadratic form φ of dimension $\geq 2i$, one has $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$.*

Proposition 7.2. *In order to prove Theorem 7.1, one only needs to show that $\mathrm{cd}_2[3](\varphi) \geq 2^n - 1$ for φ satisfying $\dim\varphi = 2^n + 3$ $(n \geq 2)$ and $\mathfrak{i}_1(\varphi) = \mathfrak{i}_2(\varphi) = 1$.*

Proof. We are reduced to the case of $i = 3$ and of φ of height ≥ 3 with $\mathfrak{i}_1(\varphi) = \mathfrak{i}_2(\varphi) = 1$ by Theorem 6.1.

So, we assume that $\dim\varphi \geq 6$. Having written $\dim\varphi = 2^n + m$ with $m \in [1,\ 2^n]$ (where $n \geq 2$), we get

$$\mathrm{cd}_2[3](\varphi) \geq \mathrm{cd}_2[2](\varphi_1) \geq \mathrm{cd}[2](2^n + m - 2) = \begin{cases} 2^n - 1 = \mathrm{cd}[3](\dim\varphi) & \text{provided that } m \geq 4; \\ 2^{n-1} - 1 \geq \mathrm{cd}[3](\dim\varphi) & \text{for } m = 1, 2 \text{ and} \\ 2^{n-1} - 1 < 2^n - 1 = \mathrm{cd}[3](\dim\varphi) & \text{for } m = 3. \end{cases}$$

So, the only problematic value of m is 3. □

Proof of Theorem 7.1. We are showing that $\mathrm{cd}_2[i](\varphi) \geq 2^n - 1$ for φ as in Proposition 7.2. Let X be the quadric of φ, T the 2-grassmannian of φ, and Y its $(2 + \mathfrak{i}_3)$-grassmannian, where $\mathfrak{i}_3 = \mathfrak{i}_3(\varphi)$ is the third Witt index of φ. We have to show that $\mathrm{cd}_2(Y) \geq 2^n - 1$.

We claim that $\mathrm{cd}_2(Y) > \mathrm{cd}_2(Y_{F(T)})$. We get the claim as a consequence of Theorem 3.2 because by Theorem 4.1, $U(Y)(2)$ is not a summand of $M(X)$.

By §4b, the complete motivic decomposition of $M(Y_{F(Y)})$ does not contain a Tate summand with a positive shift strictly below

$$\dim\varphi - 4 - 2\mathfrak{i}_3 = 2^n - 1 - 2\mathfrak{i}_3.$$

Since $\mathrm{cd}_2(Y_{F(T)}) = \dim\varphi - 4 - \mathfrak{i}_3 - 1 = 2^n - 2 - \mathfrak{i}_3$, it follows that

$$\mathrm{cd}_2(Y) \geq (2^n - 2 - \mathfrak{i}_3) + (2^n - 1 - 2\mathfrak{i}_3).$$

Therefore $\mathrm{cd}_2(Y) \geq 2^n - 1$ provided that $3\mathfrak{i}_3 \leq 2^n - 2$.

The integer $\mathfrak{i}_3$ is the first Witt index $\mathfrak{i}_1(\varphi_2)$ of the anisotropic quadratic form φ_2 (the 2-nd anisotropic kernel of φ) of dimension $2^n - 1$. It follows by §4d that $\mathfrak{i}_3 = 2^{n-1} - 1$ or $\mathfrak{i}_3 \leq 2^{n-2} - 1$. In the second case we are done and we are considering the first case below.

The equality $\mathfrak{i}_3 = 2^{n-1} - 1$ we are assuming now means that φ is a $(2^n + 3)$-dimensional anisotropic quadratic form of height 3 with the splitting pattern $(\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3) = (1, 1, 2^{n-1}-1)$. This is actually possible only for $n = 2$ and $n = 3$ (see [18, §7.2] for $n \leq 4$), but we will not use this fact because our argument will work for arbitrary n.

Note that the variety Y is now the maximal grassmannian of φ. Therefore $\mathrm{cd}_2(Y)$ can be computed as in §4c in terms of the generators $e_i \in \mathrm{Ch}^i(\bar{Y})$, $i = 0, 1, \ldots, e_{2^{n-1}+1}$.

Note that φ_2 is a $(2^n - 1)$-dimensional form of height 1. So, φ_2 is similar to a 1-codimensional subform of an anisotropic n-fold Pfister form. It follows by [3, Example 88.10] that $e_{2^{n-1}-1}$ is irrational.

As can be easily deduced from [3, Corollary 88.6], the homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(Y_{F(T)})$ is surjective in codimensions $\leq 2^{n-1} - 1$. Consequently, if both $e_{2^{n-1}}$ and $e_{2^{n-1}+1}$ are rational, then $\mathrm{cd}_2(Y_{F(T)}) = \mathrm{cd}_2(Y)$ contradicting the proved above claim. So, at least one of these two standard generators is irrational and it follows that $\mathrm{cd}_2(Y) \geq (2^{n-1} - 1) + 2^{n-1} = 2^n - 1$. □

8. Height 3

We prove (a bit more than) Conjecture 1.1 for all forms φ of height ≤ 3 in this Section.

We recall the classification of splitting patterns of quadratic forms of height 2 first (for reader's convenience, we include a proof):

Theorem 8.1 ([21, Theorem 2]). *Let φ be a non-zero anisotropic quadratic form of height ≤ 2 over a field of characteristic $\neq 2$ with a non-excellent splitting pattern. Then*

(1) *either* $\dim\varphi = 2^{n+1}$ *and* $\mathfrak{i}_1(\varphi) = 2^{n-1} = \mathfrak{i}_2(\varphi)$ *for some* $n > 0$ *or*
(2) $\dim\varphi = 2^n + 2^{n-1}$, $\mathfrak{i}_1(\varphi) = 2^{n-2}$, *and* $\mathfrak{i}_2(\varphi) = 2^{n-1}$ *for some* $n > 1$.

Proof. By [3, Theorem 84.1], the height of φ is at least the height of an anisotropic excellent form of dimension $\dim\varphi$. Moreover, for odd $\dim\varphi$ this is an equality by [3, Remark 84.6]. It follows that either $\dim\varphi = 2^n$ for some $n \geq 0$, or $\dim\varphi = 2^m - 2^{n-1}$ for some $m > n > 1$, or $\dim\varphi = 2^m - 2^n + 1$ for some $m > n > 1$. To finish, it suffices to look at the possible values of $\mathfrak{i}_1(\varphi)$ satisfying the condition of §4d together with the condition that $\dim\varphi - 2\mathfrak{i}_1(\varphi)$ is 2^r or $2^{r+1} - 1$ for some $r \geq 1$. The latter condition comes from the classical [15, Theorem 5.8] giving the list of possible dimensions of height 1 anisotropic quadratic forms. □

Theorem 8.2. *Let φ be an anisotropic quadratic form over F of height ≤ 3. For any positive integer $i \leq (\dim\varphi)/2$ one has $\mathrm{cd}_2[i](\varphi) \geq \mathrm{cd}[i](\dim\varphi)$. In particular, Conjecture 1.1 holds for all φ of height ≤ 3.*

Proof. By Theorem 6.1, we only need to consider φ of precisely height 3. Let $n := v_2(\dim\varphi)$.

Even-dimensional φ. We assume that $n \geq 1$ here. We have to show that

$$\mathrm{cd}_2[m](\varphi) \geq 2^{n-1} - 1,$$

where $m = (\dim\varphi)/2$.

If $2^{n-1}|\ \mathfrak{i}_1$, then $2^n|\ \dim\varphi_1$ and we are done. Otherwise, by §4d, $\mathfrak{i}_1 = 2^r$ for some $0 \leq r \leq n-2$. Since φ_1 is of height 2, it follows by Theorem 8.1 that $\dim\varphi = 2^n$.

If $r = n-2$ then $\mathfrak{i}_1 = 2^{n-2}$ and $\mathfrak{i}_2 = \mathfrak{i}_3 = 2^{n-3}$. It follows by [3, Corollary 83.4] that $\dim\varphi - \mathfrak{i}_1$ is a 2-power which is false. Therefore $r \leq n-3$ and we have $\mathfrak{i}_2 = 2^{n-1} - 2^{r+1}$, $\mathfrak{i}_3 = 2^r$; or $r = n-3$ and $\mathfrak{i}_2 = 2^{n-3}$, $\mathfrak{i}_3 = 2^{n-2}$. In the first case, it follows by [18, Theorem 7.7] as well as by [3, Theorem 83.3] that $U(Y_{F(X)})(\mathfrak{i}_1 + \mathfrak{i}_2)$ is a summand of $M(X_{F(X)})$, where X is the projective quadric and Y the m-grassmannian of φ. On the other hand, $U(Y)(\mathfrak{i}_1 + \mathfrak{i}_2)$ is not a summand of $M(X)$ by Theorem 4.1. It follows by [2, Theorem 1.1] that $\mathrm{cd}_2(Y_{F(X)}) < \mathrm{cd}_2(Y)$. Therefore the standard generator of maximal codimension $e_{2^{n-1}-1} \in \mathrm{Ch}(\bar{Y})$ is irrational and it follows that $\mathrm{cd}_2(Y) \geq 2^{n-1} - 1$. So, $\mathrm{cd}_2[m](\varphi) \geq 2^{n-1} - 1$ as required.

In the second case, we simply have

$$\mathrm{cd}_2(Y) = \mathrm{cd}_2[2^{n-3}+1](\varphi_1) \geq \mathrm{cd}[2^{n-3}+1](2^{n-1} + 2^{n-2}) = 2^{n-1} - 1.$$

Odd-dimensional φ. Here we assume that $n = 0$. By [3, Theorem 84.1 and Remark 84.6], the height of an anisotropic excellent quadratic form of dimension $\dim\varphi$ is 1 or 3. In the first case we have $\dim\varphi = 2^n - 1$ for some $n \geq 2$ and we need to show that $\mathrm{cd}_2[2^{n-1} - 1](\varphi) \geq 2^{n-1} - 1$.

By §4d, $\mathfrak{i}_1 = 2^r - 1$ for some $1 \leq r \leq n-1$. Moreover, $r \leq n-2$ because height of φ is 3. It follows that $\dim\varphi_1 = 2^n - 2^{r+1} + 1$. Since φ_1 is of height 2, it has an excellent splitting pattern by Theorem 8.1 so that we have $\mathfrak{i}_2 = 2^{n-1} - 2^{r+1} + 1$ and $\mathfrak{i}_3 = 2^r - 1$.

Note that $n \geq 3$ at this stage. If $n = 3$ then we are done by Theorem 7.1.

Assuming that $n \geq 4$, we claim that $U(Y_{F(X)})(\mathfrak{i}_1 + \mathfrak{i}_2)$ is a summand of $M(X_{F(X)})$, where X is the quadric and Y the maximal grassmannian of φ. For $r \leq n-3$, this is a consequence of the inequality $\mathfrak{i}_2 > \mathfrak{i}_3$ and [18, Theorem 7.7]. For the remaining case of $r = n-2$ we have $\mathfrak{i}_2 = 1$ and the above argument does not work. However, Theorem 4.1 ensures that the first shell of φ is connected with the third one. Since $\mathfrak{i}_1 = 2^r - 1 > \mathfrak{i}_2 = 1$, the first shell is not connected with the second one, and the claim follows.

Using the claim, we finish the proof of the current case the way we did it above for even-dimensional φ.

It remains to consider the case when the height of an anisotropic excellent quadratic form of dimension $\dim\varphi$ is 3. This means that $\dim\varphi = 2^{n_0} - 2^{n_1} + 2^{n_2} - 1$ for some integers $n_0 > n_1 > n_2 \geq 2$.

The first Witt index $\mathfrak{i}_1$ should satisfy §4d and in the same time be such that the height of the integer[2] $\dim\varphi_1 = \dim\varphi - 2\mathfrak{i}_1$ is 2. It follows that $\dim\varphi_1 = 2^{n_1} - 2^{n_2} + 1$ or $\dim\varphi_1 = 2^{n_0} - 2^{n_1} + 1$. In both cases we have

$$\operatorname{cd}_2[m](\varphi) \geq \operatorname{cd}_2[m_1](\varphi_1) \geq \operatorname{cd}[m_1](\dim\varphi_1) \geq \operatorname{cd}[m](\dim\varphi),$$

where $m := (\dim\varphi - 1)/2$ and $m_1 := (\dim\varphi_1 - 1)/2$. □

9. "Counter-example" with maximal grassmannian

Surprisingly, we didn't exclude $i = (\dim\varphi)/2$ in any case of Conjecture 1.1 proved so far. So, let us produce a "counter-example" to the case $i = (\dim\varphi)/2$ of Conjecture 1.1. By Theorem 7.1, i should be at least 4 and therefore $\dim\varphi$ should be at least 8. We produce it in dimension 8.

Let us find a field F and quadratic forms q and ψ such that q is 4-dimensional of discriminant a, $q_{F(\sqrt{a})}$ is anisotropic, ψ is 4-dimensional and divisible by $\langle\langle a\rangle\rangle$, and, finally, $\varphi := q\perp\psi$ is anisotropic. For instance, taking $F := k(a,b,c,d,e)$ with any field k and variables a,b,c,d,e, we can take $\psi = \langle\langle a,b\rangle\rangle$ and $q = \langle c,d,e,acde\rangle$. Then

$$\operatorname{cd}[4](\varphi) = \operatorname{cd}_2[4](\varphi) = \operatorname{cd}[2](q_{F(\sqrt{a})}) = 1 < 3 = \operatorname{cd}[4](\dim\varphi).$$

10. Fourth canonical dimension

Theorem 10.1. *Conjecture 1.1 holds for $i = 4$.*

Proposition 10.2. *It suffices to prove Theorem 10.1 only for φ of dimension $2^n + 4$ ($n \geq 3$), of height at least 4, and of Witt indexes satisfying either $\mathfrak{i}_1 = \mathfrak{i}_2 = \mathfrak{i}_3 = 1$; or $\mathfrak{i}_1 = 1, \mathfrak{i}_2 = 2$; or $\mathfrak{i}_1 = 2, \mathfrak{i}_2 = 1$. More precisely, it suffices to prove that $\operatorname{cd}_2[4](\varphi) \geq 2^n - 1$ for such φ.*

Proof. Note that Conjecture 1.1 for $i = 4$ is only about quadratic forms φ of dimension ≥ 9. We may assume that $\mathfrak{i}_1 \leq 2$ (Theorem 6.1) and that the height of φ is at least 4 (Theorem 8.2). Moreover, we may assume that $\mathfrak{i}_1 + \mathfrak{i}_2 + \mathfrak{i}_3 = 3$ or $\mathfrak{i}_1 + \mathfrak{i}_2 = 3$ (Theorem 7.1). Therefore, we have either $\mathfrak{i}_1 = \mathfrak{i}_2 = \mathfrak{i}_3 = 1$; or $\mathfrak{i}_1 = 1, \mathfrak{i}_2 = 2$; or $\mathfrak{i}_1 = 2, \mathfrak{i}_2 = 1$.

Let us write $\dim\varphi = 2^n + m$ with $n \geq 3$ and $1 \leq m \leq 2^n$. Assuming that $\mathfrak{i}_1 = 1$, we have

$$\operatorname{cd}_2[4](\varphi) \geq \operatorname{cd}[3](2^n + m - 2) = 2^n - 1 = \operatorname{cd}[4](\dim\varphi)$$

for $m \geq 5$. On the other hand,

$$\operatorname{cd}_2[4](\varphi) \geq \operatorname{cd}[3](2^n + m - 2) = 2^{n-1} - 1 = \operatorname{cd}[4](\dim\varphi)$$

for $m \leq 3$. So, the only problematic value of m is 4.

Assuming that $\mathfrak{i}_1 = 2$, we have

$$\operatorname{cd}_2[4](\varphi) \geq \operatorname{cd}[2](2^n + m - 4) = 2^n - 1 = \operatorname{cd}[4](\dim\varphi)$$

for $m \geq 6$. On the other hand,

$$\operatorname{cd}_2[4](\varphi) \geq \operatorname{cd}[2](2^n + m - 4) = 2^{n-1} - 1 = \operatorname{cd}[4](\dim\varphi)$$

[2] As in [3, §84], by the *height of a positive integer* we mean the height of an anisotropic excellent quadratic form of dimension equal this integer.

for $m \leq 3$. Moreover, since $\mathfrak{i}_1 = 2$, m is necessarily even (§4d). So, the only problematic value of m is again 4. □

Proof of Theorem 10.1. Let φ be a quadratic form as in Proposition 10.2. Let r be the integer $\in \{3, 4\}$ such that $\mathfrak{i}_1 + \cdots + \mathfrak{i}_{r-1} = 3$ (more concretely, $r := 3$ if $\mathfrak{i}_1 + \mathfrak{i}_2 = 3$, $r := 4$ if $\mathfrak{i}_1 + \mathfrak{i}_2 + \mathfrak{i}_3 = 3$). Let X be the quadric, T the 3-grassmannian, and Y the $(3 + \mathfrak{i}_r)$-grassmannian of φ. Since $\operatorname{cd}_2[4](\varphi) = \operatorname{cd}_2(Y)$, it suffices to prove that $\operatorname{cd}_2(Y) \geq 2^n - 1$.

By Theorem 4.1, the motive $U(Y)(3)$ is not a summand of $M(X)$. It follows by Theorem 3.2 that $\operatorname{cd}_2(Y) > \operatorname{cd}_2(Y_{F(T)})$.

Now, using §4b in the standard way, we get that

$$\operatorname{cd}_2(Y) \geq \operatorname{cd}_2(Y_{F(T)}) + (\dim \varphi - 2(\mathfrak{i}_1 + \cdots + \mathfrak{i}_r)) = (2^n - 3 - \mathfrak{i}_r) + (2^n - 2 - 2\mathfrak{i}_r) = 2^{n+1} - 5 - 3\mathfrak{i}_r.$$

So, the inequality $\operatorname{cd}_2(Y) \geq 2^n - 1$ holds if $2^{n+1} - 5 - 3\mathfrak{i}_r \geq 2^n - 1$, or, equivalently, if

$$2^n \geq 3\mathfrak{i}_r + 4. \tag{10.3}$$

Since the integer $\mathfrak{i}_r$ is the first Witt index of the quadratic form φ_{r-1} of dimension $\dim \varphi_{r-1} = \dim \varphi - 6 = 2^n - 2$, we have $\mathfrak{i}_r = 2^{n-1} - 2$ or $\mathfrak{i}_r \leq 2^{n-2} - 2$ or $\mathfrak{i}_r = 1$ (the last case is not included in the previous one if $n = 3$). The inequality (10.3) does not hold only in the case of $\mathfrak{i}_r = 2^{n-1} - 2$ which we consider now.

Recall that now our anisotropic quadratic form φ is of dimension $2^n + 4$ ($n \geq 3$) and has the splitting pattern

$$\text{either } (1, 1, 1, 2^{n-1} - 2, 1), \text{ or } (1, 2, 2^{n-1} - 2, 1), \text{ or } (2, 1, 2^{n-1} - 2, 1).$$

Let $d \in F^\times$ represents the discriminant of φ. We evidently have $\varphi_{F(\sqrt{d})} \in I^n$. It follows that the Clifford algebra $C(\varphi)$ is Brauer-equivalent to a quaternion algebra (c, d) with some $c \in F^\times$. Let $\psi := \varphi \perp c\,\langle\langle d \rangle\rangle$. Then $\operatorname{disc}(\psi)$ is trivial and it follows by [3, Lemma 14.2] that the Clifford invariant of ψ is trivial as well, so that $\psi \in I^3$. Let us show that $\psi \in I^n$. We know this already for $n = 3$. To show this for $n \geq 4$, it suffices to show that ψ_L is hyperbolic for any extension field L/F such that $\dim(\psi_L)_{\mathrm{an}} \leq 2^{n-1}$. Since $\dim \psi = 2^n + 6$, the condition on L ensures that $\mathfrak{i}_0(\psi_L) \geq 2^{n-2} + 3$. Since φ is a subform in ψ of codimension 2, $\mathfrak{i}_0(\varphi_L) \geq 2^{n-2} + 1$ which is ≥ 4 because $n \geq 4$. It follows that $\mathfrak{i}_0(\varphi_L) \geq 4$ and therefore $\geq 2^{n-1} + 1$ so that $\dim(\varphi_L)_{\mathrm{an}} \leq 2$ and $\dim(\psi_L)_{\mathrm{an}} \leq 4$. Since the discriminant and the Clifford invariant of ψ_L are trivial, it follows that ψ_L is hyperbolic.

We have shown that $\psi \in I^n$. On the other hand, $2^n + 2 \leq \dim \psi_{\mathrm{an}} \leq 2^n + 6$ so that for $n \geq 4$ we get a contradiction with §4e.

We proved that none of the above splitting patterns of φ is possible in the case of $n \geq 4$. It remains to consider the case of $n = 3$, that is, of $\dim \varphi = 12$. The splitting patterns of 12-dimensional anisotropic quadratic forms have been classified in [18, §7.3]. In particular, it has been shown there that only the first of our three splitting patterns is possible. For φ of this possible splitting pattern $(1, 1, 1, 2, 1)$, the above procedure provides us with an anisotropic quadratic form $\psi' := \psi_{\mathrm{an}} \in I^3$ of dimension 14 or 12 such that for any extension field L/F the condition $\mathfrak{i}_0(\varphi_L) \geq 4$ holds if and only if $\mathfrak{i}_0(\psi'_L) \geq 4$ is hyperbolic. It follows that $\operatorname{cd}_2[4](\varphi) = \operatorname{cd}_2[4](\psi')$. Since the height of ψ' is ≤ 3, it follows by Theorem 8.2 that $\operatorname{cd}_2[4](\psi') \geq \operatorname{cd}[4](\dim \psi') = 7 = 2^n - 1$. □

11. Fifth canonical dimension

Theorem 11.1. *Conjecture 1.1 holds for* $i = 5$.

Proposition 11.2. *It suffices to prove Theorem 11.1 only for* φ *of height at least* 4 *and with* $\mathfrak{i}_1 + \cdots + \mathfrak{i}_r = 4$ *for some* r*, having one of the following types:*

(1) $\dim\varphi = 2^n + 5$ *(*$n \geq 3$*) and* $\mathfrak{i}_1 = 1$*;*
(2) $\dim\varphi = 2^n + 6$ *(*$n \geq 3$*) and* $\mathfrak{i}_1 = 2$*;*
(3) $\dim\varphi = 2^n + 7$ *(*$n \geq 3$*) and* $\mathfrak{i}_1 = 3$*.*

More precisely, it suffices to prove that $\mathrm{cd}_2[4](\varphi) \geq 2^n - 1$ *for above* φ*.*

Proof. Note that Conjecture 1.1 for $i = 5$ is only about quadratic forms φ of dimension ≥ 11. We may assume that $\mathfrak{i}_1 \leq 3$ (Theorem 6.1) and that the height of φ is at least 4 (Theorem 8.2). Also we may assume that $\mathfrak{i}_1 + \cdots + \mathfrak{i}_r = 4$ for some r (Theorem 10.1).

Let us write $\dim\varphi = 2^n + m$ with $n \geq 3$ and $1 \leq m \leq 2^n$.

Assuming that $\mathfrak{i}_1 = 1$, we have

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[4](2^n + m - 2) = 2^n - 1 = \mathrm{cd}[5](\dim\varphi)$$

for $m \geq 6$. On the other hand,

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[4](2^n + m - 2) = 2^{n-1} - 1 = \mathrm{cd}[5](\dim\varphi)$$

for $m \leq 4$. So, the only problematic value of m is 5.

Assuming that $\mathfrak{i}_1 = 2$, we have

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[3](2^n + m - 4) = 2^n - 1 = \mathrm{cd}[5](\dim\varphi)$$

for $m \geq 7$. On the other hand,

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[3](2^n + m - 4) = 2^{n-1} - 1 = \mathrm{cd}[5](\dim\varphi)$$

for $m \leq 4$. Moreover, since $\mathfrak{i}_1 = 2$, m is necessarily even (§4d). So, the only problematic value of m is 6.

Finally, assuming that $\mathfrak{i}_1 = 3$, we have

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[2](2^n + m - 6) = 2^n - 1 = \mathrm{cd}[5](\dim\varphi)$$

for $m \geq 8$. On the other hand,

$$\mathrm{cd}_2[5](\varphi) \geq \mathrm{cd}[2](2^n + m - 6) = 2^{n-1} - 1 = \mathrm{cd}[5](\dim\varphi)$$

for $m \leq 4$. Moreover, since $\mathfrak{i}_1 = 3$, m is necessarily odd (§4d). So, the only problematic values of m are 5 and 7. Since 3 cannot be the first Witt index of an anisotropic quadratic form of dimension $2^n + 5$ (§4d again), the value 5 is not possible for m. □

Proof of Theorem 11.1. Let φ be a quadratic form as in Proposition 11.2. Let r be the integer such that $\mathfrak{i}_1 + \cdots + \mathfrak{i}_{r-1} = 4$. Let X be the quadric, T the 4-grassmannian, and Y the $(4 + \mathfrak{i}_r)$-grassmannian of φ. Since $\mathrm{cd}_2[5](\varphi) = \mathrm{cd}_2(Y)$, it suffices to prove that $\mathrm{cd}_2(Y) \geq 2^n - 1$.

By Theorem 4.1, the motive $U(Y)(4)$ is not a summand of $M(X)$. It follows by Theorem 3.2 that $\mathrm{cd}_2(Y) > \mathrm{cd}_2(Y_{F(T)})$.

Now, using §4b in the standard way, we get that

$$\mathrm{cd}_2(Y) \geq \mathrm{cd}_2(Y_{F(T)}) + (\dim\varphi - 2(\mathfrak{i}_1 + \cdots + \mathfrak{i}_r)) \geq$$
$$(2^n + m - 9 - \mathfrak{i}_r) + (2^n + m - 8 - 2\mathfrak{i}_r) = 2^{n+1} + 2m - 17 - 3\mathfrak{i}_r.$$

So, the inequality $\mathrm{cd}_2(Y) \geq 2^n - 1$ holds if $2^{n+1} + 2m - 17 - 3\mathfrak{i}_r \geq 2^n - 1$, or, equivalently, if

$$2^n \geq 3\mathfrak{i}_r + 16 - 2m. \tag{11.3}$$

Since the integer $\mathfrak{i}_r$ is the first Witt index of the quadratic form φ_{r-1} of dimension $2^n + m - 8$, we have $\mathfrak{i}_r = 2^{n-1} + m - 8$ or $\mathfrak{i}_r \leq 2^{n-2} + m - 8$. For $n = 3$ and $m = 6$, there is an additional case of $\mathfrak{i}_r = 1$. The inequality 11.3 does not hold only in the case of $\mathfrak{i}_r = 2^{n-1} + m - 8$ which we consider now.

Let us start with the case of $m = 5$. So, φ is of dimension $2^n + 5$ and has the splitting pattern $(\dots, 2^{n-1} - 3, 1)$.

First we consider the case of $n = 3$. In this case we have $\mathrm{cd}_2(Y_{F(T)}) = 3$, $\mathrm{cd}_2(Y) \geq 6$, and §4b tells us that in the complete decomposition of $M(Y_{F(Y)})$ there is only one Tate summand with the shift 3. On the other hand, if $\mathrm{cd}_2(Y) = 6$, then $U(Y)_{F(T)}$ contains summands $U(Y_{F(T)})$ and $U(Y_{F(T)})(3)$ so that there are two Tate summands with the shift 3 in the complete decomposition of $M(Y_{F(Y)})$. It follows that $\mathrm{cd}_2(Y) \geq 7$ and we are done in the case of $n = 3$ and $m = 5$.

In the case of $n \geq 4$ and $m = 5$, the splitting pattern of φ is impossible. Indeed, the anisotropic part of a (2^n+6)-dimensional quadratic form of trivial discriminant containing φ is in I^n and has dimension $2^n + 6$ or $2^n + 4$.

We go ahead to the case $m = 7$. Now φ is of dimension $2^n + 7$ and has the splitting pattern $(3, 1, 2^{n-1} - 1)$. This is only possible for $n = 3$, but anyway, the height of φ is 3 so that we don't need to do anything more here.

The remaining value of m is 6 so that $\dim\varphi = 2^n + 6$ now. The splitting pattern of φ is either $(2, 1, 1, 2^{n-1} - 2, 1)$ or $(2, 2, 2^{n-1} - 2, 1)$. Adding to φ an appropriate binary quadratic form of discriminant $\mathrm{disc}(\varphi)$, we get a $(2^n + 8)$-dimensional quadratic form ψ lying in I^3 and therefore in I^n. The anisotropic part of ψ has dimension $2^n + 8$, $2^n + 6$ or $2^n + 4$ and it follows that n is 3 or 4. Note that for any field extension L/F, the condition $\mathfrak{i}_0(\varphi_L) \geq 5$ is equivalent to $\mathfrak{i}_0(\psi_L) \geq 5$ so that $\mathrm{cd}_2[5](\varphi) = \mathrm{cd}_2[5](\psi)$.

If $n = 4$, then ψ is anisotropic (of dimension 24) and of height 2. Therefore we have $\mathrm{cd}_2[5](\psi) \geq \mathrm{cd}[5](24) = 15$ and the case is closed.

If $n = 3$, then the anisotropic part ψ' of ψ has dimension 12, 14, or 16. If $\dim\psi' = 12$, then $\mathrm{cd}_2[5](\psi) = \mathrm{cd}_2[3](\psi') \geq \mathrm{cd}[3](12) = 7$. If $\dim\psi' = 14$, then $\mathrm{cd}_2[5](\psi) = \mathrm{cd}_2[4](\psi') \geq \mathrm{cd}[4](14) = 7$. Finally, if $\dim\psi' = 16$, i.e., if ψ is anisotropic, then either the height of ψ is ≤ 3 or $\mathfrak{i}_1(\psi) = 1$. If the height is ≤ 3, then $\mathrm{cd}_2[5](\psi) \geq \mathrm{cd}[5](16) = 7$. If the first Witt index is 1, then $\mathrm{cd}_2[5](\psi) \geq \mathrm{cd}_2[4](\psi_1) \geq \mathrm{cd}[4](14) = 7$. □

Corollary 11.4. *Conjecture 1.1 holds in full for φ of dimension ≤ 13.*

Proof. We only need to consider $\mathrm{cd}_2[6](\varphi)$ for a 13-dimensional φ. But $\mathrm{cd}[6](13) = 1$ so that the statement to prove is trivial. □

Remark 11.5. To prove Conjecture 1.1 for 14-dimensional φ, one "only" needs to check that $\mathrm{cd}_2[6](\varphi) \geq 7$.

12. Final comments

The material of this section has been added on the suggestion of the editors.

The following proposition justifies appearance of excellent forms in the statement of Conjecture 1.1. It also answers a question raised by H. Bermudez during my talk at the International Conference on the Algebraic and Arithmetic Theory of Quadratic Forms (Puerto Natales, Patagonia, Chile) in December 2013.

Proposition 12.1. *Let φ be an anisotropic quadratic form over F such that for any integer i with $1 \leq i < (\dim \varphi)/2$, the i-th canonical dimension of φ is minimal among the i-th canonical dimensions of anisotropic quadratic forms (over field extensions of F) of dimension* $\dim \varphi$*.*

Then

(1) *the higher Witt indexes of φ are excellent , i.e., φ has the same height and the same higher Witt indexes as any anisotropic excellent quadratic form of the same dimension;*
(2) $\mathrm{cd}[i](\varphi) = \mathrm{cd}[i](\dim \varphi)$*, i.e. Conjecture 1.1 holds for quadratic forms of dimension* $\dim \varphi$*;*
(3) *the quadric of φ has excellent* motivic decomposition type*;*
(4) *assuming an open* [8, Conjecture 1.8]*, φ is excellent.*

The statement of (3) will be explained in the proof. Since we do not know if such φ exists (in arbitrary dimension), (2) does not prove Conjecture 1.1. If the i-th canonical dimension $\mathrm{cd}[i](\varphi)$ of a given anisotropic quadratic form φ is minimal for some value of i, it is not necessarily minimal for other values of i. For instance, for any $r \geq 2$ and any positive $m < 2^{r-1}$, we may find a field F and an m-dimensional quadratic form ψ over F such that the even Clifford algebra of ψ is a division algebra and ψ is a subform of an anisotropic r-fold Pfister form π. Then the i-th canonical dimension $\mathrm{cd}[i](\varphi)$ of the complement φ of ψ in π is minimal for $i = 1, \ldots, \mathfrak{j}_1(\varphi)$. For the remaining values of i however, $\mathrm{cd}[i](\varphi)$ coincides with $\mathrm{cd}[i - \mathfrak{j}_1(\varphi)](\psi)$ which is equal to $\dim X_{i-\mathfrak{j}_1(\varphi)}(\psi)$ by [13]. In particular, $\mathrm{cd}[i](\varphi)$ is not minimal in general because $\mathrm{cd}[i](\dim \varphi) = \mathrm{cd}[i - \mathfrak{j}_1(\varphi)](\dim \psi)$.

Proof of Proposition 12.1. We write $\mathfrak{j}_r$ for $\mathfrak{j}_r(\varphi)$. Since $\mathrm{cd}[1](\varphi)$ is minimal, the first Witt index of φ is excellent and (2) holds for i up to $\mathfrak{j}_1$ by the results listed in §5.

If we already know for some $r \geq 1$ that the first $r-1$ higher Witt indexes of φ are excellent and (2) holds for i up to $\mathfrak{j}_{r-1}$, the inequality $\mathrm{cd}[\mathfrak{j}_{r-1}+1](\varphi) \geq \mathrm{cd}[1](\varphi_{r-1})$ (which is an equality for φ replaced by an anisotropic excellent form of the same dimension) tells us that $\mathfrak{j}_r = \mathfrak{j}_1(\varphi_{r-1})$ is excellent and (2) holds for i up to $\mathfrak{j}_r$.

We proved (1) and (2) at this point. As a byproduct, we see that the above inequality is in fact an equality, which means by Theorem 3.2 that a shift (and therefore precisely $\mathfrak{j}_r - \mathfrak{j}_{r-1}$ shifts) of $U(X_{\mathfrak{j}_r})$ appear(s) in the complete motivic decomposition of the quadric. Having this for every r and counting the ranks of the motives over an algebraic closure, we see that each undecomposable summand of the motive of the quadric is binary, i.e. becomes over an algebraic closure a sum of two Tate motives. More precisely, every indecomposable summand looks over an algebraic closure precisely the same as the corresponding summand in the complete motivic decomposition of an anisotropic excellent quadric of the same dimension. This is what (3) means.

Finally, [8, Conjecture 1.8] produces Pfister forms out of the binary motives and allows one to show that φ is excellent. In more details, since $U(X_1)$ is binary, [8, Conjecture 1.8] implies that φ is a neighbor of a Pfister form π. By similar reason, the complement of φ in π is also a Pfister neighbor. Continuing this way, we eventually see that φ is excellent. □

Example 12.2. To visualize the statement of Conjecture 1.1, it is probably a good idea to draw the graph of the function $i \mapsto \operatorname{cd}[i](n)$ for some concrete value of n. For $n = 60 = 2^6 - 2^2$, the function is constantly $31 = 2^{6-1} - 1$ on the interval $[1,\ 28]$ and takes the value $1 = 2^{2-1} - 1$ at 29. As for arbitrary n, it is piecewise constant (with values given by some powers of 2 minus 1) and decreasing. Conjecture 1.1 claims that for any 60-dimensional anisotropic φ, the graph of the function $i \mapsto \operatorname{cd}[i](\varphi)$ is over the graph just described. We know that it is under the parabola $i \mapsto \dim X_i(\varphi) = i(i-1)/2 + i(60-2i)$. In contrast with the above lower bound, this piece of the parabola (constituting the upper bound for $\operatorname{cd}[i](\varphi)$) is not monotone: it increases until 19 and decreases after 20.

One may view Conjecture 1.1 as an analogue of the Outer Excellent Connection Theorem for quadrics, where the quadrics are replaced by higher orthogonal grassmannian. Note that according to Theorem 3.1, Conjecture 1.1 is a statement about the structure of the Chow motives of higher orthogonal grassmannians. As such, it clearly affects our understanding of their Chow groups. Finally, orthogonal grassmannians constitute a special and important case of a flag variety under a semisimple algebraic group; Conjecture 1.1 is to consider in this general context.

References

[1] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[2] De Clercq, C. A going down theorem for Grothendieck Chow motives. *Quart. J. Math. 64* (2013), 721–728.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2008.

[4] Garibaldi, S., Petrov, V., and Semenov, N. Shells of twisted flag varieties and non-decomposibility of the Rost invariant. arXiv:1012.2451v2 [math.AG] (30 Jun 2011), 48 pages.

[5] Hoffmann, D. W. Isotropy of quadratic forms over the function field of a quadric. *Math. Z. 220*, 3 (1995), 461–476.

[6] Karpenko, N. A. Minimal canonical dimensions of quadratic forms. Linear Algebraic Groups and Related Structures (preprint server) 509 (2013, Sep 1), 7 pages.

[7] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[8] Karpenko, N. A. Characterization of minimal Pfister neighbors via Rost projectors. *J. Pure Appl. Algebra 160*, 2-3 (2001), 195–227.

[9] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[10] Karpenko, N. A. Hyperbolicity of orthogonal involutions. *Doc. Math. Extra Volume: Andrei A. Suslin's Sixtieth Birthday* (2010), 371–389 (electronic).

[11] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.
[12] Karpenko, N. A. Incompressibility of orthogonal grassmannians. *C. R. Math. Acad. Sci. Paris 349*, 21-22 (2011), 1131–1134.
[13] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.
[14] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.
[15] Knebusch, M. Generic splitting of quadratic forms. I. *Proc. London Math. Soc. (3) 33*, 1 (1976), 65–93.
[16] Merkurjev, A. S. Essential dimension: a survey. *Transform. Groups 18*, 2 (2013), 415–481.
[17] Scully, S. On the splitting of quasilinear p-forms. arXiv:1210.7836v2 [math.RA] (1 Nov 2012), 29 pages. To appear in *J. Reine Angew. Math.*
[18] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.
[19] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).
[20] Vishik, A. Excellent connections in the motives of quadrics. *Ann. Sci. Éc. Norm. Supér. (4) 44*, 1 (2011), 183–195.
[21] Vishik, A. S. On the dimensions of quadratic forms. *Dokl. Akad. Nauk 373*, 4 (2000), 445–447.

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