# INCOMPRESSIBILITY OF PRODUCTS

NIKITA A. KARPENKO

Abstract. We show that the conjectural criterion of $p$-incompressibility for products of projective homogeneous varieties in terms of the factors, previously known in a few special cases only, holds in general. We identify the properties of projective homogeneous varieties actually needed for the proof to go through. For instance, generically split (non-homogeneous) varieties also satisfy these properties.

Let $F$ be a field. A smooth complete irreducible $F$-variety $X$ is *incompressible*, if every rational self-map $X \dashrightarrow X$ is dominant. This means that $\operatorname{cdim} X = \dim X$, where the *canonical dimension* $\operatorname{cdim} X$ is defined as the minimum of $\dim Y$ for $Y$ running over closed irreducible subvarieties of $X$ admitting a rational map $X \dashrightarrow Y$.

For the whole exposition, let $p$ be a fixed prime number. *Canonical $p$-dimension* $\operatorname{cdim}_p X$ is defined as the minimum of $\dim Y$ for $Y$ running over closed irreducible subvarieties of $X$ admitting a degree 0 correspondence $X \overset{p'}{\rightsquigarrow} Y$ of $p$-prime multiplicity. The variety $X$ is *$p$-incompressible*, if every degree 0 self-correspondence $X \overset{p'}{\rightsquigarrow} X$ of $p$-prime multiplicity is dominant, i.e., if $\operatorname{cdim}_p X = \dim X$. The rational equivalence class of the closure of the graph of a rational map is a degree 0 correspondence of multiplicity 1; therefore a $p$-incompressible (for at least one $p$) variety is incompressible.

Studying canonical $p$-dimension, instead of the integral Chow group CH, it is more appropriate to use the Chow group Ch with coefficients in $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Multiplicities of correspondences as well as degrees of 0-cycles take then values in $\mathbb{F}_p$. We also consider the Chow motives with coefficients in $\mathbb{F}_p$, see [2, Chapter XII].

Now we are going to introduce a class of varieties, called *nice* here, for which we can prove that the following criterion holds (see Theorem 9): the product $X \times Y$ of $F$-varieties $X$ and $Y$ is $p$-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are $p$-incompressible.

A smooth complete variety $X$ is *nice*, if it has the following two properties:

(i) The variety $X$ is *A-trivial* (cf. [11, Definition 2.3]), that is, for any field extension $L/F$ with $X(L) \neq \emptyset$, the degree homomorphism $\deg\colon \operatorname{Ch}_0(X_L) \to \mathbb{F}_p$ is an isomorphism.

(ii) For any field extension $L/F$, one has $\operatorname{cdim}_p X_L \geq d$, where $d$ is the minimal integer such that there exist an element $a \in \operatorname{Ch}_d X_L$ and an element $b \in \operatorname{Ch}^d(X_{L(X)})$ with $\deg(a_{L(X)} \cdot b) = 1$ (see Remarks 3 and 4).

*Date*: January 31, 2015. Revised: February 8, 2015.

*Key words and phrases.* Algebraic groups; projective homogeneous varieties; Severi-Brauer varieties; Chow groups and motives; canonical dimension and incompressibility. *Mathematical Subject Classification (2010):* 20G15; 14C25.

This work has been supported by a Start-Up Grant of the University of Alberta and a Discovery Grant from the National Science and Engineering Board of Canada.

**Remark 1.** The definition of "nice" depends on the prime $p$. We should probably better say "$p$-nice", but we keep saying "nice" for short. The same applies to "$A$-trivial" and to "split" (introduced below). On the other hand, we do not abbreviate "$p$-incompressible".

**Remark 2.** A nice variety remains nice under any base field extension. On the other hand, it is not clear if the product of two nice varieties is necessarily nice.

**Remark 3.** Property (ii), referring to the function field of $X_L$, is well-defined because any $A$-trivial variety is geometrically integral, see [11, Remark 2.4]. In particular, any nice variety is geometrically integral.

**Remark 4.** The opposite to the inequality in (ii) holds for any smooth complete variety $X$ (cf. [9, Proof of Theorem 5.8, part "$\leq$"]). Indeed, take the minimal $d$ such that there exist $a \in \mathrm{Ch}_d X$, and $b \in \mathrm{Ch}^d(X_{F(X)})$ with $\deg(a_{F(X)} \cdot b) = 1$. We may assume that $a = [Y]$ and $b = [Z]$ for closed subvarieties $Y \subset X$ and $Z \subset X_{F(X)}$. Since the product $[Y_{F(X)}] \cdot [Z] \in \mathrm{Ch}(X_{F(X)})$, which is a 0-cycle class of degree 1, can be represented by a 0-cycle with support on the intersection $Y_{F(X)} \cap Z$ (see [3, §8.1]), the variety $Y_{F(X)}$ has a 0-cycle of degree 1, that is, there exists a degree 0 correspondence $X \rightsquigarrow Y$ of multiplicity 1 (see [2, Page 328] concerning the relation between correspondences and 0-cycles). Therefore $\mathrm{cdim}_p X \leq \dim Y = d$.

Here is our basic example of nice varieties:

**Example 5.** Any projective homogeneous (under an action of a semi-simple affine algebraic group) variety over a $p$-special field is nice: see [11, Example 2.5] for (i) and [6, Proposition 6.1] for (ii). A field $F$ is *$p$-special*, if it has no finite extension fields of degree prime to $p$. The condition that $F$ is $p$-special is needed for (ii) only.

A smooth complete variety is *split*, if its motive decomposes into a finite direct sum of Tate motives. By *Tate motive*, we mean an arbitrary shift of the motive of the point $\mathrm{Spec}\, F$. For instance, an (absolutely) cellular variety is split, [2, Corollary 66.4].

A smooth complete geometrically irreducible $F$-variety is *generically split*, if for any field extension $L/F$ with $X(L) \neq \emptyset$, the $L$-variety $X_L$ is split.

**Example 6.** Any generically split variety is nice. Indeed, (i) holds by [9, discussion after Remark 5.6] and (ii) holds by [9, Theorem 5.8 with Remark 5.6].

The direct product of two projective homogeneous varieties is also projective homogeneous and therefore – over a $p$-special field – nice. Similarly, the direct product of two generically split varieties is generically split (and nice). The mixed product (over a $p$-special field) turns out to be nice as well:

**Example 7.** Over a $p$-special field, the direct product $X$ of a projective homogeneous variety by a generically split one is nice. Indeed, $X$ is, clearly, $A$-trivial. And property (ii) can be obtained for $X$ in the same way as it is obtained for a projective homogeneous variety in [6, Proposition 6.1]. The upper motive $U(X)$, used in the proof of [6, Proposition 6.1], is defined for $X$ in [7]; [5, Theorem 5.1 and Proposition 5.2], also used in the proof of [6, Proposition 6.1], can be proved for $X$ by almost literal repetition of their proofs; the same is valid for [7, Theorem 1.1], used in the proof of [5, Proposition 5.2].

The following well-known criterion of $p$-incompressibility for projective homogeneous varieties actually holds for arbitrary $A$-trivial varieties:

**Lemma 8.** *An $A$-trivial variety $X$ is $p$-incompressible if and only if* $\operatorname{mult}\rho = \operatorname{mult}\rho^t$ *for any degree* 0 *correspondence $\rho\colon X \rightsquigarrow X$, where $\rho^t$ is the transpose of $\rho$. In particular, this criterion holds for any nice variety $X$.*

*Proof.* We almost repeat the proof of [5, Lemma 2.7].

If $X$ is $p$-compressible, there exists a correspondence $\alpha : X \rightsquigarrow Y$ of degree 0 and multiplicity 1 to a proper closed subvariety $Y \subset X$. Considering $\alpha$ as a correspondence $X \rightsquigarrow X$, we have $\operatorname{mult}\alpha = 1$ and $\operatorname{mult}\alpha^t = 0$. Therefore the "if" part of Lemma 8 holds for arbitrary smooth complete irreducible varieties $X$, not only for $A$-trivial ones.

The other way round, suppose that we are given a degree 0 correspondence $\alpha\colon X \rightsquigarrow X$ with $\operatorname{mult}\alpha \neq \operatorname{mult}\alpha^t$. Adding a multiple of the diagonal class and multiplying by an element of $\mathbb{F}_p$, we may achieve that $\operatorname{mult}\alpha = 1$ and $\operatorname{mult}\alpha^t = 0$. In this case the pull-back of $\alpha$ with respect to the morphism $X_{F(X)} \to X \times X$ induced by the generic point of the second factor of the product $X \times X$, is a 0-cycle class of degree 0. Since $X$ is $A$-trivial, the degree homomorphism $\operatorname{Ch}_0 X_{F(X)} \to \mathbb{F}_p$ is an isomorphism. Therefore the pull-back of $\alpha$ is 0. By the continuity property of Chow groups [2, Proposition 52.9], there exists a non-empty open subset $U \subset X$ such that the pull-back of $\alpha$ to $X \times U$ is already 0. By the localization sequence [2, Proposition 57.9], it follows that $\alpha$ is the push-forward of some degree 0 correspondence $\beta : X \rightsquigarrow Y \in \operatorname{Ch}_{\dim X}(X \times Y)$, where $Y$ is the proper closed subset $Y := X \setminus U$ of $X$. Since $\operatorname{mult}\beta = \operatorname{mult}\alpha = 1$, the variety $X$ is $p$-compressible. □

The main result of this note is the "$\geq$" part of equality (10) in the following theorem:

**Theorem 9.** *Let $X$ and $Y$ be nice $F$-varieties such that the product $X \times Y$ is also nice. The variety $X \times Y$ is $p$-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are $p$-incompressible. Moreover,*

$$\operatorname{cdim}_p(X \times Y) = \operatorname{cdim}_p X_{F(Y)} + \operatorname{cdim}_p Y_{F(X)} \tag{10}$$

*provided that at least one of the three varieties $X_{F(Y)}$, $Y_{F(X)}$, $X \times Y$ is $p$-incompressible.*

**Corollary 11.** *The product $X \times Y$ of projective homogeneous $F$-varieties $X$ and $Y$ is $p$-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are $p$-incompressible. Moreover,* (10) *holds provided that at least one of the varieties $X_{F(Y)}$, $Y_{F(X)}$, $X \times Y$ is $p$-incompressible.*

*Proof.* Since canonical $p$-dimension of a variety does not change under any base field extension of degree prime to $p$ (see [15, Proposition 1.5]), we may assume that $F$ is $p$-special. By Example 5, $X$, $Y$, and $X \times Y$ are nice in this case so that Theorem 9 applies. □

Partial cases of Corollary 11, dealing with some special types of projective homogeneous varieties, have been recently proved in [8] and [4]. For an older result in this direction see Example 13 below.

The $p$-incompressibility criterion, given in Theorem 9 for nice products of two nice varieties, immediately generalizes to finite products of arbitrary length:

**Corollary 12.** *For $n \geq 1$, let $X_1, \dots, X_n$ be $F$-varieties such that every sub-product of the product $X := X_1 \times \dots \times X_n$ is nice. Then $X$ is $p$-incompressible if and only if for every $i = 1, \dots, n$ the variety $(X_i)_{F(X_1 \times \dots \times X_{i-1} \times X_{i+1} \times X_n)}$ is $p$-incompressible. The criterion also holds if for any $i = 1, \dots, n$ the variety $X_i$ is projective homogeneous or generically split.*

*Proof.* Assuming that the statement holds for some $n \geq 1$, we prove it for $n+1$. Set $X := X_1 \times \dots \times X_n$ and $Y := X_{n+1}$. If $X \times Y = X_1 \times \dots \times X_{n+1}$ is $p$-incompressible, $X_{F(Y)}$ and $Y_{F(X)}$ are $p$-incompressible, and it follows by induction hypothesis that the variety $(X_i)_{F(X_1 \times \dots \times X_{i-1} \times X_{i+1} \times X_{n+1})}$ is $p$-incompressible for any $i = 1, \dots, n+1$.

The other way round, if $(X_i)_{F(X_1 \times \dots \times X_{i-1} \times X_{i+1} \times X_{n+1})}$ is $p$-incompressible for any $i$, then, in particular, $Y_{F(X)}$ is $p$-incompressible and – by induction hypothesis – $X_{F(Y)}$ is $p$-incompressible. It follows that $X \times Y$ is $p$-incompressible. The first statement is proved.

Since any finite direct product of projective homogeneous or generically split varieties over a $p$-special field is nice (see Example 7), the second statement follows. □

**Example 13.** For purpose of computing the essential dimension of finite groups, Corollary 12 for Severi-Brauer varieties $X_1, \dots, X_n$ has been obtained in [10]. A second and simpler proof has been given in [8]. The third proof, given here (see the proof of Theorem 9), is particularly simple. The result has numerous further applications, see, e.g., [13, 14].

**Example 14.** For purpose of computing the essential dimension of representations of finite groups, introduced in [12], Corollary 12 for Weil transfers of generalized Severi-Brauer varieties has been obtained in [8] under assumption that the corresponding central simple algebras are *balanced*. Corollary 12 shows that this assumption is superfluous. Another area of applications for this result is provided in [1].

*Proof of Theorem 9.* In order to prove Theorem 9 in whole, we only need to prove equality (10). We start the prove of its (more difficult) "$\geq$" part now. If the variety $X \times Y$ is $p$-incompressible, the "$\geq$" part is however trivial. We therefore assume that the $F(X)$-variety $Y_{F(X)}$ is $p$-incompressible, that is, $\operatorname{cdim}_p Y_{F(X)} = \dim Y$.

Let $d$ be the minimal integer such that there exist $a \in \operatorname{Ch}_d(X \times Y)$ and $b \in \operatorname{Ch}^d(X \times Y)_{F(X \times Y)}$ with $\deg(a_{F(X \times Y)} \cdot b) = 1$. Slightly abusing but lightening notation, we will sometimes write $a$ instead of $a_{F(X \times Y)}$ in the last formula or in similar situations.

Since the product $X \times Y$ is nice, we have $\operatorname{cdim}_p(X \times Y) \geq d$. Our aim is to show that $d \geq \operatorname{cdim}_p X_{F(Y)} + \dim Y$.

Let $\alpha \in \operatorname{Ch}_d(X \times Y \times X \times Y)$ be the push-forward of $a$ under the diagonal morphism of $X \times Y$. Note that $\alpha = (a \times [X] \times [Y]) \cdot \Delta$, where $\Delta \in \operatorname{Ch}_{\dim(X \times Y)}(X \times Y \times X \times Y)$ is the diagonal class.

Let $\beta$ be a preimage of $b$ under the flat pull-back

$$\operatorname{Ch}^d\big((X \times Y) \times (X \times Y)\big) \twoheadrightarrow \operatorname{Ch}^d(X \times Y)_{F(X \times Y)}, \tag{15}$$

along the morphism induced by the generic point of the first factor of the product

$$(X \times Y) \times (X \times Y).$$

For surjectivity of (15) see [2, Corollary 57.11].

Let $\delta \in \operatorname{Ch}_{\dim Y}(Y_{F(X)} \times X_{F(X)} \times Y_{F(X)})$ be the class of the graph of the closed imbedding

$$in \colon Y_{F(X)} \hookrightarrow X_{F(X)} \times Y_{F(X)}$$

induced by the closed rational point $\mathbf{pt}_X$ on $X_{F(X)}$ given by the generic point of $X$.

Finally, let $\gamma \in \mathrm{Ch}^{\dim Y}(X \times Y \times Y)$ be the class of the graph of the projection

$$pr_Y \colon X \times Y \to Y.$$

We consider the elements $\alpha$, $\beta$, $\delta$, and $\gamma$ as correspondences and take their composition $\rho$ (over the field $F(X)$) in the following order:

$$\rho \colon Y_{F(X)} \overset{\delta}{\rightsquigarrow} X_{F(X)} \times Y_{F(X)} \overset{\beta}{\rightsquigarrow} X_{F(X)} \times Y_{F(X)} \overset{\alpha}{\rightsquigarrow} X_{F(X)} \times Y_{F(X)} \overset{\gamma}{\rightsquigarrow} Y_{F(X)}.$$

Let $\mathbf{pt}_Y$ be the rational point on $Y_{F(Y)}$ given by the generic point of $Y$. As usual, we abbreviate $[\mathbf{pt}_Y]_{F(Y)(X)} \in \mathrm{Ch}_0 Y_{F(X \times Y)}$ as $[\mathbf{pt}_Y]$. A direct computation shows that

$$\rho_*([\mathbf{pt}_Y]) = [\mathbf{pt}_Y],$$

where $\rho_* \colon \mathrm{Ch}\, Y_{F(X \times Y)} \to \mathrm{Ch}\, Y_{F(X \times Y)}$ is the homomorphism induced by $\rho$. Indeed,

$$[\mathbf{pt}_Y] \overset{\delta_* = in_*}{\longmapsto} [\mathbf{pt}_X] \times [\mathbf{pt}_Y] \overset{\beta_*}{\longmapsto} b \overset{\alpha_*}{\longmapsto} [\mathbf{pt}_X] \times [\mathbf{pt}_Y] \overset{\gamma_* = pr_{Y*}}{\longmapsto} [\mathbf{pt}_Y],$$

where the image under $\beta_*$ is computed like in [2, Proposition 62.4(2)].

On the other hand, $\rho_*([\mathbf{pt}_Y]) = (\mathrm{mult}\,\rho)[\mathbf{pt}_Y]$. Therefore $\mathrm{mult}\,\rho = 1$. Since the $A$-trivial $F(X)$-variety $Y_{F(X)}$ is $p$-incompressible, it follows by Lemma 8 that $\mathrm{mult}\,\rho^t = 1$. On the other hand, $\rho_*([Y]) = (\mathrm{mult}\,\rho^t)[Y]$, showing that $\rho_*([Y]) = [Y]$. We therefore have

$$[Y] \overset{\delta_* = in_*}{\longmapsto} [\mathbf{pt}_X] \times [Y] \overset{\beta_*}{\longmapsto} \tilde{b} \overset{\alpha_*}{\longmapsto} a \cdot \tilde{b} \overset{\gamma_* = pr_{Y*}}{\longmapsto} [Y]$$

for some $\tilde{b} \in \mathrm{Ch}^{d - \dim Y}(X \times Y)_{F(X)}$. By commutativity of push-forward and flat pull-back for the cartesian square

$$\begin{array}{ccc} X \times Y & \xrightarrow{pr_Y} & Y \\ \uparrow & & \uparrow \\ X_{F(Y)} & \longrightarrow & \mathrm{Spec}\, F(Y) \end{array}$$

it follows that $\deg(a' \cdot b') = 1$, where $a' \in \mathrm{Ch}_{d - \dim Y} X_{F(Y)}$ is the pull-back of the element $a \in \mathrm{Ch}_d(X \times Y)$ with respect to the morphism $X_{F(Y)} \to X \times Y$ given by the generic point of $Y$, and where $b' \in \mathrm{Ch}^{d - \dim Y} X_{F(X)(Y)} = \mathrm{Ch}^{d - \dim Y} X_{F(Y)(X)}$ is the pull-back of $\tilde{b} \in \mathrm{Ch}^{d - \dim Y}(X \times Y)_{F(X)}$. By Remark 4 we have $d - \dim Y \geq \mathrm{cdim}_p X_{F(Y)}$, that is, $d \geq \mathrm{cdim}_p X_{F(Y)} + \dim Y$. The "$\geq$" part of equality (10) is proved.

The proof of the "$\leq$" part, given in [8, Lemma 3.4] for projective homogeneous $X$ and $Y$, also works in our current settings. For reader's convenience, let us reproduce it. As in [8, Lemma 3.4], we prove the more general inequality

$$\mathrm{cdim}_p(X \times Y) \leq \mathrm{cdim}_p X + \mathrm{cdim}_p Y_{F(X)}$$

without any $p$-incompressibility assumption (on $X_{F(Y)}$, on $Y_{F(X)}$, or on $X \times Y$).

We set $x := \mathrm{cdim}_p X$ and $y := \mathrm{cdim}_p Y_{F(X)}$. Since the variety $X$ is nice, we can find $a_X \in \mathrm{Ch}_x X$ and $b_X \in \mathrm{Ch}^x X_{F(X)}$ with $\deg(a_X \cdot b_X) = 1$. Similarly, since the variety $Y_{F(X)}$ is nice, we can find $a_Y \in \mathrm{Ch}_y Y_{F(X)}$ and $b_Y \in \mathrm{Ch}^y Y_{F(X)(Y)}$ with $\deg(a_Y \cdot b_Y) = 1$. Let $\alpha_Y \in \mathrm{Ch}_{\dim X + y}(X \times Y)$ be a preimage of $a_Y$ under the pull-back along the morphism $Y_{F(X)} \to X \times Y$ induced by the generic point of $X$. We set

$$a := (a_X \times [Y]) \cdot \alpha_Y \in \mathrm{Ch}_{x+y}(X \times Y) \quad \text{and} \quad b := b_X \times b_Y \in \mathrm{Ch}^{x+y}(X \times Y)_{F(X \times Y)}.$$

We have the relation $\deg(a \cdot b) = \deg(a_X \cdot b_X) \cdot \deg(a_Y \cdot b_Y) = 1$ showing by Remark 4 that $\operatorname{cdim}_p(X \times Y) \leq x + y$. □

## References


[1] Biswas, I., Dhillon, A., and Hoffmann, N. On the essential dimension of coherent sheaves. arXiv:1306.6432v2 [math.AG] (2 Dec 2014), 17 pages.

[2] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[3] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[4] Karpenko, N. A. Incompressibility of products by Grassmannians of isotropic subspaces. Linear Algebraic Groups and Related Structures (preprint server) 541 (2014, Sept 23), 12 pages.

[5] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[6] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.

[7] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[8] Karpenko, N. A. Incompressibility of products of Weil transfers of generalized Severi-Brauer varieties. *Math. Z.* (2014), 1–11. DOI 10.1007/s00209-014-1393-4.

[9] Karpenko, N. A., and Merkurjev, A. S. Canonical $p$-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.

[10] Karpenko, N. A., and Merkurjev, A. S. Essential dimension of finite $p$-groups. *Invent. Math. 172*, 3 (2008), 491–508.

[11] Karpenko, N. A., and Merkurjev, A. S. On standard norm varieties. *Ann. Sci. Éc. Norm. Supér. (4) 46*, 1 (2013), 175–214.

[12] Karpenko, N. A., and Reichstein, Z. A numerical invariant for linear representations of finite groups. Linear Algebraic Groups and Related Structures (preprint server) 534 (2014, May 15, revised: 2014, June 18), 24 pages.

[13] Lötscher, R., MacDonald, M., Meyer, A., and Reichstein, Z. Essential dimension of algebraic tori. *J. Reine Angew. Math. 677* (2013), 1–13.

[14] Lötscher, R., MacDonald, M., Meyer, A., and Reichstein, Z. Essential $p$-dimension of algebraic groups whose connected component is a torus. *Algebra Number Theory 7*, 8 (2013), 1817–1840.

[15] Merkurjev, A. S. Essential dimension. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.


Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
*E-mail address*: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko