ESSENTIAL DIMENSION OF QUADRICS

NIKITA KARPENKO AND ALEXANDER MERKURJEV

To the memory of Oleg Izhboldin

Abstract. Let X be an anisotropic projective quadric over a field F of characteristic not 2. The essential dimension $\dim_{es}(X)$ of X, as defined by Oleg Izhboldin, is

$$\dim_{es}(X) = \dim(X) - i(X) + 1 ,$$

where $i(X)$ is the first Witt index of X (i.e., the Witt index of X over its function field).

Let Y be a complete (possibly singular) algebraic variety over F with all closed points of even degree and such that Y has a closed point of odd degree over $F(X)$. Our main theorem states that $\dim_{es}(X) \leq \dim(Y)$ and that in the case $\dim_{es}(X) = \dim(Y)$ the quadric X is isotropic over $F(Y)$.

Applying the main theorem to a projective quadric Y, we get a proof of Izhboldin's conjecture stated as follows: if an anisotropic quadric Y becomes isotropic over $F(X)$, then $\dim_{es}(X) \leq \dim_{es}(Y)$, and the equality holds if and only if X is isotropic over $F(Y)$. We also solve Knebusch's problem by proving that the smallest transcendence degree of a generic splitting field of a quadric X is equal to $\dim_{es}(X)$.

Let (V, φ) be a non-degenerate quadratic form of dimension at least 2 over a field F of characteristic not 2 and let $X = Q(\varphi)$ be the quadric hypersurface given by the equation $\varphi(x) = 0$ in the projective space $\mathbb{P}(V)$. We say that the quadric X is *anisotropic* if φ is an anisotropic quadratic form. By Springer's theorem, every closed point of an anisotropic quadric X has even degree. Is it possible to compress X rationally, i.e., to find a rational morphism $X \to Y$ to a variety Y of smaller dimension with all closed points of even degree?

The quadratic form φ is isotropic over the function field $F(X)$, hence, by the general theory of quadratic forms, $\varphi_{F(X)}$ is isomorphic to $\psi \perp k\mathbb{H}$ for some anisotropic quadratic form ψ over $F(X)$ and some $k \geq 1$, where $\mathbb{H}$ stays for the hyperbolic plane. The number k is called the *first Witt index* of φ (or X), and we denote it by $i(\varphi)$ (or $i(X)$). Let $V' \subset V$ be a subspace of codimension $i(X) - 1$. Since $V' \otimes F(X)$ intersects nontrivially a totally isotropic subspace of $V \otimes F(X)$, the anisotropic quadric $X' = Q(\varphi|_{V'})$ becomes isotropic over $F(X)$, i.e., X compresses to the subvariety X' of dimension $\dim(X) - i(X) + 1$. The latter integer is denoted $\dim_{es}(X)$ and called the *essential dimension* of X.

We prove in the paper (Corollary 3.4) that an anisotropic quadric X cannot be compressed to a variety Y of dimension smaller than $\dim_{es}(X)$ with all

Date: December 2002.

The second author was supported in part by NSF Grant #0098111.

closed points of even degree. Moreover, if there is a rational morphism $X \to Y$ with $\dim(Y) = \dim_{es}(X)$, then there is a rational morphism $Y \to X$, i.e., the quadric X is isotropic over $F(Y)$. Applying this result to a projective quadric Y, we get a proof of Izhboldin's conjecture (Theorem 4.1) stated as follows (cf. [4]): if an anisotropic quadric Y becomes isotropic over $F(X)$, then $\dim_{es}(X) \le \dim_{es}(Y)$, and the equality holds if and only if X is isotropic over $F(Y)$.

We work with correspondences rather than with rational morphisms. The main result of the paper (Theorem 3.1) is formulated in terms of correspondences that provide a more flexible tool for study of relations between varieties.

A *field* in the paper is an arbitrary field of characteristic not 2 (the characteristic restriction is important only there where quadratic forms are involved). By *scheme* we mean a separated scheme of finite type over a field, and by *variety* an integral scheme. We write $\mathrm{CH}_d(Y)$ for the *Chow group* of rational equivalence classes of dimension d algebraic cycles on a scheme Y.

1. First Witt index of generic subforms

Let φ be a (non-degenerate) quadratic form over F. Recall that $\varphi \simeq \varphi_{an} \perp k\mathbb{H}$ for an anisotropic form φ_{an}. The integer k is the *Witt index* $i_W(\varphi)$ of φ.

For a field extension L/F, φ_L denotes the form $\varphi \otimes_F L$. If $\dim \varphi \ge 3$ or $\dim \varphi = 2$ and φ is anisotropic, we write $F(\varphi)$ for the function field $F(X)$ of the (integral) quadric X defined by φ. The first Witt index $i(\varphi)$ of an anisotropic form φ is defined as $i_W(\varphi_{F(\varphi)}) \ge 1$.

By a *subform* of a quadratic form φ we mean the restriction of φ on a linear subspace of the vector space of φ and by a *subquadric* of the quadric X defined by φ we mean the closed subscheme of X given by a subform of φ. For the reader's convenience we collect some basic properties of the Witt indices in the following

Lemma 1.1. *Let φ be an anisotropic quadratic form over F.*

1. *The first Witt index $i(\varphi)$ coincides with the minimal Witt index of φ_E, when E runs over all field extension of F such that the form φ_E is isotropic.*
2. *For any field extension L/F such that φ_L is anisotropic, we have $i(\varphi_L) \ge i(\varphi)$.*
3. *For an r-codimensional subform ψ of φ and every field extension E/F, one has $i_W(\psi_E) \ge i_W(\varphi_E) - r$ and therefore $i(\psi) \ge i(\varphi) - r$.*

Proof. The first statement is proven in [7] and the second follows from the first one. The intersection of a maximal isotropic subspace U (of dimension $i_W(\varphi_E)$) of the form φ_E with the space of the subform ψ_E is of codimension at most r in U, whence the third statement. □

We are going to determine the first Witt index of certain subforms of a given anisotropic quadratic form. These subforms are generic in a sense (defined over certain purely transcendental extensions of the base field), at least their first Witt indices turn out to be the minimal possible ones. The construction of these subforms is borrowed from [3, proof of lemma 7.9] (where a different property of these subforms is studied).

Proposition 1.2. *Let φ be an anisotropic quadratic F-form, and let n be an integer such that $0 \leq n \leq \dim\varphi - 2$. There exists a purely transcendental field extension $\tilde{F}/F$ and an n-codimensional subform ψ of $\varphi_{\tilde{F}}$ such that*

$$i(\psi) = \begin{cases} i(\varphi) - n, & \text{if } i(\varphi) > n; \\ 1, & \text{if } i(\varphi) \leq n. \end{cases}$$

Proof. It suffices to give a proof for $n = 1$ in which case $\dim\varphi \geq 3$. Let t be an indeterminate. We consider the quadratic $F(t)$-form $\eta = \varphi_{F(t)} \perp \langle -t \rangle$ and let $\tilde{F}$ be the function field of η over $F(t)$. The field extension $\tilde{F}/F$ is clearly purely transcendental. Moreover, the anisotropic form $\varphi_{\tilde{F}}$ represents t, therefore $\varphi_{\tilde{F}} \simeq \psi \perp \langle t \rangle$ for certain 1-codimensional subform ψ of $\varphi_{\tilde{F}}$ over $\tilde{F}$.

We are going to determine the first Witt index of ψ. We set $i = i(\varphi)$. First of all, by Lemma 1.1(3), we have $i(\psi) \geq i - 1$. Let φ' be the anisotropic part of the form $\varphi_{F(\varphi)}$. We write τ for the form $\varphi' \perp \langle -t \rangle$ over $F(\varphi)(t)$.

There are following isomorphisms of $\tilde{F}(\varphi)$-forms (we omit the subscript $\tilde{F}(\varphi)$ in the formula):

$$\psi \perp \mathbb{H} \simeq \psi \perp \langle t \rangle \perp \langle -t \rangle \simeq \varphi \perp \langle -t \rangle \simeq \varphi' \perp \langle -t \rangle \perp i\mathbb{H} = \tau \perp i\mathbb{H}.$$

Cancelling one copy of $\mathbb{H}$, we get $\psi \simeq \tau \perp (i-1)\mathbb{H}$ over $\tilde{F}(\varphi)$. Note that the form $\tau_{\tilde{F}(\varphi)}$ is anisotropic because the field extension $\tilde{F}(\varphi)/F(\varphi)(t)$ is purely transcendental (by reason of the isotropy of the form $\eta_{F(\varphi)(t)}$). Therefore the Witt index of $\psi_{\tilde{F}(\varphi)}$ is $i - 1$. If $i - 1$ is positive, then $i(\psi) \leq i - 1$ by Lemma 1.1(1), and we are done with this case. Otherwise $i = 1$, $\psi \simeq \tau$ over $\tilde{F}(\varphi)$ and by Lemma 1.1(2),

$$1 \leq i(\psi) \leq i(\psi_{\tilde{F}(\varphi)}) = i(\tau_{\tilde{F}(\varphi)}). \tag{1}$$

The field extension $F(\varphi)(t)(\tau)/F(\varphi)$ is clearly purely transcendental. The form η is isotropic over $F(\varphi)(t)(\tau)$, hence the field $\tilde{F}(\varphi)(\tau) = F(\varphi)(t)(\tau)(\eta)$ is purely transcendental over $F(\varphi)(t)(\tau)$ and therefore over $F(\varphi)$. It follows that the form φ' remains anisotropic over $\tilde{F}(\varphi)(\tau)$. The form φ' is a subform of τ of codimension 1 over $F(\varphi)(t)$, hence by Lemma 1.1(3),

$$1 \leq i_W(\tau_{\tilde{F}(\varphi)(\tau)}) \leq i_W(\varphi'_{\tilde{F}(\varphi)(\tau)}) + 1 = 1$$

and therefore, $i(\tau_{\tilde{F}(\varphi)}) = 1$ and $i(\psi) = 1$ by (1). □

Remark 1.3. In the case $i(\varphi) > 1$, the first Witt index of *every* 1-codimensional subform is known to be $i(\varphi) - 1$. This result is due to A. Vishik [11, cor. 3] which we do not use in this paper. It readily follows from Theorem 4.1.

2. Correspondences

Let X and Y be schemes over a field F. Suppose that X is equidimensional and set $d = \dim(X)$. A *correspondence from X to Y*, denoted $\alpha\colon X \rightsquigarrow Y$, is an element $\alpha \in \mathrm{CH}_d(X \times Y)$. A correspondence α is called *prime* if α is represented by a prime (integral) cycle. Every correspondence is a linear combination of prime correspondences with integer coefficients.

Let $\alpha\colon X \rightsquigarrow Y$ be a correspondence. Assume that X is a variety and Y is complete. The projection morphism $p\colon X \times Y \to X$ is proper and hence the push-forward homomorphism

$$p_*\colon \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_d(X) = \mathbb{Z} \cdot [X]$$

is defined [2, § 1.4]. The number $\mathrm{mult}(\alpha) \in \mathbb{Z}$ such that $p_*(\alpha) = \mathrm{mult}(\alpha) \cdot [X]$ is called the *multiplicity* of α. Clearly, $\mathrm{mult}(\alpha + \beta) = \mathrm{mult}(\alpha) + \mathrm{mult}(\beta)$ for any two correspondences $\alpha, \beta\colon X \rightsquigarrow Y$.

A correspondence $\alpha\colon \operatorname{Spec} F \to Y$ is represented by a 0-cycle z on Y. We set $\deg(z) = \mathrm{mult}(\alpha)$. This coincides with the usual notion of degree for 0-cycles as defined in [2, def. 1.4].

The image of a correspondence $\alpha\colon X \rightsquigarrow Y$ under the pull-back homomorphism

$$\mathrm{CH}_d(X \times Y) \to \mathrm{CH}_0(Y_{F(X)})$$

with respect to the flat morphism $Y_{F(X)} \to X \times Y$ is represented by a 0-cycle on $Y_{F(X)}$. The degree of this cycle is equal to $\mathrm{mult}(\alpha)$ (see [6, lemma 1.4]).

Lemma 2.1. *Let $\tilde{F}/F$ be a purely transcendental field extension. Then*

$$\deg \mathrm{CH}_0(Y) = \deg \mathrm{CH}_0(Y_{\tilde{F}}) .$$

Proof. It suffices to consider the case where $\tilde{F}$ is the function field of the affine line $\mathbb{A}^1$. The statement follows from the fact that the restriction homomorphism $\mathrm{CH}_*(Y) \to \mathrm{CH}_*(Y_{F(\mathbb{A}^1)})$ is surjective (cf. [5, proof of prop. 3.12]) as the composite of the surjections

$$\mathrm{CH}_*(Y) \to \mathrm{CH}_{*+1}(Y \times \mathbb{A}^1) \quad \text{and} \quad \mathrm{CH}_{*+1}(Y \times \mathbb{A}^1) \to \mathrm{CH}_*(Y_{F(\mathbb{A}^1)})$$

(for the surjectivity of the first map see [2, prop. 1.9]). □

Let X and Y be varieties over F and $\dim(X) = d$. The generic point of a multiplicity $r > 0$ prime d-dimensional cycle $Z \subset X \times Y$ defines a degree r closed point of the generic fiber $Y_{F(X)}$ of the projection $X \times Y \to X$ and vise versa. Hence the following two sets are naturally bijective for every $r > 0$:

1) multiplicity r prime d-dimensional cycles on $X \times Y$;
2) closed points of $Y_{F(X)}$ of degree r.

A rational morphism $X \to Y$ defines a multiplicity 1 prime correspondence $X \rightsquigarrow Y$ as the closure of its graph. Conversely, a multiplicity 1 prime cycle $Z \subset X \times Y$ is birational to X and therefore the projection to Y defines a rational map $X \sim Z \to Y$. Hence there are natural bijections between the sets of:

0) rational morphisms $X \to Y$;
1) multiplicity 1 prime d-dimensional cycles on $X \times Y$;
2) rational points of $Y_{F(X)}$.

A multiplicity r prime correspondence $X \rightsquigarrow Y$ can be viewed as a "generically r-valued map" between X and Y.

Let $g : Y \to Y'$ be a morphism of complete schemes. The image β of a correspondence $\alpha\colon X \rightsquigarrow Y$ under the push-forward homomorphism

$$(\mathrm{id}_X \times g)_*\colon \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_d(X \times Y')$$

is a correspondence from X to Y'. The following statement is a consequence of functoriality of the push-forward homomorphisms:

Lemma 2.2. $\mathrm{mult}(\beta) = \mathrm{mult}(\alpha)$. □

Let $X' \subset X$ be a closed subvariety such that the embedding $i\colon X' \hookrightarrow X$ is regular of codimension r [2, B.7.1]. Then for every scheme Y, the embedding $i \times \mathrm{id}_Y\colon X' \times Y \hookrightarrow X \times Y$ is also regular of codimension r, hence the pull-back homomorphism

$$(i \times \mathrm{id}_Y)^*\colon \mathrm{CH}_d(X \times Y) \to \mathrm{CH}_{d-r}(X' \times Y)$$

is defined [2, § 6]. The pull-back γ of a correspondence $\alpha : X \rightsquigarrow Y$ is a correspondence from X' to Y.

Lemma 2.3. $\mathrm{mult}(\gamma) = \mathrm{mult}(\alpha)$. □

Proof. The statement follows from the commutativity of the diagram [2, th. 6.2]:

$$\begin{array}{ccc}
\mathrm{CH}_d(X \times Y) & \xrightarrow{(i\times\mathrm{id}_Y)^*} & \mathrm{CH}_{d-r}(X' \times Y) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p'_*} \\
\mathrm{CH}_d(X) & \xrightarrow{i^*} & \mathrm{CH}_{d-r}(X'),
\end{array}$$

where p and p' are the projections. □

Let $\alpha\colon X \rightsquigarrow Y$ be a correspondence between varieties of dimension d. We write α^t for the element in $\mathrm{CH}_d(Y \times X)$ corresponding to α under the exchange isomorphism $X \times Y \simeq Y \times X$. The correspondence $\alpha^t : Y \rightsquigarrow X$ is called the *transpose* of α.

3. Main theorem

In this section X is an anisotropic projective quadric over a field F. We recall that the essential dimension $\dim_{es}(X)$ of X is defined as the integer $\dim(X) - i(X) + 1$.

Theorem 3.1. *Let X be an anisotropic projective F-quadric and let Y be a complete F-variety with all closed points of even degree. Suppose Y has a closed point of odd degree over $F(X)$. Then*

(1) $\dim_{es}(X) \leq \dim(Y)$*;*

(2) *if, moreover,* $\dim_{es}(X) = \dim(Y)$*, then* X *is isotropic over* $F(Y)$*.*

Proof. A closed point of Y over $F(X)$ of odd degree gives rise to a prime correspondence $\alpha\colon X \rightsquigarrow Y$ of odd multiplicity. By Springer's theorem [8, ch. VII, th. 2.3], to prove the statement **(2)** it is sufficient to find a closed point of $X_{F(Y)}$ of odd degree or equivalently, to find an odd multiplicity correspondence $Y \rightsquigarrow X$.

Assume first that $i(X) = 1$, so that $\dim_{es}(X) = \dim(X)$. We prove both statements simultaneously by induction on $n = \dim(X) + \dim(Y)$.

If $n = 0$, i.e., X and Y are of dimension zero, we have $X = \operatorname{Spec} K$ and $Y = \operatorname{Spec} L$, where K and L are field extensions of F with $[K : F] = 2$ and $[L : F]$ even. Taking the push-forward to $\operatorname{Spec} F$ of the correspondence α we get the formula

$$[K : F] \cdot \operatorname{mult}(\alpha) = [L : F] \cdot \operatorname{mult}(\alpha^t).$$

Since $\operatorname{mult}(\alpha)$ is odd, $\alpha^t\colon Y \rightsquigarrow X$ is a correspondence of odd multiplicity.

Assume that $n > 0$ and let d be the dimension of X. We are going to prove **(2)**, so that we have $\dim(Y) = d > 0$. It is sufficient to show that $\operatorname{mult}(\alpha^t)$ is odd. Assume that the multiplicity of α^t is even. Let $x \in X$ be a closed point of degree 2. Since the multiplicity of the correspondence $Y \times x\colon Y \rightsquigarrow X$ is 2 and the multiplicity of $x \times Y : X \rightsquigarrow Y$ is zero, we can modify α by an appropriate multiple of $x \times Y$ and therefore assume that $\operatorname{mult}(\alpha)$ is odd and $\operatorname{mult}(\alpha^t) = 0$. Hence the degree of the pull-back of α^t on $X_{F(Y)}$ is zero. By [5, prop. 2.6] or [10], the degree homomorphism

$$\deg\colon \operatorname{CH}_0(X_{F(Y)}) \to \mathbb{Z}$$

is injective. Therefore there is a nonempty open subset $U \subset Y$ such that the restriction of α on $X \times U$ is trivial. Write Y' for the reduced scheme $Y \setminus U$, $i\colon X \times Y' \to X \times Y$ and $j\colon X \times U \to X \times Y$ for the closed and open embeddings respectively. The sequence

$$\operatorname{CH}_d(X \times Y') \xrightarrow{i_*} \operatorname{CH}_d(X \times Y) \xrightarrow{j^*} \operatorname{CH}_d(X \times U)$$

is exact [2, prop. 1.8]. Hence there exists $\alpha' \in \operatorname{CH}_d(X \times Y')$ such that $i_*(\alpha') = \alpha$. We can view α' as a correspondence $X \rightsquigarrow Y'$. By Lemma 2.2, $\operatorname{mult}(\alpha') = \operatorname{mult}(\alpha)$, hence $\operatorname{mult}(\alpha')$ is odd. Since α' is an integral linear combination of prime correspondences, we can find a prime correspondence $\beta\colon X \rightsquigarrow Y'$ of odd multiplicity, i.e., Y' has a closed point of odd degree over $F(X)$. The class β is represented by a prime cycle, hence we may assume that Y' is irreducible. Since $\dim(Y') < \dim(Y) = \dim(X) = \dim_{es}(X)$, by induction hypothesis, we get a contradiction with the statement **(1)**.

In order to prove **(1)** assume that $\dim(Y) < \dim(X)$. Let $Z \subset X \times Y$ be a prime cycle representing α. Since $\operatorname{mult}(\alpha)$ is odd, the field extension $F(X) \hookrightarrow F(Z)$ is of odd degree. The restriction of the projection $X \times Y \to Y$ gives a proper morphism $Z \to Y$. Replacing Y by the image of this morphism, we come to the situation where $Z \to Y$ is a surjection and so, the function field $F(Z)$ is a field extension of $F(Y)$.

In view of Proposition 1.2, extending the scalars to a purely transcendental extension $\tilde{F}$ of F, we can find a subquadric X' of X of the same dimension as Y having $i(X') = 1$. We note that according to Lemma 2.1, the hypothesis on X and Y is still satisfied over $\tilde{F}$. By Lemma 2.3, the pull-back of α with respect to the regular embedding $X' \times Y \hookrightarrow X \times Y$ produces an odd multiplicity correspondence $X' \rightsquigarrow Y$. Since $\dim(X') < \dim(X)$, by the induction hypothesis, the statement **(2)** holds for X' and Y, that is, there exists an odd multiplicity correspondence $\beta\colon Y \rightsquigarrow X'$. We compose β with the embedding $X' \hookrightarrow X$ to produce an odd multiplicity (in fact, of the same multiplicity as β) correspondence $\gamma\colon Y \rightsquigarrow X$ (Lemma 2.2). We may assume that γ is prime. Let $T \subset Y \times X$ be a prime cycle representing γ. Since the multiplicity of γ is odd, the projection $T \to Y$ is surjective, so that $F(T)$ is a field extension of $F(Y)$ of odd degree.

Using the odd multiplicity prime correspondences $\alpha\colon X \rightsquigarrow Y$ and $\gamma\colon Y \rightsquigarrow X$, we are going to construct an odd multiplicity correspondence $\delta\colon X \rightsquigarrow X$ with even $\operatorname{mult}(\delta^t)$ getting this way a contradiction with

Theorem 3.2 ([6, th. 6.4])**.** *Let X be an anisotropic quadric with $i(X) = 1$. Then for every correspondence $\delta : X \rightsquigarrow X$, one has* $\operatorname{mult}(\delta) \equiv \operatorname{mult}(\delta^t) \pmod 2$.

Note that in the case where Y is smooth we can simply take for δ the composite of the correspondences α and γ (cf. [6, proof of prop. 7.1]).

Lemma 3.3. *Let $F \hookrightarrow L$ and $F \hookrightarrow E$ be two field extensions with odd degree $[L : F]$. Then there is a field K and field extensions $L \hookrightarrow K$ and $E \hookrightarrow K$ such that $[K : E]$ is odd.*

Proof. We may assume that L is generated over F by one element, say θ. Let $f \in F[t]$ be the minimal polynomial of θ (of odd degree). Choose an odd degree irreducible polynomial $g \in E[t]$ dividing f and set $K = E[t]/gE[t]$. □

By Lemma 3.3 applied to the field extensions $F(T)$ and $F(Z)$ of $F(Y)$, we can find a field extension K of $F(T)$ and $F(Z)$ such that $[K : F(Z)]$ is odd. Let S be a projective variety over F which is a model of the field extension K/F. Replacing S by the closure of the graph of the rational morphism $S \to Z \times T$, we come to the situation where the rational morphisms $S \to Z$ and $S \to T$ are regular. Let f be the composite of $S \to Z$ with $Z \to X$ and g be the composite of $S \to T$ and $T \to X$. We write δ for the correspondence $X \rightsquigarrow X$ given by the image of the morphism $(f, g)\colon S \to X \times X$. The multiplicity

$$\operatorname{mult}(\delta) = [F(S) : F(X)] = [F(S) : F(Z)] \cdot [F(Z) : F(X)]$$

is odd and the multiplicity of the transpose of δ is zero since g is not surjective as $\dim T = \dim Y < \dim X$, a contradiction.

We have proven Theorem 3.1 in the case $i(X) = 1$. Consider now the general case (the first Witt index of X is arbitrary). Let X' be a subquadric of X with $\dim(X') = \dim_{es}(X') = \dim_{es}(X)$ which we may find after extending the scalars to a purely transcendental extension according to Proposition 1.2. By

Lemma 2.3, the pull-back $\beta\colon X' \rightsquigarrow Y$ of α with respect to the embedding of $X' \times Y$ into $X \times Y$ is an odd multiplicity correspondence. Therefore $\dim X' \le \dim Y$ by the first part of the proof. If $\dim X' = \dim Y$, then again by the first part of the proof, X' and hence X have rational points over $F(Y)$. □

As the first corollary of the main theorem we get that an anisotropic quadric X cannot be compressed to a variety Y of dimension smaller than $\dim_{es}(X)$ with all closed points of even degree:

Corollary 3.4. *Let X be an anisotropic projective F-quadric and let Y be a complete F-variety with all closed points of even degree. If $\dim_{es}(X) > \dim(Y)$, then there are no rational morphisms $X \to Y$.* □

Remark 3.5. For X and Y as in part **(2)** of Theorem 3.1, assume additionally that $\dim(X) = \dim_{es}(X)$, i.e., $i(X) = 1$. In the proof of Theorem 3.1, it is shown that $\operatorname{mult}(\alpha^t)$ is odd for every odd multiplicity correspondence $\alpha\colon X \rightsquigarrow Y$.

We have also the following more precise version of Theorem 3.1:

Corollary 3.6. *Let X and Y be as in Theorem 3.1. Then there exists a closed subvariety $Y' \subset Y$ such that*

(i) *$\dim(Y') = \dim_{es}(X)$;*
(ii) *$Y'_{F(X)}$ possesses a closed point of odd degree;*
(iii) *$X_{F(Y')}$ is isotropic.*

Proof. Let $X' \subset X$ be a subquadric with $\dim(X') = \dim_{es}(X)$. Then, by Theorem 4.1, $\dim_{es}(X') = \dim(X')$. An odd degree closed point on $Y_{F(X)}$ gives an odd multiplicity correspondence $X \rightsquigarrow Y$ which in turn gives an odd multiplicity correspondence $X' \rightsquigarrow Y$. We may assume that the latter correspondence is prime and take a prime cycle $Z \subset X' \times Y$ representing it. We define Y' as the image of the proper morphism $Z \to Y$. Clearly, $\dim(Y') \le \dim(Z) = \dim(X') = \dim_{es}(X)$. On the other hand, Z gives an odd multiplicity correspondence $X' \rightsquigarrow Y'$, therefore $\dim(Y') \ge \dim(X')$ by Theorem 3.1, and the condition (i) of Corollary 3.6 is satisfied. Moreover, $Y'_{F(X')}$ has a closed point of odd degree. Since the field $F(X \times X')$ is purely transcendental over $F(X')$ as well as over $F(X)$, Lemma 2.1 shows that there is an odd degree closed point on $Y'_{F(X)}$, that is, the condition (ii) of Corollary 3.6 is satisfied. Finally the quadric $X'_{F(Y')}$ is isotropic by Theorem 3.1; therefore $X_{F(Y')}$ is isotropic. □

4. Application to the algebraic theory of quadratic forms

Now we apply Theorem 3.1 to a special (but may be the most interesting) case where the variety Y is also a projective quadric:

Theorem 4.1. *Let X and Y be anisotropic quadrics over F and suppose that Y is isotropic over $F(X)$. Then*

(1) $\dim_{es}(X) \leq \dim_{es}(Y)$*;*
(2) *moreover, the equality* $\dim_{es}(X) = \dim_{es}(Y)$ *holds if and only if* X *is isotropic over* $F(Y)$.

Proof. Let us choose a subquadric $Y' \subset Y$ with $\dim(Y') = \dim_{es}(Y)$ (we can do it over a purely transcendental extension of the base field by Proposition 1.2). Since Y' becomes isotropic over $F(Y)$ and Y is isotropic over $F(X)$, Y' is isotropic over $F(X)$. According to Theorem 3.1, $\dim_{es}(X) \leq \dim(Y')$. Moreover, in the case of equality, X is isotropic over $F(Y')$ and hence over $F(Y)$. Conversely, if X is isotropic over $F(Y)$, interchanging the roles of X and Y, we get as above the inequality $\dim_{es}(Y) \leq \dim_{es}(X)$, hence the equality holds. □

We have the following upper bound for the Witt index of Y over $F(X)$.

Corollary 4.2. *Let* X *and* Y *be anisotropic quadrics over* F *and suppose that* Y *is isotropic over* $F(X)$*. Then*

$$i_{\mathrm{W}}(Y_{F(X)}) - i(Y) \leq \dim_{es}(Y) - \dim_{es}(X)\ .$$

Proof. If $\dim_{es}(X) = 0$, the statement is trivial. Otherwise, let Y' be a subquadric of Y of dimension $\dim_{es}(X) - 1$. Since $\dim_{es}(Y') \leq \dim(Y') < \dim_{es}(X)$, the quadric Y' remains anisotropic over $F(X)$ by the first part of Theorem 4.1. Therefore, by Lemma 1.1(3), $i_{\mathrm{W}}(Y_{F(X)}) \leq \operatorname{codim}_Y(Y') = \dim(Y) - \dim_{es}(X) + 1$, whence the inequality. □

Let φ be an anisotropic quadratic form over F of dimension at least 2 and let X be the quadric given by φ. A field extension K/F is called a *generic splitting field of* X *(and of* φ*)* if X is isotropic over K and for every field extension L/F with X_L isotropic, there is an F-place from K to L. In [7, §4] M. Knebusch raised the problem to determine the smallest transcendence degree of a generic splitting field of a given quadric X (we thank J. Arason for pointing out this question).

Theorem 4.3. *The smallest transcendence degree of a generic splitting field of a quadric* X *is equal to* $\dim_{es}(X)$.

Proof. Let X' be a subquadric of X of dimension $\dim_{es}(X)$. Since X' has a point over every field extension L/F with X_L isotropic, the field $F(X')$ is a generic splitting field of X of transcendence degree $\dim_{es}(X)$.

Let K/F be a generic splitting field of the quadric X. We will show that

$$\operatorname{tr.deg}(K/F) \geq \dim_{es}(X). \tag{2}$$

Replace K by a finitely generated subextension of K/F that splits X. Let Y' be a projective variety over F which is a model of the field extension K/F. Since X is isotropic over K, there is a rational morphism $Y' \to X$. Let Y be the closure of the graph of this morphism. Then Y is also a projective model of K/F and every closed point of Y has even degree since there is a morphism $Y \to X$. Since X is isotropic over $F(X)$, there is a place from K to $F(X)$ over

F. The valuation ring of this place has a center $y \in Y$ and therefore, the place gives a morphism $\operatorname{Spec} F(X) \to Y$ with the image $\{y\}$. In particular, Y has a rational point over $F(X)$. By Theorem 3.1, $\dim(Y) \geq \dim_{es}(X)$, whence the inequality (2). $\square$

We write $W(F)$ for the Witt ring of a field F and I for the fundamental ideal of $W(F)$, i.e., for the ideal of classes of even dimensional forms. For every $n \geq 1$ the ideal I^n is additively generated by the classes of n-fold Pfister forms. If φ is a nonzero anisotropic form such that $\varphi \in I^n$ then by the Arason-Pfister theorem [9, th. 5.6], $\dim(\varphi) \geq 2^n$. The following theorem was proved by A. Vishik. We give a simpler proof of this statement due to D. Hoffmann.

Theorem 4.4. *Let φ be an anisotropic quadratic form such that $\varphi \in I^n$ and $\dim(\varphi) > 2^n$. Then $\dim(\varphi) \geq 2^n + 2^{n-1}$.*

Proof. We can write the class of φ in $W(F)$ in the form $\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_m$, where each φ_i is similar to an anisotropic n-fold Pfister form. Assume that the statement is not true (in particular, $n \geq 2$) and choose a counterexample φ of the smallest dimension with the smallest number m of the φ_i's (over all fields).

Since the anisotropic part φ' of the form $\varphi_{F(\varphi)}$ has dimension smaller than φ, we must have $\dim(\varphi') \leq 2^n$. Then for the quadric Y given by φ we have

$$i(Y) \geq (\dim(\varphi) - 2^n)/2 = (\dim(Y) - 2^n)/2 + 1$$

and therefore,

$$\dim_{es}(Y) = \dim(Y) - i(Y) + 1 \leq \dim(Y)/2 + 2^{n-1} < 2^n + 2^{n-2} - 1 \tag{3}$$

since by assumption $\dim(\varphi) < 2^n + 2^{n-1}$.

We are going to prove that all the φ_i are pairwise similar. Assume that there are two non-similar forms φ_i and φ_j. Consider the anisotropic part ψ of the form $\varphi_i \perp \varphi_j$. Note that ψ is not similar to an n-fold Pfister form by the choice of m.

We claim that $\dim(\psi) \geq 2^n + 2^{n-1}$. By a theorem of Elman-Lam [1, th. 4.5], the Witt index of the form $\varphi_i \perp \varphi_j$ is either 0 of 2^s, where s is the linkage number of φ_i and φ_j. We have $s \neq n$ since the forms φ_i and φ_j are non-similar and $s \neq n-1$ since ψ is not similar to an n-fold Pfister form. Thus, $s \leq n-2$ and therefore, $\dim(\psi) \geq 2^n + 2^{n-1}$. The claim is proved.

It follows from [1, th. 3.2] that ψ is not similar to an $(n+1)$-fold Pfister form. Therefore, by [9, th. 5.4], ψ is not hyperbolic over $F(\psi)$. The Arason-Pfister theorem implies that dimension of the anisotropic part of $\psi_{F(\psi)}$ is at least 2^n. Hence $i(\psi) \leq (\dim(\psi) - 2^n)/2$ and therefore, for the quadric X given by ψ we have

$$\dim_{es}(X) = \dim(\psi) - i(\psi) - 1 \geq \dim(\psi)/2 + 2^{n-1} - 1 \geq 2^n + 2^{n-2} - 1. \tag{4}$$

It follows from Theorem 4.1, (3), and (4) that φ is anisotropic over $F(\psi)$, so that φ over $F(\psi)$ is also a minimal counterexample. But dimension of the anisotropic part of ψ (i.e., of $\varphi_i \perp \varphi_j$) over $F(\psi)$ is smaller than $\dim(\psi)$.

Iterating the construction we come to the situation when dimension of the anisotropic part of $\varphi_i \perp \varphi_j$ becomes smaller than $2^n + 2^{n-1}$, a contradiction to the claim.

We have proved that all the φ_i are similar to some n-fold Pfister form ρ, so that the class of φ in the Witt ring is divisible by ρ and therefore $\dim(\varphi)$ is divisible by 2^n [9, th. 5.4(iv)] since φ is anisotropic. This contradicts the inequality $2^n < \dim \varphi < 2^n + 2^{n-1}$. $\square$

References

[1] R. Elman and T. Y. Lam, *Pfister forms and K-theory of fields*, J. Algebra **23** (1972), 181–213.

[2] W. Fulton, *Intersection Theory*, Springer-Verlag, Berlin, 1984.

[3] O. T. Izhboldin, *Fields of u-invariant 9*, Ann. Math. **154** (2001), no. 3, 529–587.

[4] O. T. Izhboldin, *Virtual Pfister neighbors and first Witt index*, Preprint (2000).

[5] N. A. Karpenko, *Algebro-geometric invariants of quadratic forms*, Algebra i Analiz **2** (1990), no. 1, 141–162 (in Russian); English translation: St. Petersburg Math. J. **2** (1991), no. 1, 119–138.

[6] N. A. Karpenko, *On anisotropy of orthogonal involutions*, J. Ramanujan Math. Soc. **15** (2000), no. 1, 1–22.

[7] M. Knebusch, *Generic splitting of quadratic forms. I*, Proc. London Math. Soc. (3), **33** (1976), no. 1, 65–93.

[8] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin, Reading, Massachusetts, 1973.

[9] W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[10] R. G. Swan, *Zero cycles on quadric hypersurfaces*, Proc. Amer. Math. Soc. **107** (1989), no. 1, 43–46.

[11] A. Vishik, *Direct summands in the motives of quadrics*, Preprint, 1999.

Laboratoire des Mathématiques, Faculté des Sciences, Université d'Artois, Rue Jean Souvraz SP 18, 62307 Lens Cedex, France

Department of Mathematics, University of California, Los Angeles, CA 90095-1555

E-mail address: `karpenko@euler.univ-artois.fr`

E-mail address: `merkurev@math.ucla.edu`