HYPERBOLICITY OF HERMITIAN FORMS
OVER Biquaternion Algebras

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Abstract. We show that a non-hyperbolic hermitian form over a biquaternion algebra over a field of characteristic \( \neq 2 \) remains non-hyperbolic over a generic splitting field of the algebra.

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1. Introduction

Throughout this note (besides of §3 and §4) \( F \) is a field of characteristic \( \neq 2 \). The basic reference for the staff related to involutions on central simple algebras is [12]. The degree \( \deg A \) of a (finite dimensional) central simple \( F \)-algebra \( A \) is the integer \( \sqrt{\dim_F A} \); the index \( \text{ind} A \) of \( A \) is the degree of a central division algebra Brauer-equivalent to \( A \).

Conjecture 1.1. Let \( A \) be a central simple \( F \)-algebra endowed with an orthogonal involution \( \sigma \). If \( \sigma \) becomes hyperbolic over the function field of the Severi-Brauer variety of \( A \), then \( \sigma \) is hyperbolic (over \( F \)).

In a stronger version of Conjecture 1.1, each of two words “hyperbolic” is replaced by “isotropic”, cf. [10, Conjecture 5.2].

Here is the complete list of indices \( \text{ind} A \) and coindices \( \text{coind} A = \deg A / \text{ind} A \) of \( A \) for which Conjecture 1.1 is known (over an arbitrary field of characteristic \( \neq 2 \)), given in the chronological order:

- \( \text{ind} A = 1 \) — trivial;

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\begin{itemize}
  \item \text{coind} A = 1 \ (\text{the stronger version}) — [10, Theorem 5.3];
  \item \text{ind} A = 2 \ (\text{the stronger version}) — [15, Corollary 3.4];
  \item \text{coind} A \ \text{odd} — [6, appendix by Zainoulline] and independently [7, Theorem 3.3];
  \item \text{ind} A = 4 \text{ and coind} A = 2 — [18, Proposition 3].
\end{itemize}

Let us note that Conjecture 1.1 for any given \((A, \sigma)\) with \text{coind} A = 2 implies the stronger version \text{version} of Conjecture 1.1 for this \((A, \sigma)\): indeed, by [7, Theorem 3.3], if \text{coind} A = 2 and \(\sigma\) becomes isotropic over the function field of the Severi-Brauer variety, then \(\sigma\) becomes hyperbolic over this function field and the weaker version applies. In particular, the stronger version of Conjecture 1.1 is known for \(A\) with \text{ind} A = 4 and \text{coind} A = 2 (the last case in the above list, also discussed below).

The main result of this note is as follows:

**Theorem 1.2 (Main Theorem).** Conjecture 1.1 holds for any \(A\) of index 4.

Sivatski’s proof of the case with \text{deg} A = 8 and \text{ind} A = 4, mentioned above, is based on the following theorem, due to Laghribi:

**Theorem 1.3** ([13, Théorème 4]). Let \(\varphi\) be a quadratic form of dimension 8 and of trivial discriminant. Assume that the index of the Clifford algebra \(C\) of \(\varphi\) is 4. If \(\varphi\) becomes isotropic over the function field \(F(X_1)\) of the Severi-Brauer variety \(X_1\) of \(C\), then \(\varphi\) is isotropic (over \(F\)).

The following alternate proof of Theorem 1.3, given by Vishik, is a prototype of our proof of Main Theorem (Theorem 1.2). Let \(Y\) be the projective quadric of \(\varphi\) and let \(X_2\) be the Albert quadric of the biquaternion division algebra Brauer-equivalent to \(C\). Assume that \(\varphi_{F(X_1)}\) is isotropic. Then for any field extension \(E/F\), the Witt index of \(\varphi_E\) is at least 2 if and only if \(X_2(E) \neq \emptyset\). By [19, Theorem 4.15] and since the Chow motive \(M(X_2)\) of \(X_2\) is indecomposable, it follows that the motive \(M(X_2)(1)\) is a summand of the motive of \(Y\). The complement summand of \(M(Y)\) is then given by a Rost projector on \(Y\) in the sense of Definition 5.1. Since \(\dim Y + 1\) is not a power of 2, it follows that \(Y\) is isotropic (cf. [5, Corollary 80.11]).

In §4 we produce a replacement of [19, Theorem 4.15] (used right above to split off the summand \(M(X_2)(1)\) from the motive of \(Y\)) valid for more general (as projective quadrics) algebraic varieties. In §5 we reproduce some recent results due to Rost concerning the modulo 2 Rost correspondences and Rost projectors on more general (as projective quadrics) varieties. In §6 we apply some standard motivic decompositions of projective homogeneous varieties to certain varieties related to an index 4 central simple algebra with an isotropic orthogonal involution. Finally, in §7 we prove Main Theorem (Theorem 1.2).

### 2. Notation

Unlike [5], we understand under a \textit{variety} a separated scheme of finite type over a field. Let \(D\) be a central simple \(F\)-algebra. The \(F\)-dimension of any right ideal in \(D\) is divisible by \text{deg} \(D\); the quotient is the \textit{reduced dimension} of the ideal. For any integer \(i\), we write \(X(i; D)\) for the generalized Severi-Brauer variety of the right ideals in \(D\) of reduced dimension \(i\). In particular, \(X(0; D) = \text{Spec} F = X(\text{deg} D; D)\) and \(X(i, D) = \emptyset\) for \(i < 0\) and for \(i > \text{deg} D\).
More generally, let $V$ be a right $D$-module. The $F$-dimension of $V$ is then divisible by $\deg D$ and the quotient $\text{rdim} V = \dim_F V / \deg D$ is called the reduced dimension of $V$. For any integer $i$, we write $X(i; V)$ for the variety of the $D$-submodules in $V$ of reduced dimension $i$ (non-empty iff $0 \leq i \leq \text{rdim} V$). For a finite sequence of integers $i_1, \ldots, i_r$, we write $X(i_1 \subset \cdots \subset i_r; V)$ for the variety of flags of the $D$-submodules in $V$ of reduced dimensions $i_1, \ldots, i_r$ (non-empty iff $0 \leq i_1 \leq \cdots \leq i_r \leq \text{rdim} V$).

Now we additionally assume that $D$ is endowed with an orthogonal involution $\tau$. Then we write $X(i; (D, \tau))$ for the variety of the totally isotropic right ideals in $D$ of reduced dimension $i$ (non-empty iff $0 \leq i \leq \deg D / 2$).

If moreover $V$ is endowed with a hermitian form $h$, we write $X(i; (V, h))$ for the variety of the totally isotropic $D$-submodules in $V$ of reduced dimension $i$ and we write $X(i_1 \subset \cdots \subset i_r; (V, h))$ for the variety of flags of the totally isotropic $D$-submodules in $V$ of reduced dimensions $i_1, \ldots, i_r$.

Let us observe that all the varieties introduced in this section are projective homogeneous.

3. Krull-Schmidt principle

The characteristic of the base field $F$ is arbitrary in this section.

Our basic reference for Chow groups and Chow motives (including notation) is [5]. We fix an associative unital commutative ring $\Lambda$ (we will take $\Lambda = F_2$ in the application) and for a variety $X$ we write $\text{CH}(X; \Lambda)$ for its Chow group with coefficients in $\Lambda$. Our category of motives is the category $\text{CM}(F, \Lambda)$ of graded Chow motives with coefficients in $\Lambda$, [5, definition of §64]. By a sum of motives we always mean the direct sum.

We shall often assume that our coefficient ring $\Lambda$ is finite. This simplifies significantly the situation (and is sufficient for our application). For instance, for a finite $\Lambda$, the endomorphism rings of finite sums of Tate motives are also finite and the following easy statement applies:

**Lemma 3.1.** An appropriate power of any element of any finite associative (not necessarily commutative) ring is idempotent.

**Proof.** Since the ring is finite, for any its element $x$ we have $x^a = x^{a+b}$ for some $a \geq 1$ and $b \geq 1$. It follows that $x^{ab}$ is an idempotent.  

Let $X$ be a smooth complete variety over $F$. We call $X$ split, if its motive $M(X) \in \text{CM}(F, \Lambda)$ is a finite sum of Tate motives. We call $X$ geometrically split, if it splits over a field extension of $F$. We say that $X$ satisfies the nilpotence principle, if for any field extension $E/F$ the kernel of the change of field homomorphism $\text{End}(M(X)) \to \text{End}(M(X)_E)$ consists of nilpotents. Any projective homogeneous variety is geometrically split and satisfies the nilpotence principle, [3, Theorem 8.2].

**Corollary 3.2.** Assume that the coefficient ring $\Lambda$ is finite. Let $X$ be a geometrically split variety satisfying the nilpotence principle. Then an appropriate power of any endomorphism of the motive of $X$ is a projector.
Proof. Let \( \bar{F}/F \) be a field extension such that the motive \( M(X)_F \) is a finite sum of Tate motives. Let \( f \) be an endomorphism of \( M(X) \). Since our coefficient ring \( \Lambda \) is finite, it follows that the ring \( \text{End}(M(X)_F) \) is finite. Therefore a power of \( f_F \) is idempotent by Lemma 3.1, and we may assume that \( f_F \) is idempotent. Since \( X \) satisfies the nilpotence principle, the element \( \varepsilon := f^2 - f \) is nilpotent. Let \( n \) be a positive integer such that \( \varepsilon^n = 0 = n\varepsilon \). Then \( (f + \varepsilon)^{n^\alpha} = f^{n^\alpha} \) because the binomial coefficients \( \binom{n^\alpha}{i} \) for \( i < n \) are divisible by \( n \). Therefore \( f^{n^\alpha} \) is a projector. \( \square \)

Lemma 3.3 (cf. [4, Theorem 28]). Assume that the coefficient ring \( \Lambda \) is finite. Let \( X \) be a geometrically split variety satisfying the nilpotence principle and let \( p \in \text{End}(M(X)) \) be a projector. Then the motive \((X, p)\) decomposes into a finite sum of indecomposable motives.

Proof. If \((X, p)\) does not decompose this way, we get an infinite sequence

\[ p_0 = p, \ p_1, \ p_2, \ldots \in \text{End}(M(X)) \]

of pairwise distinct projectors such that \( p_i \circ p_j = p_j \circ p_i \) for any \( i < j \).

Let \( \bar{F}/F \) be a splitting field of \( X \). Since the ring \( \text{End}(M(X)_{\bar{F}}) \) is finite, we have \( (p_i)_{\bar{F}} = (p_j)_{\bar{F}} \) for some \( i < j \). The difference \( p_i - p_j \) is nilpotent and idempotent, therefore \( p_i = p_j \). \( \square \)

A (non necessarily commutative) ring is called local, if the sum of any two non-invertible elements differs from 1 in the ring. Since the sum of two nilpotents is never 1, we have

Lemma 3.4. A ring, where each non-invertible element is nilpotent, is local. In particular, by Corollary 3.2, so is the ring \( \text{End}(M(X)) \), if \( \Lambda \) is finite and \( X \) is a geometrically split variety satisfying the nilpotence principle and such that the motive \( M(X) \) is indecomposable. \( \square \)

Theorem 3.5 ([1, Theorem 3.6 of Chapter I]). If the endomorphism ring of any indecomposable object of a pseudo-abelian category is local, then any object of this category has at most one decomposition in a finite direct sum of indecomposable objects.

We say that the Krull-Schmidt principle holds for a given pseudo-abelian category, if every object of the category has one and unique decomposition in a finite direct sum of indecomposable objects. In the sequel, we are constantly using the following statement which is an immediate consequence of Lemmas 3.3 and 3.4 and the above theorem:

Corollary 3.6 (cf. [4, Corollary 35]). Assume that the coefficient ring \( \Lambda \) is finite. The Krull-Schmidt principle holds for the pseudo-abelian Tate subcategory in \( \text{CM}(F, \Lambda) \) generated by the motives of the geometrically cellular \( F \)-varieties satisfying the nilpotence principle. \( \square \)

Remark 3.7. Replacing the Chow groups \( \text{CH}(-; \Lambda) \) by the reduced Chow groups \( \overline{\text{CH}}(-; \Lambda) \) (cf. [5, §72]) in the definition of the category \( \text{CM}(F, \Lambda) \), we get a “simplified” motivic category \( \overline{\text{CM}}(F, \Lambda) \) (which is still sufficient for the main purpose of this paper). Working within this category, we do not need the nilpotence principle any more. In particular, the Krull-Schmidt principle holds (with a simpler proof) for the pseudo-abelian Tate subcategory in \( \overline{\text{CM}}(F, \Lambda) \) generated by the motives of the geometrically cellular \( F \)-varieties.
4. Splitting off a motivic summand

The characteristic of the base field $F$ is still arbitrary in this section.

In this section we assume that the coefficient ring $\Lambda$ is connected. We shall often assume that $\Lambda$ is finite.

Before climbing to the main result of this section (which is Proposition 4.5), let us do some warm up.

**Definition 4.1.** A motive $M \in \text{CM}(F, \Lambda)$ is called **outer**, if $\text{CH}^0(M; \Lambda) \neq 0$. A summand $M$ of the motive of a smooth complete variety is called **outer**, if this summand is an outer motive.

For instance, the motive of any smooth complete non-empty variety is outer and an outer summand of itself.

Given a correspondence $\alpha \in \text{CH}^{\dim X}(X \times Y; \Lambda)$ between some smooth complete irreducible varieties $X$ and $Y$, we write $\text{mult} \alpha \in \Lambda$ for the multiplicity of $\alpha$, [5, definition of §75]. Multiplicity of a composition of two correspondences is the product of multiplicities of the composed correspondences (cf. [10, Corollary 1.7]). In particular, multiplicity of a projector is idempotent and therefore $\in \{0, 1\}$ because the coefficient ring $\Lambda$ is connected.

**Lemma 4.2.** Let $X$ be a smooth complete irreducible variety. The motive $(X, p)$ given by a projector $p \in \text{CH}^{\dim X}(X \times X; \Lambda)$ is outer iff $\text{mult} p = 1$.

**Proof.** The group $\text{CH}^0((X, p); \Lambda)$ (defined as $\text{Hom}((X, p), \Lambda)$) is the image of the endomorphism of $\text{CH}^0(X; \Lambda) = \Lambda \cdot [X]$ given by the multiplication by $\text{mult} p$. \hfill $\square$

**Lemma 4.3.** Assume that a motive $M$ decomposes into a sum of Tate motives. Then $M$ is outer iff the Tate motive $\Lambda$ is present in the decomposition.

**Proof.** For any $i \in \mathbb{Z}$ we have: $\text{CH}^0(\Lambda(i); \Lambda) \neq 0$ iff $i = 0$.

The following statement generalizes (the finite coefficient version of) [19, Corollary 3.9]:

**Lemma 4.4.** Assume that the coefficient ring $\Lambda$ is finite. Let $X$ and $Y$ be smooth complete irreducible varieties such that there exist multiplicity 1 correspondences

$$\alpha \in \text{CH}^{\dim X}(X \times Y; \Lambda) \quad \text{and} \quad \beta \in \text{CH}^{\dim Y}(Y \times X; \Lambda).$$

Assume that $X$ is geometrically split and satisfies the nilpotence principle. Then there is an outer summand of $M(X)$ isomorphic to an outer summand of $M(Y)$. Moreover, for any outer summand $M_X$ of $M(X)$ and any outer summand $M_Y$ of $M(Y)$, there is an outer summand of $M_X$ isomorphic to an outer summand of $M_Y$.

**Proof.** By Corollary 3.2, the composition $p := (\beta \circ \alpha)^n$ for some $n \geq 1$ is a projector. Therefore $q := (\alpha \circ \beta)^{2n}$ is also a projector and the summand $(X, p)$ of $M(X)$ is isomorphic to the summand $(Y, q)$ of $M(Y)$: mutually inverse isomorphisms are, say,

$$\alpha \circ (\beta \circ \alpha)^{2^n} : (X, p) \to (Y, q) \quad \text{and} \quad \beta \circ (\alpha \circ \beta)^{2^{n-1}} : (Y, q) \to (X, p).$$

Since $\text{mult} p = (\text{mult} \beta \cdot \text{mult} \alpha)^n = 1$ and similarly $\text{mult} q = 1$, the summand $(X, p)$ of $M(X)$ and the summand $(Y, q)$ of $M(Y)$ are outer by Lemma 4.2.
We have proved the first statement of Lemma 4.4. As to the second statement, let
\[ p' \in \text{CH}_{\dim X}(X \times X; \Lambda) \quad \text{and} \quad q' \in \text{CH}_{\dim Y}(Y \times Y; \Lambda) \]
be projectors such that \( M_X = (X, p') \) and \( M_Y = (Y, q') \). Replacing \( \alpha \) and \( \beta \) by \( q' \circ \alpha \circ p' \) and \( p' \circ \beta \circ q' \), we get isomorphic outer motives \((X, p)\) and \((Y, q)\) which are summands of \( M_X \) and \( M_Y \). \( \square \)

Here comes the needed replacement of [19, Theorem 4.15]:

**Proposition 4.5.** Assume that the coefficient ring \( \Lambda \) is finite. Let \( X \) be a geometrically irreducible variety satisfying the nilpotence principle and let \( M \) be a motive. Assume that there exists a field extension \( E/F \) such that

1. the \( E \)-motive \( M(X)_E \in \text{CM}(E, \Lambda) \) of the \( E \)-variety \( X_E \) is indecomposable (this implies that the \( F \)-motive \( M(X) \in \text{CM}(F, \Lambda) \) is indecomposable);
2. the field extension \( E(X)/F(X) \) is purely transcendental;
3. the motive \( M(X)_E \) is a summand of the motive \( M_E \).

Then the motive \( M(X) \) is a summand of the motive \( M \).

**Proof.** We may assume that \( M = (Y, p, n) \) for some irreducible smooth complete \( F \)-variety \( Y \), a projector \( p \in \text{CH}_{\dim Y}(Y \times Y; \Lambda) \), and an integer \( n \).

By the assumption (3), we have morphisms of motives \( f : M(X)_E \to M_E \) and \( g : M_E \to M(X)_E \) with \( g \circ f = \text{id}_{M(X)_E} \).

Let \( \xi : \text{Spec} F(X) \to X \) be the generic point of the (irreducible) variety \( X \). For any \( F \)-scheme \( Z \), we write \( \xi_Z \) for the morphism \( \xi_Z = (\xi \times \text{id}_Z) : Z_{F(X)} = \text{Spec}_{F(X)} Z \to X \times Z \).

Note that for any \( \alpha \in \text{CH}(X \times Z) \), the image \( \xi^*_Z(\alpha) \in \text{CH}(Z_{F(X)}) \) of \( \alpha \) under the pull-back homomorphism \( \xi^*_Z : \text{CH}(X \times Z, \Lambda) \to \text{CH}(Z_{F(X)}, \Lambda) \) coincides with the composition of correspondences \( \alpha \circ [\xi] \), [5, Proposition 62.4(2)], where \( [\xi] \in \text{CH}_0(X_{F(X)}, \Lambda) \) is the class of the point \( \xi : X \times Z \to X \times Z \).

\[
(\ast) \quad \xi^*_Z(\alpha) = \alpha \circ [\xi].
\]

In the commutative square
\[
\begin{array}{ccc}
\text{CH}(X_E \times Y_E; \Lambda) & \xrightarrow{\xi^*_Y} & \text{CH}(Y_E(X); \Lambda) \\
\text{res}_{E/F} & & \text{res}_{E/F} \\
\text{CH}(X \times Y; \Lambda) & \xrightarrow{\xi^*_Y} & \text{CH}(Y_{F(X)}; \Lambda)
\end{array}
\]

the change of field homomorphism \( \text{res}_{E/F} \) is surjective\(^1\) because of the assumption (2) by the homotopy invariance of Chow groups [5, Theorem 57.13] and by the localization property of Chow groups [5, Proposition 57.11]. Moreover, the pull-back homomorphism \( \xi^*_Y \) is surjective by [5, Proposition 57.11]. It follows that there exists an element \( f' \in \text{CH}(X \times Y; \Lambda) \) such that \( \xi^*_Y(f_E) = \xi^*_Y(f) \).

The multiplicity \( \text{mult}(g \circ f) \) of the correspondence given by the composition \( g \circ f = \text{id}_{M(X)_E} \) is equal to 1. The multiplicity \( \text{mult}(g \circ f') \) coincides with \( \text{mult}(g \circ f) \) Indeed,\(^1\)

\(^1\)In fact, \( \text{res}_{E/F} \) is even an isomorphism, but we do not need its injectivity (which can be obtained with a help of specialization).
Therefore $-\lambda X < d > X$ no more need to assume that the nilpotence principle is no more needed in the proof of the weaker version. Because of that, there is a special correspondence on $\Lambda$ which is still sufficient for our application. The nilpotence position and composition. In particular, if $\rho$ is a Rost correspondence, then its both symmetrizations $\rho^t \circ \rho$ and $\rho \circ \rho^t$ are (symmetric) Rost correspondences. Writing $\rho_{F(X)}$ as in Definition 5.1, we have $(\rho^t \circ \rho)_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ (and $(\rho \circ \rho^t)_{F(X)} = \chi_2 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$).

Remark 4.6. Replacing $\text{CM}(F, \Lambda)$ by $\overline{\text{CM}}(F, \Lambda)$ in Proposition 4.5, we get a weaker version of Proposition 4.5 which is still sufficient for our application. The nilpotence principle is no more needed in the proof of the weaker version. Because of that, there is no more need to assume that $X$ satisfies the nilpotence principle.

5. ROST CORRESPONDENCES

In this section, $X$ stands for a smooth complete absolutely irreducible variety of dimension $d > 0$.

The coefficient ring $\Lambda$ of the motivic category is $\mathbb{F}_2$ in this section. We write $\text{Ch}(-)$ for the Chow group $\text{CH}(-; \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$. We write $\deg_{X/F}$ for the degree homomorphism $\text{Ch}_0(X) \to \mathbb{F}_2$.

Definition 5.1. An element $\rho \in \text{Ch}_d(X \times X)$ is called a Rost correspondence (on $X$), if $\rho_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \text{Ch}_0(X_{F(X)})$ of degree 1. A Rost projector is a Rost correspondence which is a projector.

Remark 5.2. Our definition of a Rost correspondence differs from the definition of a special correspondence in [16]. Our definition is weaker in the sense that a special correspondence on $X$ (which is an element of the integral Chow group $\text{CH}_d(X \times X)$) considered modulo 2 is a Rost correspondence but not any Rost correspondence is obtained this way. This difference gives a reason to reproduce below some results of [16].

Remark 5.3. Clearly, the set of all Rost correspondences on $X$ is stable under transposition and composition. In particular, if $\rho$ is a Rost correspondence, then its both symmetrizations $\rho^t \circ \rho$ and $\rho \circ \rho^t$ are (symmetric) Rost correspondences. Writing $\rho_{F(X)}$ as in Definition 5.1, we have $(\rho^t \circ \rho)_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$ (and $(\rho \circ \rho^t)_{F(X)} = \chi_2 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2$).
Lemma 5.4. Assume that the variety $X$ is projective homogeneous. Let $\rho \in \text{Ch}_d(X \times X)$ be a projector. If there exists a field extension $E/F$ such that $\rho_E = \chi_1 \times [X_E] + [X_E] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \text{Ch}_0(X_E)$ of degree 1, then $\rho$ is a Rost projector.

Proof. According to [3, Theorem 7.5], there exist some integer $n \geq 0$ and for $i = 1, \ldots, n$ some integers $r_i > 0$ and some projective homogeneous varieties $X_i$ satisfying $\dim X_i + r_i < d$ such that for $M = \bigoplus_{i=1}^n M(X_i)(r_i)$ the motive $M(X)_{F(X)}$ decomposes as $F_2 \oplus M \oplus F_2(d)$. Since there is no non-zero morphism between different summands of this decomposition, the ring $\text{End} M(X)$ decomposes in the product of rings

$$\text{End} F_2 \times \text{End} M \times \text{End} F_2(d) = F_2 \times \text{End} M \times F_2.$$ 

Let $\chi \in \text{Ch}_0(X_{F(X)})$ be a 0-cycle class of degree 1. We set

$$\rho' = \chi \times [X_{F(X)}] + [X_{F(X)}] \times \chi \in F_2 \times F_2$$
$$\subset F_2 \times \text{End} M \times F_2 = \text{End} M(X)_{F(X)} = \text{Ch}_d(X_{F(X)} \times X_{F(X)})$$

and we show that $\rho_{F(X)} = \rho'$. The difference $\varepsilon = \rho_{F(X)} - \rho'$ vanishes over $E(X)$. Therefore $\varepsilon$ is a nilpotent element of $\text{End} M$. Choosing a positive integer $m$ with $\varepsilon^m = 0$, we get

$$\rho_{F(X)} = \rho_{F(X)}^m = (\rho' + \varepsilon)^m = (\rho')^m + \varepsilon^m = (\rho')^m = \rho'.$$

\square

Lemma 5.5. Let $\rho \in \text{Ch}_d(X \times X)$ be a projector. The motive $(X, \rho)$ is isomorphic to $F_2 \oplus F_2(d)$ if and only if $\rho = \chi_1 \times [X] + [X] \times \chi_2$ for some 0-cycle classes $\chi_1, \chi_2 \in \text{Ch}_0(X)$ of degree 1.

Proof. A morphism $F_2 \oplus F_2(d) \to (X, \rho)$ is given by some

$$f \in \text{Hom}(F_2, M(X)) = \text{Ch}_0(X) \quad \text{and} \quad f' \in \text{Hom}(F_2(d), M(X)) = \text{Ch}_d(X).$$

A morphism in the inverse direction is given by some

$$g \in \text{Hom}(M(X), F_2) = \text{Ch}^0(X) \quad \text{and} \quad g' \in \text{Hom}(M(X), F_2(d)) = \text{Ch}^d(X).$$

The two morphisms are mutually inverse isomorphisms if $\rho = f \times g + f' \times g'$ and $\deg_{X/F}(fg) = 1 = \deg_{X/F}(f'g')$. The degree condition means that $f' = [X] = g$ and $\deg_{X/F}(f) = 1 = \deg_{X/F}(g')$. \square

Corollary 5.6. If $X$ is projective homogeneous and $\rho$ is a projector on $X$ such that

$$(X, \rho)_E \simeq F_2 \oplus F_2(d)$$

for some field extension $E/F$, then $\rho$ is a Rost projector. \square

A smooth complete variety is called anisotropic, if the degree of its any closed point is even.

Lemma 5.7 ([16, Lemma 9.2], cf. [17, proof of Lemma 6.2]). Assume that $X$ is anisotropic and possesses a Rost correspondence $\rho$. Then for any integer $i \neq d$ and any elements $\alpha \in \text{Ch}_i(X)$ and $\beta \in \text{Ch}^i(X_{F(X)})$, the image of the product $\alpha_{F(X)} \cdot \beta \in \text{Ch}_0(X_{F(X)})$ under the degree homomorphism $\deg_{X_{F(X)}/F(X)} : \text{Ch}_0(X_{F(X)}) \to F_2$ is 0.
Proof. Let $\gamma \in \text{Ch}^i(X \times X)$ be a preimage of $\beta$ under the surjection

$$\xi_X^i : \text{Ch}^i(X \times X) \to \text{Ch}^i(\text{Spec } F(X) \times X)$$

(where $\xi_X^i$ is as defined in the proof of Proposition 4.5). We consider the 0-cycle class

$$\delta = \rho \cdot ([X] \times \alpha) \cdot \gamma \in \text{Ch}_0(X \times X).$$

Since $X$ is anisotropic, so is $X \times X$, and it follows that $\text{deg}_{(X \times X)/F} \delta = 0$. Therefore it suffices to show that $\text{deg}_{(X \times X)/F} \delta = \text{deg}_{X_{F(X)}/F(X)}(\alpha_{F(X)} \cdot \beta)$.

We have $\text{deg}_{(X \times X)/F} \delta = \text{deg}_{X_{F(X)}/F(X)}(\delta_{F(X)})$ and

$$\delta_{F(X)} = (\chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2) \cdot (([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} = ([\chi_1] \times [X_{F(X)}]) \cdot ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}$$

(because $i \neq d$) where $\chi_1, \chi_2 \in \text{Ch}_0(X_{F(X)})$ are as in Definition 5.1. For the first projection $\text{pr}_1 : X_{F(X)} \times X_{F(X)} \to X_{F(X)}$ we have

$$\text{deg}_{(X \times X)/F} \delta_{F(X)} = \text{deg}_{X_{F(X)}/F(X)}(\text{pr}_1)_*(\delta_{F(X)})$$

and by the projection formula

$$(\text{pr}_1)_*(\delta_{F(X)}) = [\chi_1] \cdot (\text{pr}_1)_*([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)}).$$

Finally,

$$(\text{pr}_1)_*([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} = \text{mult} \left( ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} \right) \cdot [X_{F(X)}]$$

and

$$\text{mult} \left( ([X_{F(X)}] \times \alpha_{F(X)}) \cdot \gamma_{F(X)} \right) = \text{mult} \left( ([X] \times \alpha) \cdot \gamma \right).$$

Since $\text{mult} \chi = \text{deg}_{X_{F(X)}/F(X)} \xi_X^i(\chi)$ for any element $\chi \in \text{Ch}_0(X \times X)$ by [5, Lemma 75.1], it follows that

$$\text{mult} \left( ([X] \times \alpha) \cdot \gamma \right) = \text{deg}(\alpha_{F(X)} \cdot \beta).$$

For anisotropic $X$, we consider the homomorphism $\text{deg}/2 : \text{Ch}_0(X) \to \mathbb{F}_2$ induced by the homomorphism $\text{CH}_0(X) \to \mathbb{Z}$, $\alpha \mapsto \text{deg}(\alpha)/2$.

Corollary 5.8. Assume that $X$ is anisotropic and possesses a Rost correspondence. Then for any integer $i \neq d$ and any elements $\alpha \in \text{Ch}_i(X)$ and $\beta \in \text{Ch}^i(X)$ with $\beta_{F(X)} = 0$ one has

$$(\text{deg}/2)(\alpha \cdot \beta) = 0.$$
We use once again the condition that $i \not\in \{0, d\}$.

Let $\chi_1$ and $\chi_2$ be as in Definition 5.1. Let $\chi'_1, \chi'_2 \in \text{CH}_0(X_{F(X)})$ be integral representatives of $\chi_1$ and $\chi_2$. Then $\rho_{F(X)} = \chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2 + 2\gamma$ for some $\gamma \in \text{CH}_d(X_{F(X)} \times X_{F(X)})$. Therefore (since $i \not\in \{0, d\}$)

\[(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \rho_{F(X)} = 2(\alpha'_{F(X)} \times \beta'_{F(X)}) \cdot \gamma.\]

Applying the projection $pr_1$ onto the first factor and the projection formula, we get twice the element $(\alpha'_{F(X)} \cdot (pr_1)_*([X_{F(X)}] \times \beta'_{F(X)}) \cdot \gamma)$ whose degree is even by Lemma 5.7 (here we use once again the condition that $i \not= d$).

**Lemma 5.10.** Assume that $X$ is anisotropic and possesses a Rost correspondence $\rho$. Then $(\deg/2)(\rho^2) = 1$.

**Proof.** Let $\chi_1$ and $\chi_2$ be as in Definition 5.1. Let $\chi'_1, \chi'_2 \in \text{CH}_0(X_E)$ be integral representatives of $\chi_1$ and $\chi_2$. The degrees of $\chi'_1$ and $\chi'_2$ are odd. Therefore, the degree of the cycle class

\[(\chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2)^2 = 2(\chi'_1 \times \chi'_2) \in \text{CH}_0(X_{F(X)} \times X_{F(X)})\]

is not divisible by 4.

Let $\rho' \in \text{CH}_d(X \times X)$ be an integral representative of $\rho$. Since $\rho'_{F(X)} = \chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2$ modulo 2, $(\rho'_{F(X)})^2$ is $(\chi'_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi'_2)^2$ modulo 4. Therefore $(\deg/2)(\rho^2) = 1$. \hfill \Box

**Theorem 5.11** ([16, Theorem 9.1], see also [17, proof of Lemma 6.2]). Let $X$ be an anisotropic smooth complete absolutely irreducible variety of dimension $d > 0$ over a field $F$ of characteristic $\not= 2$ possessing a Rost correspondence. Then the degree of the highest Chern class $c_d(-T_X)$, where $T_X$ is the tangent bundle on $X$, is not divisible by 4.

**Proof.** In this paper, we write $c_\bullet(-T_X)$ for the total Chern class $c_\bullet \in \text{Ch}(X)$ in the Chow group with coefficient in $\mathbb{F}_2$. It suffices to show that $(\deg/2)(c_d(-T_X)) = 1$.

Let $\text{Sq}^X : \text{Ch}(X) \to \text{Ch}(X)$ be the modulo 2 homological Steenrod operation, [5, §59]. We have a commutative diagram

```
\begin{align*}
\text{Ch}_d(X \times X) & \xrightarrow{(pr_1)_*} \text{Ch}_0(X) \\
\text{Ch}_d(X) & \xrightarrow{\text{sq}^X} \text{Ch}_0(X) \times \text{Ch}_0(X) \\
\text{Ch}_0(X) & \xrightarrow{(pr_1)_*} \text{Ch}_0(X) \\
\text{deg}/2 \quad \text{deg}/2 & \xrightarrow{\text{deg}/2} \text{deg}/2
\end{align*}
```
Since \((pr_1)_\ast (\rho) = [X]\) and \(Sq^X_\ast ([X]) = c_d(-T_X)\) [5, formula (60.1)], it suffices to show that
\[
(deg/2)\left( Sq^X_{d\times X}(\rho) \right) = 1.
\]

We have \(Sq^X_{d\times X} = c_\ast (-T_{X\times X}) \cdot Sq^X_{d\times X}\), where \(Sq^\ast\) is the cohomological Steenrod operation, [5, §61]. Therefore
\[
Sq^X_{d\times X}(\rho) = \sum_{i=0}^{d} c_{d-i}(-T_{X\times X}) \cdot Sq^i_{X\times X}(\rho).
\]

The summand with \(i = d\) is \(Sq^d_{X\times X}(\rho) = \rho^2\) by [5, Theorem 61.13]. By Lemma 5.10, its image under \(deg/2\) is 1.

Since \(c_\ast (-T_{X\times X}) = c_\ast (-T_X) \cdot c_\ast (-T_X)\) and \(Sq^0 = \text{id}\), the summand with \(i = 0\) is
\[
\left( \sum_{j=0}^{d} c_j(-T_X) \cdot c_{d-j}(-T_X) \right) \cdot \rho.
\]

Its image under \(deg/2\) is 0 because
\[
(deg/2)\left( (c_0(-T_X) \times c_d(-T_X)) \cdot \rho \right) = (deg/2)(c_d(-T_X))
= (deg/2)\left( (c_d(-T_X) \times c_0(-T_X)) \cdot \rho \right)
\]
while for \(j \notin \{0, d\}\), we have \((deg/2)\left( (c_j(-T_X) \times c_{d-j}(-T_X)) \cdot \rho \right) = 0\) by Corollary 5.9.

Finally, for any \(i\) with \(0 < i < d\) the \(i\)th summand is the sum
\[
\sum_{j=0}^{d-i} (c_j(-T_X) \times c_{d-i-j}(-T_X)) \cdot Sq^i_{X\times X}(\rho).
\]

We shall show that for any \(j\) the image of the \(j\)th summand under \(deg/2\) is 0. Note that the image under \(deg/2\) coincides with the image under the composition \((deg/2) \circ (pr_1)_\ast\) and also under the composition \((deg/2) \circ (pr_2)_\ast\) (look at the above commutative diagram). By the projection formula we have
\[
(pr_1)_\ast \left( (c_j(-T_X) \times c_{d-i-j}(-T_X)) \cdot Sq^i_{X\times X}(\rho) \right) =
\]
\[
\left( [X] \times c_{d-i-j}(-T_X) \right) \cdot Sq^i_{X\times X}(\rho)
\]
and the image under \(deg/2\) is 0 for positive \(j\) by Corollary 5.8 applied to \(\alpha = c_j(-T_X)\) and \(\beta = (pr_1)_\ast \left( ([X] \times c_{d-i-j}(-T_X)) \cdot Sq^i_{X\times X}(\rho) \right)\). Corollary 5.8 can be indeed applied, because since \(\rho_{F(X)} = \chi_1 \times [X_{F(X)}] + [X_{F(X)}] \times \chi_2\) and \(i > 0\), we have \(Sq^i_{X\times X}(\rho)_{F(X)} = 0\) and therefore \(\beta_{F(X)} = 0\).

For \(j = 0\) we use the projection formula for \(pr_2\) and Corollary 5.8 with \(\alpha = c_{d-i}(-T_X)\) and \(\beta = (pr_2)_\ast \left( Sq^i_{X\times X}(\rho) \right)\).

\[\square\]

**Remark 5.12.** The reason of the characteristic exclusion in Theorem 5.11 is that its proof makes use of Steenrod operations on Chow groups with coefficients in \(\mathbb{F}_2\) which are not available in characteristic 2.
We would like to mention

**Lemma 5.13** ([16, Lemma 9.10]). Let $X$ be an anisotropic smooth complete equidimensional variety over a field of arbitrary characteristic. If $\dim X + 1$ is not a power of 2, then the degree of the integral 0-cycle class $c_{\dim X}(-T_X) \in \text{CH}_0(X)$ is divisible by 4.

6. Motivic decompositions of some isotropic varieties

The coefficient ring $\Lambda$ is arbitrary in the first half of this section and $\Lambda = \mathbb{F}_2$ in the second half (starting from Lemma 6.3). Throughout this section, $D$ is a biquaternion division algebra, $X_1$ is the Severi-Brauer variety $X(1; D)$ of $D$, and $X_2$ is the generalized Severi-Brauer variety $X(2; D)$.

We say that motives $M$ and $N$ are quasi-isomorphic and write $M \simeq N$, if there exist decompositions $M \simeq M_1 \oplus \cdots \oplus M_m$ and $N \simeq N_1 \oplus \cdots \oplus N_n$ such that

$$M_1(i_1) \oplus \cdots \oplus M_m(i_m) \simeq N_1(j_1) \oplus \cdots \oplus N_n(j_n)$$

for some (twist) integers $i_1, \ldots, i_m$ and $j_1, \ldots, j_n$.

**Lemma 6.1.** Tensor product of any finite number of copies of $M(X_1)$ by any finite number of copies of $M(X_2)$ is quasi-isomorphic to a sum of finite number of copies of $M(X_1)$ and $M(X_2)$. (In other words, the Tate subcategory of $\text{CM}(F, \Lambda)$ generated by $M(X_1)$ and $M(X_2)$ is a tensor subcategory, cf. [4, Theorem 24].)

**Proof.** The product $X_1 \times X_1$ is a projective bundle over $X_1$. Therefore $M(X_1) \otimes M(X_1)$ is quasi-isomorphic to a sum of several copies of $M(X_1)$, [5, Theorem 63.10].

The product $X_2 \times X_1$ is a grassmann bundle over $X_1$. Therefore $M(X_2) \otimes M(X_1)$ is also quasi-isomorphic to a sum of several copies of $M(X_1)$.

Finally, by [4], the motive of the product $X_2 \times X_2$ quasi-decomposes into two copies of $M(X_2)$ and one copy of the motive of the flag variety $X(1 \subset 2 \subset 3; D)$. The projections

$$X(1 \subset 2 \subset 3; D) \to X(1 \subset 2; D) \to X(1; D) = X_1$$

are grassmann bundles. Consequently, $M(X(1 \subset 2 \subset 3; D))$ is quasi-isomorphic to a sum of several copies of $M(X_1)$.

For the rest of this section, we fix an orthogonal involution on the biquaternion algebra $D$.

**Lemma 6.2.** Let $n$ be a positive integer. Let $h$ be a hyperbolic hermitian form on the right $D$-module $D^{2n}$ and let $Y$ be the variety $X(4n; (D^{2n}, h))$ (of the maximal totally isotropic submodules). Then the motive $M(Y)$ is isomorphic to a finite sum of several twisted copies of the motives $M(X_1), M(X_2)$, and $\Lambda$ including one non-twisted copy of $\Lambda$.

**Proof.** By [9, Следствие 15.9] (cf. [3]), the motive of the variety $Y$ is quasi-isomorphic to the motive of the “total” variety

$$X(**; D^n) = \prod_{i \in \mathbb{Z}} X(i; D^n) = \prod_{i=0}^{4n} X(i; D^n)$$

of $D$-submodules in $D^n$. Furthermore, $M(X(**; D^n)) \approx M(X(**; D))^{\otimes n}$ by [9, Следствие 10.10] (cf. [3]). We finish by Lemma 6.1. The non-twisted copy of $\Lambda$ is obtained as $\Lambda = M(X(0; D^n)) = M(X(0; D))^{\otimes n}$. 

□
As before, we write Ch(−) for the Chow group CH(−; \mathbb{F}_2) with coefficients in \mathbb{F}_2. We recall that a smooth complete variety is called anisotropic, if the degree of its any closed point is even (the empty variety is anisotropic).

**Lemma 6.3.** Let Z be an anisotropic F-variety with a projector p ∈ CH_{\dim Z}(Z × Z) such that the motive (Z, p) \in \text{CM}(L, \mathbb{F}_2) for a field extension L/F is isomorphic to a finite sum of Tate motives. Then the number of the Tate summands is even. In particular, the motive in CM(F, \mathbb{F}_2) of any anisotropic F-variety does not contain a Tate summand.

*Proof.* Mutually inverse isomorphisms between (Z, p) \in \text{CM}(L, \mathbb{F}_2) and Tate motives, are given by two sequences of homogeneous elements a_1, ..., a_n and b_1, ..., b_n in Ch(Z_L) with p_L = a_1 × b_1 + ... + a_n × b_n and such that for any i, j = 1, ..., n the degree deg(a_ib_j) is 0 for i ≠ j and 1 ∈ \mathbb{F}_2 for i = j. The pull-back of p via the diagonal morphism of Z is therefore a 0-cycle class on Z of degree n (modulo 2). □

**Lemma 6.4.** Let n be a positive integer. Let h' be a hermitian form on the right D-module D^n such that h' is anisotropic for any finite odd degree field extension L/F. Let h be the hermitian form on the right D-module D^{n+2} which is the orthogonal sum of h' and a hyperbolic D-plane. Let Y' be the variety of totally isotropic submodules of D^{n+2} of reduced dimension 4 (= ind D). Then the complete motivic decomposition of M(Y') \subseteq \text{CM}(F, \mathbb{F}_2) (cf. Corollary 3.6) contains one summand \mathbb{F}_2, one summand \mathbb{F}_2(\text{dim} Y'), and does not contain any other Tate motive.

*Proof.* According to [9, Следствие 15.9], M(Y') is quasi-isomorphic to the sum of the motives of the products

\[X(i \subset j; D) \times X(j - i; (D^n, h'))\]

where i, j run over all integers (the product is non-empty iff 0 ≤ i ≤ j ≤ 4). The choices i = j = 0 and i = j = 4 give the summands \mathbb{F}_2 and \mathbb{F}_2(\text{dim} Y'). The variety obtained by any other choice of i, j is anisotropic (the variety with i = 0, j = 4 is anisotropic by the assumption involving the odd degree field extensions), and we are done by Lemma 6.3. □

**Lemma 6.5.** The motives M(X_1) and M(X_2) in the category \text{CM}(F, \mathbb{F}_2) are indecomposable.

*Proof.* Since the varieties X_1 and X_2 satisfy the nilpotence principle, it suffices to show that the reduced Chow groups \overline{\text{Ch}}_{\dim X_1}(X_1 × X_1) and \overline{\text{Ch}}_{\dim X_2}(X_2 × X_2) do not contain non-trivial projectors. The statement on X_1 is a consequence of [8, Предложение 2.1.1].

The variety X_2 is isomorphic to the Albert quadric of D, [5, §16.A]. By [19, §7.3], the splitting pattern ([5, definition of §25]) of the Albert form of a biquaternion division algebra is (0, 1, 3). It follows by [5, Theorem 73.26] that in the case of decomposable M(X_2), one of the summands is given by a Rost projector on X_2. As already mentioned, there can’t be a Rost projector on X_2 because X_2 is anisotropic and \text{dim} X_2 + 1 = 5 is not a power of 2. Here one may refer to [5, Corollary 80.11], or to a more general Theorem 5.11, or to a simpler argument for this specific X_2 which uses [11, Lemma 6.5] and computation of K_0(X_2). □
7. Proof of Main Theorem

We fix a central simple algebra $A$ of index 4 with a non-hyperbolic orthogonal involution $\sigma$. We assume that $\sigma$ becomes hyperbolic over the function field of the Severi-Brauer variety of $A$ and we are looking for a contradiction.

According to [7, Theorem 3.3], $\text{coind } A = 2n$ for some integer $n \geq 1$. We assume that Main Theorem (Theorem 1.2) is already proven for all algebras (over all fields) of index 4 and of coindex $< 2n$.

Let $D$ be a biquaternion division algebra Brauer-equivalent to $A$. Let $X_1$ be the Severi-Brauer variety of $D$. Let us fix an orthogonal involution on $D$ and an isomorphism of $F$-algebras $A \simeq \text{End}_D(D^{2n})$. Let $h$ be a hermitian form on the right $D$-module $D^{2n}$ such that $\sigma$ is adjoint to $h$. Then $h_{F(X_1)}$ is hyperbolic. Since the anisotropic kernel of $h$ also becomes hyperbolic over $F(X_1)$, our induction hypothesis ensures that $h$ is anisotropic. Moreover, $h_L$ is hyperbolic for any field extension $L/F$ such that $h_L$ is isotropic. It follows by [2, Proposition 1.2] that $h_L$ is anisotropic for any finite odd degree field extension $L/F$.

Let $Y$ be the variety of totally isotropic submodules in $D^{2n}$ of reduced dimension $4n$. (The variety $Y$ is a twisted form of the variety of maximal totally isotropic subspaces of a quadratic form studied in [5, Chapter XVI].) It is isomorphic to the variety of totally isotropic right ideals in $A$ of reduced dimension $4n$. Since $\sigma$ is hyperbolic over $F(X_1)$ and the field $F$ is algebraically closed in $F(X_1)$ (because the variety $X_1$ is geometrically integral), the discriminant of $\sigma$ is trivial. Therefore the variety $Y$ has two connected components $Y_+$ and $Y_-$ corresponding to the components $C_+$ and $C_-$ (cf. [5, Theorem 8.10]) of the Clifford algebra $C(A, \sigma)$.

Since $\sigma_{F(X_1)}$ is hyperbolic, $Y(F(X_1)) \neq \emptyset$. Since the varieties $Y_+$ and $Y_-$ become isomorphic over $F(X_1)$, each of them has an $F(X_1)$-point.

The central simple algebras $C_+$ and $C_-$ are related with $A$ by the formula [12, (9.14)]:

$$[C_+] + [C_-] = [A] \in \text{Br}(F).$$

Since $[C_+]_{F(X_1)} = [C_-]_{F(X_1)} = 0 \in \text{Br}(F(X_1))$, we have $[C_+], [C_-] \in \{0, [A]\}$ and it follows that $[C_+] = [A], [C_-] = 0$ up to exchange of the indices $+, -$.

By the index reduction formula for the varieties $Y_+$ and $Y_-$ of [14, page 594], we have (up to exchange of the indices $+, -$): $\text{ind } D_{F(Y_+)} = 4$, $\text{ind } D_{F(Y_-)} = 1$. We replace $Y$ by the component of $Y$ whose function field does not reduce the index of $D$.

The coefficient ring $\Lambda$ is $\mathbb{F}_2$ in this section. As before, we set $X_2 = X(2, D)$.

Lemma 7.1. The motive of $Y$ decomposes as $R_1 \oplus R_2$, where $R_1$ is quasi-isomorphic to a finite sum of several copies of the motives $M(X_1)$ and $M(X_2)$, and where $(R_2)_{F(Y)}$ is isomorphic to a finite sum of Tate motives including one exemplar of $\mathbb{F}_2$.

Proof. According to Lemma 6.2, the motive $M(Y)_{F(Y)}$ is isomorphic to a sum of several twisted copies of $M(X_1)$, $M(X_2)$, and $\mathbb{F}_2$ including a copy of $\mathbb{F}_2$. If for some $i = 1, 2$ there is at least one copy of $M(X_i)$ (with a twist $j \in \mathbb{Z}$) in the decomposition, let us apply Proposition 4.5 taking as $X$ the variety $X_i$, taking as $M$ the motive $M(Y)(-j)$, and taking as $E$ the function field $F(Y)$.

The condition (3) of Proposition 4.5 is fulfilled.
Since $D_E$ is still a division algebra, the condition (1) is fulfilled by Lemma 6.5. Since $	ext{ind} D_{F(X)} < 4$, the hermitian form $h_{F(X)}$ is hyperbolic by [15]; therefore the variety $Y_{F(X)}$ is rational and the condition (2) is fulfilled.

It follows that $M(X_i)$ is a summand of $M(Y)(-j)$. Let now $M$ be the complement summand of $M(Y)(-j)$. By Corollary 3.6, the complete decomposition of $M_{F(Y)}$ is the complete decomposition of $M(Y)(-j)_{F(Y)}$ with one copy of $M(X_i)$ erased. If $M_{F(Y)}$ contains one more copy of a twist of $M(X_i)$ (for some $i = 1, 2$), we once again apply Proposition 4.5 to the variety $X_i$ and an appropriate twist of $M$. Doing this until we can, we get the desired decomposition in the end. 

Now let us consider a minimal right $D$-submodule $V \subset D^{2n}$ such that $V$ becomes isotropic over a finite odd degree field extension of $F(Y)$. We set $v = \dim_D V$. Clearly, $2 \leq v \leq n + 1$. \footnote{One probably always have $v = n + 1$ here, but we do not need to know the precise value of $v$.} For $n > 1$, let $Y'$ be the variety of totally isotropic submodules in $V$ of reduced dimension 4 (that is, of “$D$-dimension” 1). For $n = 1$ we set $Y' = Y$.

The variety $Y'$ has dimension $\dim Y' = 16v - 26$ which is even (and positive). Moreover, the variety $Y'$ is anisotropic (because the hermitian form $h$ is anisotropic and remains anisotropic over any finite odd degree field extension of the base field). Surprisingly, we can however prove the following

**Lemma 7.2.** There is a Rost projector (Definition 5.1) on $Y'$.

**Proof.** By the construction of $Y'$, there exists a correspondence of odd multiplicity (that is, of multiplicity $1 \in \mathbb{F}_2$) $\alpha \in \text{Ch}_{\dim Y'}(Y' \times Y')$. On the other hand, since $h_{F(Y')}$ is isotropic, $h_{F(Y')}$ is hyperbolic and therefore there exist a rational map $Y' \dashrightarrow Y$ and a multiplicity 1 correspondence $\beta \in \text{Ch}_{\dim Y'}(Y' \times Y)$. Since the summand $R_2$ of $M(Y)$ given by Lemma 7.1 is outer (cf. Definition 4.1 and Lemma 4.3), by Lemma 4.4 there is an outer summand of $M(Y')$ isomorphic to a summand of $R_2$.

Let $\rho \in \text{Ch}_{\dim Y'}(Y' \times Y')$ be the projector giving this summand. We claim that $\rho$ is a Rost projector. We prove the claim by showing that the motive $(Y', \rho)_F$ is isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y')$, cf. Corollary 5.6, where $\tilde{F}/F(Y)$ is a finite odd degree field extension such that $V$ becomes isotropic over $\tilde{F}$.

Since $(R_2)_{F(Y)}$ is a finite sum of Tate motives, the motive $(Y', \rho)_F$ is also a finite sum of Tate motives. Since $(Y', \rho)_F$ is outer, the Tate motive $\mathbb{F}_2$ is included (Lemma 4.3). Now, by the minimal choice of $V$, the hermitian form $(h|_V)_F$ satisfies the condition on $h$ in Lemma 6.4: $(h|_V)_F$ is an orthogonal sum of a hyperbolic $D_{\tilde{F}}$-plane and a hermitian form $h'$ such that $h'_{F}$ is anisotropic for any finite odd degree field extension $L/\tilde{F}$ of the base field $\tilde{F}$ (otherwise any $D$-hyperplane $V' \subset V$ would become isotropic over some odd degree extension of $F(Y)$). Therefore the complete motivic decomposition of $Y'_F$ (see Corollary 3.6) has one copy of $\mathbb{F}_2$, one copy of $\mathbb{F}_2(\dim Y')$, and no other Tate summands. By Corollary 3.6 and anisotropy of the variety $Y'$ (see Lemma 6.3), it follows that

$$(Y', \rho)_F \simeq \mathbb{F}_2 \oplus \mathbb{F}_2(\dim Y').$$

Lemma 7.2 contradicts to the general results of §5 (namely, to Theorem 5.11 and Lemma 5.13) thus proving Main Theorem (Theorem 1.2). We can avoid the use of Lemma 5.13 by showing that $\text{deg} c_{\dim Y'}(-T_{Y'})$ is divisible by 16 for our variety $Y'$. Indeed, let $K$ be
the field $F(t_1, \ldots, t_{4v})$ of rational functions over $F$ in $4v$ variables. Let us consider the (generic) diagonal quadratic form $\langle t_1, \ldots, t_{4v} \rangle$ on the $K$-vector space $K^{4v}$. Let $Y''$ be the variety of 4-dimensional totally isotropic subspaces in $K^{4v}$. The degree of any closed point on $Y''$ is divisible by $2^4$. In particular, the integer $\deg c_\dim Y'(-TY')$ is divisible by $2^4$. Since over an algebraic closure $\overline{K}$ of $K$ the varieties $Y'$ and $Y''$ become isomorphic, we have

$$\deg c_\dim Y'(-TY') = \deg c_\dim Y''(-TY'').$$

**References**


