Cohomological Invariants of Homogeneous Varieties (with Applications to Quadratic Forms and Central Simple Algebras)

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Preface

Let X be a projective homogeneous variety over a field F. We study the Chow group $\mathrm{CH}^*(X)$ of algebraic cycles on X modulo rational equivalence graded by codimension of cycles. Especially, we are interested in the Chow group $\mathrm{CH}^2(X)$ of 2-codimensional cycles because of its connection with the relative Galois cohomology group $H^3(F(X)/F, \mathbb{Z}/2)$, where $F(X)$ is the function field of X. We compute these groups for various types of X and apply the results to problems in the theories of quadratic forms (e.g. isotropy over function fields of homogeneous varieties) and central simple algebras (decomposability, common splitting fields).

Each chapter is written as an independent article with an abstract and an introduction.

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Contents

CHAPTER 1

Codimension 2 cycles on Severi-Brauer varieties

For a given sequence of integers $(n_i)_{i=1}^{\infty}$, we consider all the central simple algebras A (over all fields) satisfying the condition $\operatorname{ind} A^{\otimes i} = n_i$ and find among them an algebra having the biggest torsion in the second Chow group CH^2 of the corresponding Severi-Brauer variety ("biggest" means that it can be mapped epimorphically onto each other).

We give a description of this biggest torsion in the general case (via the gamma-filtration) and find out when (i.e. for which sequences $(n_i)_{i=1}^{\infty}$) it is non-trivial. We also make an explicit computation in some special situations: e.g. in the situation of algebras of a square-free exponent e the biggest torsion turns out to be (cyclic) of order e.

As an application we prove indecomposability for certain algebras of a prime exponent.

0. Introduction

We consider finite dimensional central simple algebras over fields. Let A be such an algebra and $X = \mathrm{SB}(A)$ the corresponding Severi-Brauer variety ([**6**, §1]). We are interested to describe the torsion in the second Chow group $\mathrm{CH}^2(X)$ of 2-codimensional cycles on X modulo rational equivalence (the question seems more natural if one takes in account that the groups $\mathrm{CH}^0(X)$ and $\mathrm{CH}^1(X)$ never have a torsion). Here are some preliminary observations. The group $\operatorname{Tors}\mathrm{CH}^2(X)$ is finite and annihilated by $\operatorname{ind} A$. Further, if A' is another algebra Brauer equivalent to A and $X' = \mathrm{SB}(A')$ then by [**58**, Lemma 1.12] or [**37**, Corollary 1.3.2]

$$\operatorname{Tors}\mathrm{CH}^2(X) \simeq \operatorname{Tors}\mathrm{CH}^2(X')\ .$$

Finally, if

$$A = \bigotimes_p A_p$$

is the decomposition of an algebra A into the tensor product of its primary components and $X_p = \mathrm{SB}(A_p)$ for each prime p then

$$\operatorname{Tors}\mathrm{CH}^2(X) \simeq \bigoplus_p \operatorname{Tors}\mathrm{CH}^2(X_p)$$

or in other words, the p-primary part of the group $\operatorname{Tors}\mathrm{CH}^2(X)$ is isomorphic to $\operatorname{Tors}\mathrm{CH}^2(X_p)$ (Proposition 1.3).

Summarizing, we see that the problem to compute $\operatorname{Tors}\mathrm{CH}^2(X)$ for all algebras reduces itself to the case of primary division algebras.

Now consider the Grothendieck group $K(X) = K_0(X)$ together with the gamma-filtration (Definition 2.6):

$$K(X) = \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots \ .$$

One has a canonical epimorphism (see the proof of Corollary 2.15)

$$\Gamma^{2/3}K(X) \twoheadrightarrow \mathrm{CH}^2(X)$$

of the quotient

$$\Gamma^{2/3}K(X) = \Gamma^2 K(X)/\Gamma^3 K(X)\ .$$

We consider the group $\Gamma^{2/3}K(X)$ as an upper bound for $\mathrm{CH}^2(X)$ and will show that in the primary case this upper bound is in certain sense the least one.

To formulate it precisely, let us call the sequence $(\operatorname{ind} A^{\otimes i})_{i=1}^{\infty}$ the *behaviour* of A. A sequence of integers $(n_i)_{i=1}^{\infty}$ will be called a *$\big((p\text{-})$primary$\big)$ behaviour* if it is the behaviour of a $\big((p\text{-})$primary$\big)$ algebra.

Suppose that A is a division algebra. The Grothendieck group $K(X)$ depends only on the behaviour of A (Theorem 3.1). Moreover, $K(X)$ together with the gamma-filtration (and the group $\Gamma^{2/3}K(X)$ in particular) depend only on the behaviour (Corollary 3.2) and our main observation is (Theorem 3.13):

For any primary behaviour (and any given field) there exists a division algebra $\widetilde{A}$ (over an extension of the field) of the given behaviour for which the canonical epimorphism $\Gamma^{2/3}K(\widetilde{X}) \twoheadrightarrow \mathrm{CH}^2(\widetilde{X})$ with $\widetilde{X} = \mathrm{SB}(\widetilde{A})$ is bijective.

The construction of the algebra $\widetilde{A}$ is rather simple (Definition 3.12). We take a division algebra (over a suitable extension of the field) of the index as in the given behaviour and of the exponent coinciding with the index. After that we pass to the function field of a product of certain generalized Severi-Brauer varieties in order to change the behaviour in the way prescribed.

Since the groups $\Gamma^{2/3}K(X)$ and $\mathrm{CH}^2(X)$ have the same rank (Proposition 2.14) (rank 1 if X is a Severi-Brauer variety of dimension at least 2), we also have an epimorphism of the torsion subgroups

$$\operatorname{Tors}\Gamma^{2/3}K(X) \twoheadrightarrow \operatorname{Tors}\mathrm{CH}^2(X)$$

which is moreover bijective iff $\Gamma^{2/3}K(X) \twoheadrightarrow \mathrm{CH}^2(X)$ is (Corollary 2.15). So, formulating the main observation we may replace (and we do replace) both the groups $\Gamma^{2/3}K(X)$ and $\mathrm{CH}^2(X)$ by their torsion subgroups.

The gamma-filtration for a Severi-Brauer variety X and the group

$$\operatorname{Tors}\Gamma^{2/3}K(X)$$

in particular are from the so to say "algebro-geometrical" point of view very easy to compute (Propositions 4.1, 4.6, 4.10): $K(X)$ is a subring of $K(\mathbb{P})$ where $\mathbb{P}$ is a $\dim X$-dimensional projective space and the Chern classes on X

with values in K (Definition 2.1, Remark 2.2) needed to determine the gamma-filtration are simply the restrictions of the Chern classes on $K(\mathbb{P})$. However to get the answer in a final form (say, to find the canonical decomposition of the finite abelian group $\operatorname{Tors}\Gamma^{2/3}K(X)$ for any primary behaviour) further calculations are required which can be done e.g. by computer in every particular situation (i.e. for every particular behaviour) but seem to be not easy in the general case. Our main efforts in this direction are made in Propositions 4.7, 4.9, 4.13 and 4.14 where we firstly find out when this group is non-trivial (Propositions 4.7, 4.9) and after that describe a wide class of situations when this group is cyclic and compute its order (Propositions 4.13, 4.14, see also Example 4.15).

To the structure of the chapter.

In §1 we reduce the problem of computation of $\operatorname{Tors}\operatorname{CH}^2(\operatorname{SB}(A))$ for an arbitrary central simple algebra A to the case when $\operatorname{ind} A$ is a power of a prime. In §2 we recall and partially prove certain general facts on the Chern classes (with various values) and on the gamma-filtration. In §3 we make the main observation. In §4 we investigate the group $\Gamma^{2/3}K$ for various primary behaviours.

In §5 we consider algebras of prime exponent. We show that the group $\operatorname{Tors}\operatorname{CH}^2(X)$, where $X = \operatorname{SB}(A)$ for an algebra A of a prime exponent p, is (cyclic) of order p or trivial (Proposition 5.1). Moreover, if A *decomposes* (into a tensor product of two smaller algebras), then $\operatorname{Tors}\operatorname{CH}^2(X) = 0$ (Proposition 5.3). However, the torsion group is non-trivial if A is a "generic" division algebra of index p^n and exponent p (see Example 4.12 for the definition) with $n \geq 2$ for an odd p and $n \geq 3$ for $p = 2$ (Proposition 5.1). Thus we obtain a wide family of indecomposable algebras (Corollary 5.4) which can be constructed over an extension of any given field (without any restriction on the characteristic in particular). Here is a list of some articles where the question of indecomposability for central simple algebras was considered previously: [**4, 73, 83, 26, 35**]. The method of [**35**] is close to but different from the one presented here; it does not cover the case $p = 2$.

Some additional notations concerning filtrations on $K(X)$ are introduced in § 2.

1. Reduction to the primary case

In this §, A is a central simple algebra over a field F, A_p (for every prime number p) stays for the p-primary component of A, finally

$$X = \operatorname{SB}(A) \quad \text{and} \quad X_p = \operatorname{SB}(A_p)\ .$$

For an abelian group C, we denote by C_p its p-primary part.

Let E/F be a finite field extension. Consider the homomorphisms

$$\operatorname{res}_{E/F}\colon \operatorname{CH}^2(X) \to \operatorname{CH}^2(X_E) \quad \text{and} \quad N_{E/F}\colon \operatorname{CH}^2(X_E) \to \operatorname{CH}^2(X)\ .$$

The projection formula shows that the composition $N_{E/F} \circ \mathrm{res}_{E/F}$ coincides with the multiplication by $[E : F]$.

LEMMA 1.1. *The composition* $\mathrm{res}_{E/F} \circ N_{E/F}$ *coincides with the multiplication by* $[E : F]$ *as well.*

PROOF. Consider the homomorphisms $\mathrm{res}_{E/F}$ and $N_{E/F}$ on the Grothendieck groups $K(X)$ and $K(X_E)$. Since these groups are torsion-free and have the same rank (Theorem 3.1) and since the composition $N_{E/F} \circ \mathrm{res}_{E/F}$ is the multiplication by $[E : F]$, the composition taken in the other order is the multiplication by $[E : F]$ as well. Since the second Chow group coincides with the second successive quotient of the topological filtration on the Grothendieck group (see e.g. [**32**, §3.1]), we are done. □

COROLLARY 1.2. *If* $[E : F]$ *is not divisible by a given prime number* p*, then*

$$\mathrm{CH}^2(X)_p \simeq \mathrm{CH}^2(X_E)_p\ .$$

□

PROPOSITION 1.3. *For every prime* p*, the* p*-primary part of the group* $\mathrm{CH}^2(X)$ *coincides with the torsion of* $\mathrm{CH}^2(X_p)$*.*

PROOF. Fix a prime p and a finite field extension E/F of degree prime to p such that the algebra A_E is Brauer equivalent to $(A_p)_E$. We have

$$\mathrm{CH}^2(X)_p \simeq \mathrm{CH}^2(X_E)_p \simeq \mathrm{CH}^2((X_p)_E)_p \simeq \mathrm{CH}^2(X_p)_p$$

(for the first and the third steps, we use the corollary). Since $\mathrm{Tors}\,\mathrm{CH}^2(X_p)$ is annihilated by $\mathrm{ind}\,A_p$, we finally get

$$\mathrm{CH}^2(X_p)_p = \mathrm{Tors}\,\mathrm{CH}^2(X_p)\ .$$

□

2. Chern classes and gamma-filtration

In this §, we are working with the category of smooth projective irreducible algebraic varieties over a fixed field. The Grothendieck ring K is considered as a contravariant functor on this category.

DEFINITION 2.1 (Chern classes with values in K). The total Chern class c_t is a homomorphism of functors

$$c_t : K^+ \longrightarrow K[[t]]^\times$$

(where the left-hand side is the additive group of the ring K while the right-hand side is the multiplicative group of series in one variable t over K) satisfying the following property: if $\xi \in K(X)$ is a class of an invertible sheaf on a variety X then

$$c_t(\xi) = 1 + (\xi - 1)t\ .$$

One defines the Chern classes $c^i : K \to K$ by putting

$$c_t = \sum_{i=0}^{\infty} c^i \cdot t^i \ .$$

REMARK 2.2. Usually, one does *not* use the name "Chern classes" for the maps c^i defined above e.g. since unlike the Chern classes 2.7, 2.8, and 2.11 they do not satisfy the rule $c^1(\xi \cdot \eta) = c^1(\xi) + c^1(\eta)$ for classes of invertible sheaves ξ and η.

PROPOSITION 2.3. *Chern classes with values in K are unique.*

PROOF. Follows from the

LEMMA 2.4 (**Splitting principle**, [**52**, Proposition 5.6]). *For any variety X and any $x \in K(X)$ there exists a morphism $f : Y \to X$ such that:*

1. *f is a composition of some projective bundle morphisms;*
2. *$f^*(x) \in K(Y)$ is a linear combination (with integral coefficients) of classes of some invertible sheaves.*

To obtain uniqueness of the Chern classes just note that the homomorphism $f^* : K(X) \to K(Y)$ in the lemma is injective. □

PROPOSITION 2.5. *Chern classes with values in K exist.*

PROOF. Here is the way of constructing due to Grothendieck with the original notations ([**52**, Theorem 3.10 and §8]).

Take a variety X. First one constructs a homomorphism

$$\lambda_t : K^+ \longrightarrow K[[t]]^\times$$

by sending the class of a locally free sheaf $\mathcal{E}$ to

$$\lambda_t([\mathcal{E}]) = \sum_{i=0}^{\infty} [\Lambda^i \mathcal{E}] \cdot t^i$$

where $\Lambda^i \mathcal{E}$ is the i-th exterior power of $\mathcal{E}$.

After that one considers another homomorphism

$$\gamma_t : K^+ \longrightarrow K[[t]]^\times$$

namely,

$$\gamma_t = \lambda_{\frac{t}{1-t}}$$

(this γ_t gave the name of the gamma-filtration).

Finally, one puts

$$c_t = \gamma_t \circ (\mathrm{id} - \mathrm{rk})$$

where $\mathrm{rk} : K(X) \to \mathbb{Z}$ is the rank homomorphism (followed by the inclusion $\mathbb{Z} \hookrightarrow K(X)$ more precisely). □

DEFINITION 2.6 (Gamma-filtration). The gamma-filtration

$$K(X) \supset \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots$$

is the smallest ring filtration on $K(X)$ such that $\Gamma^0 K(X) = K(X)$ and

$$c^i\big(K(X)\big) \subset \Gamma^i K(X) \ \text{ for all } \ i \geq 1 \ .$$

In other words, for every $l \geq 0$, $\Gamma^l K(X)$ is the subgroup of $K(X)$ generated by all the products

$$c^{i_1}(x_1) \dots c^{i_r}(x_r) \ \text{ with } \ x_j \in K(X) \ \text{ and } \ \sum_{i=1}^{r} i_j \geq l$$

(it might be not immediately clear but it is nevertheless easy to see that the group $K(X) = \Gamma^0 K(X)$ is really also generated by these products).

In particular, $\Gamma^1 K(X) = \operatorname{Ker}(\operatorname{rk} : K(X) \to \mathbb{Z})$.

We denote by $\mathrm{G}^*\Gamma K(X)$ the adjoint graded ring.

DEFINITION 2.7 (Chern classes with values in $\mathrm{G}^*\Gamma K$). For any variety X, we call the induced maps

$$c^i : K(X) \to \mathrm{G}^i\Gamma K(X)$$

the Chern classes with values in $\mathrm{G}^*\Gamma K$. The total Chern class c_t is the homomorphism

$$c_t : K(X)^+ \longrightarrow \left(\sum_{i=0}^{\infty} \mathrm{G}^i\Gamma K(X) \cdot t^i\right)^{\times}$$

It is a morphism of functors and

$$c_t(\xi) = 1 + (\xi - 1)t$$

for a class $\xi \in K(X)$ of an invertible sheaf on X ($(\xi - 1)$ is considered as an element of $\mathrm{G}^1\Gamma K(X)$ in the last formula).

Side by side with the gamma-filtration we consider the topological filtration on $K(X)$ (in fact defined on $K_0'(X)$) ([**69**, §7]):

$$K(X) = \mathrm{T}^0 K(X) \supset \mathrm{T}^1 K(X) \supset \dots \ .$$

Note that

$$\mathrm{T}^1 K(X) = \operatorname{Ker}(\operatorname{rk} : K(X) \to \mathbb{Z}) = \Gamma^1 K(X) \ .$$

We will denote by $\mathrm{G}^*\mathrm{T}K(X)$ the adjoint graded ring.

DEFINITION 2.8 (Chern classes with values in $\mathrm{G}^*\mathrm{T}K$). The total Chern class c_t is a homomorphism of functors

$$c_t : K^+ \longrightarrow \left(\sum_{i=0}^{\infty} \mathrm{G}^i\mathrm{T}K \cdot t^i\right)^{\times}$$

satisfying the property:

$$c_t(\xi) = 1 + (\xi - 1)t \ .$$

One defines the Chern classes $c^i : K \to \mathrm{G}^i\mathrm{T}K$ by putting

$$c_t = \sum_{i=0}^{\infty} c^i \cdot t^i .$$

Proposition 2.9. *Chern classes with values in* $\mathrm{G}^*\mathrm{T}K$ *are unique.*

Proof. Follows from the splitting principle (Lemma 2.4) since the homomorphism

$$f^* : \mathrm{G}^*\mathrm{T}K(X) \to \mathrm{G}^*\mathrm{T}K(Y)$$

is injective ([**13**, Lemma 3.8 of Chapter V]). □

Proposition 2.10. *Chern classes with values in* $\mathrm{G}^*\mathrm{T}K$ *exist.*

Proof. Simply compose the Chern classes with values in CH^* (Definition 2.11) with the canonical epimorphism $\mathrm{CH}^* \twoheadrightarrow \mathrm{G}^*\mathrm{T}K$ mapping a class $[Z] \in \mathrm{CH}^*(X)$ of a simple cycle $Z \subset X$ to the class of the structure sheaf $\mathcal{O}_Z$ of Z prolonged to X by 0. □

Definition 2.11 (Chern classes with values in CH^*). We repeat Definition 2.8, replacing $\mathrm{G}^*\mathrm{T}K$ by CH^*. In the formula $c_t(\xi) = 1 + (\xi - 1)t$, we consider $(\xi - 1)$ as an element of $\mathrm{CH}^1(X)$ via the canonical isomorphism $\mathrm{CH}^1(X) \simeq \mathrm{G}^1\mathrm{T}K(X)$ ([**69**, §7.5]) described in the proof of Proposition 2.10.

Proposition 2.12. *Chern classes with values in* CH^* *are unique.*

Proof. Follows from the splitting principle (Lemma 2.4) since the homomorphism

$$f^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y)$$

is injective. □

Proposition 2.13 ([**12**, §3.2]). *Chern classes with values in* CH^* *exist.*

Now we establish certain connections between the gamma-filtration and the topological one.

Proposition 2.14. *For any variety* X*,*
1. $\Gamma^i K(X) \subset \mathrm{T}^i K(X)$ *for all* i*;*
2. $\Gamma^i K(X) = \mathrm{T}^i K(X)$ *for* $i \leq 2$*;*
3. $\Gamma^i K(X) \otimes \mathbb{Q} = \mathrm{T}^i K(X) \otimes \mathbb{Q}$ *for all* i*.*

Proof. 1. [**13**, Theorem 3.9 of Chapter V].
2. We only need to manage the case $i = 2$.
There are canonical isomorphisms

$$\mathrm{G}^1\Gamma K(X) \simeq \mathrm{Pic}(X) \text{ ([\textbf{13}, Remark 1 in §3 of Chapter IV]) ;}$$
$$\mathrm{CH}^1(X) \simeq \mathrm{G}^1\mathrm{T}K(X) \text{ ([\textbf{69}, §7.5])}$$

(the definition of the second map is given in the proof of Proposition 2.10).

Since $\Gamma^2 K(X) \subset \mathrm{T}^2 K(X)$ we have a surjection $\mathrm{G}^1\Gamma K(X) \twoheadrightarrow \mathrm{G}^1\mathrm{T}K(X)$ which gives an epimorphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$. But the latter map is an isomorphism ([**14**, Corollary 6.16]). Thus $\Gamma^2 K(X) = \mathrm{T}^2 K(X)$.

3. [**13**, Proposition 5.5 of Chapter VI]. □

Corollary 2.15. *One has an exact sequence*

$$0 \to \mathrm{T}^3K(X)/\Gamma^3K(X) \to \operatorname{Tors} \mathrm{G}^2\Gamma K(X) \to \operatorname{Tors} \mathrm{CH}^2(X) \to 0\ .$$

Proof. The equality $\Gamma^2K(X) = \mathrm{T}^2K(X)$ and the inclusion $\Gamma^3K(X) \subset \mathrm{T}^3K(X)$ stated in the proposition give an exact sequence

$$0 \to \mathrm{T}^3K(X)/\Gamma^3K(X) \to \mathrm{G}^2\Gamma K(X) \to \mathrm{G}^2\mathrm{T}K(X) \to 0\ .$$

Consider the commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\operatorname{Tors} \mathrm{G}^2\Gamma K(X) & \xrightarrow{(1)} & \operatorname{Tors} \mathrm{G}^2\mathrm{T}K(X) \\
\downarrow & & \downarrow \\
\mathrm{G}^2\Gamma K(X) & \xrightarrow{(2)} & \mathrm{G}^2\mathrm{T}K(X) \\
\downarrow & & \downarrow \\
\mathrm{G}^2\Gamma K(X)/\operatorname{Tors} & \xrightarrow{(3)} & \mathrm{G}^2\mathrm{T}K(X)/\operatorname{Tors} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The map (2) is surjective. Hence the map (3) is surjective as well. Since by the proposition the map (3)$\otimes\mathbb{Q}$ is bijective, the map (3) itself is bijective as well. Thus (1) is surjective and the kernels of (1) and (2) coincide. So, we get the exact sequence

$$0 \to \mathrm{T}^3K(X)/\Gamma^3K(X) \to \operatorname{Tors} \mathrm{G}^2\Gamma K(X) \to \operatorname{Tors} \mathrm{G}^2\mathrm{T}K(X) \to 0\ .$$

Finally, the canonical map $\mathrm{CH}^2(X) \to \mathrm{G}^2\mathrm{T}K(X)$ is an isomorphism (see e.g. [**32**, §3.1], the definition of the map is given in the proof of Proposition 2.10). □

As a corollary of the uniqueness assertion of Proposition 2.9, we get a connection between Chern classes with different values:

Lemma 2.16. *The following diagram of maps commutes:*

$$\begin{array}{ccc}
K(X) & \xrightarrow{c^i} & \mathrm{CH}^i(X) \\
\Big\downarrow{\scriptstyle c^i} & & \Big\downarrow{\scriptstyle can.} \\
\mathrm{G}^i\Gamma K(X) & \longrightarrow & \mathrm{G}^i\mathrm{T}K(X)
\end{array}$$

Proof. Both the compositions are Chern classes with values in $\mathrm{G}^*\mathrm{T}K$ (Definition 2.8) which are unique (Proposition 2.9). □

Remark 2.17. One can formulate a criterion (which will be used later) for when the gamma-filtration coincides with the topological one. It is clear from the very definition (Definition 2.6) that for any variety X the ring $\mathrm{G}^*\Gamma K(X)$ is generated by the Chern classes (with values in $\mathrm{G}^*\Gamma K$). So, if the gamma and the topological filtrations are the same, the ring G^*TK is generated by the Chern classes (with values in G^*TK this time) as well.

The other way round, if G^*TK is generated by the Chern classes, then the homomorphism $G^*\Gamma K(X) \to G^*TK(X)$ is surjective whence the filtrations coincide.

3. Main observation

From now on, X denotes the Severi-Brauer variety corresponding to a central simple algebra A over a field F.

Denote by $\mathbb{P}$ the projective space $X_{\bar{F}}$ where $\bar{F}$ is an algebraic closure of F and let $\xi \in K(\mathbb{P})$ be the class of $\mathcal{O}_{\mathbb{P}}(-1)$. The ring $K(\mathbb{P})$ is generated by ξ subject to only one relation:$(\xi - 1)^n = 0$ where $n = \dim X + 1 = \deg A$. We consider the restriction map $K(X) \to K(\mathbb{P})$ which is a ring homomorphism commuting with the Chern classes 2.1.

Theorem 3.1 ([**69**, §8, Theorem 4.1]). *The map $K(X) \to K(\mathbb{P})$ is injective; its image is additively generated by $(\operatorname{ind} A^{\otimes i}) \cdot \xi^i$ $(i \geq 0)$.* □

Corollary 3.2. *For a division algebra A, the group $K(X)$ together with the gamma-filtration depends only on the behaviour of A.* □

Proposition 3.3 ([**33**, Theorem 1]). *If $\operatorname{ind} A = \exp A$ for an algebra A then (for any $l \geq 0$) the l-th term of the topological filtration $T^l K(X)$ is generated by all*

$$\frac{\operatorname{ind} A}{(i, \operatorname{ind} A)}(\xi - 1)^i \ \text{ with } \ l \leq i < \deg A$$

where $(\cdot, \cdot)$ denotes the greatest common divisor. In particular, the group $G^*TK(X)$ is torsion-free.

Proposition 3.4. *If A is a primary algebra then*

$$\frac{\operatorname{ind} A}{(i, \operatorname{ind} A)}(\xi - 1)^i \in \Gamma^i K(X) \ \text{ for any } \ i \geq 0\ .$$

Proof. Put $n = \operatorname{ind} A$. For $n\xi \in K(X)$ we have:

$$c_t(n\xi) = c_t(\xi)^n = \big(1 + (\xi - 1)t\big)^n$$

where c_t is the total Chern class with values in K (the last equality holds by Definition 2.1). Whence

$$c^i(n\xi) = \binom{n}{i}(\xi - 1)^i \in \Gamma^i K(X)\ .$$

In particular,

$$(\xi - 1)^n \in \Gamma^n K(X)$$

thereby for the rest of the proof we may assume that $i \leq n$. Moreover,

$$n^i(\xi - 1)^i = c^1(n\xi)^i \in \Gamma^i K(X)\ .$$

The last observation is

LEMMA 3.5. *If n is a power of a prime p and $n \geq i \geq 0$ then*

$$\left(n^i, \binom{n}{i}\right) = \frac{n}{(i, n)} \ .$$

Moreover, if $i \neq 0$ then $v_p \binom{n}{i} = v_p(n) - v_p(i)$, where $v_p(i)$ is the multiplicity of p in i.

PROOF. The case $i = 0$ is evident. Suppose that $i \neq 0$. If $1 \leq j < n$ then $v_p(j) < v_p(n)$ and so $v_p(n - j) = v_p(j)$. Hence

$$v_p \left(\frac{(n-1)(n-2)\dots(n-(i-1))}{1 \cdot 2 \dots (i-1)} \right) = 0$$

and

$$v_p \binom{n}{i} = v_p \left(\frac{n}{i}\right) = v_p \left(\frac{n}{(i,n)}\right) \ .$$

□

□

COROLLARY 3.6. *If A is a primary algebra and* $\operatorname{ind} A = \exp A$ *then the gamma-filtration on $K(X)$ coincides with the topological one.*

□

THEOREM 3.7. *Let A be as in Corollary 3.6, $X = \mathrm{SB}(A)$. Let $Y_1, \dots, Y_m$ be some generalized Severi-Brauer varieties (*[**7**, §4]*) of some algebras which are (Brauer equivalent to) some tensor powers of A.*

The gamma-filtration on the Grothendieck group of the variety X over the function field $F(Y_1 \times \dots \times Y_m)$ coincides with the topological one. In particular, the epimorphism of Corollary 2.15

$$\operatorname{Tors} \mathrm{G}^2 \Gamma K \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2$$

for this variety is bijective.

PROOF. For every Y_i the product $X \times Y_i$ is a Grassman bundle over X (with respect to the first projection) (Corollary 6.4). Hence $\mathrm{CH}^*(X \times Y_i)$ is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes of a locally free sheaf (see e.g. [**12**, Proposition 14.6.5] or [**43**, Theorem 3.2]). Taking the product of all $X \times Y_i$ over X we obtain that

$$\mathrm{CH}^*(X \times Y_1 \times \dots \times Y_m)$$

is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes (of some locally free sheaves).

The homomorphism of $\mathrm{CH}^*(X)$-algebras

$$\mathrm{CH}^*(X \times Y_1 \times \dots \times Y_m) \to \mathrm{CH}^*(X_{F(Y_1 \times \dots \times Y_m)})$$

(given by the pull-back) is surjective (see e.g. [**39**, Theorem 3.1]). Whence the right-hand side is generated as a $\mathrm{CH}^*(X)$-algebra by the Chern classes as well.

Using the epimorphism $\mathrm{CH}^* \twoheadrightarrow \mathrm{G}^*\mathrm{T}K$ of the Chow ring onto the adjoint graded Grothendieck ring we obtain the same statement as in the previous paragraph but for $\mathrm{G}^*\mathrm{T}K$ instead of CH^* by meaning the Chern classes with values in $\mathrm{G}^*\mathrm{T}K$ this time.

The gamma-filtration on $K(X)$ coincides with the topological one (Corollary 3.6) and therefore the ring $\mathrm{G}^*\mathrm{T}K(X)$ is generated by the Chern classes (Remark 2.17). Consequently, $\mathrm{G}^*\mathrm{T}K(X_{F(Y_1\times\cdots\times Y_m)})$ is generated by the Chern classes as a ring, not only as a $\mathrm{G}^*\mathrm{T}K(X)$-algebra. It means that the gamma-filtration on $K(X_{F(Y_1\times\cdots\times Y_m)})$ coincides with the topological one (Remark 2.17). $\square$

Definition 3.8. Let A be a p-primary algebra. The sequence of integers

$$\left(\log_p \operatorname{ind} A^{\otimes p^i}\right)_{i=0}^{\log_p \exp A}$$

is called the *reduced behaviour* of A.

Example 3.9. The reduced behaviour of a p-primary algebra A with

$$\operatorname{ind} A = \exp A = p^n$$

is $n,\ n-1,\ n-2,\ \dots,\ 1,\ 0$.

Proof. Suppose that $n > 0$. By [**3**, Lemma 7 on Page 76], $\operatorname{ind} A^{\otimes p} < \operatorname{ind} A$. Moreover, $\operatorname{ind} A^{\otimes p} \geq \exp A^{\otimes p} = p^{n-1}$. Thus $\operatorname{ind} A^{\otimes p} = p^{n-1}$. $\square$

Lemma 3.10. *The behaviour of a primary algebra is completely determined by its reduced behaviour. The reduced behaviour of an algebra is a finite strong decreasing sequence of integers with 0 in the end. Any finite strong decreasing sequence of integers with 0 in the end is for any prime p the reduced behaviour of a p-primary division algebra.*

Proof. Let A be a p-primary algebra. If i is an integer prime to p then the splitting fields of the algebra A are the same as the splitting fields of the algebra $A^{\otimes i}$. Therefore $\operatorname{ind} A^{\otimes i} = \operatorname{ind} A$, what proves the first sentence of the lemma.

If in addition $\operatorname{ind} A \neq 1$ then $\operatorname{ind} A^{\otimes p} < \operatorname{ind} A$ ([**3**, Lemma 7 on Page 76]). It proves the second sentence.

Finally, fix a sequence $n_0 > n_1 > \cdots > n_m = 0$ and a prime p. A construction of a p-primary algebra having the reduced behaviour $(n_i)_{i=0}^m$ is given in [**78**, Construction 2.8]. This construction involves function fields of usual Severi-Brauer varieties only. We describe another known construction which involves function fields of generalized Severi-Brauer varieties as well and is more suitable for our purposes.

We start with a division algebra A (over a suitable field) for which

$$\operatorname{ind} A = \exp A = p^{n_0}\ .$$

For each $i = 1, 2, \dots, m$ consider the generalized Severi-Brauer variety

$$Y_i = \mathrm{SB}(p^{n_i}, A^{\otimes p^i})$$

(we define here $\mathrm{SB}(p^{n_i}, A^{\otimes p^i})$ to be the variety of rank p^{n_i} *left* ideals in $A^{\otimes p^i}$; its function field is a generic extension making the index of $A^{\otimes p^i}$ to be equal to p^{n_i}).

Finally, we denote the function field $F(Y_1 \times \cdots \times Y_m)$ by $\widetilde{F}$ and put $\widetilde{A} = A_{\widetilde{F}}$. Using the index reduction formula [**7**, Theorem 5] or an improved version of this formula [**60**, Formula I] one can easily show that the algebra $\widetilde{A}$ has the reduced behaviour $(n_i)_{i=0}^m$. □

Remark 3.11. In the construction described in the proof above, it is not necessary to use all of the varieties Y_i: if $n_i = n_{i-1} - 1$ for some i then the variety Y_i can be omitted.

Definition 3.12. We refer to an algebra $\widetilde{A}$ constructed like as in the above proof (with taking the remark into account) as to a *"generic" p-primary division algebra of the reduced behaviour* $(n_i)_{i=0}^m$. Note that it can be constructed over an extension of any given field.

Theorem 3.13. *Fix a prime p and a reduced behaviour. If $\widetilde{A}$ is a "generic" p-primary division algebra of the given reduced behaviour (Definition 3.12) then the epimorphism of Corollary 2.15*

$$\operatorname{Tors} \mathrm{G}^2\Gamma K(\widetilde{X}) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(\widetilde{X}) \quad \textit{(where } \widetilde{X} = \mathrm{SB}(\widetilde{A})\textit{)}$$

is bijective. If A is an arbitrary p-primary algebra of the same reduced behaviour as $\widetilde{A}$ then there exists an epimorphism

$$\operatorname{Tors} \mathrm{CH}^2(\widetilde{X}) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(X)\ .$$

Proof. The first part follows from Theorem 3.7 and from the definition of "generic" algebras (Definition 3.12). The second part follows from Corollary 2.15, Corollary 3.2 and from the first one. □

4. Computation of gamma-filtration

Remind that we have put forever $X = \mathrm{SB}(A)$.

Proposition 4.1. *Let A be a p-primary algebra and $(n_i)_{i=0}^m$ its reduced behaviour. For any $l \geq 0$, the group $\Gamma^l K(X)$ is generated by all the products*

$$(*) \qquad \prod_{i=0}^m \frac{p^{n_i}}{(j_i, p^{n_i})}(\xi^{p^i} - 1)^{j_i} \quad \textit{with } j_i \geq 0 \textit{ and } \sum_{i=0}^m j_i \geq l$$

where $\xi = [\mathcal{O}(-1)] \in K(\mathbb{P})$.

Proof. The formula

$$c^j(p^{n_i}\xi^{p^i}) = \binom{p^{n_i}}{j}(\xi^{p^i} - 1)^j \ \text{ where } 0 \leq j \leq p^{n_i}$$

and Lemma 3.5 show that for any $j \geq 0$ (even for $j > p^{n_i}$) the element $\frac{p^{n_i}}{(j,p^{n_i})}(\xi^{p^i} - 1)^j$ lies in $\Gamma^j K(X)$ (compare with the proof of Proposition 3.4). Therefore, each product $(*)$ lies in $\Gamma^l K(X)$.

For the opposite inclusion we need

LEMMA 4.2. *Consider the polynomials over $\mathbb{Z}$ in one variable ζ. For any integers j, $r \geq 0$ the polynomial $(\zeta^r - 1)^j$ is equal to a sum*

$$\sum_{s \geq j} a_s (\zeta - 1)^s$$

with integers a_s such that $s \cdot a_s$ is a multiple of $j \cdot r$.

PROOF. It is clear that

$$(\zeta^r - 1)^j = \sum_{s=j}^{j \cdot r} a_s (\zeta - 1)^s$$

for some (uniquely determined) $a_s \in \mathbb{Z}$. Taking the derivative we obtain the statement on the coefficients. ☐

As follows from Theorem 3.1, the additive group $K(X)$ is generated by all $p^{n_i} \xi^{rp^i}$ with $0 \leq i \leq m$ and $r \geq 0$. We have:

$$c^j(p^{n_i} \xi^{rp^i}) = \binom{p^{n_i}}{j} (\xi^{rp^i} - 1)^j = \sum_{s \geq j} \binom{p^{n_i}}{j} \cdot a_s \cdot (\xi^{p^i} - 1)^s .$$

Since by the lemma $j \mid s \cdot a_s$ and $v_p\binom{p^{n_i}}{j}$ equals $n_i - v_p(j)$ or ∞ (Lemma 3.5), the coefficient

$$\binom{p^{n_i}}{j} \cdot a_s \text{ is divisible by } \frac{p^{n_i}}{(s, p^{n_i})} .$$

Thus the Chern class $c^j(p^{n_i} \xi^{rp^i})$ is a linear combination (with integral coefficients) of

$$\frac{p^{n_i}}{(s, p^{n_i})} (\xi^{p^i} - 1)^s \text{ with } s \geq j .$$

☐

DEFINITION 4.3. Fix a prime number p, a reduced behaviour $(n_i)_{i=0}^m$, and consider a polynomial ring $\mathbb{Z}[\zeta]$. Let $\mathbb{K} \subset \mathbb{Z}[\zeta]$ be the additive subgroup generated by all $p^{n_i} \cdot \zeta^{r \cdot p^i}$ where $0 \leq i \leq m$ and $r \geq 0$. Consider a filtration Γ on $\mathbb{K}$ defined by the formula of Proposition 4.1: for any $l \geq 0$, the group $\Gamma^l \mathbb{K}$ is generated by all the products

$$(*) \qquad \prod_{i=0}^m \frac{p^{n_i}}{(j_i, p^{n_i})} (\zeta^{p^i} - 1)^{j_i} \text{ with } j_i \geq 0 \text{ and } \sum_{i=0}^m j_i \geq l .$$

Note that $\mathbb{K}$ is a ring and that for any $l_1, l_2 \geq 0$ one has $\Gamma^{l_1} \mathbb{K} \cdot \Gamma^{l_2} \mathbb{K} \subset \Gamma^{l_1 + l_2} \mathbb{K}$.

LEMMA 4.4. *In the notation of the definition, one has:*

1. $\mathbb{K} = \Gamma^0 \mathbb{K}$;
2. *for any $l \geq 0$, $\mathbb{K} \supset \Gamma^l \mathbb{K} \supset \Gamma^{l+1} \mathbb{K}$;*
3. *if n is a multiple of p^{n_0}, then $\Gamma^n \mathbb{K} = (\zeta - 1)^n \mathbb{K}$.*

PROOF. 1. The inclusion $\mathbb{K} \subset \Gamma^0\mathbb{K}$ is evident. The inverse inclusion is a particular case of the second statement of the lemma.
2. The inclusion $\Gamma^l\mathbb{K} \supset \Gamma^{l+1}\mathbb{K}$ is evident. Fix an arbitrary $l \geq 0$. We prove the inclusion $\mathbb{K} \supset \Gamma^l\mathbb{K}$ using the third statement of the lemma. Choose a multiple n of p^{n_0} such that $n > l$. Since $\Gamma^l\mathbb{K} \supset \Gamma^n\mathbb{K} = (\zeta - 1)^n\mathbb{K} \subset \mathbb{K}$, it suffices to show that $\mathbb{K}/(\zeta - 1)^n\mathbb{K} \supset \Gamma^l\mathbb{K}/(\zeta - 1)^n\mathbb{K}$. Find a p-primary algebra A (over an appropriate field F) having the reduced behaviour $(n_i)_{i=0}^m$ (see Lemma 3.10 for the existence of A). The latter inclusion follows now from Proposition 4.1.
3. Since $(\zeta - 1)^n \in \Gamma^n\mathbb{K}$, the inclusion $\supset$ holds. One also sees immediately from the definition that any polynomial $f \in \Gamma^n\mathbb{K}$ is of the kind $f = (\zeta - 1)^n \cdot h$ where $h \in \mathbb{Z}[\zeta]$. We have to prove that $h \in \mathbb{K}$. It is a consequence of the following

LEMMA 4.5. *Let f, g, and h are polynomials from $\mathbb{Z}[\zeta]$ such that $f = g \cdot h$ and suppose that the free coefficient of g equals ± 1. If f and g lie in $\mathbb{K}$ then h lies in $\mathbb{K}$ as well.*

PROOF. Let
$$f = \sum_{i \geq 0} f_i\zeta^i , \qquad g = \sum_{i \geq 0} g_i\zeta^i , \qquad h = \sum_{i \geq 0} h_i\zeta^i .$$
We prove that $h_i\zeta^i \in \mathbb{K}$ using an induction on i. There is no problem with the base of the induction: $h_0 \in \mathbb{K}$ for any integral h_0. Suppose that $h_1\zeta, \ldots, h_{i-1}\zeta^{i-1} \in \mathbb{K}$. Polynomial f is equal to the product of g and h; because of that we have:
$$f_i\zeta^i = \pm h_i\zeta^i + g_1\zeta \cdot h_{i-1}\zeta^{i-1} + \cdots + g_{i-1}\zeta^{i-1} \cdot h_1\zeta + g_i\zeta^i \cdot h_0 .$$
Since $g \in \mathbb{K}$, every its monomial $g_j\zeta^j$ is in $\mathbb{K}$ as well (see the definition of $\mathbb{K}$). By the same reason, $f_i\zeta^i \in \mathbb{K}$. Hence $h_i\zeta^i \in \mathbb{K}$. □

□

If now A is a p-primary algebra of the reduced behaviour $(n_i)_{i=0}^m$, the ring homomorphism $\mathbb{K} \to K(X)$ mapping ζ to ξ respects the filtrations and thereby induces a homomorphism of graded groups
$$\mathrm{G}^*\Gamma\mathbb{K} \to \mathrm{G}^*\Gamma K(X) .$$

PROPOSITION 4.6. *For every $0 \leq l < \deg A$ the group homomorphism*
$$\mathrm{G}^l\Gamma\mathbb{K} \to \mathrm{G}^l\Gamma K(X)$$
is bijective.

PROOF. It is evidently surjective by Proposition 4.1. To see the rest, put $n = \deg A$. By Lemma 4.4 and Theorem 3.1, the ring homomorphism $\phi : \Gamma^{0/n}\mathbb{K} \to K(X)$ is bijective. Consider the induced filtration on $\Gamma^{0/n}\mathbb{K}$. We know that the bijective ring homomorphism ϕ respects the filtrations and is surjective on the successive quotients. Thus it is bijective on the successive quotients. □

The proposition gives in particular a description of the group $\mathrm{G}^2\Gamma K(X)$ for any p-primary algebra A of the reduced behaviour $(n_i)_{i=0}^m$. We want to find out when this group has a non-trivial torsion. We start with the case of an odd prime.

PROPOSITION 4.7. *Let A be a p-primary algebra with an odd p. The group $\mathrm{G}^2\Gamma K(X)$ has a torsion iff $\operatorname{ind} A > \exp A$.*

PROOF. See Proposition 3.3 with Corollary 3.6 for the "only if" part.

Suppose that $\operatorname{ind} A > \exp A$. Then in the reduced behaviour $(n_i)_{i=0}^m$ of A one has:

$$n_s \le n_{s-1} - 2 \text{ for some } s\ .$$

Using Proposition 4.6, we shall work with $\mathrm{G}^2\Gamma\mathbb{K}$ instead of $\mathrm{G}^2\Gamma K(X)$.

Consider the element

$$x = p^{n_{s-1}-2}(\zeta^{p^s} - 1)^2 - p^{n_{s-1}}(\zeta^{p^{s-1}} - 1)^2 \in \Gamma^2\mathbb{K}\ .$$

Since x is divisible by $(\zeta - 1)^3$ in the polynomial ring $\mathbb{Z}[\zeta]$, since moreover $p^{n_0}(\zeta - 1)^3 \in \Gamma^3\mathbb{K}$ and $p^{n_0} f(\zeta) \in \mathbb{K}$ for any polynomial $f(\zeta)$ by Definition 4.3, one sees that a multiple of x lies in $\Gamma^3\mathbb{K}$. So, for our purposes it suffices to show that x itself is not in $\Gamma^3\mathbb{K}$.

Let us act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo $p^{n_{s-1}-1}$. We have:

$$x \equiv p^{n_{s-1}-2}(\zeta^{p^s} - 1)^2\ .$$

Consider a generator of $\Gamma^3\mathbb{K}$ (Definition 4.3):

$$(*) \qquad \prod_{i=0}^{m} \frac{p^{n_i}}{(j_i, p^{n_i})}(\zeta^{p^i} - 1)^{j_i} \text{ where } j_i \ge 0 \text{ and } \sum_{i=0}^{m} j_i \ge 3\ .$$

We state that

$$(*) \equiv (\zeta^{p^s} - 1)^3 \cdot f(\zeta^{p^s})$$

where f is a polynomial. If we would manage to show it, we could proceed as follows. Suppose that $x \in \Gamma^3\mathbb{K}$. Then

$$p^{n_{s-1}-2}(\zeta^{p^s} - 1)^2 = (\zeta^{p^s} - 1)^3 \cdot f(\zeta^{p^s}) + p^{n_{s-1}-1} \cdot g(\zeta^{p^s})$$

for some polynomials f and g. Canceling by $p^{n_{s-1}-2}$ and $(\zeta^{p^s} - 1)^2$ and substituting $t = \zeta^{p^s} - 1$ we get:

$$1 = tf_0(t) + pg_0(t) \in \mathbb{Z}[t]$$

what is a contradiction because t and p do not generate the unit ideal in the polynomial ring $\mathbb{Z}[t]$.

It remains to show that

$$(*) \equiv (\zeta^{p^s} - 1)^3 \cdot f(\zeta^{p^s})\ .$$

If for all $i < s$ the number j_i in the product $(*)$ equals 0 then even the exact equality (not only the congruence) holds. Suppose that $j_i \ne 0$ for some $i < s$. Write down this j_i as $j_i = p^r \cdot j$ with j prime to p. If $n_i - r \ge n_{s-1} - 1$ then

$$\frac{p^{n_i}}{(j_i, p^{n_i})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_i - r < n_{s-1} - 1$. We have:

$$r > n_i - n_{s-1} + 1 \geq (s-1) - i + 1 = s - i\ .$$

In order to proceed we need

LEMMA 4.8. *In a polynomial ring* $\mathbb{Z}[t]$, *there is a congruence*

$$(t-1)^{p^k} \equiv (t^p - 1)^{p^{k-1}} \mod p^k$$

for any prime p *and any integer* $k > 0$.

PROOF. Induction on k starting from $k = 1$:

$$(t-1)^{p^{k+1}} = \Big((t-1)^{p^k}\Big)^p =$$
$$= \Big((t^p-1)^{p^{k-1}} + p^k \cdot f(t)\Big)^p \equiv (t^p - 1)^{p^k} \mod p^{k+1}$$

($f(t)$ is a polynomial, it exists by the induction hypothesis). □

According to the lemma we have:

$$(\zeta^{p^i} - 1)^{p^r} \equiv (\zeta^{p^s} - 1)^{p^{r-s+i}} \mod p^{r-s+i+1}\ .$$

Hence

$$\frac{p^{n_i}}{(j_i, p^{n_i})}(\zeta^{p^i} - 1)^{j_i} \equiv \frac{p^{n_i}}{(j_i, p^{n_i})}(\zeta^{p^s} - 1)^{p^{r-s+i}\cdot j} \mod p^{n_i - s + i + 1}\ .$$

Since $p^{r-s+i} \cdot j \geq p \geq 3$ and $n_i - s + i + 1 \geq n_{s-1} - 1$ we are done. □

The analogous statement in the case $p = 2$ looks out a little bit more complicated:

PROPOSITION 4.9. *Let* A *be a 2-primary algebra. The group* $\mathrm{G}^2\Gamma K(X)$ *has a torsion iff* $\operatorname{ind} A > \exp A$ *and the reduced behaviour of* A *is not of the kind*

$$n,\ n-1,\ \ldots,\ 3,\ 2,\ 0\ .$$

PROOF. We start with the "only if" part. The case $\operatorname{ind} A = \exp A$ is covered by Proposition 3.3 with Corollary 3.6. Suppose that A has the reduced behaviour

$$n,\ n-1,\ \ldots,\ 3,\ 2,\ 0\ .$$

Using the same method as in [**33**] one can show that the whole adjoint graded group is torsion-free. Namely, a formula like one of [**33**, Proposition] states:

$$|\operatorname{Tors} \mathrm{G}^*\Gamma K(X)| = \frac{|\mathrm{G}^*\Gamma K(\mathbb{P}) / \operatorname{Im} \mathrm{G}^*\Gamma K(X)|}{|K(\mathbb{P})/K(X)|}$$

where $|\ .\ |$ denotes the order of a group. Since we know the behaviour of A we can compute that

$$|K(\mathbb{P})/K(X)| = \frac{1}{2}\prod_{i=0}^{2^n-1} \frac{2^n}{(i, 2^n)}$$

(to avoid unnecessary complications we assume here that A is a division algebra). On the other hand, Proposition 3.4 shows that

$$|\mathrm{G}^i\Gamma K(\mathbb{P})/\operatorname{Im}\mathrm{G}^i\Gamma K(X)| \leq \frac{2^n}{(i,2^n)} \text{ for any } i\ .$$

Moreover,

$$|\mathrm{G}^1\Gamma K(\mathbb{P})/\operatorname{Im}\mathrm{G}^1\Gamma K(X)| \leq 2^{n-1}$$

because $\xi^{2^{n-1}} - 1 \in \Gamma^1 K(X)$ (see also the computation of $\mathrm{CH}^1(X)$ in [**6**, §2]) and therefore

$$|\mathrm{G}^*\Gamma K(\mathbb{P})/\operatorname{Im}\mathrm{G}^*\Gamma K(X)| \leq \frac{1}{2}\prod_{i=0}^{2^n-1}\frac{2^n}{(i,2^n)}\ .$$

Thus, $|\operatorname{Tors}\mathrm{G}^*\Gamma K(X)| = 1$.

Now we "correct" the "if" proof of the previous proposition in order to match the current 2-primary situation. Suppose that we have an algebra A for which existence of the torsion is stated. Then in the reduced behaviour $(n_i)_{i=0}^m$ of A we have:

$$n_s \leq n_{s-1} - 2 \text{ and } n_{s-1} \geq 3 \text{ for some } s\ .$$

Consider the element

$$x = 2^{n_{s-1}-3}(\zeta^{2^s} - 1)^2 - 2^{n_{s-1}-1}(\zeta^{2^{s-1}} - 1)^2 \in \Gamma^2\mathbb{K}$$

where $\mathbb{K}$ is as in Definition 4.3. Since in $\mathbb{Z}[\zeta]$ the polynomial x is divisible by $(\zeta-1)^3$, it is clear that a multiple of x lies in $\Gamma^3\mathbb{K}$. So, for our purposes it suffices to show that x itself is not in $\Gamma^3\mathbb{K}$.

Let us act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo $2^{n_{s-1}-2}$. We have:

$$x \equiv 2^{n_{s-1}-3}(\zeta^{2^s} - 1)^2\ .$$

Consider a generator of $\Gamma^3\mathbb{K}$ given in Proposition 4.1:

$$(*) \qquad \prod_{i=0}^m \frac{2^{n_i}}{(j_i,2^{n_i})}(\zeta^{2^i} - 1)^{j_i} \text{ where } j_i \geq 0 \text{ and } \sum_{i=0}^m j_i \geq 3\ .$$

We state that

$$(*) \equiv (\zeta^{2^s} - 1)^3 \cdot f(\zeta^{2^s})$$

where f is a polynomial. If we would manage to show it we could proceed in the same manner as in the proof of the previous proposition.

If for all $i < s$ the number j_i in the product $(*)$ equals 0 then even the exact equality (not only the congruence) holds. Suppose that $j_i \neq 0$ for some $i < s$. Write down this j_i as $j_i = 2^r \cdot j$ with j prime to 2. If $n_i - r \geq n_{s-1} - 2$ then

$$\frac{2^{n_i}}{(j_i,2^{n_i})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_i - r < n_{s-1} - 2$. We have:

$$r > n_i - n_{s-1} + 2 \geq (s-1) - i + 2 = s - i + 1\ .$$

According to Lemma 4.8, we have:

$$(\zeta^{2^i}-1)^{2^r} \equiv (\zeta^{2^s}-1)^{2^{r-s+i}} \mod 2^{r-s+i+1} .$$

Hence

$$\frac{2^{n_i}}{(j_i,2^{n_i})}(\zeta^{2^i}-1)^{j_i} \equiv \frac{2^{n_i}}{(j_i,2^{n_i})}(\zeta^{2^s}-1)^{2^{r-s+i}\cdot j} \mod 2^{n_i-s+i+1} .$$

Since $2^{r-s+i}\cdot j \geq 2^2 \geq 3$ and $n_i - s + i + 1 \geq n_{s-1} - 1$ we are done. $\square$

Now we want to reduce the number of generators of the filtration of Definition 4.3.

PROPOSITION 4.10. *In the notation of Definition 4.3, for every $l \geq 0$, the group $\Gamma^l\mathbb{K}$ is in fact also generated by a reduced number of the products $(*)$, namely by the products satisfying the additional condition: $j_i = 0$ for every i such that $n_i = n_{i-1} - 1$.*

PROOF. Fix some i such that $n_i = n_{i-1} - 1$. One has

$$(\zeta^{p^i}-1)^j = \sum_{s\geq j} a_s \cdot (\zeta^{p^{i-1}}-1)^s$$

for some integers a_s with $j\cdot p \mid s\cdot a_s$ (Lemma 4.2). Consequently,

$$\frac{p^{n_i}}{(j,p^{n_i})}(\zeta^{p^i}-1)^j = \sum_{s\geq j} a_s \cdot \frac{p^{n_i}}{(j,p^{n_i})}(\zeta^{p^{i-1}}-1)^s = \sum_{s\geq j} b_s \cdot \frac{p^{n_{i-1}}}{(s,p^{n_{i-1}})}(\zeta^{p^{i-1}}-1)^s$$

for some integers b_s. $\square$

Using the proposition, we compute the group $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$ explicitly in a special situation. The situation we mean is described in the following

DEFINITION 4.11. We say that a reduced behaviour $(n_i)_{i=0}^m$ "makes (exactly) one jump" iff there exists exactly one s such that $n_s \leq n_{s-1} - 2$.

EXAMPLE 4.12. Fix a prime p and integers $n > m \geq 1$. One can define a "generic" division algebra $\widetilde{A}$ of index p^n and exponent p^m in spirit of Definition 3.12: take a division algebra A of index and exponent p^n, put $Y = \mathrm{SB}(A^{\otimes p^m})$ and $\widetilde{A} = A_{F(Y)}$.

The resulting algebra $\widetilde{A}$ can be also obtained as a "generic" p-primary division algebra of the reduced behaviour

$$n,\ n-1,\ \ldots,\ n-m+2,\ n-m+1,\ 0\ .$$

In particular, it is an example of an algebra with reduced behaviour "making one jump".

PROPOSITION 4.13. *Let A be a p-primary algebra with an odd p and suppose that the reduced behaviour $(n_i)_{i=0}^m$ of A "makes one jump". Then the torsion in $\mathrm{G}^2\Gamma K(X)$ is a cyclic group of order p to the power*

$$\min\{s,\ n_0 - n_s - s\}$$

where s is the subscript for which $n_s \leq n_{s-1} - 2$.

Proof. We work with $\mathbb{K}$ instead of $K(X)$ (see Proposition 4.6). According to Proposition 4.10, for any $l \geq 0$, the group $\Gamma^l\mathbb{K}$ is generated by the products:

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta-1)^{j_0} \cdot \frac{p^{n_s}}{(j_s, p^{n_s})}(\zeta^{p^s}-1)^{j_s} \text{ with } j_i \geq 0 \text{ and } j_0 + j_s \geq l \; .$$

In particular, residue classes in the quotient $\mathrm{G}^2\Gamma\mathbb{K}$ of the following three elements

$$\begin{aligned} u &= p^{n_0}(\zeta-1)^2 \; ; \\ v &= p^{n_0}(\zeta-1) \cdot p^{n_s}(\zeta^{p^s}-1) \; ; \\ w &= p^{n_s}(\zeta^{p^s}-1)^2 \end{aligned}$$

of $\Gamma^2\mathbb{K}$ generate the quotient. The second one can be excluded: the difference $v - p^{n_s+s}u$ is in $\Gamma^3\mathbb{K}$ since it is divisible by $p^{n_0}(\zeta-1)^3$ in $\mathbb{Z}[\zeta]$ and thereby can be written as a linear combination (with integral coefficients)of the polynomials

$$p^{n_0}(\zeta-1)^3, \;\; p^{n_0}(\zeta-1)^4, \;\; \ldots \quad \in \Gamma^3\mathbb{K} \; .$$

Since the classes of u and w in the quotient have infinite order, any torsion element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ of the kind $x = u - kw$ or $x = ku - w$ with an integer k (if exists) generates the torsion subgroup. Consider two cases: if $n_0 \geq n_s + 2s$ then we put

$$x = u - p^{n_0 - n_s - 2s} w \; ;$$

otherwise we put

$$x = p^{n_s + 2s - n_0} u - w \; .$$

The element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ is evidently a torsion element. We finish the proof when we show that x has order p^s in the first case and order $p^{n_0 - n_s - s}$ in the second. In both cases it means the same:

$$p^{n_0+s}(\zeta-1)^2 - p^{n_0-s}(\zeta^{p^s}-1)^2 \in \Gamma^3\mathbb{K} \tag{1}$$

and

$$p^{n_0+s-1}(\zeta-1)^2 - p^{n_0-s-1}(\zeta^{p^s}-1)^2 \notin \Gamma^3\mathbb{K} \; . \tag{2}$$

In order to avoid repetition of some boring computations, we prove the inclusion (1) in the following "tricky" way. Consider a ring $\mathbb{K}'$ with a filtration Γ constructed as in Definition 4.3 for the reduced behaviour $(n_0, n_0-1, \ldots, 1, 0)$. The ring $\mathbb{K}'$ is contained in $\mathbb{K}$ and this inclusion respects the filtrations. The element of (1) is in $\Gamma^2\mathbb{K}'$ and there is a multiple of it lying in $\Gamma^3\mathbb{K}'$. Since $\mathrm{G}^*\Gamma\mathbb{K}'$ is torsion-free (Propositions 4.6 and 3.3), this element lies even in $\Gamma^3\mathbb{K}'$. Hence (1).

The proof of (2) goes parallel to the proof of Proposition 4.7 and does not contain any new idea. Let us act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo p^{n_0-s}. The element we are interested in is congruent to

$$p^{n_0-s-1}(\zeta^{p^s}-1)^2 \; .$$

Consider a generator of $\Gamma^3\mathbb{K}$:

$$(*) \qquad \frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta-1)^{j_0} \cdot \frac{p^{n_s}}{(j_s, p^{n_s})}(\zeta^{p^s}-1)^{j_s} \text{ with } j_i \geq 0 \text{ and } j_0 + j_s \geq 3 .$$

The proof is completed when we show that

$$(*) \equiv (\zeta^{p^s}-1)^3 \cdot f(\zeta^{p^s})$$

where f is a polynomial (compare with the proof of Proposition 4.7).

If $j_0 = 0$ then even the exact equality (not only the congruence) holds. Suppose that $j_0 \neq 0$. Write down j_0 as $j_0 = p^r \cdot j$ with j prime to p. If $n_0 - r \geq n_0 - s$ then

$$\frac{p^{n_0}}{(j_0, p^{n_0})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_0 - r < n_0 - s$, i.e. that $r > s$. According to Lemma 4.8, we have:

$$(\zeta-1)^{p^r} \equiv (\zeta^{p^s}-1)^{p^{r-s}} \mod p^{r-s+1} .$$

Hence

$$\frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta-1)^{j_0} \equiv \frac{p^{n_0}}{(j_0, p^{n_0})}(\zeta^{p^s}-1)^{p^{r-s}\cdot j} \mod p^{n_0-s+1} .$$

Since $p^{r-s} \cdot j \geq p \geq 3$ and $n_0 - s + 1 \geq n_0 - s$, we are done. $\square$

PROPOSITION 4.14. *Let A be a 2-primary algebra. Suppose that the reduced behaviour $(n_i)_{i=0}^m$ of A “makes one jump” and let s be the subscript for which $n_s \leq n_{s-1} - 2$. The group* $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$ *is cyclic; its order equals p to the power*

$$\begin{cases} \min\{s,\ n_0 - n_s - s\} & \text{if } n_s > 0; \\ \min\{s,\ n_0 - s - 1\} & \text{if } n_s = 0. \end{cases}$$

PROOF. We describe here only the changes which should be made in order to adopt the previous proof to the 2-primary case.

First suppose that $n_s > 0$.

The quotient $\mathrm{G}^2\Gamma\mathbb{K}$ has three generators:

$$\begin{aligned} u &= 2^{n_0-1}(\zeta-1)^2 ; \\ v &= 2^{n_0}(\zeta-1) \cdot 2^{n_s}(\zeta^{2^s}-1) ; \\ w &= 2^{n_s-1}(\zeta^{2^s}-1)^2 . \end{aligned}$$

The second one can be evidently excluded.

If $n_0 \geq n_s + 2s$ then we put

$$x = u - 2^{n_0-n_s-2s}w ;$$

otherwise we put

$$x = 2^{n_s+2s-n_0}u - w .$$

The element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ generates the torsion subgroup. To verify the statement on its order we have to check that

$$2^{n_0+s-1}(\zeta-1)^2 - 2^{n_0-s-1}(\zeta^{2^s}-1)^2 \in \Gamma^3\mathbb{K} \tag{1}$$

and

$$2^{n_0+s-2}(\zeta-1)^2 - 2^{n_0-s-2}(\zeta^{2^s}-1)^2 \notin \Gamma^3\mathbb{K} \tag{2}$$

The inclusion (1) can be done in the same way as previously.

Let us do (2). We act in the polynomial ring $\mathbb{Z}[\zeta]$ modulo 2^{n_0-s-1}. The element we are interested in is congruent to

$$2^{n_0-s-2}(\zeta^{2^s}-1)^2 \ .$$

Consider a generator of $\Gamma^3\mathbb{K}$:

$$(*) \qquad \frac{2^{n_0}}{(j_0, 2^{n_0})}(\zeta-1)^{j_0} \cdot \frac{2^{n_s}}{(j_s, 2^{n_s})}(\zeta^{2^s}-1)^{j_s} \ \text{ with } \ j_i \geq 0 \ \text{ and } \ j_0 + j_s \geq 3 \ .$$

The proof is complete when we show that

$$(*) \equiv (\zeta^{2^s}-1)^3 \cdot f(\zeta^{2^s})$$

where f is a polynomial.

If $j_0 = 0$ then even the exact equality (not only the congruence) holds. Suppose that $j_0 \neq 0$. Write down j_0 as $j_0 = 2^r \cdot j$ with odd j. If $n_0 - r \geq n_0 - s - 1$ then

$$\frac{2^{n_0}}{(j_0, 2^{n_0})} \equiv 0$$

and hence $(*) \equiv 0$. So, assume that $n_0 - r < n_0 - s - 1$, i.e. that $r > s + 1$. According to Lemma 4.8, we have:

$$(\zeta-1)^{2^r} \equiv (\zeta^{2^s}-1)^{2^{r-s}} \mod 2^{r-s+1} \ .$$

Hence

$$\frac{2^{n_0}}{(j_0, 2^{n_0})}(\zeta-1)^{j_0} \equiv \frac{2^{n_0}}{(j_0, 2^{n_0})}(\zeta^{2^s}-1)^{2^{r-s}\cdot j} \mod 2^{n_0-s+1} \ .$$

Since $2^{r-s} \cdot j \geq 2^2 \geq 3$ and $n_0 - s + 1 \geq n_0 - s - 1$ we are done.

Now suppose that $n_s = 0$.

The generators of $\mathrm{G}^2\Gamma\mathbb{K}$ are:

$$\begin{aligned} u &= 2^{n_0-1}(\zeta-1)^2 \ ; \\ v &= 2^{n_0}(\zeta-1) \cdot (\zeta^{2^s}-1) \ ; \\ w &= (\zeta^{2^s}-1)^2 \ . \end{aligned}$$

The second one can be evidently excluded.

If $n_0 \geq 2s + 1$ then we put

$$x = u - 2^{n_0-2s-1}w \ ;$$

otherwise we put

$$x = 2^{2s+1-n_0}u - w \ .$$

The element $x \in \mathrm{G}^2\Gamma\mathbb{K}$ generates the torsion subgroup. To verify the statement on its order we have to check that

$$2^{n_0+s-1}(\zeta-1)^2 - 2^{n_0-s-1}(\zeta^{2^s}-1)^2 \in \Gamma^3\mathbb{K} \tag{1}$$

and

$$2^{n_0+s-2}(\zeta-1)^2 - 2^{n_0-s-2}(\zeta^{2^s}-1)^2 \notin \Gamma^3\mathbb{K} \tag{2}$$

But it was done already (the assumption $n_s > 0$ was not in use). □

EXAMPLE 4.15. Let $\widetilde{A}$ be a "generic" division algebra of index p^n and exponent p^m (Example 4.12). Put $\widetilde{X} = \mathrm{SB}(\widetilde{A})$. From Theorem 3.13 and Propositions 4.13 and 4.14, it follows that $\operatorname{Tors}\mathrm{CH}^2(\widetilde{X})$ is a cyclic group of order p to the power

$$\begin{cases} \min\{m,\ n-m\} & \text{for an odd } p; \\ \min\{m,\ n-m-1\} & \text{for } p=2. \end{cases}$$

5. Algebras of prime exponent

Applying Theorem 3.13 and Propositions 4.13 and 4.14 to the case of a prime exponent we can state

PROPOSITION 5.1. *Let A be an algebra of a prime exponent p. Then the group* $\operatorname{Tors}\mathrm{CH}^2(X)$ *is trivial or (cyclic) of order p. It is trivial if* $\operatorname{ind} A = p$ *or* $\operatorname{ind} A \mid 4$. *It is not if A is a "generic" division algebra of index p^n and exponent p (see Definition 3.12 or Example 4.12) where $n \geq 2$ in the case of odd p and $n \geq 3$ in the case when $p = 2$.* □

COROLLARY 5.2. *Let A be an algebra of a square-free exponent e. The group* $\operatorname{Tors}\mathrm{CH}^2(X)$ *is (cyclic) of order dividing e; moreover, there exists an algebra $\widetilde{A}$ of the exponent e with* $\operatorname{Tors}\mathrm{CH}^2(\widetilde{X})$ *of order e.*

PROOF. Follows from Propositions 1.3 and 5.1. □

It would be interesting to list all algebras A of prime exponent with trivial $\operatorname{Tors}\mathrm{CH}^2(X)$. We can only describe a class of such algebras. In [**34**] it was shown that any *decomposable* (into a tensor product of two smaller algebras) division algebra of index p^2 and exponent p has no torsion in $\mathrm{CH}^2(X)$ (in fact, there is no torsion in the whole graded group $\mathrm{G}^*\mathrm{T}K(X)$ ([**34**, Theorem 1])). The 2-analogy of this fact was obtained in [**36**, Corollary 3.1]: any decomposable division algebra of index 2^3 and exponent 2 has no torsion in $\mathrm{CH}^2(X)$ (although non-trivial torsion may exist in $\mathrm{G}^*\mathrm{T}K(X)$). These facts can be generalized as follows:

PROPOSITION 5.3. *Let A be a division algebra of prime exponent. If A decomposes then the group* $\mathrm{CH}^2(X)$ *is torsion-free.*

PROOF. First consider the case when $p \neq 2$. We have a surjection

$$\operatorname{Tors}\mathrm{G}^2\Gamma K(X) \twoheadrightarrow \operatorname{Tors}\mathrm{G}^2\mathrm{T}K(X) \simeq \operatorname{Tors}\mathrm{CH}^2(X)\ .$$

The group from the left-hand side is cyclic (Proposition 4.13), its generator is represented by the element

$$x = p^n(\xi - 1)^2 - p^{n-2}(\xi^p - 1)^2 \in \Gamma^2 K(X) = \mathrm{T}^2 K(X)$$

where $p^n = \operatorname{ind} A$.

Let $A = A_1 \otimes A_2$ be the decomposition of A into a product of two smaller algebras. Assume that the base field F has no extensions of degree prime to p (otherwise we can replace F by a maximal extension of prime to p degree; such a change has no effect on $\mathrm{CH}^2(X)$, compare with Corollary 1.2). Take an extension E/F of degree $[E : F] = p^{n-2}$ such that $\operatorname{ind}(A_1)_E = \operatorname{ind}(A_2)_E = p$ (one can obtain E/F by taking first an extension E_1/F of degree $[E_1 : F] = (\operatorname{ind} A_1)/p$ for which $\operatorname{ind}(A_1)_{E_1} = p$ and extending E_1 to E in such a way that $[E : E_1] = (\operatorname{ind} A_2)/p$ and $\operatorname{ind}(A_2)_E = p$). Consider an element

$$y = p^2(\xi - 1)^2 - (\xi^p - 1)^2 \in \mathrm{T}^2 K(X_E) .$$

Since the algebra A_E is Brauer equivalent to a decomposable division algebra of index p^2 the group $\mathrm{CH}^2(X_E)$ is torsion-free ([**34**, Theorem 1]). Hence, $y \in \mathrm{T}^3 K(X)$. Taking the transfer of y we get:

$$N_{E/F}(y) = p^n(\xi - 1)^2 - p^{n-2}(\xi^p - 1)^2 = x \in \mathrm{T}^3 K(X) .$$

Consequently $\operatorname{Tors} \mathrm{CH}^2(X) = 0$.

Now consider the case $p = 2$.

If $\operatorname{ind} A = 4$ then $\operatorname{Tors} \mathrm{CH}^2(X) = 0$ (see e.g. Proposition 4.9 or use the Albert theorem and [**34**, Theorem 1]). Suppose that $\operatorname{ind} A \geq 8$.

The group $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$ is cyclic (Proposition 4.14), its generator is represented by the element

$$x = 2^{n-1}(\xi - 1)^2 - 2^{n-3}(\xi^2 - 1)^2 \in \Gamma^2 K(X) = \mathrm{T}^2 K(X)$$

where $2^n = \operatorname{ind} A$.

Let $A = A_1 \otimes A_2$ be the decomposition of A into a product of two smaller algebras and $\operatorname{ind} A_1 \geq \operatorname{ind} A_2$. Assume that the base field F has no extensions of odd degree. Take an extension E/F of degree $[E : F] = 2^{n-3}$ such that $\operatorname{ind}(A_1)_E = 4$ and $\operatorname{ind}(A_2)_E = 2$ (one can obtain E/F by taking first an extension E_1/F of degree $[E_1 : F] = (\operatorname{ind} A_1)/4$ for which $\operatorname{ind}(A_1)_{E_1} = 4$ and extending E_1 to E in such a way that $[E : E_1] = (\operatorname{ind} A_2)/2$ and $\operatorname{ind}(A_2)_E = 2$). Consider an element

$$y = 2^2(\xi - 1)^2 - (\xi^2 - 1)^2 \in \mathrm{T}^2 K(X_E) .$$

Since the algebra A_E is Brauer equivalent to a decomposable division algebra of index 2^3 the group $\mathrm{CH}^2(X_E)$ is torsion-free ([**36**, Corollary 3.1]). Hence, $y \in \mathrm{T}^3 K(X)$. Taking the transfer of y we get:

$$N_{E/F}(y) = 2^{n-1}(\xi - 1)^2 - 2^{n-3}(\xi^2 - 1)^2 = x \in \mathrm{T}^3 K(X) .$$

Consequently $\operatorname{Tors} \mathrm{CH}^2(X) = 0$. □

COROLLARY 5.4. *A "generic" algebra of prime exponent p and index p^n (Example 4.12) is always indecomposable excluding the Albert case: $p = 2 = n$.*

PROOF. Follows from Propositions 5.1 and 5.3. □

6. Appendix

This § is included because we do not have an appropriate reference for Corollary 6.4. A particular case of Corollary 6.4 is proved in [**64**, Proposition 4.7].

We start with certain preliminary observations concerning functors of points of algebraic varieties (schemes).

Let F be a field. Denote by F-$\mathfrak{alg}$ the category of commutative associative unital F-algebras. One refers to a covariant functor from F-$\mathfrak{alg}$ to the category of sets as to an *F-functor.*

Let X be a scheme over F. For any $R \in F$-$\mathfrak{alg}$ the *set of R-points $X(R)$* of X is by definition the set $\mathrm{Mor}_F(\operatorname{Spec} R, X)$ of morphisms of schemes over F. This set is evidentely functorial in R, so we obtain an F-functor X called the *functor of points* of the scheme X. A morphism of F-schemes $f\colon X \to Y$ gives a natural transformation of their functors of points.

PROPOSITION 6.1 ([**63**, Proposition 2 in §6 of Chapter 2]). *Let X and Y be F-schemes and let $\phi\colon X \to Y$ be a natural transformation of their functors of points. There exists a unique morphism of F-schemes $f\colon X \to Y$ inducing ϕ.*

COROLLARY 6.2. *Two F-schemes X and Y are isomorphic iff there exists a natural transformation of the F-functors $\phi : X \to Y$ such that for every $R \in F$-$\mathfrak{alg}$ the map of sets $\phi(R)\colon X(R) \to Y(R)$ is bijective.*

If additionally we are given morphisms $X \to Z$ and $Y \to Z$ to one more F-scheme Z, then the schemes X and Y are isomorphic over Z iff there exists a natural transformation ϕ as above commuting with the natural transformations to the F-functor Z. □

We need a couple more of natural definitions and trivial remarks.

Let $\mathcal{F}$ be an F-functor supplied with a natural transformation $\mathcal{F} \to \mathcal{G}$ to another F-functor $\mathcal{G}$. For any $R \in F$-$\mathfrak{alg}$, the fibre of $\mathcal{F}$ over an R-point x of $\mathcal{G}$ is by definition the inverse image of x with respect to the map $\mathcal{F}(R) \to \mathcal{G}(R)$; let us denote it by $\mathcal{F}_x$.

Let $\mathcal{F}'$ be one more F-functor supplied with a natural transformation to $\mathcal{G}$. Giving a natural transformation $\mathcal{F} \to \mathcal{F}'$ over $\mathcal{G}$ is equivalent to giving a collection of maps of sets $\mathcal{F}_x \to \mathcal{F}'_x$ for every $R \in F$-$\mathfrak{alg}$ and every $x \in \mathcal{G}(R)$ satisfying the evident functorial property: if $R \to S$ is a homomorphism in F-$\mathfrak{alg}$, $x \in \mathcal{G}(R)$, and if $y \in \mathcal{G}(S)$ is the image of x with respect to the map $\mathcal{G}(R) \to \mathcal{G}(S)$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_x & \longrightarrow & \mathcal{F}'_x \\ \downarrow & & \downarrow \\ \mathcal{F}_y & \longrightarrow & \mathcal{F}'_y \end{array}$$

Now everything is prepared to prove

PROPOSITION 6.3. *Let A be a central simple algebra over a field F, X its Severi-Brauer variety, and Y a generalized Severi-Brauer variety* $\mathrm{SB}(n, A)$ *with some $n \geq 0$ (see Remark 6.6).*

The product $X \times Y$ considered over X (via the first projection) is isomorphic (as a scheme over X) to the Grassman bundle $\mathbb{\Gamma}(n, \mathcal{V})$ "of n-dimensonal subspaces" of the canonical vector bundle $\mathcal{V}$ on X (see e.g. [**80**, Page 94] *for a definition of the canonical vector bundle on a Severi-Brauer variety).*

PROOF. It suffices to show that for every $R \in F\text{-}\mathfrak{alg}$ and every $x \in X(R)$ there is a natural bijection of the sets $(X \times Y)_x$ and $\mathbb{\Gamma}(n, \mathcal{V})_x$. First of all we give descriptions of the sets of R-points of the varieties under consideration (these descriptions are in fact the most natural definitions of the varieties, see e.g. [**38**]).

The set $Y(R)$ consists of *left* ideals J of the R-algebra $A_R = A \otimes_F R$ having the following two properties:

1. the exact sequence of A_R-modules
$$0 \to J \to A_R \to A_R/J \to 0$$
splits (in particular, J is a projective R-module);
2. $\operatorname{rk} J = n$ where $\operatorname{rk} J$ is the R-rank of J devided by $\deg A$.

Analogously, the set $X(R)$ consists of right ideals I of A_R such that the sequence $0 \to I \to A_R \to A_R/I \to 0$ splits and $\operatorname{rk} I = 1$. For the rest of the proof, we fix R, an ideal I like that, and we set $x = I \in X(R)$. Note that $A_R = \operatorname{End}_R I$.

The fiber $\mathcal{V}_x$ of $\mathcal{V}$ over x is I; $\mathbb{\Gamma}(n, \mathcal{V})_x$ is the set of R-submodules N of I such that the sequence
$$0 \to N \to I \to I/N \to 0$$
splits and $\operatorname{rk}_R N = n$.

Now it is clear that the Morita theory ([**11**, Theorem 4.29]) gives a canonical bijection of the sets $(X \times Y)_x = Y(R)$ and $\mathbb{\Gamma}(n, \mathcal{V})_x$: $N \in \mathbb{\Gamma}(n, \mathcal{V})_x$ corresponds to the left ideal $\operatorname{Hom}_R(I, N)$ of $(\operatorname{End}_R I)^{\mathrm{op}} = A_R$, where $(\operatorname{End}_R I)^{\mathrm{op}}$ is the opposite algebra. □

COROLLARY 6.4. *In the condition of the proposition, put $Y_m = \mathrm{SB}(n, A^{\otimes m})$ for any $m > 0$. Then $X \times Y_m$ is a Grassman bundle over X.*

PROOF. Two ways of proving are possible: one can adopt the proof of the proposition to this new setting, or one can argue as follows.

Put $X_m = \mathrm{SB}(A^{\otimes m})$ and consider the morphism of varieties $X \to X_m$ given by the following natural transformation of their functors of points: for every $R \in F\text{-}\mathfrak{alg}$ the map $X(R) \to X_m(R)$ puts an ideal $I \in X(R)$ to its m-th tensor (over R) power $I^{\otimes m} \in X_m(R)$. In the cartesian square

$$\begin{array}{ccc} X \times Y_m & \longrightarrow & X_m \times Y_m \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_m \end{array}$$

the right arrow is a Grassman bundle by the proposition. Hence the left arrow is a Grassman bundle as well. □

REMARK 6.5. It is possible to "spread out" the statement of the corollary a little bit replacing $A^{\otimes m}$ by any Brauer equivalent central simple F-algebra.

REMARK 6.6. In contrast to [**7**, §2] and in contrast to the definition of $\mathrm{SB}(A)$, we define here $\mathrm{SB}(n, A)$ to be the variety of rank n *left* ideals in A. This variety is canonically isomorphic to the variety of rank $\deg A - n$ *right* ideals in A [**42**, §1 of Chapter I]. Of course, it is also isomorphic to the variety of rank n right ideals in the opposite algebra A^{op}.

CHAPTER 2

Codimension 2 cycles on products of Severi-Brauer varieties

We study the Chow group of 2-codimensional cycles on products of n Severi-Brauer varieties ($n \geq 2$). We analyze more detailed

- the product of a biquaternion variety and a conic;
- the product of two Severi-Brauer surfaces.

0. Introduction

In Chapter 1, we study the Chow group CH^2 for one Severi-Brauer variety. Here, using the same methods, we study the same group for a direct product of Severi-Brauer varieties. The motivation for doing this work is given by the following result of O. Izhboldin ([**24**]): the function field of a Severi-Brauer variety is universally excellent if and only if the index of the corresponding algebra is not divisible by 4. To prove this result, he needed a certain information on the group CH^2 of the product of a biquaternion variety and a conic (Theorem 6.1).

Certain results on the group CH^2 of a product of Severi-Brauer varieties was obtained in [**66**]. Its connection with the 3d Galois cohomology group ([**66**, Theorem 4.1]) was established and an example of product of three conics with torsion in CH^2 was constructed ([**66**, Remark 6.1]).

The main and general result of this Chapter is Theorem 5.5 (with Corollary 5.6). We apply it to products of two small-dimensional varieties (Theorems 6.1 and 7.1); this way we obtain, in particular, new examples of torsion in CH^2.

We use the following terminology and notation. By saying "A is an algebra", we always mean that A is a central simple algebra over a field. For an algebra A over a field F, we denote by $[A]$ its class in the Brauer group $\mathrm{Br}(F)$ of F; $\exp A$ stays for the exponent, $\deg A$ for the degree and $\operatorname{ind} A$ for the index of A.

The Severi-Brauer variety of an algebra A is denoted by $\mathrm{SB}(A)$. A *variety* is always a smooth projective algebraic variety over a field; a *sheaf* over X is an $\mathcal{O}_X$-module. The Grothendieck ring of a variety X is denoted by $K(X)$;

$$K(X) = \Gamma^0 K(X) \supset \Gamma^1 K(X) \supset \dots \quad \text{and} \quad K(X) = \mathrm{T}^0 K(X) \supset \mathrm{T}^1 K(X) \supset \dots$$

are respectively the gamma-filtration and the topological filtration on $K(X)$; we use the notation $\mathrm{G}^*\Gamma K(X)$ and $\mathrm{G}^*\mathrm{T}K(X)$ for the adjoint graded rings of these filtrations. There are certain relations between $\mathrm{G}^*\Gamma K(X)$, $\mathrm{G}^*\mathrm{T}K(X)$, and the Chow ring $\mathrm{CH}^*(X)$ we use here; they can be found in §2 of Chapter 1.

1. Two preliminary results

PROPOSITION 1.1. *Let $A_1, \dots, A_n$ and $B_1, \dots, B_m$ be algebras over a field F such that the subgroups in $\mathrm{Br}(F)$ generated by $[A_1]$, ..., $[A_n]$ and by $[B_1]$, ..., $[B_m]$ coincide. Then*

$$\mathrm{Tors}\,\mathrm{CH}^2\left(\mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)\right) \simeq \mathrm{Tors}\,\mathrm{CH}^2\left(\mathrm{SB}(B_1) \times \cdots \times \mathrm{SB}(B_m)\right) .$$

PROOF. Set $X = \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)$, $Y_1 = \mathrm{SB}(B_1)$. It suffices to show that

$$\mathrm{Tors}\,\mathrm{CH}^2(X) \simeq \mathrm{Tors}\,\mathrm{CH}^2(X \times Y_1) .$$

Since $X \times Y_1 \to X$ is a projective space bundle (Proposition 5.3), one has ([**14**, §2 of Appendix A])

$$\mathrm{CH}^2(X \times Y_1) \simeq \mathrm{CH}^2(X) \oplus \cdots \oplus \mathrm{CH}^{2-\dim Y_1}(X) .$$

The last observation is: for all $i < 2$, the group $\mathrm{CH}^i(X)$ has no torsion (for $i = 1$ see [**74**, Lemme 6.3, (i)]). □

Let p be a prime. For an algebra A as well as for an abelian group A, we are going to denote by $A\{p\}$ the p-primary part of A.

PROPOSITION 1.2. *Let $A_1, \dots, A_n$ be algebras over a field. One has*

$$\mathrm{CH}^2\left(\mathrm{SB}(A_1)\times\cdots\times\mathrm{SB}(A_n)\right)\{p\} \simeq \mathrm{Tors}\,\mathrm{CH}^2\left(\mathrm{SB}(A_1\{p\})\times\cdots\times\mathrm{SB}(A_n\{p\})\right) .$$

PROOF. For $n = 1$, the assertion is proved in Proposition 1.3 of Chapter 1. The same proof works for $n > 1$. □

2. Grothendieck group of product of Severi-Brauer varieties

Let $A_1, \dots, A_n$ be algebras over a field F, let $X_1, \dots, X_n$ be their Severi-Brauer varieties, and $X = X_1 \times \cdots \times X_n$. Fix a separable closure $\bar{F}$ of F and put $\bar{X}_i = (X_i)_{\bar{F}}$ for each i. The varieties $\bar{X}_i$ are (isomorphic to) projective spaces; denote by ξ_i the class in $K(\bar{X})$ of the tautological sheaf of the projective space bundle

$$\bar{X} \to \prod_{j \neq i} \bar{X}_j .$$

The ring $K(\bar{X})$ is generated by the elements $\xi_1, \dots, \xi_n$ subject to the relations

$$(\xi_1 - 1)^{\deg A_1} = \cdots = (\xi_n - 1)^{\deg A_n} = 0 .$$

Consider the restriction $K(X) \to K(\bar{X})$ which is a ring homomorphism.

THEOREM 2.1. *The homomorphism $K(X) \to K(\bar{X})$ is injective; its image is additively generated by the elements*

$$\mathrm{ind}(A_1^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n}) \cdot \xi_1^{j_1} \cdots \xi_n^{j_n}$$

with $0 \le j_1 < \deg A_1$, ... , $0 \le j_n < \deg A_n$.

PROOF. Use a generalized Peyre's version [**66**, Proposition 3.1] of Quillen's computation of K-theory for Severi-Brauer schemes [**69**, Theorem 4.1 of §8] n times. □

Corollary 2.2. *For algebras* $A_1, \dots, A_n$ *of fixed degrees, the ring* $K(X)$ *with the gamma-filtration (and in particular the group* $\operatorname{Tors} \mathrm{G}^2\Gamma K(X)$*) depend only on the numbers* $\operatorname{ind}(A_1^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n})$.

Proof. By the theorem, the numbers determine $K(X)$ completely as a subring in $K(\bar{X})$. The Chern classes with values in K (Definition 2.1 of Chapter 1) for X, which determine the gamma-filtration (Definition 2.6 of Chapter 1), are the restrictions of the Chern classes for $\bar{X}$. □

3. Disjoint varieties and disjoint algebras

Definition 3.1. Let $X_1, \dots, X_n$ be arbitrary varieties over a field. We say that they are *disjoint* if the ring homomorphism

$$K(X_1) \otimes \cdots \otimes K(X_n) \to K(X_1 \times \cdots \times X_n) ,$$

induced by the pull-back homomorphisms

$$pr_i^* \colon K(X_i) \to K(X_1 \times \cdots \times X_n)$$

with respect to the projections $pr_i \colon X_1 \times \cdots \times X_n \to X_i$, is an isomorphism.

Proposition 3.2. *Let* $X_1, \dots, X_n$ *be disjoint varieties. The gamma-filtration on* $K(X_1 \times \cdots \times X_n)$ *coincides with the filtration induced by the gamma-filtrations on* $K(X_1), \dots, K(X_n)$.

Proof. Denote by X the product $X_1 \times \cdots \times X_n$ and by $\tilde{\Gamma}$ the induced filtration, where for each $l \geq 0$, the term $\tilde{\Gamma}^l K(X)$ is going to be the subgroup of $K(X)$ generated by the products

$$pr_1^* \Gamma^{l_1} K(X_1) \cdots pr_n^* \Gamma^{l_n} K(X_n)$$

for all $l_1, \dots, l_n \geq 0$ with $l_1 + \cdots + l_n \geq l$. Since a pull-back homomorphism respects the gamma-filtration, one has an inclusion $\tilde{\Gamma}^l K(X) \subset \Gamma^l K(X)$. Let us prove the inverse inclusion. Since the gamma-filtration Γ on $K(X)$ is the smallest ring filtration having the properties $\Gamma^0 K(X) = K(X)$ and $c^l(x) \in \Gamma^l K(X)$ for all $x \in K(X)$ and $l \geq 1$, where c^l is the l-th Chern class with values in K (Definition 2.1 of Chapter 1), it suffices to show that

$$(*) \qquad c^l(x) \in \tilde{\Gamma}^l K(X) .$$

Since the varieties $X_1, \dots, X_n$ are disjoint, the additive group of $K(X)$ is generated by the products

$$(**) \qquad x = pr_1^*(x_1) \cdots pr_n^*(x_n)$$

where $x_i \in K(X_i)$ is the class of a locally free sheaf. Therefore it suffices to check the inclusion $(*)$ only for x of the form $(**)$. Since c^l commutes with pr_i^*, one has

$$c^l\big(pr_i^*(x_i)\big) \in \tilde{\Gamma}^l K(X) ,$$

and the last step of the proof is

LEMMA 3.3. *Let $n, m, l \geq 0$. There exists a $\mathbb{Z}$-polynomial $f_l\big((\sigma_i), (\tau_j)\big)$, where $\sigma_1, \dots, \sigma_n$ and $\tau_1, \dots, \tau_m$ are variables, having two following properties:*

- *if $x, y \in K(X)$ are classes of locally free sheaves over a variety X, the Chern class $c^l(x \cdot y)$ is equal to $f_l\big(c^i(x), c^j(y)\big)$;*
- *if one puts $\deg \sigma_i = i$ and $\deg \tau_j = j$, the degree of every monomial of f_l is at least l.*

PROOF. By the splitting principle ([**52**, Proposition 5.6]), it suffices to consider the case where

$$x = \xi_1 + \dots + \xi_n, \quad y = \eta_1 + \dots + \eta_m$$

with the classes of invertible sheaves ξ_i, η_j. For the total Chern class c_t (Definition 2.1 of Chapter 1), one has

$$c_t(x) = c_t(\sum_{i=1}^{n} \xi_i) = \prod_{i=1}^{n} \big(1 + (\xi_i - 1)t\big) = \prod_{i=1}^{n}(1 + a_i t) \text{ where } a_i = \xi_i - 1;$$

$$c_t(y) = c_t(\sum_{j=1}^{m} \eta_j) = \prod_{j=1}^{m} \big(1 + (\eta_j - 1)t\big) = \prod_{j=1}^{m}(1 + b_j t) \text{ where } b_j = \eta_j - 1;$$

$$c_t(xy) = c_t(\sum_{i,j} \xi_i\eta_j) = \prod_{i,j} \big(1 + (\xi_i\eta_j - 1)t\big) = \prod_{i,j} \big(1 + (a_i b_j + a_i + b_j)t\big) .$$

The class $c^l(xy)$ is (by definition) the coefficient of t^l in $c_t(xy)$. This coefficient is evidently a polynomial in $a_1, \dots, a_n$ and $b_1, \dots, b_m$ symmetric with respect to the variables (a_i) and also symmetric with respect to the variables (b_j) (notice that the degree of each monomial is at least l). Consequently, by the main theorem on the symmetric polynomials, $c^l(xy) = f_l\big((\sigma_i), (\tau_j)\big)$ for a polynomial f_l, where $(\sigma_i)_{i=1}^n$ are the standard symmetric polynomials for (a_i) (σ_i is a homogeneous polynomial of degree i) and $(\tau_j)_{j=1}^m$ are the standard symmetric polynomials for (b_j). The assertion of the lemma concerning the degree is evidently satisfied. Finally, note that $\sigma_i = c^i(x)$ and $\tau_j = c^j(y)$. □

□

COROLLARY 3.4. *Let $X_1, \dots, X_n$ be varieties with finitely generated Grothendieck groups (for instance, Severi-Brauer varieties). If the varieties are disjoint and the groups $G^*\Gamma K(X_1)$, ..., $G^*\Gamma K(X_n)$ are torsion-free, then the group $G^*\Gamma K(X_1 \times \dots \times X_n)$ is torsion-free as well.*

PROOF. According to the proposition, the natural homomorphism

$$G^*\Gamma K(X_1) \otimes \dots \otimes G^*\Gamma K(X_n) \to G^*\Gamma K(X_1 \times \dots \times X_n)$$

is surjective. By our assumption, the group on the left-hand side is finitely generated and torsion-free; so, it is a free abelian group of finite rank. This rank coincides with the rank of the group on the right-hand side, because the varieties are disjoint. □

No we are going to understand what the condition of being disjoint means for Severi-Brauer varieties.

Definition 3.5. Let $A_1, \dots, A_n$ be algebras over a field. We say that they are *disjoint* if

$$\operatorname{ind}(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n}) = \operatorname{ind} A_1^{\otimes j_1} \cdots \operatorname{ind} A_n^{\otimes j_n} \quad \text{for all } j_1, \dots, j_n \geq 0.$$

Proposition 3.6. *Algebras $A_1, \dots, A_n$ are disjoint if and only if their Severi-Brauer varieties are disjoint.*

Proof. Since, for an arbitrary algebra A, there is a canonical isomorphism $K(A) = \operatorname{ind} A \cdot \mathbb{Z}$, where $K(A)$ denotes the Grothendieck group of the algebra, the algebras are disjoint if and only if the maps

$$K(A_1^{\otimes j_1}) \otimes \dots \otimes K(A_n^{\otimes j_n}) \to K(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n})$$

are isomorphisms for all $0 \leq j_1 < \deg A_1, \dots, 0 \leq j_n < \deg A_n$. Taking the direct sum over all $j_1, \dots, j_n$, we obtain the map

$$\left(\coprod_{j_1=0}^{\deg A_1 - 1} K(A_1^{\otimes j_1})\right) \otimes \dots \otimes \left(\coprod_{j_n=0}^{\deg A_n - 1} K(A_n^{\otimes j_n})\right) \longrightarrow$$
$$\longrightarrow \coprod_{j_1=0}^{\deg A_1 - 1} \dots \coprod_{j_n=0}^{\deg A_n - 1} K(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n}) .$$

Identifying the factors of the product on the left-hand side with

$$K\big(\operatorname{SB}(A_1)\big), \dots, K\big(\operatorname{SB}(A_n)\big)$$

and the direct sum on the right-hand side with

$$K\big(\operatorname{SB}(A_1) \times \dots \times \operatorname{SB}(A_n)\big)$$

by Theorem 2.1, one obtains on the place of the arrow the homomorphism of Definition 3.1. $\square$

4. “Generic” varieties

Definition 4.1. Let us say that a variety X is *“generic”*, if the gamma-filtration on $K(X)$ coincides with the topological filtration.

Lemma 4.2. *If* $\operatorname{Tors} \mathrm{G}^*\Gamma K(X) = 0$ *(for an arbitrary variety X), then X is “generic”.*

Proof. To see that the filtrations coincide, it suffices to show that the homomorphism

$$\alpha \colon \mathrm{G}^*\Gamma K(X) \to \mathrm{G}^*\mathrm{T} K(X) ,$$

induced by the inclusion of the filtrations, is injective. Since $\alpha \otimes \mathbb{Q}$ is bijective ([**13**, Proposition 5.5 of Chapter VI]), the kernel of α contains only elements of finite order. Therefore, α is really injective if the group $\mathrm{G}^*\Gamma K(X)$ has no torsion. $\square$

Lemma 4.3. *Let $\mathcal{G} \to X$ be a grassmanian bundle. If X is “generic”, the variety $\mathcal{G}$ is “generic” as well.*

PROOF. Since $\mathcal{G}$ is a grassmanian bundle over X, the $\mathrm{CH}^*(X)$-algebra $\mathrm{CH}^*(\mathcal{G})$ is generated by the Chern classes (with values in CH^*) (see [**12**, Proposition 14.6.5] or [**43**, Theorem 3.2]). Using the natural epimorphism $\mathrm{CH}^* \to \mathrm{G}^*\mathrm{T}K$, one obtains the same result for $\mathrm{G}^*\mathrm{T}K$: the $\mathrm{G}^*\mathrm{T}K(X)$-algebra $\mathrm{G}^*\mathrm{T}K(\mathcal{G})$ is generated by the Chern classes (with values in $\mathrm{G}^*\mathrm{T}K$). Since X is "generic", the ring $\mathrm{G}^*\mathrm{T}K(X)$ itself is generated by the Chern classes (Remark 2.17 of Chapter 1). Consequently, $\mathrm{G}^*\mathrm{T}K(\mathcal{G})$ is generated by the Chern classes not only as algebra but also as a ring. That means $\mathcal{G}$ is "generic" (Remark 2.17 of Chapter 1). $\square$

LEMMA 4.4. *Let $X \to Y$ be a smooth morphism of varieties and let $\tilde{X}$ be its generic fiber. If X is "generic", the variety $\tilde{X}$ (it is a variety over the function field of Y) is also "generic".*

PROOF. The morphism (of schemes) $\tilde{X} \to X$ induces a homomorphism of Grothendieck groups $K(X) \to K(\tilde{X})$, respecting the both filtrations, and a homomorphism of Chow groups $\mathrm{CH}^*(X) \to \mathrm{CH}^*(\tilde{X})$ which is surjective (Proposition 4.1 of Chapter 5, see also [**39**, Theorem 3.1]). Consequently, the homomorphism

$$\mathrm{G}^*\mathrm{T}K(X) \to \mathrm{G}^*\mathrm{T}K(\tilde{X})$$

is also surjective, and therefore, for every l, the group $\mathrm{T}^l K(X)$ is mapped surjectively onto $\mathrm{T}^l K(\tilde{X})$. Since $\mathrm{T}^l K(X) = \Gamma^l K(X)$, it follows that $\mathrm{T}^l K(\tilde{X}) \subset \Gamma^l K(\tilde{X})$. The inverse inclusion is always true. $\square$

COROLLARY 4.5. *Let X and Y be varieties over a field F such that the projection $X \times Y \to X$ is a grassmanian bundle. If X is "generic", then $X_{F(Y)}$ is also "generic".*

PROOF. The variety $X \times Y$ is "generic" according to Lemma 4.3; therefore the variety $X_{F(Y)}$ is "generic" by Lemma 4.4. $\square$

5. "Generic" algebras

PROPOSITION 5.1. *Let A be a primary algebra (i.e. $\deg A$ is a power of a prime). Suppose that*

- *either* $\operatorname{ind} A = \exp A$
- *or* $\operatorname{ind} A = 2^n$ *and* $\operatorname{ind} A^{\otimes 2^{n-2}} = 4$ *($n \geq 2$)*

(an example of such A is a biquaternion algebra). Then the group $\mathrm{G}^*\Gamma\big(\mathrm{SB}(A)\big)$ is torsion-free.

PROOF. For algebras of the first type see Proposition 3.3 and Corollary 3.6 of Chapter 1; for the second type see the proof of Proposition 4.9 of Chapter 1. $\square$

COROLLARY 5.2. *Let $A_1, \dots, A_n$ be disjoint algebras and suppose that each A_i satisfies the condition of Proposition 5.1. Then for the product X of their Severi-Brauer varieties, one has:* $\operatorname{Tors} \mathrm{G}^*\Gamma K(X) = 0$*; in particular,* $\operatorname{Tors} \mathrm{CH}^2(X) = 0$.

Proof. It is a straightforward consequence of the proposition with Corollary 3.4 and Proposition 3.6. $\square$

For an algebra B and an integer $r \geq 0$, denote by $\mathrm{SB}(r, B)$ the *generalized Severi-Brauer variety* of rank r right ideals in B ([**7**, §2]). In particular, $\mathrm{SB}(1, B) = \mathrm{SB}(B)$.

Proposition 5.3. *Let $A_1, \ldots, A_n$ and B be algebras over a field, let $X = \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n)$ and let $Y = \mathrm{SB}(r, B)$ with certain $r \geq 0$.*

If the Brauer class $[B]$ of the algebra B belongs to the group generated by $[A_1], \ldots, [A_n]$, then the projection $X \times Y \to X$ is an r-grassmanian.

Proof. We may assume that

$$B \simeq A_1^{\otimes j_1} \otimes \cdots \otimes A_n^{\otimes j_n}$$

with some $j_1, \ldots, j_n \geq 0$. Consider the cartesian square

$$\begin{array}{ccc} X \times Y & \longrightarrow & T \times Y \\ \Big\downarrow & & \Big\downarrow \\ X & \longrightarrow & T \end{array}$$

where $T = \mathrm{SB}(B)$ and where the morphism $X \to T$ is given by tensor product of ideals. The arrow on the right-hand side (that is the projection $T \times Y \to T$) is an r-grassmanian by Proposition 6.3 of Chapter 1. Therefore, the projection $X \times Y \to Y$ (that is the left-hand side arrow) is an r-grassmanian as well. $\square$

Definition 5.4. We call a collection of algebras $\tilde{A}_1, \ldots, \tilde{A}_n$ *"generic"*, if it can be obtained by the following procedure. One starts with disjoint algebras $A_1, \ldots, A_n$ over a field F such that each A_i satisfies the condition of Proposition 5.1. Then one takes F-algebras $B_1, \ldots, B_m$ such that their classes in $\mathrm{Br}(F)$ belong to the subgroup generated by $[A_1], \ldots, [A_n]$. Finally, one takes as Y a direct product of some generalized Severi-Brauer of algebras $B_1, \ldots, B_m$ and one puts $\tilde{A}_i = (A_i)_{F(Y)}$ for all $i = 1, \ldots, n$.

Theorem 5.5. *If a collection of algebras $\tilde{A}_1, \ldots, \tilde{A}_n$ is "generic", then the product $\tilde{X}$ of their Severi-Brauer varieties is a "generic" variety (Definition 4.1); in particular, the epimorphism*

$$\mathrm{Tors}\, \mathrm{G}^2 \Gamma K(\tilde{X}) \to \mathrm{Tors}\, \mathrm{CH}^2(\tilde{X})$$

is bijective in this case.

Proof. Let $A_1, \ldots, A_n$ be algebras used in construction of our "generic" collection (Definition 5.4). Put $X_i = \mathrm{SB}(A_i)$ for $i = 1, \ldots, n$ and let $X = X_1 \times \cdots \times X_n$. According to Corollary 5.2, the group $\mathrm{G}^* \Gamma K(X)$ is torsion-free. In particular, the variety X is "generic" (Lemma 4.2).

Now, let Y be the direct product of generalized Severi-Brauer varieties, used in the construction of our generic collection. By Proposition 5.3, the projection $X \times Y \to X$ is a fiber product (over X) of grassmanians. Therefore, using Corollary 4.5 m times, one proves that the variety $\tilde{X} = X_{F(Y)}$ is "generic". $\square$

COROLLARY 5.6. *Let* $A_1, \dots, A_n$ *be arbitrary algebras and let* X *be the product of their Severi-Brauer varieties. Let* $\tilde{A}_1, \dots, \tilde{A}_n$ *be a "generic" collection of algebras such that* $\deg \tilde{A}_i = \deg A_i$ *and*

$$\operatorname{ind}(\tilde{A}_1^{\otimes j_1} \otimes \dots \otimes \tilde{A}_n^{\otimes j_n}) = \operatorname{ind}(A_1^{\otimes j_1} \otimes \dots \otimes A_n^{\otimes j_n})$$

for all i *and all* $j_1, \dots, j_n$. *Then the group* $\operatorname{Tors} \mathrm{CH}^2(X)$ *is isomorphic to a factorgroup of* $\operatorname{Tors} \mathrm{CH}^2(\tilde{X})$.

PROOF. By the theorem, there is an isomorphism

$$\operatorname{Tors} \mathrm{CH}^2(\tilde{X}) \simeq \operatorname{Tors} \mathrm{G}^2\Gamma K(\tilde{X}) ;$$

by Corollary 2.2, one has

$$\operatorname{Tors} \mathrm{G}^2\Gamma K(\tilde{X}) \simeq \operatorname{Tors} \mathrm{G}^2\Gamma K(X) ;$$

finally, we always have a surjection (Corollary 2.15 of Chapter 1)

$$\operatorname{Tors} \mathrm{G}^2\Gamma K(X) \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(X) .$$

$\square$

6. Biquaternion variety times conic

A Severi-Brauer variety of a biquaternion algebra is called *biquaternion variety* here.

THEOREM 6.1. *Let* X *be a biquaternion variety,* Y *be a conic (over the same field) and* A, B *be the corresponding algebras (*B *is a quaternion algebra).*

1. *The torsion in the group* $\mathrm{CH}^2(X \times Y)$ *is either trivial, or of order 2.*
2. *If the torsion is non-trivial, then*

$$(*) \qquad \operatorname{ind} A = \operatorname{ind}(A \otimes B) = 4 \quad \textit{and} \quad \operatorname{ind} B = 2 .$$

3. *If the collection* A, B *is "generic" (Definition 5.4) and satisfies the condition* $(*)$, *then the torsion is not trivial.*

PROOF. If $\operatorname{ind} B \neq 2$, i.e. if B is split, then we know from Proposition 1.1 that $\operatorname{Tors} \mathrm{CH}^2(X \times Y) \simeq \mathrm{CH}^2(X)$; hereby the latter group is torsion-free ([**34**, Corollary]).

If $\operatorname{ind} A \neq 4$, than A is Brauer-equivalent to a quaternion algebra A'; denoting by X' its Severi-Brauer variety, one gets (Proposition 1.1)

$$\operatorname{Tors} \mathrm{CH}^2(X \times Y) \simeq \operatorname{Tors} \mathrm{CH}^2(X' \times Y) .$$

Since $\dim(X' \times Y) = 2$, the group

$$\mathrm{G}^2\Gamma K(X' \times Y) = \Gamma^2 K(X' \times Y) \subset K(X' \times Y)$$

has no torsion. It follows that in this case $\operatorname{Tors} \mathrm{CH}^2(X \times Y) = 0$ as well.

Let C be a division algebra Brauer-equivalent to the product $A \otimes B$; $T = \mathrm{SB}(C)$. Using Proposition 1.1 once again, we have

$$\operatorname{Tors} \mathrm{CH}^2(X \times Y) \simeq \operatorname{Tors} \mathrm{CH}^2(T \times Y) .$$

If $\operatorname{ind}(A \otimes B) \leq 2$, then $\dim T \times Y \leq 2$ and we are done in the same way as above.

If $\operatorname{ind}(A \otimes B) = 8$, then the algebras A, B are disjoint and Corollary 5.2 shows that $\operatorname{Tors}\operatorname{CH}^2(X \times Y) = 0$.

The rest is served by

PROPOSITION 6.2. *Suppose that a biquaternion algebra A and a quaternion algebra B are division algebras and* $\operatorname{ind}(A \otimes B) = 4$. *For X, Y as above, one has:* $\operatorname{Tors} \mathrm{G}^2\Gamma K(X \times Y) \simeq \mathbb{Z}/2$.

PROOF. Put $K = K(X \times Y)$, $\bar{K} = K(\bar{X} \times \bar{Y})$. The commutative ring $\bar{K}$ is generated by elements ξ, η subject to the relations $(\xi - 1)^4 = 0 = (\eta - 1)^2$ (see §2). In particular, the additive group of $\bar{K}$ is a free abelian group generated by the elements $\xi^i \eta^j$, $i = 0, 1, 2, 3$, $j = 0, 1$. We are also going to use another system of generators: $f^i g^j$, $i = 0, 1, 2, 3$, $j = 0, 1$, where $f = \xi - 1$, $g = \eta - 1$.

For each l, the l-th term $\Gamma^l \bar{K}$ of the gamma-filtration on $\bar{K}$ is generated by the products $f^i g^j$ with $i + j \geq l$. In particular, $\mathrm{G}^l\Gamma\bar{K}$ is an abelian group freely generated by the residue classes of the products $f^i g^j$ with $i + j = l$.

LEMMA 6.3. *The subring $K \subset \bar{K}$ is additively generated by the elements*

$$1,\ 4\xi,\ \xi^2,\ 4\xi^3,\ 2\eta,\ 4\xi\eta,\ 2\xi^2\eta,\ 4\xi^3\eta\ .$$

PROOF. It is a particular case of Theorem 2.1. □

LEMMA 6.4. *The following elements are also generators of the additive group of K:*

$$1,\ 2f - f^2,\ 2g,\ 2f^2,\ 4fg,\ 4f^3,\ \boxed{2f^2g},\ 4f^3g$$

(the singled out element is going to produce the torsion — see Corollary 6.9).

PROOF. A straightforward verification. □

LEMMA 6.5. *There are the following inclusions:*

$$\Gamma^1 K \ni 2f - f^2,\ 2g\ ;$$
$$\Gamma^2 K \ni 2f^2,\ 4fg,\ \boxed{2f^2g}\ ;$$
$$\Gamma^3 K \ni 4f^3,\ 2 \cdot \boxed{2f^2g}\ ;$$
$$\Gamma^4 K \ni 4f^3g\ .$$

PROOF. The assertion on $\Gamma^1 K$ is evident.

Since $2f^2,\ 4fg \in K \cap \Gamma^2\bar{K}$ and the restriction homomorphism $\mathrm{G}^1\Gamma K \to \mathrm{G}^1\Gamma\bar{K}$ is injective ([**74**, Lemme 6.3, (i)]), the assertion on $\Gamma^2 K$ holds (a direct verification (see the rest of the proof) is also easy).

Finally, one has:

$$c_t(4\xi) = (1 + ft)^4 \quad \Rightarrow \quad c^3(4\xi) = 4f^3 \quad \Rightarrow \quad 4f^3 \in \Gamma^3 K\ ;$$
$$2f^2 \in \Gamma^2 K \text{ and } 2g \in \Gamma^1 K \quad \Rightarrow \quad 4f^2g = (2f^2) \cdot (2g) \in \Gamma^3 K\ ;$$
$$c^4(4\xi\eta) = (\xi\eta - 1)^4 = \big((f+1)(g+1) - 1\big)^4 = 4f^3g \in \Gamma^4 K\ .$$

□

COROLLARY 6.6. *Denote by* α^* *the restriction homomorphism*

$$\mathrm{G}^*\Gamma K \to \mathrm{G}^*\Gamma \bar{K} .$$

For all $i > 0$*, one has:* $\operatorname{Im}\alpha^i \subset 2\mathrm{G}^i\Gamma\bar{K}$.

PROOF. According to the lemma, the group $\mathrm{G}^1\Gamma K$ is generated by the residue classes of the elements $2f - f^2$ and $2g$; their images in $\mathrm{G}^1\Gamma\bar{K}$ are really divisible by 2. So,the assertion of the corollary for $i = 1$ is proved.

Since the elements of $\Gamma^2 K$, $\Gamma^3 K$ and $\Gamma^4 K$, listed in the lemma, generate $\Gamma^2 K$ and are divisible by 2 in $\bar{K}$, we obtain the assertion for $i \geq 2$ (use the absence of torsion in $\mathrm{G}^*\Gamma\bar{K}$). □

COROLLARY 6.7. $\#(\operatorname{Tors}\mathrm{G}^*\Gamma K) \leq 2$.

PROOF. Since the group $\mathrm{G}^*\Gamma\bar{K}$ is torsion-free, $\operatorname{Tors}\mathrm{G}^*\Gamma K \subset \operatorname{Ker}\alpha^*$. We are going to show that $\#(\operatorname{Ker}\alpha^*) \leq 2$, using the following formula ([**33**, Proposition]):

$$\#(\operatorname{Ker}\alpha^*) = \#(\operatorname{Coker}\alpha^*)/\#(\bar{K}/K) .$$

It is easy to calculate that $\#(\bar{K}/K) = 2^{10}$. According to the lemma,

$$\#(\operatorname{Coker}\alpha^*) \leq 2^{11} .$$

□

LEMMA 6.8. $2f^2g \notin \Gamma^3 K$.

PROOF. It suffices to show that $\operatorname{Im}\alpha^3 \subset 4\mathrm{G}^3\Gamma\bar{K}$.

The group $\operatorname{Im}\alpha^3$ is generated by the subgroup $\operatorname{Im}\alpha^1 \cdot \operatorname{Im}\alpha^2$ and by the subset $\alpha^3(c^3 K)$, where c^3 is the 3d Chern class with values in $\mathrm{G}^*\Gamma K$ (Definition 2.7 of Chapter 1). Since $\operatorname{Im}\alpha^i \subset 2\mathrm{G}^i\Gamma\bar{K}$ for $i > 0$ by Corollary 6.6, one has: $\operatorname{Im}\alpha^1 \cdot \operatorname{Im}\alpha^2 \subset 4\mathrm{G}^3\Gamma\bar{K}$. Therefore, it suffices to verify that $\alpha^3\big(c^3(S)\big) \subset 4\mathrm{G}^3\Gamma\bar{K}$ for a system S of generators of the additive group of K. The verification is trivial if we take as S the system of generators of Lemma 6.3. □

COROLLARY 6.9. *The residue of* $2f^2g$ *in* $\mathrm{G}^2\Gamma K$ *has order 2 and generates the torsion subgroup.*

PROOF. The residue is of order 2 by Lemmas 6.8 and 6.5. It generates the whole torsion subgroup (not only in $\mathrm{G}^3\Gamma K$ but also in $G^*\Gamma K$) by Corollary 6.7. □

The proofs of the theorem and of the proposition are complete. □

□

REMARK 6.10. In the condition of the theorem, denote the base field by F and suppose that there exists a quadratic extension L/F (or, more generally, an extension of degree not divisible by 4) such that the algebra A_L is no more a division algebra and the algebra B_L is split. In this case, $f^2g \in \mathrm{T}^3K(X_L \times Y_L)$;

using the norm map, we obtain: $2f^2g \in \mathrm{T}^3K(X\times Y)$, i.e. $\operatorname{Tors}\mathrm{CH}^2(X\times Y) = 0$.

Therefore, if A, B are such that $\operatorname{Tors}\mathrm{CH}^2(X\times Y) \neq 0$ (for example, if A, B form a "generic" collection (Theorem 6.1)), there are no extensions like that. The first example of this phenomenon is constructed in [**51**].

7. Product of two Severi-Brauer surfaces

A *Severi-Brauer surface* is a Severi-Brauer variety of dimension 2.

THEOREM 7.1. *Let X, Y be Severi-Brauer surfaces over a field and let A, B be the corresponding algebras.*

1. *The torsion in the group $\mathrm{CH}^2(X \times Y)$ is either trivial, or of order 3.*
2. *If the torsion is not trivial, then*

$$(*) \qquad \operatorname{ind} A = \operatorname{ind} B = \operatorname{ind}(A\otimes B) = \operatorname{ind}(A\otimes B^{\mathrm{o}}) = 3$$

where B^{o} is the algebra opposite to B.

3. *If the collection A, B is "generic" (Definition 5.4) and satisfies the condition $(*)$, then the torsion is not trivial.*

PROOF. If at least one of the algebras A, B, $A\otimes B$, $A\otimes B^{\mathrm{o}}$ is split, then there exists an algebra C of degree 3 such that its class $[C]$ in the Brauer group generates the same subgroup as $[A]$ and $[B]$ (together). According to Proposition 1.1, in this case, the group $\operatorname{Tors}\mathrm{CH}^2(X \times Y)$ is isomorphic to the group $\operatorname{Tors}\mathrm{CH}^2\big(\mathrm{SB}(C)\big)$ which is trivial by [**33**, Corollary], or also by Lemma 2.4 of Chapter 4.

If $\operatorname{ind}(A\otimes B) = \operatorname{ind}(A\otimes B^{\mathrm{o}}) = 9$, then the algebras A, B are disjoint and one can use Corollary 5.2.

Put $Y^{\mathrm{o}} = \mathrm{SB}(B^{\mathrm{o}})$. Since by Proposition 1.1

$$\operatorname{Tors}\mathrm{CH}^2(X \times Y) \simeq \operatorname{Tors}\mathrm{CH}^2(X \times Y^{\mathrm{o}}) ,$$

it suffices to consider only one of the two following cases:

- $\operatorname{ind}(A\otimes B) = 3$ and $\operatorname{ind}(A\otimes B^{\mathrm{o}}) = 9$;
- $\operatorname{ind}(A\otimes B) = 9$ and $\operatorname{ind}(A\otimes B^{\mathrm{o}}) = 3$.

LEMMA 7.2. *If* $\operatorname{ind} A = \operatorname{ind} B = \operatorname{ind}(A\otimes B) = 3$ *and* $\operatorname{ind}(A\otimes B^{\mathrm{o}}) = 9$, *then* $\operatorname{Tors}\mathrm{G}^2\Gamma K(X \times Y) = 0$.

PROOF. Put $K = K(X\times Y)$, $\bar K = K(\bar X\times\bar Y)$. The commutative ring $\bar K$ is generated by elements ξ, η subject to the relations $(\xi - 1)^3 = 0 = (\eta - 1)^3$ (see §2). In particular, the additive group of $\bar K$ is an abelian group freely generated by the elements $\xi^i\eta^j$, $i, j = 0, 1, 2$. We also are going to use another system of generators: f^ig^j, $i, j = 0, 1, 2$, where $f = \xi - 1$, $g = \eta - 1$.

For every l, the l-th term $\Gamma^l\bar K$ of the gamma-filtration on $\bar K$ is generated by the products f^ig^j with $i + j \geq l$.

The condition of the lemma implies that

$$\operatorname{ind} A^{\otimes 2} = \operatorname{ind} B^{\otimes 2} = \operatorname{ind}(A^{\otimes 2} \otimes B^{\otimes 2}) = 3 \text{ and}$$
$$\operatorname{ind}(A \otimes B^{\otimes 2}) = \operatorname{ind}(A^{\otimes 2} \otimes B) = 9 .$$

So, according to Theorem 2.1, the subring $K \subset \bar{K}$ is additively generated by

$$1,\ 3\xi,\ 3\xi^2,\ 3\eta,\ 3\xi\eta,\ 9\xi^2\eta,\ 3\eta^2,\ 9\xi\eta^2,\ 3\xi^2\eta^2 .$$

We are also going to use another system of generators:

$$1,\ 3f,\ 3g,\ 3f^2,\ 3fg,\ 3g^2,\ 9f^2g,\ 3f^2g + 3fg^2 + 6f^2g^2,\ 9f^2g^2 .$$

Now it is evident that the intersection $K \cap \Gamma^3\bar{K}$ is generated by

$$9f^2g,\ 3f^2g + 3fg^2 + 6f^2g^2, \text{ and } 9f^2g^2 .$$

To prove that the group $\mathrm{G}^2\Gamma K$ is torsion-free, it suffices to verify that these three elements belong to $\Gamma^3 K$.

Since $3f^2,\ 3g^2 \in \Gamma^2 K$, and $3g \in \Gamma^1 K$, one has:

$$9f^2g = (3f^2)\cdot(3g) \in \Gamma^3 K,\ \ 9f^2g^2 = (3f^2)\cdot(3g^2) \in \Gamma^4 K .$$

The last element coincides with a 3-d Chern class:

$$c^3(3\xi\eta) = (\xi\eta - 1)^3 = \big((f+1)(g+1) - 1\big)^3 = (fg + f + g)^3 =$$
$$3fg(f+g)^2 + (f+g)^3 = 6f^2g^2 + 3f^2g + 3fg^2 .$$

□

We finish the proof of the theorem by

Proposition 7.3. *If* $\operatorname{ind} A = \operatorname{ind} B = \operatorname{ind}(A \otimes B) = \operatorname{ind}(A \otimes B^{\mathrm{o}}) = 3$, *then* $\operatorname{Tors}\mathrm{G}^2\Gamma K(X \times Y) \simeq \mathbb{Z}/3$.

Proof. We use the notation introduced in the beginning of the proof of the last lemma.

Lemma 7.4. *The subring $K \subset \bar{K}$ is now generated by 1 and $3\bar{K}$. Moreover,*

$$\Gamma^1 K = 3\Gamma^1\bar{K} ;$$
$$\Gamma^2 K = 3\Gamma^2\bar{K} ;$$
$$\Gamma^3 K \ni 3f^2g - 3fg^2,\ 3f^2g + 3fg^2 + 6f^2g^2 ;$$
$$\Gamma^4 K \ni 9f^2g^2 .$$

Proof. The assertion about $\Gamma^1 K$ is trivial. The assertion about $\Gamma^2 K$ is caused by injectivity of the restriction homomorphism $\mathrm{G}^1\Gamma K \to \mathrm{G}^1\Gamma\bar{K}$ ([**74**, Lemme 6.3, (i)]); $9f^2g^2 \in \Gamma^4 K$ because $3f^2, 3g^2 \in \Gamma^2 K$.

To prove the assertion about $\Gamma^3 K$, let us compute the 3d Chern class

$$c^3(\xi^2\eta) = (\xi^2\eta - 1)^3 = \big((f+1)^2(g+1) - 1\big)^3 = 27f^2g^2 + 12f^2g + 6fg^2 .$$

Since $9f^2g, 9fg^2 \in \Gamma^3 K$, we conclude that $3f^2g - 3fg^2 \in \Gamma^3 K$.

Finally, as we have already computed in the proof of Lemma 7.2,

$$3f^2g + 3fg^2 + 6f^2g^2 = c^3(3\xi\eta) \in \Gamma^3 K .$$

□

Corollary 7.5. $\#(\operatorname{Tors} \mathrm{G}^*\Gamma K) \le 3$.

Proof. Analogously to Corollary 6.7. Now one has (Lemma 7.4):

$$\#(\bar{K}/K) = 3^8 \quad \text{and} \quad \#(\operatorname{Coker} \alpha^*) \le 3^9 .$$

□

Lemma 7.6. $3f^2g^2 \notin \Gamma^3 K$.

Proof. Let us define a homomorphism $\phi_9 \colon \bar{K} \to \mathbb{Z}/9$ as follows: write an arbitrary element $x \in \bar{K}$ as a linear combination

$$x = \sum_{i,j=0}^{2} a_{ij} f^i g^j \quad \text{with } a_{ij} \in \mathbb{Z},$$

put $\phi(x) = a_{21} + a_{12} - a_{22}$ and define $\phi_9(x)$ as the residue of $\phi(x)$ modulo 9.

Since $\phi_9(3f^2g^2) \ne 0$, it suffices to show that $\phi_9(\Gamma^3 K) = 0$.

A priori, the group $\Gamma^3 K$ is generated by $\Gamma^1 K \cdot \Gamma^2 K$, $c^3(S)$ et $c^4(S)$ where

$$S = 1,\ 3\xi,\ 3\xi^2,\ 3\eta,\ 3\xi\eta,\ 3\xi^2\eta,\ 3\eta^2,\ 3\xi\eta^2,\ 3\xi^2\eta^2 .$$

Hereby, $c^4(s) = 0$ for all $s \in S$; thus one can eliminate $c^4(S)$ from the list of generators.

Since

$$\Gamma^1 K \cdot \Gamma^2 K \subset \Gamma^1 K \cdot \Gamma^1 K \subset 9\bar{K} \quad \text{(Lemma 7.4)},$$

one has: $\phi_9(\Gamma^1 K \cdot \Gamma^2 K) = 0$.

It remains $c^3(S)$. For $s = 1, 3\xi, 3\xi^2, 3\eta$, and $3\eta^2$, the value $\phi(s)$ is already 0. The following calculations show that $\phi_9\big(c^3(s)\big) = 0$ for the other four elements $s \in S$ as well:

$$c^3(3\xi\eta) = (\xi\eta - 1)^3 = \big((f+1)(g+1) - 1\big)^3 = \big(fg + (f+g)\big)^3 = \\ = 3fg(f+g)^2 + (f+g)^3 = \boxed{6f^2g^2 + 3f^2g + 3fg^2}\ ;$$

$$c^3(3\xi^2\eta^2) = (\xi^2\eta^2 - 1)^3 = \big((f+1)^2(g+1)^2 - 1\big)^3 = \big((f^2+4fg+g^2) + 2(f+g)\big)^3 = \\ = 12(f^2 + 4fg + g^2)(f+g)^2 + 8(f+g)^3 = \boxed{120f^2g^2 + 24f^2g + 24fg^2}\ ;$$

$$c^3(3\xi^2\eta) = (\xi^2\eta - 1)^3 = \big((f+1)^2(g+1) - 1\big)^3 = \big(f(f+2g) + (2f+g)\big)^3 = \\ = 3f(f+2g)(2f+g)^2 + (2f+g)^3 = \boxed{27f^2g^2 + 12f^2g + 6fg^2}\ ;$$

$$c^3(3\xi\eta^2) = \boxed{27f^2g^2 + 6f^2g + 12fg^2}\ .$$

□

According to Lemma 7.4, we have $3f^2g^2 \in \Gamma^2 K$. The residue class of the element $3f^2g^2$ in $\mathrm{G}^2\Gamma K$ has order 3 by Lemmas 7.4 and 7.6. Therefore, by Corollary 7.5, it generates the whole torsion subgroup of $\mathrm{G}^2\Gamma K$. So, the proof of Proposition 7.3 is complete. $\square$

Proposition 7.3 completes the proof of Theorem 7.1. $\square$

CHAPTER 3

Isotropy of virtual Albert forms over function fields of quadrics

Let F be a field of characteristic different from 2 and ϕ be a virtual Albert form over F, i.e. an anisotropic 6-dimensional quadratic form over F which is still anisotropic over the field $F(\sqrt{d_\pm \phi})$. We give a complete description of the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$. This completes the series of works ([**16**], [**44**], [**45**], [**49**], [**54**]) where the question was considered previously.

Results of this Chapter are obtained in joint work with Oleg Izhboldin.

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. In the case when $\dim \phi \leq 5$ the problem was completely solved in [**15**] and [**75**]. For 6-dimensional quadratic forms, the problem was studied by D. W. Hoffmann ([**16**]), A. Laghribi ([**44**], [**45**]), D. Leep ([**49**]), and A. S. Merkurjev ([**54**]) and was solved fully except for the following two cases (see [**44**] and [**45**]):

- $\dim \psi = 4$, $d_\pm \psi \neq 1$, and $\operatorname{ind}(C_0(\phi)) = 2$;
- $\dim \psi = 4$, $d_\pm \psi \neq 1$, $\operatorname{ind}(C_0(\phi)) = 4$, and $d_\pm \phi = d_\pm \psi$.

In this Chapter the second case is studied completely. Our result (Theorem 5.1) and results of Laghribi, Leep, and Merkurjev give rise to the following

Theorem. *Let ϕ be a 6-dimensional quadratic form with $\operatorname{ind}(C_0(\phi)) = 4$. In the case when $\psi \notin GP_2(F)$, the quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ. In the case when $\psi \in GP_2(F)$, the form $\phi_{F(\psi)}$ is isotropic if and only if a 3-dimensional subform of ψ is similar to a subform of ϕ.*

We deduce Theorem 5.1 from a result on 8-dimensional forms (Proposition 4.1) which also has an independent value: together with [**47**], it gives rise to Theorem 4.4 answering the question about isotropy of an 8-dimensional quadratic form ϕ with $\det \phi = 1$ and $\operatorname{ind}(C(\phi)) = 8$ over function fields of quadrics.

1. Terminology, notation, and backgrounds

1.1. Quadratic forms. By $\phi \perp \psi$, $\phi \simeq \psi$, and $[\phi]$ we denote orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F respectively. To simplify notation we will write $\phi+\psi$ instead of $[\phi]+[\psi]$. For a quadratic form ϕ of dimension n, we set $d_\pm \phi = (-1)^{n(n-1)/2} \det \phi$. We consider $d_\pm \phi$ as an element of F^*/F^{*2}. The maximal ideal of $W(F)$ consisting of the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We denote by $\langle\langle a_1, \dots, a_n \rangle\rangle$ the n-fold Pfister form

$$\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$$

and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms we denote by $GP_n(F)$. For any field extension L/F, we put $\phi_L = \phi \otimes L$, $W(L/F) = \mathrm{Ker}(W(F) \to W(L))$, and $I^n(L/F) = \mathrm{Ker}(I^n(F) \to I^n(L))$.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ if $\dim \phi \geq 3$; $F(\phi) = F(\sqrt{d})$ if $\dim \phi = 2$ and $d = d_\pm \phi \neq 1$; and $F(\phi) = F$ otherwise.

Let $\psi \in GP_2(F)$ and ψ_0 be a 3-dimensional subform of ψ. Then $\psi_{F(\psi_0)}$ and $(\psi_0)_{F(\psi)}$ are isotropic. Hence for any quadratic form ϕ, the isotropy of $\phi_{F(\psi)}$ is equivalent to isotropy of $\phi_{F(\psi_0)}$. Thus, to give a complete description of the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$, it is sufficient to consider the case where $\psi \notin GP_2(F)$.

We say that a quadratic form ϕ is a *Pfister neighbor* if for some n there exists $\pi \in P_n(F)$ such that ϕ is similar to a subform of π and $\dim \phi > 2^{n-1}$.

Let ϕ be a quadratic form of dimension 2^n. We say that ϕ^* is a *half-neighbor* of ϕ, if $\dim \phi^* = 2^n$ and there exists $k \in F^*$ such that $\phi^* \equiv k\phi \pmod{I^{n+1}(F)}$.

1.2. Algebras. Let A be a central simple algebra over F. By $\deg(A)$, $\mathrm{ind}(A)$, $[A]$, and $\exp(A)$ we denote respectively the degree of A, the Schur index of A, the class of A in the Brauer group $\mathrm{Br}(F)$, and the order of $[A]$ in the Brauer group. By $\mathrm{SB}(A)$ we denote the Severi–Brauer variety of A. If an algebra B has the form $B = A \times A$, we set $\mathrm{ind}\, B = \mathrm{ind}\, A$.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ. By $C_0(\phi)$ we denote the even part of $C(\phi)$. If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Hence we get a well defined element $[C(\phi)]$ of $\mathrm{Br}_2(F)$ which we denote by $c(\phi)$.

1.3. Quadratic forms of dimension 6. Let ϕ be an anisotropic quadratic form of dimension 6 and let $d = d_\pm \phi$. If $d = 1$, then ϕ is an Albert form. In this case the problem of isotropy of ϕ over the function field of a quadratic form ψ is completely solved ([**49**], [**54**]): in the case when $\psi \notin GP_2(\psi)$, the form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform in ϕ.

Suppose now that $d \neq 1$. Then $C_0(\phi)$ is a central simple algebra over $L = F(\sqrt{d})$. In this case we have the following classification of anisotropic 6-dimension forms:

Type 1 is defined by one of the following equivalent conditions:

- $\operatorname{ind}(C_0(\phi)) = 1$;
- ϕ_L is hyperbolic;
- ϕ has the form $\langle\langle d\rangle\rangle\, \mu$ where $\dim \mu = 3$;
- ϕ is a Pfister neighbor.

Type 2 is defined by one of the following equivalent conditions:

- $\operatorname{ind}(C_0(\phi)) = 2$;
- ϕ_L is isotropic but not hyperbolic;
- ϕ is similar to a form of the kind $\langle\langle a, b\rangle\rangle \perp c\,\langle\langle d\rangle\rangle$, where $\langle\langle a, b\rangle\rangle_L$ is not isotropic.

Type 3 is defined by one of the following equivalent conditions:

- $\operatorname{ind}(C_0(\phi)) = 4$;
- ϕ_L is anisotropic;

The quadratic form of the type 3 is called a *virtual Albert form.*

For the quadratic forms ϕ of type 1 (i.e. for the Pfister neighbors), the problem of isotropy $\phi_{F(\psi)}$ is completely solved by Arason–Pfister subform theorem. The case of quadratic forms of type 2 was studied by D. Hoffmann in [**16**]: he found the conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic excepting the case $\dim \psi = 4$. [1]

The case of the quadratic forms ϕ of type 3 (virtual Albert forms) was studied completely by A. Laghribi in [**44, 45**] except for the case where $\dim \psi = 4$ and $d_\pm\, \psi \neq 1$. In this chapter we complete the investigation of isotropy of virtual Albert forms over the function field of a quadric.

1.4. Cohomology groups. By $H^*(F)$ we denote the graded ring of Galois cohomology

$$H^*(F, \mathbb{Z}/2\mathbb{Z}) = H^*(\mathrm{Gal(F_{sep}/F)}, \mathbb{Z}/2\mathbb{Z}).$$

For any field extension L/F, we set $H^*(L/F) = \operatorname{Ker}(H^*(F) \to H^*(L))$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \operatorname{Br}_2(F)$. Thus any element $a \in F^*$ gives rise to an element of $H^1(F)$; it is denoted by (a). The cup product $(a_1) \cup \cdots \cup (a_n)$ is denoted by $(a_1, \ldots, a_n)$.

For $n = 0, 1, 2$ there is a homomorphism $e^n : I^n(F) \to H^n(F)$ defined as follows: $e^0(\phi) = \dim \phi \pmod 2$, $e^1(\phi) = d_\pm\, \phi$, and $e^2(\phi) = c(\phi)$. Moreover, there exists a homomorphism $e^3 : I^3(F) \to H^3(F)$ which is uniquely determined by the condition $e^3(\langle\langle a_1, a_2, a_3\rangle\rangle) = (a_1, a_2, a_3)$ (see [**5**]). The homomorphism e^n is surjective and $\operatorname{Ker} e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [**53**], [**62**], and [**71**]).

[1] The case where ϕ is of type 2 and $\dim \psi = 4$ is studied in Chapter 4.

1.5. K-theory and Chow groups. In §2 we use the following notation.

Let X be a smooth algebraic F-variety. The Grothendieck ring of X is denoted by $K(X)$. This ring is supplied with the filtration "by codimension of support" (which respects the multiplication); the adjoint graded ring is denoted by $\mathrm{G}^*\mathrm{K}(\mathrm{X})$. There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $\mathrm{G}^*\mathrm{K}(\mathrm{X})$; its kernel consists only of torsion elements and is trivial in the 0-th, 1-st and 2-nd graded components ([**81**, §9]).

We fix a separable closure $\bar{F}$ of the ground field F and denote by $\bar{X}$ the variety $X_{\bar{F}}$. The image of the restriction homomorphism $\mathrm{G}^*\mathrm{K}(\mathrm{X}) \to \mathrm{G}^*\mathrm{K}(\bar{\mathrm{X}})$ is denoted by $\bar{\mathrm{G}}^*K(X)$.

We denote by $|S|$ the order of a finite set S.

2. Computation of $H^3(F(\mathrm{SB}(A) \times \mathrm{SB}(B))/F)$

THEOREM 2.1. *Let A and B be biquaternion division F-algebras such that* $\mathrm{ind}(A\otimes B) = 8$. *Suppose that there exists a quadratic extension L/F such that A_L and B_L are not division algebras. Then*

$$H^3(F(\mathrm{SB}(A) \times \mathrm{SB}(B))/F) = [A] \cdot H^1(F) + [B] \cdot H^1(F).$$

PROOF. We put $X = \mathrm{SB}(A) \times \mathrm{SB}(B)$.

LEMMA 2.2 ([**33**, Proposition 2]).

$$|\operatorname{Tors} \mathrm{G}^*\mathrm{K}(\mathrm{X})| = \frac{|\mathrm{G}^*\mathrm{K}(\bar{\mathrm{X}})/\bar{\mathrm{G}}^*\mathrm{K}(\mathrm{X})|}{|\mathrm{K}(\bar{\mathrm{X}})/\mathrm{K}(\mathrm{X})|} .$$

LEMMA 2.3. $|K(\bar{X})/K(X)| = 2^{28}$.

PROOF. Applying [**69**, §8, Theorem 4.1], one gets an isomorphism

$$K(X) \simeq K(F)^{\oplus 4} \oplus K(A)^{\oplus 4} \oplus K(B)^{\oplus 4} \oplus K(A \otimes B)^{\oplus 4}$$

which shows that

$$|K(\bar{X})/K(X)| = (\mathrm{ind}\, A)^4 \cdot (\mathrm{ind}\, B)^4 \cdot (\mathrm{ind}\, A \otimes B)^4 = 2^{28} .$$

□

The variety $\overline{\mathrm{SB}(A)}$ is a projective space; denote by f the class of a hyperplane in $\mathrm{G}^1\mathrm{K}(\overline{\mathrm{SB}(\mathrm{A})})$.

LEMMA 2.4. *For any $i \geq 0$, the group $\bar{\mathrm{G}}^iK(\mathrm{SB}(A_L))$ contains*

- f^i, *if i is even,*
- $2f^i$, *if i is odd.*

PROOF. By [**33**, Lemma 3], for any i, one has an inclusion

$$\bar{\mathrm{G}}^iK(\mathrm{SB}(A_L)) \ni \frac{\mathrm{ind}\, A_L}{(i, \mathrm{ind}\, A_L)} f^i$$

where $(\cdot\,, \cdot)$ denotes the greatest common divisor. Since $\mathrm{ind}\, A_L = 2$, the statement follows. □

LEMMA 2.5. *$\bar{\mathrm{G}}^1K(\mathrm{SB}(A)) \ni 2f$.*

PROOF. By the computation [**6**, §2] of the Picard group of a Severi-Brauer variety, one knows that

$$\bar{\mathrm{G}}^1K(\mathrm{SB}(A)) \ni (\exp A)f\ .$$

Since $\exp A = 2$, the statement follows. □

The variety $\overline{\mathrm{SB}(B)}$ is a projective space; denote by g the class of a hyperplane in $\mathrm{G}^1\mathrm{K}(\overline{\mathrm{SB(B)}})$.

COROLLARY 2.6. *For any $i, j \geq 0$, the group $\bar{\mathrm{G}}^{i+j}K(X)$ contains*

- *$f^i \times g^j$, if $i = j = 0$;*
- *$2(f^i \times g^j)$, if*
  i and j are even or
  $i = 0$, $j = 1$ or
  $i = 1$, $j = 0$;
- *$4(f^i \times g^j)$, if*
  $i + j$ is odd or
  $i = j = 1$;
- *$8(f^i \times g^j)$ for any i, j.*

PROOF. The case $i = j = 0$ is evident.

If i and j are even, then $f^i \in \bar{\mathrm{G}}^iK(\mathrm{SB}(A_L))$ and $g^j \in \bar{\mathrm{G}}^jK(\mathrm{SB}(B_L))$ by Lemma 2.4. Thus $f^i \times g^j \in \bar{\mathrm{G}}^{i+j}K(X_L)$ and the transfer argument shows that $2(f^i \times g^j) \in \bar{\mathrm{G}}^{i+j}K(X)$.

By Lemma 2.5, $\bar{\mathrm{G}}^1K(\mathrm{SB}(A)) \ni 2f$ and $\bar{\mathrm{G}}^1K(\mathrm{SB}(B)) \ni 2g$. Therefore $\bar{\mathrm{G}}^1K(X)$ contains $2(f \times 1)$ and $2(1 \times g)$; moreover, $\bar{\mathrm{G}}^2K(X) \ni 4(f \times g)$.

If $i+j$ is odd, then $2(f^i \times g^j) \in \bar{\mathrm{G}}^{i+j}K(X_L)$ by Lemma 2.4 and the transfer argument shows that $4(f^i \times g^j) \in \bar{\mathrm{G}}^{i+j}K(X)$.

Since there exists a field extension of degree 8 simultaneously splitting the algebras A and B, the inclusion $8(f^i \times g^j) \in \bar{\mathrm{G}}^{i+j}K(X)$ holds for any i, j. □

COROLLARY 2.7. *$|\mathrm{G}^*\mathrm{K}(\bar{\mathrm{X}})/\bar{\mathrm{G}}^*\mathrm{K}(\mathrm{X})| \leq 2^{28}$.*

PROOF. Since $\overline{\mathrm{SB}(A)}$ and $\overline{\mathrm{SB}(B)}$ are projective spaces, $\mathrm{G}^*\mathrm{K}(\bar{\mathrm{X}})$ is an abelian group freely generated by $f^i \times g^j$ with $i, j = 0, 1, 2, 3$. By Lemma 2.6, we know that the following multiples of these generators are in $\bar{\mathrm{G}}^*K(X)$:

$$\begin{array}{llll}
2^0 \cdot (f^0 \times g^0), & 2^1 \cdot (f^0 \times g^1), & 2^1 \cdot (f^0 \times g^2), & 2^2 \cdot (f^0 \times g^3), \\
2^1 \cdot (f^1 \times g^0), & 2^2 \cdot (f^1 \times g^1), & 2^2 \cdot (f^1 \times g^2), & 2^3 \cdot (f^1 \times g^3), \\
2^1 \cdot (f^2 \times g^0), & 2^2 \cdot (f^2 \times g^1), & 2^1 \cdot (f^2 \times g^2), & 2^2 \cdot (f^2 \times g^3), \\
2^2 \cdot (f^3 \times g^0), & 2^3 \cdot (f^3 \times g^1), & 2^2 \cdot (f^3 \times g^2), & 2^3 \cdot (f^3 \times g^3).
\end{array}$$

Taking the product of the coefficients, we get 2^{28}. □

COROLLARY 2.8. *$\mathrm{Tors}\,\mathrm{G}^*\mathrm{K}(\mathrm{X}) = 0$.*

PROOF. Follows from Lemma 2.2, Lemma 2.3 and Corollary 2.7. □

Since the Chow group $\mathrm{CH}^2(X)$ is isomorphic to $\mathrm{G}^2\mathrm{K}(\mathrm{X})$ ([**81**, §9]), we also get

COROLLARY 2.9. $\operatorname{Tors}\mathrm{CH}^2(X) = 0$. □

To complete the proof of Theorem 2.1, we apply [**66**, Theorem 4.1 with Remark 4.1]. By that result, there is a monomorphism

$$\frac{H^3(F(X)/F)}{[A]\cdot H^1(F) + [B]\cdot H^1(F)} \hookrightarrow \operatorname{Tors}\mathrm{CH}^2(X)$$

and so, by Corollary 2.9, we are done. □

3. Computation of $H^3(F(X_\psi \times \mathrm{SB}(D))/F)$

THEOREM 3.1. *Let $\psi = \langle -a, -b, ab, d\rangle$ be an anisotropic quadratic form over F with $d \notin F^{*2}$. Let $D = (a,b)\otimes(u,v)\otimes(d,s)$ be a division 3-quaternion algebra over F. Then the group*

$$H^3(F(X_\psi \times \mathrm{SB}(D))/F)$$

is equal to

$$\{e^3(\psi\,\langle\langle k\rangle\rangle) \mid k \in F^* \text{ is such that } \psi\,\langle\langle k\rangle\rangle \in GP_3(F)\} + [D]H^1(F).$$

PROOF. Put $\widehat{F} = F((t))$. Consider two biquaternion algebras $A = (a,b)\otimes(d,t)$ and $B = (d,st)\otimes(u,v)$ over $\widehat{F}$.

Since D is a division algebra, it follows that $\operatorname{ind}(D) = 8$. Therefore

$$\operatorname{ind}((a,b)\otimes(u,v))_{F(\sqrt{d})} = \operatorname{ind} D_{F(\sqrt{d})} = 4\ .$$

Hence $(a,b)_{F(\sqrt{d})}$ and $(u,v)_{F(\sqrt{d})}$ are division $F(\sqrt{d})$-algebras. By Tignol's theorem [**83**, Proposition 2.4], A and B are division $\widehat{F}$-algebras as well.

Since $[A\otimes B] = [D_{\widehat{F}}]$ in $\mathrm{Br}(\widehat{F})$, we have $\operatorname{ind}(A\otimes_{\widehat{F}} B) = \operatorname{ind}(D) = 8$. Since $A_{\widehat{F}(\sqrt{d})}$ and $B_{\widehat{F}(\sqrt{d})}$ are not division algebras, the conditions of Theorem 2.1 hold for the field $\widehat{F}$ and algebras A and B over $\widehat{F}$. Therefore

$$H^3(\widehat{F}(\mathrm{SB}(A)\times\mathrm{SB}(B))/\widehat{F}) = [A]H^1(\widehat{F}) + [B]H^1(\widehat{F}).$$

Let $E = \widehat{F}(\mathrm{SB}(A)\times\mathrm{SB}(B))$. Clearly $[A_E] = [B_E] = 0$. Hence $[D_E] = [A_E] + [B_E] = 0$. Thus $\mathrm{SB}(D)_E$ is a rational variety.

Since $[A_E] = 0$, the Albert form $\langle -a, -b, ab, d, t, -dt\rangle_E$ of the biquaternion algebra $A_E = ((a,b)\otimes(d,t))_E$ is hyperbolic. Hence $\langle -a,-b,ab,d\rangle_E = \langle dt, -d\rangle_E$ in the Witt ring $W(E)$. Therefore ψ_E is isotropic. Hence $(X_\psi)_E$ is a rational variety.

Let $Y = X_\psi \times \mathrm{SB}(D)$. Since $(X_\psi)_E$ and $\mathrm{SB}(D)_E$ are rational, it follows that Y_E is rational. Hence $E(Y)/E$ is a purely transcendental extension. Therefore $H^3(E(Y)/F) = H^3(E/F)$. We have $H^3(F(Y)/F) \subset H^3(E(Y)/F) = H^3(E/F)$.

Let $u \in H^3(F(X_\psi \times \mathrm{SB}(D))/F) = H^3(F(Y)/F)$. To prove the theorem, it is enough to show that u can be written in the form

$$u = e^3(\psi \langle\langle k \rangle\rangle) + [D] \cup (r)$$

with some $k, r \in F^*$.

Since $H^3(F(Y)/F) \subset H^3(E/F)$, it follows that

$$u_{\widehat{F}} \in H^3(E/\widehat{F}) = H^3(\widehat{F}(\mathrm{SB}(A) \times \mathrm{SB}(B))/\widehat{F}) = [A]H^1(\widehat{F}) + [B]H^1(\widehat{F}).$$

Since $[A] + [B] = [D_{\widehat{F}}]$, we have $u_{\widehat{F}} \in [A]H^1(\widehat{F}) + [D_{\widehat{F}}]H^1(\widehat{F})$. Hence there are $\alpha, \beta \in \widehat{F}$ such that $u_{\widehat{F}} = [A] \cup (\alpha) + [D_{\widehat{F}}] \cup (\beta)$. Since $\widehat{F}^*/\widehat{F}^{*2} \simeq F^*/F^{*2} \times \{1, t\}$, we can suppose that $\alpha = kt^i$ and $\beta = rt^j$, where $k, r \in F^*$ and $i, j \in \{0, 1\}$. We have

$$\begin{aligned} u_{\widehat{F}} &= [A] \cup (\alpha) + [D_{\widehat{F}}] \cup (\beta) = \\ &= ((a, b) + (d, t)) \cup (kt^i) + [D] \cup (rt^j) = \\ &= ((a, b, k) + [D] \cup (r)) + (t) \cup (i(a, b) + (d, k(-1)^i) + j[D]). \end{aligned}$$

Using well-known isomorphism $H^i(F((t))) = H^i(F) \oplus H^{i-1}(F)$, we have

$$u = (a, b, k) + [D] \cup (r)$$

and

$$i(a, b) + (d, k(-1)^i) + j[D] = 0.$$

We claim that $j = 0$. Indeed, if $j \neq 0$ then $j = 1$ and hence $[D] = i(a, b) + (d, k(-1)^i)$. Therefore $[D] = [(a, b^i) \otimes (d, k(-1)^i)]$. Thus $\mathrm{ind}(D) \leq 4$, a contradiction.

So $j = 0$, and we have $i(a, b) + (d, k(-1)^i) = 0$. Thereby $i(a, b)_{F(\sqrt{d})} = 0$. Since $(a, b)_{F(\sqrt{d})} \neq 0$, it follows that $i = 0$.

Since $i(a, b) + (d, k(-1)^i) = 0$ and $i = 0$, we have $(d, k) = 0$. Hence $\langle\langle d, k \rangle\rangle = 0$ in $W(F)$. Since $\psi = \langle -a, -b, ab, d \rangle = \langle\langle a, b \rangle\rangle - \langle\langle d \rangle\rangle$, we have

$$\psi \langle\langle k \rangle\rangle = (\langle\langle a, b \rangle\rangle - \langle\langle d \rangle\rangle) \langle\langle k \rangle\rangle = \langle\langle a, b, k \rangle\rangle - \langle\langle d, k \rangle\rangle = \langle\langle a, b, k \rangle\rangle .$$

Therefore $\psi \langle\langle k \rangle\rangle \in GP_3(F)$ and $e^3(\psi \langle\langle k \rangle\rangle) = (a, b, k)$.

Hence the element $u = (a, b, k) + [D] \cup (r)$ belongs to the set

$$\{e^3(\psi \langle\langle k \rangle\rangle) \mid k \text{ is such that } \psi \langle\langle k \rangle\rangle \in GP_3(F)\} + [D]H^1(F).$$

$\square$

PROPOSITION 3.2. *In the notation of Theorem 3.1, let $\xi \in I^2(F)$ be a quadratic form such that $c(\xi) = [D]$. Then for an arbitrary element $\pi \in I^3(F(X_\psi \times \mathrm{SB}(D))/F)$ there are $k_1, k_2 \in F^*$ such that*

$$\pi \equiv \psi \langle\langle k_1 \rangle\rangle + \xi \langle\langle k_2 \rangle\rangle \pmod{I^4(F)}$$

PROOF. Obviously $e^3(\pi) \in H^3(F(X_\psi \times \mathrm{SB}(D))/F)$. It follows from Theorem 3.1 that there are $k_1, k_2 \in F^*$ such that $e^3(\pi) = e^3(\psi \langle\langle k_1 \rangle\rangle) + [D] \cup (k_2)$. Clearly $[D] \cup (k_2) = e^2(\xi) \cup e^1(\langle\langle k_2 \rangle\rangle) = e^3(\xi \langle\langle k_2 \rangle\rangle)$. Hence $e^3(\pi) = e^3(\psi \langle\langle k_1 \rangle\rangle) + e^3(\xi \langle\langle k_2 \rangle\rangle)$. Since $\mathrm{Ker}(e^3 : I^3(F) \to H^3(F)) = I^4(F)$, we have $\pi \equiv \psi \langle\langle k_1 \rangle\rangle + \xi \langle\langle k_2 \rangle\rangle \pmod{I^4(F)}$. $\square$

4. 8-dimensional quadratic forms

Proposition 4.1. *Let ϕ be an 8-dimensional quadratic form with $d_\pm \phi = 1$ and $\operatorname{ind} C(\phi) = 8$. Let ψ be a 4-dimensional quadratic form with $d_\pm \psi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic. Then there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.*

Proof. Changing ψ by a coefficient we can assume that $\psi = \langle -a, -b, ab, d\rangle$ with $d \notin F^{*2}$. Clearly $C_0(\psi) = (a, b)_{F(\sqrt{d})}$. Since $\operatorname{ind} C(\phi) = 8$, there exists a 3-quaternion division algebra D such that $c(\phi) = [D]$.

Lemma 4.2. *There exist $u, v, s \in F^*$ such that $D = (a, b) \otimes (u, v) \otimes (d, s)$.*

Proof. Since $\phi_{F(\psi)}$ is isotropic, it follows that $D_{F(\psi)}$ is not a division algebra. The index reduction formula [**55**] shows that $\operatorname{ind}(C_0(\psi) \otimes_F D) = 2$. Therefore $\operatorname{ind}((a, b) \otimes_F D)_{F(\sqrt{d})} = 2$. Hence there are $u, v, s \in F^*$ such that $[(a, b)\otimes_F D] = [(u, v)\otimes_F (d, s)]$ in $\operatorname{Br}_2(F)$. Hence $[D] = [(a, b)\otimes_F (u, v)\otimes_F (d, s)]$. Since $\deg D = 8$, we have $D = (a, b) \otimes (u, v) \otimes (d, s)$. □

Consider the quadratic form

$$\gamma = \langle -a, -b, ab, d\rangle \perp -s \langle -u, -v, uv, d\rangle .$$

One can verify that $d_\pm \gamma = 1$ and $c(\gamma) = [D]$. Hence $c(\phi) = [D] = c(\gamma)$. Therefore $\phi + \gamma \in I^3(F)$.

Lemma 4.3. *$\phi + \gamma \in I^3(F(X_\psi \times SB(D))/F)$.*

Proof. Let $E = F(X_\psi \times SB(D))$. Since $\phi + \gamma \in I^3(F)$, it is sufficient to verify that ϕ_E and γ_E are hyperbolic. Obviously $[D_E] = 0$, and the form ψ_E is isotropic. Since $c(\phi_E) = c(\gamma_E) = [D_E] = 0$ and $\dim \phi = \dim \gamma = 8$, we have $\phi_E, \gamma_E \in GP_3(E)$. Hence it is sufficient to prove that ϕ_E and γ_E are isotropic. Since $\phi_{F(\psi)}$ and ψ_E are isotropic, ϕ_E is isotropic as well. Since $\psi \subset \gamma$ and ψ_E is isotropic, we see that γ_E is isotropic. □

Now we can complete the proof of Proposition 4.1. By Proposition 3.2 and Lemma 4.3, there exist $k_1, k_2 \in F^*$ such that

$$\phi + \gamma \equiv \psi \langle\langle k_1\rangle\rangle + \phi \langle\langle k_2\rangle\rangle \pmod{I^4(F)}.$$

Let $\rho = -s \langle -u, -v, uv, d\rangle$. We have $\gamma = \psi + \rho$. Hence

$$\phi + \psi + \rho \equiv \psi - k_1\psi + \phi - k_2\phi \pmod{I^4(F)}.$$

Thus $k_1\psi + \rho \equiv -k_2\phi \pmod{I^4(F)}$. Hence $\psi + k_1\rho \equiv -k_1k_2\phi \pmod{I^4(F)}$. We finish the proof by setting $\phi^* = \psi \perp k_1\rho$. □

Theorem 4.4. *Let ϕ be an 8-dimensional quadratic form with $d_\pm \phi = 1$ and $\operatorname{ind} C(\phi) = 8$. Let ψ be a quadratic form of dimension ≥ 4 such that $\psi \notin GP_2(F)$. The following conditions are equivalent:*

- *$\phi_{F(\psi)}$ is isotropic;*
- *there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.*

PROOF. The case $\dim\psi = 4$ is Proposition 4.1. In the case $\dim\psi \neq 4$ the statement was proved by Laghribi in [**47**] and [**45**]. $\square$

5. Main theorem

THEOREM 5.1. *Let ϕ be a virtual Albert form (i.e. a 6-dimensional quadratic form with $d_\pm \phi \notin F^{*2}$ and $\operatorname{ind}(C_0(\phi)) = 4$). Let ψ be a 4-dimensional quadratic form such that $d_\pm \psi \neq 1$. The following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic;*
(2) *ψ is similar to a subform in ϕ.*

PROOF. (1)⇒(2). Let $d = d_\pm \phi$. Consider the 8-dimensional quadratic form $\xi = \phi_{\widehat{F}} \perp t\,\langle\langle d\rangle\rangle$ over the field $\widehat{F} = F((t))$. Since $d_\pm \phi \notin F^{*2}$ and $\operatorname{ind}(C_0(\phi)) = 4$, one has $\operatorname{ind}(C(\xi)) = 8$. Clearly $\xi_{\widehat{F}(\psi)}$ is isotropic. It follows from Proposition 4.1 that there exists a quadratic form ξ^* over $\widehat{F}$ such that ξ and ξ^* are half-neighbors and $\psi_{\widehat{F}} \subset \xi^*$.

LEMMA 5.2. *ξ^* is similar to ξ.*

PROOF. Since ξ and ξ^* are half-neighbors, there exists $k \in \widehat{F}$ such that $\xi \equiv k\xi^* \pmod{I^4(\widehat{F})}$. By Springer's theorem one can write $k\xi^*$ in the form $k\xi^* = \mu_0 \perp t\mu_1$, where quadratic forms μ_0 and μ_1 are defined over F. We have

$$\phi \perp t\,\langle\langle d\rangle\rangle = \xi \equiv k\xi^* = \mu_0 \perp t\mu_1 \pmod{I^4(\widehat{F})}.$$

Hence $\phi \equiv \mu_0 \pmod{I^3(F)}$, $\langle\langle d\rangle\rangle \equiv \mu_1 \pmod{I^3(F)}$, and $\phi + \langle\langle d\rangle\rangle \equiv \mu_0 + \mu_1 \pmod{I^4(F)}$. Therefore $\operatorname{ind} C_0(\mu_0) = \operatorname{ind} C_0(\phi) \geq 4$. Hence $\dim\mu_0 \geq 6$. Therefore $\dim\mu_1 \leq 2$. By Arason–Pfister Hauptsatz the condition $\langle\langle d\rangle\rangle \equiv \mu_1 \pmod{I^3(F)}$ implies that $\mu_1 = \langle\langle d\rangle\rangle$. Hence $\phi \equiv \mu_0 \pmod{I^4(F)}$. Applying Arason–Pfister Hauptsatz once again, we have $\phi = \mu_0$. Therefore $\xi = k\xi^*$. $\square$

Now we return to the proof of Theorem 5.1. Since ψ is similar to a subform in ξ^*, and ξ^* is similar to ξ, it follows that ψ is similar to a subform in $\xi = \phi \perp t\,\langle\langle d\rangle\rangle$. Thus ψ is similar to a subform of ϕ by the following obvious observation.

LEMMA 5.3. *Let ψ, γ_0 and γ_1 be anisotropic quadratic forms over F. The following conditions are equivalent:*

a) *$\psi_{F((t))}$ is similar to a subform in $\gamma_0 \perp t\gamma_1$,*
b) *ψ is similar either to a subform in γ_0 or to a subform in γ_1.* $\square$

Thus we have proved that condition (1) of Theorem 5.1 implies condition (2). On the other hand, condition (2) obviously implies condition (1). The proof of Theorem 5.1 is complete. $\square$

THEOREM 5.4. *Let ϕ be a virtual Albert form and let $\psi \notin GP_2(F)$. The quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform in ϕ.*

PROOF. This theorem was proved by A. Laghribi in the following cases ([**44**], [**45**]):

- $\dim\psi \neq 4$;

- $\dim\psi = 4$, $d_\pm\,\psi \neq d_\pm\,\phi$.

Thus we can suppose that $\dim\psi = 4$. To complete the proof it is sufficient to apply Theorem 5.1. □

In the special case which was not covered by the results of A. Laghribi, we get the following

COROLLARY 5.5. *Let ϕ be a virtual Albert form and ψ be a 4-dimensional form such that $d_\pm\,\psi = d_\pm\,\phi$. Then $\phi_{F(\psi)}$ is anisotropic.*

PROOF. If ψ is similar to a subform in ϕ, then ϕ is isotropic, a contradiction. Therefore ψ is not similar to a subform in ϕ. By Theorem 5.1, it means that $\phi_{F(\psi)}$ is anisotropic. □

Together with results described in §1, Theorem 5.4 gives rise to the following

COROLLARY 5.6. *Let ϕ be a 6-dimensional quadratic form with $\operatorname{ind} C_0(\phi) = 4$. In the case when $\psi \notin GP_2(F)$, the quadratic form $\phi_{F(\psi)}$ is isotropic if and only if ψ is similar to a subform of ϕ. In the case when $\psi \in GP_2(F)$, the form $\phi_{F(\psi)}$ is isotropic if and only if a 3-dimensional subform of ψ is similar to a subform of ϕ.* □

CHAPTER 4

Isotropy of 6-dimensional quadratic forms over function fields of quadrics

Let F be a field of characteristic different from 2 and ϕ be an anisotropic 6-dimensional quadratic form over F. We study the last open cases in the problem of describing the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$.

Results of this Chapter are obtained in joint work with Oleg Izhboldin.

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic.

More precisely, one studies the question whether the isotropy of ϕ over $F(\psi)$ is standard in a sense. In this chapter we will use the following definition of "*standard isotropy*":

Definition. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. We say that the isotropy of $\phi_{F(\psi)}$ is *standard*, if at least one of the following conditions holds:

- ψ is similar to a subform in ϕ;
- there exists a subform $\phi_0 \subset \phi$ with the following two properties:
  - the form ϕ_0 is a Pfister neighbor,
  - the form $(\phi_0)_{F(\psi)}$ is isotropic.

Otherwise, we say that the isotropy is *non-standard*.

In the case when $\dim \phi \leq 5$, the isotropy of $\phi_{F(\psi)}$ is always standard ([**75**], [**15**]). For 6-dimensional quadratic forms, the problem was studied by A. S. Merkurjev ([**54**]), D. Leep ([**49**]), D. W. Hoffmann ([**16**]), A. Laghribi ([**44**], [**45**]), and in Chapter 3. It was proved that isotropy of a 6-dimensional quadratic form ϕ over the function field of a quadratic form ψ is always standard except (possibly) the following case (see [**44**] and Chapter 3):

- $\dim \psi = 4$, $d_{\pm} \psi \neq 1$, $d_{\pm} \phi \neq 1$, and $\operatorname{ind} C_0(\phi) = 2$.

In this Chapter we study the isotropy of $\phi_{F(\psi)}$ for quadratic forms ϕ and ψ satisfying these conditions (with $\dim \phi = 6$).

Note that the condition $\operatorname{ind} C_0(\phi) = 2$ implies that there exist $a, b, c, d \in F^*$ such that ϕ is similar to the form $\langle\langle a, b\rangle\rangle \perp -c \langle\langle d\rangle\rangle$. Since ϕ can be replaced by a similar form, we can assume that $\phi = \langle\langle a, b\rangle\rangle \perp -c \langle\langle d\rangle\rangle$. Note that in this case $[C_0(\phi)] = [(a,b)_{F(\sqrt{d})}] = [C_0(\rho)]$, where ρ is defined as follows: $\rho = \langle -a, -b, ab, d\rangle$.

Since $\dim\psi = 4$, there exist $u, v, \delta \in F^*$ such that ψ is similar to the quadratic form $\langle -u, -v, uv, \delta\rangle$. Since $d_\pm \psi \neq 1$, we have $\delta \notin F^{*2}$. Thus our main problem is reduced to the following

Question. *Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ be anisotropic quadratic forms over F with $d, \delta \notin F^{*2}$. Suppose that $\phi_{F(\psi)}$ is isotropic. Is the isotropy standard?*

This question naturally splits into the following four cases:

(1) $d = \delta$ as elements of F^*/F^{*2},
(2) $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$,
(3) $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 2$,
(4) $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 4$.

We prove that in the cases (1), (2), and (4) the isotropy of $\phi_{F(\psi)}$ is always standard (see Theorem 8.5, Propositions 8.6 and 8.7). This statement gives rise to the following

Theorem. *Let ϕ be an anisotropic quadratic form of dimension ≤ 6 and ψ be such that $\phi_{F(\psi)}$ is isotropic. Then isotropy is standard except (possibly) the following case:* $\dim\phi = 6$, $\dim\psi = 4$, $1 \neq d_\pm \phi \neq d_\pm \psi \neq 1$, *and* $\operatorname{ind} C_0(\phi) = 2 = \operatorname{ind} C_0(\phi) \otimes_F C_0(\psi)$.

The proof of this theorem is based on a computation of the second Chow group for certain homogeneous varieties. Namely, we show that the question on the standard isotropy can be reduced to a question on the group $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$, where $\rho = \langle -a, -b, ab, d\rangle$ and X_ψ and X_ρ are the projective quadrics corresponding to ψ and ρ. In the cases (1), (2) and (4), we compute the group $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$ completely (see Theorems 5.7, 5.1, 5.8, and Lemma 7.7):

Theorem. *Let ψ and ρ be 4-dimensional quadratic forms. Then the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$ *is zero or isomorphic to* $\mathbb{Z}/2\mathbb{Z}$. *Moreover,*

- *if* $\det\psi = \det\rho$ *or if* $\operatorname{ind} C_0(\psi) \otimes_F C_0(\rho) = 4$, *then the group* $\operatorname{CH}^2(X_\psi \times X_\rho)$ *is torsion-free;*
- *in the case* $\operatorname{ind} C_0(\psi) \otimes_F C_0(\rho) = 1$, *the group* $\operatorname{CH}^2(X_\psi \times X_\rho)$ *is torsion-free if and only if ρ and ψ contain similar 3-dimensional subforms.*

It will be shown later (in Chapter 5) that in the case (3), i.e. in the case where $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 2$, the answer to our main question (whether the isotropy of $\phi_{F(\psi)}$ is always standard) is negative. Here we show that this question is equivalent to the following one (see §9): *is the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$ *trivial for any* 4*-dimensional quadratic forms ψ and ρ such that* $1 \neq \det\psi \neq \det\delta \neq 1$ *and* $\operatorname{ind} C_0(\psi) \otimes C_0(\rho) = 2$*?*

1. Terminology, notation, and backgrounds

Quadratic forms. By $\phi \perp \psi$, $\phi \simeq \psi$, and $[\phi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F. To simplify notation, we write $\phi_1+\phi_2$ instead of $[\phi_1]+[\phi_2]$. For a quadratic form ϕ of dimension n, we set $d_\pm \phi = (-1)^{n(n-1)/2} \det \phi \in F^*/F^{*2}$. The maximal ideal of $W(F)$ generated by the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form

$$\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to an n-fold Pfister form we denote by $GP_n(F)$. For any field extension L/F, we put $\phi_L = \phi \otimes_F L$, $W(L/F) = \ker(W(F) \to W(L))$, and $I^n(L/F) = \ker(I^n(F) \to I^n(L))$.

For a quadratic form ϕ and a field extension L/F, we denote by $D_L(\phi)$ the set of the non-zero values of the quadratic form ϕ_L.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ and $F(\phi, \psi) = F(X_\phi \times X_\psi)$ for quadratic forms ϕ and ψ of dimensions ≥ 3.

Algebras. We consider only finite-dimensional F-algebras.

For a simple F-algebra A, by $\mathrm{ind}(A)$ we denote the Schur index of A. For an algebra B of the form $B = A \times \cdots \times A$ with simple A, we set $\mathrm{ind}\, B = \mathrm{ind}\, A$.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ. By $C_0(\phi)$ we denote the even part of $C(\phi)$. For any collection $\rho_1, \dots, \rho_m$ of quadratic forms, the algebra $C_0(\rho_1) \otimes_F \cdots \otimes_F C_0(\rho_m)$ is of the form $A \times \cdots \times A$ with simple A. Therefore, we get a well-defined positive integer $\mathrm{ind}\, C_0(\rho_1) \otimes_F \cdots \otimes_F C_0(\rho_m)$.

If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Hence we get a well-defined element $[C(\phi)]$ in the 2-part $\mathrm{Br}_2(F)$ of the Brauer group $\mathrm{Br}(F)$ which we denote by $c(\phi)$.

Cohomology groups. By $H^*(F)$ we denote the graded ring of Galois cohomology $H^*(F, \mathbb{Z}/2\mathbb{Z}) \stackrel{\mathrm{def}}{=} H^*(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2\mathbb{Z})$. For any field extension L/F, we set $H^*(L/F) = \ker\big(H^*(F) \to H^*(L)\big)$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \mathrm{Br}_2(F)$. So any element $a \in F^*$ gives rise to an element of $H^1(F)$ which we denote by (a). The cup product $(a_1) \cup \cdots \cup (a_n)$ we denote by $(a_1, \dots, a_n)$.

For $n = 0, 1, 2$, there is a homomorphism $e^n : I^n(F) \to H^n(F)$ defined as follows: $e^0(\phi) = \dim \phi \pmod 2$, $e^1(\phi) = d_\pm \phi$, and $e^2(\phi) = c(\phi)$. Moreover there exists a homomorphism $e^3 : I^3(F) \to H^3(F)$ which is uniquely determined by the condition $e^n(\langle\langle a_1, a_2, a_3\rangle\rangle) = (a_1, a_2, a_3)$ (see [**5**]). The homomorphism e^n is surjective and $\ker e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [**53**], [**62**], and [**71**]).

We also work with the cohomology groups $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$, $(i = 0, 1, 2)$, defined by B. Kahn (see [**29**]). For any field extension L/F, we set

$$H^*(L/F, \mathbb{Q}/\mathbb{Z}(i)) = \ker\big(H^*(F, \mathbb{Q}/\mathbb{Z}(i)) \to H^*(L, \mathbb{Q}/\mathbb{Z}(i))\big) .$$

For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with the 2-part of $H^n(F, \mathbb{Q}/\mathbb{Z}(n-1))$.

K-theory and Chow groups. For a smooth algebraic F-variety X, its Grothendieck ring is denoted by $K(X)$. This ring is supplied with the filtration by codimension of support (which respects the multiplication). For a ring (or a group) with filtration A, we denote by G^*A the adjoint graded ring (resp., the adjoint graded group). There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $\mathrm{G}^*K(X)$, its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([**81**, §9]).

2. The group $H^3(F(\rho_1, \rho_2)/F)$

The main result of this § (in view of our further purposes) is Corollary 2.13.

By a *homogeneous variety* we always mean a *projective* homogeneous variety.

Proposition 2.1 ([**67**]). *For any homogeneous F-variety X, there is a natural (in X and in F) epimorphism*

$$\tau_X \colon H^3\big(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)\big) \twoheadrightarrow \operatorname{Tors}\mathrm{CH}^2(X) .$$

Proposition 2.2. *For any homogeneous varieties $X_1, \dots, X_m$ over F, the quotient*

$$\frac{H^3\big(F(X_1 \times \dots \times X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)}{H^3\big(F(X_1)/F, \mathbb{Q}/\mathbb{Z}(2)\big) + \dots + H^3\big(F(X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)}$$

is canonically isomorphic to

$$\frac{\operatorname{Tors}\mathrm{CH}^2(X_1 \times \dots \times X_m)}{pr_1^* \operatorname{Tors}\mathrm{CH}^2(X_1) + \dots + pr_m^* \operatorname{Tors}\mathrm{CH}^2(X_m)}$$

where $pr_1^*, \dots, pr_m^*$ are the pull-backs with respect to the projections $pr_1, \dots, pr_m$ of the product $X_1 \times \dots \times X_m$ on $X_1, \dots, X_m$.

Proof. Set $X = X_1 \times \dots \times X_m$. The homomorphism τ_X of Proposition 2.1 induces an epimorphism

$$f \colon \frac{H^3\big(F(X_1 \times \dots \times X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)}{H^3\big(F(X_1)/F, \mathbb{Q}/\mathbb{Z}(2)\big) + \dots + H^3\big(F(X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)} \twoheadrightarrow$$
$$\twoheadrightarrow \frac{\operatorname{Tors}\mathrm{CH}^2(X_1 \times \dots \times X_m)}{pr_1^* \operatorname{Tors}\mathrm{CH}^2(X_1) + \dots + pr_m^* \operatorname{Tors}\mathrm{CH}^2(X_m)}$$

with the kernel $\ker f = \ker\tau_X/(\ker\tau_{X_1} + \dots + \ker\tau_{X_m})$.

The kernel of τ_X is computed (for any homogeneous X) in [**57**]: let A be the separable F-algebra associated with X ([**57**, §2]) and denote by E the center of A; then $\ker \tau_X = \{N_{E/F}(\bar{x} \cup [A]) \mid \text{with } x \in E^*\}$ where $[A]$ is the class of A in the Brauer group $\mathrm{Br}(E) = H^2(E, \mathbb{Q}/\mathbb{Z}(1))$, $\bar{x}$ is the class of $x \in E^*$ in $H^1(E, \mathbb{Q}/\mathbb{Z}(1))$, $\bar{x} \cup [A] \in H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is the cup-product and $N_{E/F}$ is the norm map.

Denote by $A_1, \dots, A_m$ the separable algebras associated with $X_1, \dots, X_m$ respectively. Then $A = A_1 \times \cdots \times A_m$ and $E = E_1 \times \cdots \times E_m$. Thus for any $x \in E^*$

$$N_{E/F}(\bar{x} \cup [A]) = N_{E_1/F}(\bar{x}_1 \cup [A_1]) + \cdots + N_{E_m/F}(\bar{x}_m \cup [A_m]) ,$$

where x_i is the E_i-component of x, what proves that $\ker f = 0$. $\square$

Corollary 2.3. *Let $X_1, \dots, X_m$ and $X'_1, \dots, X'_m$ be homogeneous varieties such that X_i is stably birationally equivalent to X'_i for $i = 1, \dots, m$. The quotient*

$$\frac{\operatorname{Tors} \mathrm{CH}^2(X_1 \times \cdots \times X_m)}{pr_1^* \operatorname{Tors} \mathrm{CH}^2(X_1) + \cdots + pr_m^* \operatorname{Tors} \mathrm{CH}^2(X_m)}$$

is isomorphic to the quotient

$$\frac{\operatorname{Tors} \mathrm{CH}^2(X'_1 \times \cdots \times X'_m)}{pr_1^* \operatorname{Tors} \mathrm{CH}^2(X'_1) + \cdots + pr_m^* \operatorname{Tors} \mathrm{CH}^2(X'_m)} .$$

$\square$

Lemma 2.4. *For any homogeneous variety X of dimension ≤ 2, the group $\mathrm{CH}^2(X)$ is torsion-free.*

Proof. Since X is a homogeneous variety, $K(X)$ is a torsion-free group ([**65**]). Since $\dim X \leq 2$, the term $K(X)^{(3)}$ of the topological filtration is trivial. Hence $K(X)^{(2/3)}$ is a torsion-free group. By [**81**, §9], $\mathrm{CH}^2(X) \simeq K(X)^{(2/3)}$. Hence $\operatorname{Tors} \mathrm{CH}^2(X) = 0$. $\square$

Corollary 2.5. *Under the conditions of Corollary 2.3 suppose additionally that the varieties $X_1, \dots, X_m; X'_1, \dots, X'_m$ have the dimensions ≤ 2. Then there is an isomorphism*

$$\operatorname{Tors} \mathrm{CH}^2(X_1 \times \cdots \times X_m) \simeq \operatorname{Tors} \mathrm{CH}^2(X'_1 \times \cdots \times X'_m).$$

Proof. Obvious in view of Corollary 2.3 and Lemma 2.4. $\square$

Lemma 2.6. *Let X_1 and X_2 be homogeneous varieties. If the variety $(X_2)_{F(X_1)}$ has a rational point, then*

$$H^3(F(X_1 \times X_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X_1)/F, \mathbb{Q}/\mathbb{Z}(2)) .$$

Proof. Since the homogeneous variety $(X_2)_{F(X_1)}$ has a rational point, it is rational, i.e. the field extension $F(X_1 \times X_2)/F(X_1)$ is purely transcendental. $\square$

Corollary 2.7. *Let X_1 and X_2 be projective quadrics of the dimensions ≤ 2. If the quadric $(X_2)_{F(X_1)}$ is isotropic (e.g., if X_2 is isotropic or if $X_1 \simeq X_2$) then* $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$.

Proof. Follows from Lemma 2.6, Proposition 2.2 and Lemma 2.4. □

Lemma 2.8. *For any quadratic form ρ of dimension ≥ 3, we have*

$$2H^3(F(\rho)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$

In other words, $H^3(F(\rho)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\rho)/F)$.

Proof. Let $u \in H^3(F(\rho)/F, \mathbb{Q}/\mathbb{Z}(2))$. There exists a field extension L/F such that ρ_L is isotropic and $[L : F] \leq 2$. Since ρ_L is isotropic, $u_L = 0$. Using transfer homomorphism, we have $[L : F] \cdot u = 0$. Hence $2u = 0$. □

Corollary 2.9. *For any quadratic form ρ of dimension ≥ 3 the homomorphism $H^3(F(\rho)/F) \to \operatorname{Tors}\operatorname{CH}^2(X_\rho)$, induced by the epimorphism of Proposition 2.1, is surjective. In particular, $2\operatorname{Tors}\operatorname{CH}^2(X_\rho) = 0$.* □

Lemma 2.10. *Let ρ_1 and ρ_2 be quadratic form of dimension ≥ 3. Then*

$$2H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$

In other words, $H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\rho_1, \rho_2)/F)$.

Proof. Let ρ_1' and ρ_2' be 3-dimensional subforms in ρ_1 and ρ_2 respectively. Clearly $H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3(F(\rho_1', \rho_2')/F, \mathbb{Q}/\mathbb{Z}(2))$. Thus, replacing ρ_1 by ρ_1' and ρ_2 by ρ_2', one can reduce the proof to the case $\dim \rho_1 = \dim \rho_2 = 3$. In this case, $\dim X_{\rho_1} \times X_{\rho_2} = 2$; therefore $\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} \times X_{\rho_2}) = 0$ (Lemma 2.4). For $i = 1, 2$, the conic X_{ρ_i} is isomorphic to the Severi-Brauer variety of the algebra $C_i \overset{\text{def}}{=} C_0(\rho_i)$. Applying [**66**, Thm. 41], we obtain

$$H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = [C_1] \cup H^1(F, \mathbb{Q}/\mathbb{Z}(1)) + [C_2] \cup H^1(F, \mathbb{Q}/\mathbb{Z}(1)) \,.$$

Since $2[C_1] = 2[C_2] = 0$ in the group $H^2(F, \mathbb{Q}/\mathbb{Z}(1)) = \operatorname{Br}(F)$, it follows that $2H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. □

Corollary 2.11. *Let ρ_1 and ρ_2 be quadratic forms of dimension ≥ 3. Then the homomorphism*

$$H^3(F(\rho_1, \rho_2)/F) \to \operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} \times X_{\rho_2})$$

induced by the epimorphism of Proposition 2.1, is surjective. In particular, $2\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} \times X_{\rho_2}) = 0$. □

Corollary 2.12. *For any quadratic forms ρ_1 and ρ_2 of dimension ≥ 3, there is a natural isomorphism*

$$\frac{H^3(F(\rho_1, \rho_2)/F)}{H^3(F(\rho_1)/F) + H^3(F(\rho_2)/F)} \simeq \frac{\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} \times X_{\rho_2})}{pr_1^* \operatorname{Tors}\operatorname{CH}^2(X_{\rho_1}) + pr_2^* \operatorname{Tors}\operatorname{CH}^2(X_{\rho_2})} \,.$$

Proof. Follows from Proposition 2.2 and Lemmas 2.8 and 2.10. □

Corollary 2.13. *For any quadratic forms ρ_1 and ρ_2 with $3 \leq \dim \rho_i \leq 4$ $(i = 1, 2)$, there is a natural isomorphism*

$$\frac{H^3(F(\rho_1, \rho_2)/F)}{H^3(F(\rho_1)/F) + H^3(F(\rho_2)/F)} \simeq \operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} \times X_{\rho_2}).$$

Proof. Follows from Corollary 2.12 and Lemma 2.4. □

3. The Grothendieck group of a quadric

In this §, ρ is an $(n+2)$-dimensional quadratic form over F (where $n \geq 1$), V is the vector space of definition of ρ, $\mathbb{P}$ is the projective space of the vector space dual to V, and $X = X_\rho \subset \mathbb{P}$ is the n-dimensional projective quadric determined by ρ.

We are mainly interested in the case when $n = 2$, i.e. when X is a surface.

The even Clifford algebra $C_0(\rho)$ of the form ρ is denoted in this § by C. Let $\mathcal{U}$ be Swan's sheaf on X [**82**, §6]. It is an $C \otimes_F \mathcal{O}_X$-module locally free as $\mathcal{O}_X$-module (note that the algebra C is canonically self-opposite; thus it is not necessary to distinguish between left and right action of C).

We denote by h the class of a general hyperplane section of X, i.e. the pull-back of the class of a hyperplane with respect to the imbedding $X \hookrightarrow \mathbb{P}$. The subring of $K(X)$ generated by h is denoted by H; it coincides with the image of the pull-back homomorphism $K(\mathbb{P}) \to K(X)$. Some further evident assertions on H are collected in

Lemma 3.1. *The abelian group H is freely generated by $1, h, \ldots, h^n$. The topological filtration on $K(X)$ induces on H the filtration by powers of h, i.e. for every $0 \leq r \leq n$, the term $H^{(r)}$ is generated by all h^j with $r \leq j \leq n$. In particular, the adjoint graded group G^*H is torsion-free.* □

In the case when X splits (i.e. when ρ is hyperbolic) and $n = 2$, we refer as to a *line class* (resp., *point class*) to the class in $K(X)$ of a line (resp., of a closed rational point) lying on X.

Lemma 3.2 ([**31**]). *Suppose that X splits and* $\dim X = 2$.

1. *For any two different lines in X, their classes in $K(X)$ coincide if and only if the lines have no intersection. There are exactly two different line classes in $K(X)$.*
2. *The classes in $K(X)$ of any two closed rational points of X coincide, i.e. there is only one point class in $K(X)$.*
3. *Denote by l and l' the different line classes and by p the point class in $K(X)$. The abelian group $K(X)$ is freely generated by the elements $1, l, l', p$.*
4. *The second term $K(X)^{(2)}$ of the topological filtration on $K(X)$ is generated by p; the term $K(X)^{(1)}$ is generated by l, l', p.*
5. *The multiplication in $K(X)$ is determined by the formulas $l^2 = 0 = (l')^2$ and $l \cdot l' = p$.*
6. $h = l + l' - p$.

□

In the case when the quadric X is arbitrary (not necessary of dimension 2, not necessary split), we dispose of the following information on $K(X)$:

Lemma 3.3. 1. *The group $K(X)$ is torsion-free and, for any field extension E/F, the restriction homomorphism $K(X) \to K(X_E)$ is injective.*

2. *The class* $[\mathcal{U}(n)] \in K(X)$ *of the* n *times twisted Swan's sheaf equals*

$$2^n + 2^{n-1}h + \cdots + 2h^{n-1} + h^n \ .$$

3. *The homomorphism* $\mathfrak{u}\colon K(C) \to K(X)$ *given by the functor of taking tensor product* $\mathcal{U}(n) \otimes_C (-)$ *induces an epimorphism* $K(C) \twoheadrightarrow K(X)/H$.
4. *If* C *is a skewfield, then* $K(X) = H$.
5. *For any autoisometry* ξ *of the quadratic form* ρ, *the diagram*

$$\begin{array}{ccc} K(C) & \xrightarrow{\mathfrak{u}} & K(X) \\ \uparrow & & \uparrow \\ K(C) & \xrightarrow{\mathfrak{u}} & K(X) \end{array}$$

commutes, where the vertical maps are induced by the automorphisms of C *and of* X *given by* ξ.

PROOF. 1. Follows from [**82**, Theorem 9.1].
2. See [**32**, Lemma 3.6].
3. According to [**82**, Theorem 9.1], the functor $\mathcal{U} \otimes_C (-)$ induces an epimorphism $K(C) \to K(X)/H$. Since for any $r \in \mathbb{Z}$ (and in particular for $r = n$) the twisting by r gives an automorphism of $K(X)/H$, the functor $\mathcal{U}(n) \otimes_C (-)$ induces an epimorphism as well.
4. If C is a skewfield, then the image of this epimorphism is generated by $[\mathcal{U}(n)]$. Since $[\mathcal{U}(n)] \in H$ by Item 2, it follows that $K(X) = H$.
5. It is evident in view of the way the sheaf $\mathcal{U}$ is constructed (see [**82**, §6]). $\square$

LEMMA 3.4 ([**48**]). *The* F*-algebra* $C = C_0(\rho)$ *has the dimension* $2^{n+1} = 2^{\dim \rho - 1}$ *over* F. *Its isomorphism class depends only on the similarity class of* ρ. *Moreover,*

- *if* n *is odd, then* C *is a central simple* F*-algebra;*
- *if* n *is even, then* $C \simeq C_0(\rho') \otimes_F F(\sqrt{d_\pm \rho})$ *where* ρ' *is an arbitrary* 1*-codimensional subform of* ρ.

In particular, if ρ is an even-dimensional form of trivial discriminant, the algebra C is the direct product of two isomorphic central simple algebras; any automorphism of C should either interchange or stabilize the factors.

LEMMA 3.5. *Suppose that* $\dim \rho$ *is even and* $d_\pm \rho$ *is trivial. Let* ξ *be an autoisometry of the quadratic space* (V, ρ) *having the determinant* -1. *Then the automorphism of* C *induced by* ξ *interchanges the simple components of* C.

PROOF. Since $d_\pm \rho$ is trivial, there exists a basis $v_0, \ldots, v_{n+1}$ of V such that

$$(v_0 \cdots v_{n+1})^2 = 1 \in C \ .$$

Since $\xi(v_0) \cdots \xi(v_{n+1}) = (\det \xi) \cdot (v_0 \cdots v_{n+1}) = -v_0 \cdots v_{n+1}$, the automorphism of C induced by ξ interchanges the elements

$$e = (1 + v_0 \cdots v_{n+1})/2 \quad \text{and} \quad e' = (1 - v_0 \cdots v_{n+1})/2 \ .$$

Since e and e' are orthogonal idempotents, they lie in different simple components of C. Therefore, the components of C are interchanged. $\square$

Comparing Lemma 3.2 with Lemma 3.3 in the situation of a split quadric surface X, we get the following computation (note that here C is isomorphic to $M_2(F) \times M_2(F)$ and thus there exist exactly two, up to isomorphisms, simple C-modules; their classes are free generators of $K(C)$):

LEMMA 3.6. *Suppose that X splits and $\dim X = 2$. There exist simple C-modules M and M' such that $u = 1 + l$ and $u' = 1 + l'$ where*

$$u \stackrel{\text{def}}{=} \mathfrak{u}([M]) = [\mathcal{U}(2) \otimes_C M], \quad u' \stackrel{\text{def}}{=} \mathfrak{u}([M']) = [\mathcal{U}(2) \otimes_C M'] \in K(X) .$$

PROOF. Take as M an arbitrary simple C-module and denote by M' a (determined uniquely up to an isomorphism) simple C-module non-isomorphic to M. Since by Lemma 3.2 the elements $1, l, l', p$ generate $K(X)$, we have

$$u = a + bl + b'l' + cp$$

for certain $a, b, b', c \in \mathbb{Z}$. Now we are going to show that

$$u' = a + b'l + bl' + cp .$$

Let ξ be an autoisometry of the quadratic space (V, ρ) having determinant -1. By Lemma 3.5, the induced by ξ automorphism of $K(C)$ interchanges $[M]$ and $[M']$. Thus, by Item 5 of Lemma 3.3, the induced by ξ automorphism of $K(X)$ interchanges u and u'.

Since ρ splits, there exist 2-dimensional totally isotropic subspaces W and W' of V with 1-dimensional intersection and an autoisometry ξ of (V, ρ) having the determinant -1 interchanging W and W'. The line classes in $K(X)$ determined by W and W' are different (Item 1 of Lemma 3.2); therefore they coincide with l and l' (or vice versa: with l' and l).

Thus, we have found an automorphism of $K(X)$ interchanging u with u' and l with l' while leaving untouched 1 (of course) and p (since all the point classes coincide). Thereby, $u' = a + b'l + bl' + cp$.

Since $2([M] + [M']) = [C] \in K(C)$, we have: $2(u + u') = [\mathcal{U}(2)]$, and so, $2(u + u') = 4 + 2h + h^2$ by Item 2 of Lemma 3.3. Since $K(X)$ is torsion-free, the last equality can be divided by 2. Replacing h by $l + l' - p$ and h^2 by $(l + l' - p)^2 = 2p$ (see Lemma 3.2), we obtain that $u + u' = 2 + l + l'$. From the other hand, $u + u' = 2a + (b + b')l + (b' + b)l' + 2c$; therefore $a = 1$, $b + b' = 1$ and $c = 0$.

We have proved that

$$u = 1 + bl + (1 - b)l' \quad \text{and} \quad u' = 1 + (1 - b)l + bl'$$

for certain $b \in \mathbb{Z}$. It remains to show that $b = 1$ or $b = 0$.

It follows from Item 3 of Lemma 3.3 that the elements $u, u', 1, h, h^2$ generate the group $K(X)$. Since $h^2 = 2p$ and $h = u + u' - 2 - p$, the elements $u, u', 1, p$ also generate $K(X)$. So, the quotient $K(X)/(\mathbb{Z} \cdot 1 + \mathbb{Z} \cdot p)$ which is according to

Item 6 of Lemma 3.2 freely generated by l, l' is also generated by u, u'. Thus, the $\mathbb{Z}$-matrix

$$\begin{pmatrix} b & 1-b \\ 1-b & b \end{pmatrix}$$

is invertible, i.e. its determinant is ± 1. Hence, $b = 1$ or $b = 0$. $\square$

4. The Grothendieck group of a product of quadrics

In this and in the next §, we work with two quadratic forms ρ_1 and ρ_2 of the dimensions ≥ 3. We use the notation of the previous § amplified by the index 1 or 2. So, for $i = 1, 2$, we have ρ_i, n_i (we are mainly interested in the case when $n_1 = 2 = n_2$), V_i, $\mathbb{P}_i$, X_i, C_i, $\mathcal{U}_i$,h_i, H_i, l_i, l'_i and p_i. We set $n = (n_1, n_2)$, $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$, $X = X_1 \times X_2$, and $C = C_1 \otimes_F C_2$.

For any $x_1 \in K(X_1)$ and $x_2 \in K(X_2)$, we denote by $x_1 \boxtimes x_2$ the product $pr_1^*(x_1) \cdot pr_2^*(x_2) \in K(X)$ where pr_1 and pr_2 are the projections of $X_1 \times X_2$ on X_1 and X_2 respectively.

Denote by H the image of the pull-back homomorphism $K(\mathbb{P}) \to K(X)$.

Lemma 4.1. *One has: $H = H_1 \boxtimes H_2 \subset K(X)$. The abelian group H is freely generated by all $h_1^{j_1} \boxtimes h_2^{j_2}$ with $0 \leq j_1 \leq n_1$ and $0 \leq j_2 \leq n_2$. Moreover, the filtration on H induced by the topological filtration on $K(X)$ looks as follows: for any $0 \leq r \leq n_1 + n_2$, the term $H^{(r)}$ is generated by all $h_1^{j_1} \boxtimes h_2^{j_2}$ with $j_1 + j_2 \geq r$. In particular, the adjoint graded group $\mathrm{G}^* H$ is torsion-free.* $\square$

The following lemma is also evident; together with Lemma 3.2, it gives a complete description of the ring with filtration $K(X)$ in the split situation.

Lemma 4.2. *If X_1 and X_2 split then the map $K(X_1) \otimes K(X_2) \to K(X)$, $x_1 \otimes x_2 \mapsto x_1 \boxtimes x_2$ is an isomorphism of rings with filtrations.* $\square$

For an $\mathcal{O}_{X_1}$-module $\mathcal{F}_1$ and an $\mathcal{O}_{X_2}$-module $\mathcal{F}_2$, we denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the tensor product $pr_1^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} pr_2^*(\mathcal{F}_2)$. The sheaf $\mathcal{U} = \mathcal{U}_1 \boxtimes \mathcal{U}_2$ has for $i = 1, 2$ the structures of a C_i-module commuting with each other. Thus, it is a C-module. Set $\mathcal{U}(n) = \mathcal{U}_1(n_1) \boxtimes \mathcal{U}_2(n_2)$. It is also a C-module. The functor of taking the tensor product $\mathcal{U}(n) \otimes_C (-)$ determines a homomorphism $\mathfrak{u}\colon K(C) \to K(X)$.

Lemma 4.3.
1. *The group $K(X)$ is torsion-free and, for any field extension E/F, the restriction homomorphism $K(X) \to K(X_E)$ is injective.*
2. *The homomorphism $\mathfrak{u}\colon K(C) \to K(X)$, defined right above, induces an epimorphism $K(C) \twoheadrightarrow K(X)/\big(K(X_1) \boxtimes K(X_2)\big)$.*
3. *If C is a skewfield, then $K(X) = H$.*

Proof. 1. This statement is valid for any homogeneous variety X ([**65**]). 2. The isomorphism $K_*(X_1) \simeq K_*(F)^{\oplus n_1} \oplus K_*(C_1)$ of [**82**, Theorem 9.1] remains bijective after changing the base F to any field extension, i.e. for any field extension E/F, the homomorphism $K_*(\operatorname{Spec} E \times X_1) \to K_*(\operatorname{Spec} E)^{\oplus n_1} \oplus K_*(\operatorname{Spec} E, C_1)$ is bijective. Therefore, for any F-variety Y, the defined in the similar way homomorphism $K_*(Y \times X_1) \to K_*(Y)^{\oplus n_1} \oplus K_*(Y, C_1)$ is bijective (compare to the proof of Proposition 4.1 of [**69**, §7]). In particular, $K(X) \simeq K(X_2)^{\oplus n_1} \oplus K(X_2, C_1)$. Computing $K(X_2)$ and $K(X_2, C_1)$ using [**82**, Theorem 9.1] once again, one gets

$$K(X) \simeq K(F)^{\oplus n_1 n_2} \oplus K(C_1)^{\oplus n_2} \oplus K(C_2)^{\oplus n_1} \oplus K(C) \; .$$

The image of $K(F)^{\oplus n_1 n_2} \oplus K(C_1)^{\oplus n_2} \oplus K(C_2)^{\oplus n_1}$ in $K(X)$ is contained in $K(X_1) \boxtimes K(X_2)$ and the homomorphism $K(C) \to K(X)$ is induced by the functor of taking tensor product $\mathcal{U} \otimes_C (-)$. Thus $\mathfrak{u}\colon K(C) \to K(X)$ is modulo $K(X_1) \boxtimes K(X_2)$ an epimorphism.
3. If the algebra C is a skewfield then the image of $\mathfrak{u}$ is contained in H; moreover, the algebras C_1 and C_2 are skewfields as well and thus $K(X_i) = H_i$ for $i = 1, 2$. □

Corollary 4.4. *If C is a skewfield, then $\mathrm{G}^* K(X)$ is torsion-free. In particular,* $\operatorname{Tors} \mathrm{CH}^2(X) = 0$.

Proof. If C is a skewfield, then $K(X) = H$ by Item 3 of Lemma 4.3. Consequently, $\operatorname{Tors} \mathrm{G}^* K(X) = \operatorname{Tors} \mathrm{G}^* H = 0$ (see Lemma 4.1). □

5. CH^2 of a product of quadrics

The notation used in this § is introduced in the beginning of the previous one. However, each of the quadratic forms ρ_1 and ρ_2 is now supposed to have the dimension 3 or 4. So, each of X_i is either a quadric surface or a conic. We are mainly interested in the case when X_1 and X_2 are surfaces.

Theorem 5.1. *Suppose that* $\dim \rho_1 = 4 = \dim \rho_2$*, i.e. that X_1 and X_2 are surfaces. If* $\det \rho_1 = \det \rho_2$*, then* $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$.

Proof. If one of the quadratic forms is isotropic, then $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$ by Corollary 2.7. In the rest of the proof we assume that ρ_1 and ρ_2 are anisotropic.

As a next step, we are going to consider the case when $\det \rho_1 = \det \rho_2 = 1$.

Lemma 5.2. *Any projective quadric surface defined by a quadratic form of determinant 1 is stably birationally equivalent to a conic.*

Proof. Suppose that we are given a quadric determined by a 4-dimensional quadratic form ρ with $\det \rho = 1$. Take the conic determined by an arbitrary 3-dimensional subform $\rho' \subset \rho$. Since ρ' becomes isotropic over $F(\rho)$ and vice versa, ρ becomes isotropic over $F(\rho')$, the quadrics given by ρ' and ρ are stably birationally equivalent. □

Suppose that $\det \rho_1 = \det \rho_2 = 1$ and choose some conics X_1' and X_2' stably birationally equivalent to X_1 and X_2 respectively. Applying Corollary 2.5, we get an isomorphism of $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2)$ onto the group $\operatorname{Tors} \mathrm{CH}^2(X_1' \times X_2')$ which is trivial by Lemma 2.4.

Therefore, we may assume that $d \neq 1$ where $d = \det \rho_1 = \det \rho_2$.

As a next step of the proof of Theorem, we consider the case when the F-algebras $C_1 \overset{\text{def}}{=} C_0(\rho_1)$ and $C_2 \overset{\text{def}}{=} C_0(\rho_2)$ are isomorphic. In this case, the forms ρ_1 and ρ_2 becomes similar over the field $F(\sqrt{d})$. Thus by a theorem of Wadsworth ([**86**, Theorem 7]), they are already similar over F. Therefore the quadrics X_1 and X_2 are isomorphic and consequently $\operatorname{Tors}\operatorname{CH}^2(X) = 0$ by Corollary 2.7.

It remains only to consider the situation when the forms ρ_1 and ρ_2 are anisotropic, $d \neq 1$ and $C_1 \not\simeq C_2$. Set $c = \operatorname{ind} C$. We have: $c = 2$ or $c = 4$.

Fix a separable closure $\bar{F}$ of the field F. For the algebra $C_{\bar{F}}$, the variety $X_{\bar{F}}$, etc. we shall use the notation $\bar{C}$, $\bar{X}$, etc.

For $i = 1, 2$, denote by M_i and M_i' the (determined uniquely up to an isomorphism and up to the order) non-isomorphic simple $\bar{C}_i$-modules. There are exactly 4 different isomorphism classes of simple C-modules; they are represented by $M_1 \boxtimes M_2$ ($M_1 \boxtimes M_2$ is by definition the tensor product $M_1 \otimes M_2$ considered as $\bar{C}$-module in the natural way), $M_1 \boxtimes M_2'$, $M_1' \boxtimes M_2$, and $M_1' \boxtimes M_2'$. Denote by m_i the class of M_i and by m_i' the class of M_i' in $K(\bar{C}_i)$. The abelian group $K(\bar{C})$ is freely generated by $m_1 \boxtimes m_2$ ($m_1 \boxtimes m_2$ is defined as follows: for $i = 1, 2$, one takes the image of $m_i \in K(C_i)$ with respect to the map $K(C_i) \to K(C)$ and than takes the product of the images in the ring $K(C)$), $m_1 \boxtimes m_2'$, $m_1' \boxtimes m_2$, and $m_1' \boxtimes m_2'$. We identify $K(C)$ with a subgroup in $K(\bar{C})$ via the restriction map $K(C) \hookrightarrow K(\bar{C})$.

Lemma 5.3. *The subgroup $K(C) \subset K(\bar{C})$ is generated by*

$$c \cdot (m_1 \boxtimes m_2 + m_1' \boxtimes m_2') \quad \textit{and} \quad c \cdot (m_1 \boxtimes m_2' + m_1' \boxtimes m_2)\ .$$

Proof. Denote by L the quadratic extension $F(\sqrt{d})$ of the field F, where $d = \det \rho_1 = \det \rho_2$. The algebra C_L is the direct product of 4 copies of a central simple L-algebra of index c. Evidently, the subgroup $K(C_L)$ of $K(\bar{C})$ is freely generated by $c \cdot m_1 \boxtimes m_2$, $c \cdot m_1 \boxtimes m_2'$, $c \cdot m_1' \boxtimes m_2$, and $c \cdot m_1' \boxtimes m_2'$.

Now we are going to determine $K(C)$ as a subgroup in $K(C_L)$. Computing the norm $N_{L/F} \colon K(C_L) \to K(C)$, we get:

$$x \overset{\text{def}}{=} N_{L/F}(c \cdot m_1 \boxtimes m_2) = c \cdot (m_1 \boxtimes m_2 + m_1' \boxtimes m_2')\ ;$$
$$x' \overset{\text{def}}{=} N_{L/F}(c \cdot m_1 \boxtimes m_2') = c \cdot (m_1 \boxtimes m_2' + m_1' \boxtimes m_2)\ .$$

Thus, the elements x and x' are in $K(C)$. Note that:

- x and x' can be included in a system of free generators of the free abelian group $K(C_L)$ (e.g. x, x', $c \cdot m_1 \boxtimes m_2$, and $c \cdot m_1 \boxtimes m_2'$);
- $K(C)$ is a free abelian group of rank 2 (because the algebra C is the direct product of two copies of a simple algebra, since for $i = 1, 2$ one has: $C_i = C_i' \otimes_F L$ for a central simple F-algebra C_i');
- $K(C)$ is a subgroup of $K(C_L)$ containing x and x'.

Consequently, $K(C)$ is generated by x and x'. □

We identify $K(X)$ with a subgroup in $K(\bar{X})$ via the restriction map (which is injective by Item 1 of Lemma 4.3). For $i = 1, 2$, let l_i, l'_i be the different line classes and p_i the point class in $K(\bar{X}_i)$ (see Lemma 3.2).

COROLLARY 5.4. *The group $K(X)$ is generated modulo H by $c \cdot (l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2)$ and $c \cdot p_1 \boxtimes p_2$.*

PROOF. According to Item 2 of Lemma 4.3, the map $\mathfrak{u}\colon K(C) \to K(X)/H$ is surjective. By Lemma 5.3, the group $K(C)$ is generated by

$$c \cdot (m_1 \boxtimes m_2 + m'_1 \boxtimes m'_2) \quad \text{and} \quad c \cdot (m_1 \boxtimes m'_2 + m'_1 \boxtimes m_2)\ .$$

Applying Lemma 3.6, we can compute the images of these generators in $K(X)$: up to the order, they are

$$\begin{gathered} c \cdot \big((1 + l_1) \boxtimes (1 + l_2) + (1 + l'_1) \boxtimes (1 + l'_2)\big) \quad \text{and} \\ c \cdot \big((1 + l_1) \boxtimes (1 + l'_2) + (1 + l'_1) \boxtimes (1 + l_2)\big)\ . \end{gathered}$$

One can modify the first expression as follows (the formulas of Lemma 3.2 are in use):

$$\begin{aligned} & c \cdot \big((1 + l_1) \boxtimes (1 + l_2) + (1 + l'_1) \boxtimes (1 + l'_2)\big) = \\ & \qquad = c \cdot \big(2 + (l_1 + l'_1) \boxtimes 1 + 1 \boxtimes (l_2 + l'_2) + l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2\big) = \\ & \qquad = c \cdot \big(2 + (h_1 + h_1^2/2) \boxtimes 1 + 1 \boxtimes (h_2 + h_2^2/2) + l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2\big) \equiv \\ & \qquad\qquad \equiv c \cdot (l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \pmod{H} \end{aligned}$$

(note that c is divisible by 2). The analogous modification can be made for the second expression as well. Thus, the group $K(X)$ is generated modulo H by $c \cdot (l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2)$ and $c \cdot (l_1 \boxtimes l'_2 + l'_1 \boxtimes l_2)$. Taking the sum of these generators, we get:

$$\begin{aligned} & c \cdot (l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) + c \cdot (l_1 \boxtimes l'_2 + l'_1 \boxtimes l_2) = \\ & \qquad = c \cdot (l_1 + l'_1) \boxtimes (l_2 + l'_2) = c \cdot (h_1 + h_1^2/2) \boxtimes (h_2 + h_2^2/2) \equiv \\ & \qquad\qquad \equiv c \cdot (h_1^2/2) \boxtimes (h_2^2/2) = c \cdot p_1 \boxtimes p_2 \end{aligned}$$

(where the congruence is modulo H). □

LEMMA 5.5. 1. *$c \cdot (l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in K(X)^{(2)}$;*
2. *$c \cdot p_1 \boxtimes p_2 \in K(X)^{(3)}$;*
3. *for any $0 \neq r \in \mathbb{Z}$, the set $r \cdot c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) + H$ has no intersection with $K(X)^{(3)}$.*

PROOF. 1. It is evident that $c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in K(\bar{X})^{(2)}$. Since $K(X)^{(2)} = K(\bar{X})^{(2)} \cap K(X)$ (see e.g. [**74**, Lemme 6.3, (i)]), we are done.

2. If we multiply the element $c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in K(X)^{(2)}$ by the element $h_1 \boxtimes 1 \in K(X)^{(1)}$, we get:

$$\begin{aligned} K(X)^{(3)} \ni c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \cdot (h_1 \boxtimes 1) = \\ = c(p_1 \boxtimes l_2 + p_1 \boxtimes l'_2) = c \cdot p_1 \boxtimes (h_2 + p_2) = \\ = c \cdot p_1 \boxtimes h_2 + c \cdot p_1 \boxtimes p_2 \ . \end{aligned}$$

Since $c \cdot p_1 \boxtimes h_2 \in H^{(3)} \in K(X)^{(3)}$, it follows that $c \cdot p_1 \boxtimes p_2 \in K(X)^{(3)}$.
3. By Lemmas 3.2 and 4.2, the abelian group $K(\bar{X})$ is freely generated by the products $x_1 \boxtimes x_2$ where x_i is one of the elements $1, l_i, l'_i, p_i$; moreover, the term $K(\bar{X})^{(3)}$ of the filtration is generated by $l_1 \boxtimes p_2$, $l'_1 \boxtimes p_2$, $p_1 \boxtimes l_2$, $p_1 \boxtimes l'_2$ and $p_1 \boxtimes p_2$. In particular, $4K(\bar{X})^{(3)} \subset H$.

Suppose that, for certain $0 \neq r \in \mathbb{Z}$, the intersection of $r \cdot c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) + H$ with $K(X)^{(3)}$ is non-empty. Then $4r \cdot c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in H$, a contradiction. □

Corollary 5.6. *Let us supply the quotient $K(X)/H$ with the filtration induced from $K(X)$. Then* $\operatorname{Tors} \mathrm{G}^2(K(X)/H) = 0$.

Proof. By Corollary 5.4 and Lemma 5.5, $\mathrm{G}^2(K(X)/H)$ is an infinite cyclic group (generated by the residue of $c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2)$). □

To finish the proof of Theorem 5.1, consider the exact sequence

$$0 \to \mathrm{G}^2 H \to \mathrm{G}^2 K(X) \to \mathrm{G}^2(K(X)/H) \to 0 \ .$$

The left-hand side term is torsion-free by Lemma 4.1 while the right-hand side term is torsion-free by Corollary 5.6. Consequently, the middle term is a torsion-free group as well. □

Theorem 5.7. *The order of the group* $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2)$ *is at most* 2.

Proof. Since $2 \operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$ by Corollary 2.11, it suffices to show that the torsion in $\mathrm{CH}^2(X_1 \times X_2)$ is a cyclic group.

By Corollary 2.7, it suffices to consider only the case when the both quadratic forms ρ_1 and ρ_2 are anisotropic.

Set as usual $X = X_1 \times X_2$, $C_i = C_0(\rho_i)$ and $C = C_1 \otimes_F C_2$. Suppose that the algebra C is simple. Then $K(C)$ is a cyclic group and therefore, by Item 2 of Lemma 4.3, the quotient $K(X)/H$ is cyclic as well. Moreover, C_1 and C_2 are division algebras (since they are simple and the quadratic forms are anisotropic) and therefore $K(X_i) = H_i$ for $i = 1, 2$ by Item 4 of Lemma 3.3. Supplying $K(X)/H$ with the filtration induced from $K(X)$, we get an exact sequence of the adjoint graded groups

$$0 \to \mathrm{G}^* H \to \mathrm{G}^* K(X) \to \mathrm{G}^*(K(X)/H) \to 0 \ .$$

Take any $r \geq 0$. Since $\mathrm{G}^r H$ is torsion-free (Lemma 4.1), $\operatorname{Tors} \mathrm{G}^r K(X)$ is mapped injectively into $G^r(K(X)/H)$. Since $K(X)/H$ is cyclic, $G^r(K(X)/H)$ is cyclic as well and thus so is also $\operatorname{Tors} \mathrm{G}^r K(X)$. In particular, the group $\operatorname{Tors} \mathrm{CH}^2(X) \simeq \operatorname{Tors} \mathrm{G}^2 K(X)$ is cyclic.

Now suppose that C is *not* simple. Then

either: $\dim X_1 = 2 = \dim X_2$ and $\det X_1 = \det X_2$,

or: for $i = 1$ or for $i = 2$, one has: $\dim X_i = 2$ and $\det X_i = 1$.

In the first case, the torsion in $\mathrm{CH}^2(X)$ is 0 by Theorem 5.1. In the second case, we replace the surface X_i by a stably birationally equivalent conic (see Lemma 5.2 and Corollary 2.5). □

THEOREM 5.8. *If* $\operatorname{ind} C_0(\rho_1) \otimes_F C_0(\rho_2) = 4$*, then* $\operatorname{Tors}\mathrm{CH}^2(X_1 \times X_2) = 0$.

PROOF. We set $C = C_0(\rho_1) \otimes_F C_0(\rho_2)$ and suppose that $\operatorname{ind} C = 4$.

If C is a simple algebra, then it is a skewfield and we are done by Corollary 4.4.

If C is *not* simple, then

either: $\dim X_1 = 2 = \dim X_2$ and $\det \rho_1 = \det \rho_2$,

or: for $i = 1$ or for $i = 2$, one has: $\dim X_i = 2$ and $\det X_i = 1$.

In the first case, the torsion in $\mathrm{CH}^2(X_1 \times X_2)$ is 0 by Theorem 5.1. In the second case, we replace the surface X_i by a stably birationally equivalent conic (see Lemma 5.2 and Corollary 2.5). □

THEOREM 5.9. *Suppose that* $\dim \rho_1 = 4$, $\det \rho_1 \neq 1$ *and that for a certain 3-dimensional subform* ρ_1' *of* ρ_1 *one has:*

$$\operatorname{ind} C_0(\rho_1) \otimes_F C_0(\rho_2) = \operatorname{ind} C_0(\rho_1') \otimes_F C_0(\rho_2)\ .$$

Then $\operatorname{Tors}\mathrm{CH}^2(X_1 \times X_2) = 0$.

PROOF. Applying the same arguments as above, we may assume that

- the forms ρ_1 and ρ_2 are anisotropic and
- one of the following alternative conditions holds:
  - the dimension of ρ_2 equals 3 or
  - the dimension of ρ_2 is 4 and $\det \rho_1 \neq \det \rho_2 \neq 1$.

We are going to show that, under the assumptions made, $\operatorname{Tors}\mathrm{G}^2 K(X_1 \times X_2) = 0$.

The algebra C is now simple; it has the index 1, 2, or 4. Set $c = \operatorname{ind} C$. The group $K(C)$ is generated by $(c/4)\cdot[C]$ where $[C] \in K(C)$ is the class of C.

Consider the case when $\dim \rho_2 = 4$.

It follows from Item 2 of Lemma 4.3 that $K(X)$ is generated modulo H by the element $(c/4)[\mathcal{U}(2,2)]$. Applying Item 2 of Lemma 3.3, one computes that $[\mathcal{U}(2,2)] = (4 + 2h_1 + h_1^2) \boxtimes (4 + 2h_2 + h_2^2) \in K(X)$. Thus, $K(X)$ is generated modulo H also by $x \overset{\text{def}}{=} (c/4)(2\cdot h_1 \boxtimes h_2^2 + 2\cdot h_1^2 \boxtimes h_2 + h_1^2 \boxtimes h_2^2)$. Since we have the exact sequence

$$0 \to \mathrm{G}^* H \to \mathrm{G}^* K(X) \to \mathrm{G}^*(K(X)/H) \to 0$$

with torsion-free $\mathrm{G}^* H$, it would suffice to show that $x \in K(X)^{(3)}$.

Consider the conic X_1' determined by ρ_1' and denote by $\mathcal{U}_1'$ Swan's sheaf on X_1'. The product $\mathcal{U}_1'(1) \boxtimes \mathcal{U}_2(2)$ of twisted Swan's sheaves has a structure of module over $C' \overset{\text{def}}{=} C_1' \otimes C_2$; its class in $K(X')$, where $X' \overset{\text{def}}{=} X_1' \times X_2$ is equal to $(2+h_1')\boxtimes(4+2h_2+h_2^2)$ where h_1' is the class in $K(X_1')$ of a hyperplane section of X_1'. Since $\operatorname{ind} C' = \operatorname{ind} C = c$, the latter product can be divided by $(4/c)$ in $K(X')$, i.e.

$$K(X') \ni x' \stackrel{\text{def}}{=} (c/4)(2\cdot 1\boxtimes h_2^2 + 2\cdot h_1'\boxtimes h_2 + h_1'\boxtimes h_2^2)\ .$$

Since $4x' \in K(X')^{(2)}$ and the group $\mathrm{G}^1K(X') = \mathrm{CH}^1(X')$ is torsion-free (see e.g. [**74**, Lemme 6.3, (i)]), it follows that $x' \in K(X')^{(2)}$. Since the image of x' with respect to the push-forward given by the closed imbedding $X' \hookrightarrow X$ coincides with x and $\operatorname{codim}_X X' = 1$, the element x is in $K(X)^{(3)}$.

Now suppose that $\dim\rho_2 = 3$.

If $c = 1$, then the quadric $(X_2)_{F(X_1)}$ is isotropic and therefore $\operatorname{Tors}\mathrm{CH}^2(X) = 0$ by Corollary 2.7. Thus we may assume that c is divisible by 2.

The group $K(X)$ is now generated modulo H by $(c/4)[\mathcal{U}(2,1)]$ and

$$[\mathcal{U}(2,1)] = (4+2h_1+h_1^2)\boxtimes(2+h_2) \in K(X)\ .$$

Thus, $K(X)$ is generated modulo H also by $x \stackrel{\text{def}}{=} (c/4)(h_1^2\boxtimes h_2)$ and it suffices to show that $x\in K(X)^{(3)}$.

The class in $K(X')$ of the product $\mathcal{U}_1'(1)\boxtimes\mathcal{U}_2(1)$ of twisted Swan's sheaves is equal this time to $(2+h_1')\boxtimes(2+h_2)$ and can be divided by $(4/c)$ in $K(X')$, i.e.

$$K(X') \ni x' \stackrel{\text{def}}{=} (c/4)(h_1'\boxtimes h_2)\ .$$

Since $x' \in K(X')^{(2)}$ and the image of x' with respect to the push-forward given by the closed imbedding $X' \hookrightarrow X$ coincides with x, the element x is in $K(X)^{(3)}$. □

COROLLARY 5.10. *If ρ_1 and ρ_2 contain similar 3-dimensional subforms, then* $\operatorname{Tors}\mathrm{CH}^2(X_1\times X_2) = 0$.

PROOF. If $\dim\rho_1 = 3$ or if $\det\rho_1 = 1$, then the quadric $(X_2)_{F(X_1)}$ is isotropic and so we are done by Corollary 2.7.

Therefore, we may assume that $\dim\rho_1 = 4$ and $\det\rho_1 \neq 1$. These are the first two conditions of Theorem 5.9. We state that also the last condition of Theorem 5.9 is satisfied. Indeed, denote by $\rho_1'\subset\rho_1$ and $\rho_2'\subset\rho_2$ the similar 3-dimensional subforms. According to Lemma 3.4, the F-algebras $C_0(\rho_1')$ and $C_0(\rho_2')$ are isomorphic and $C_0(\rho_i) = C_0(\rho_i')_{F(\sqrt{\det\rho_i})}$ for $i = 1,2$. Therefore, $\operatorname{ind} C_0(\rho_1)\otimes_F C_0(\rho_2) = 1 = \operatorname{ind} C_0(\rho_1')\otimes_F C_0(\rho_2)\ .$ □

6. The group $I^3(F(\rho,\psi)/F)$

The following assertion is obvious:

LEMMA 6.1. *Let $\rho = \langle -a,-b,ab,d\rangle$ be a quadratic form over F. For any $k\in F^*$ the following conditions are equivalent.*

(1) $k \in D_F(\langle\langle d\rangle\rangle)$;
(2) $\langle\langle a,b,k\rangle\rangle = \rho\,\langle\langle k\rangle\rangle$;
(3) $\rho\,\langle\langle k\rangle\rangle \in P_3(F)$. □

LEMMA 6.2. *Let $\rho = \langle -a, -b, ab, d\rangle$ be a quadratic form over F. Then*
1. $P_3(F(\rho)/F) = \{\langle\langle a, b, k\rangle\rangle \mid k \in D_F(\langle\langle d\rangle\rangle)\}$,
2. $H^3(F(\rho)/F) = \{(a, b, k) \mid k \in D_F(\langle\langle d\rangle\rangle)\}$.

PROOF. 1. See [**15**, Lemma 3.1].
2. Let $\rho_0 = \langle -a, -b, ab\rangle$. Clearly $H^3(F(\rho)/F) \subset H^3(F(\rho_0)/F)$. It follows from [**5**, Beweis vom Satz 5.6] that $H^3(F(\rho_0)/F) = (a, b) \cup H^1(F)$. Hence any element $u \in H^3(F(\rho)/F)$ has the form (a, b, x) where $x \in F^*$. Since $(a, b, x) \in H^3(F(\rho)/F)$, the Pfister form $\langle\langle a, b, x\rangle\rangle_{F(\rho)}$ is hyperbolic. It follows from the first assertion that there exists $k \in D_F(\langle\langle d\rangle\rangle)$ such that $\langle\langle a, b, x\rangle\rangle = \langle\langle a, b, k\rangle\rangle$. Hence $u = (a, b, x) = (a, b, k)$. □

COROLLARY 6.3. *Let $\rho_1, \dots, \rho_m$ be 4-dimensional quadratic forms over F. Then for a quadratic form ϕ the following conditions are equivalent:*
(1) $\phi \in I^3(F(\rho_1)/F) + \cdots + I^3(F(\rho_m)/F) + I^4(F)$;
(2) $\phi \in P_3(F(\rho_1)/F) + \cdots + P_3(F(\rho_m)/F) + I^4(F)$;
(3) $\phi \in I^3(F)$ *and* $e^3(\phi) \in H^3(F(\rho_1)/F) + \cdots + H^3(F(\rho_m)/F)$.

PROOF. (2)⇒(1)⇒(3). Obvious.
(3)⇒(2). Follows from Lemma 6.2. □

COROLLARY 6.4. *Let $\rho_1, \dots, \rho_m$ be 4-dimensional quadratic forms such that $H^3(F(\rho_1, \dots, \rho_m)/F) = H^3(F(\rho_1)/F) + \cdots + H^3(F(\rho_m)/F)$. Then*

$$I^3(F(\rho_1, \dots, \rho_m)/F) \subset I^3(F(\rho_1)/F) + \cdots + I^3(F(\rho_m)/F) + I^4(F).$$

□

COROLLARY 6.5. *Let $\rho = \langle -a, -b, ab, d\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ be quadratic forms over F. Then for any $\pi \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$ there exist $k_1, k_2 \in F^*$ with the following properties:*
1) $\langle\langle a, b, k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$ *and* $\langle\langle u, v, k_2\rangle\rangle = \psi\,\langle\langle k_2\rangle\rangle$;
2) $\pi \equiv \langle\langle a, b, k_1\rangle\rangle + \langle\langle u, v, k_2\rangle\rangle \pmod{I^4(F)}$.

PROOF. By Corollary 6.3, we have $\pi \in P_3(F(\rho)/F) + P_3(F(\psi)/F) + I^4(F)$. Hence there exist $\pi_1 \in P_3(F(\rho)/F)$ and $\pi_2 \in P_3(F(\psi)/F)$ such that

$$\pi \equiv \pi_1 + \pi_2 \pmod{I^4(F)}\ .$$

By Lemma 6.2, there exist $k_1, k_2 \in F^*$ such that $\pi_1 = \langle\langle a, b, k_1\rangle\rangle$ and $\pi_2 = \langle\langle u, v, k_2\rangle\rangle$. Finally, Lemma 6.1 shows that $\langle\langle a, b, k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$, $\langle\langle u, v, k_2\rangle\rangle = \psi\,\langle\langle k_2\rangle\rangle$. □

7. The case of index 1

In this §, we study the group $H^3(F(\rho, \psi)/F)$ in the case where ρ, ψ are 4-dimensional quadratic forms with non-trivial discriminants and $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 1$. In the case $d_\pm\,\rho = d_\pm\,\psi$ we obviously have $C_0(\rho) \simeq C_0(\psi)$. Hence ρ is similar to ψ (see [**86**, Theorem 7]) and hence the group $H^3(F(\rho, \psi)/F)$ coincides with $H^3(F(\rho)/F)$. So it is sufficient to study only the case where $d_\pm\,\rho \neq d_\pm\,\psi$.

Replacing ρ and ψ by similar forms, we can rewrite our conditions as follows:

1) $\rho = \langle -a, -b, ab, d\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ with $a, b, d, u, v, \delta \in F^*$;
2) d, δ, and $d\delta$ are not squares in F^*;
3) $\operatorname{ind}((a, b) \otimes_F (u, v))_{F(\sqrt{d},\sqrt{\delta})} = 1$.

During this section we will suppose that the conditions 1)–3) hold.

We define the set $\Gamma(\rho, \psi)$ as

$$\{\gamma \in I^3(F) \mid \text{there exist } l_1, l_2 \in F^* \text{ such that } \gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle\}.$$

LEMMA 7.1. *The set $\Gamma(\rho, \psi)$ is not empty.*

PROOF. Since $\operatorname{ind}((a, b) \otimes_F (u, v))_{F(\sqrt{d},\sqrt{\delta})} = 1$, there exist $s, r \in F^*$ such that $(a, b) \otimes (u, v) = (d, s) \otimes (\delta, r)$. Set $l_1 = \delta s$, $l_2 = -\delta r$. It is sufficient to verify that $\gamma \overset{\text{def}}{=} l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \in I^3(F)$. We have

$$\begin{aligned}\gamma &= \delta s\rho - \delta r\psi + \langle 1, -d\delta\rangle = \delta(s\rho - r\psi + \langle \delta, -d\rangle) = \\ &= \delta(s(\langle\langle a, b\rangle\rangle - \langle\langle d\rangle\rangle) - r(\langle\langle u, v\rangle\rangle - \langle\langle \delta\rangle\rangle) + (\langle\langle d\rangle\rangle - \langle\langle \delta\rangle\rangle)) = \\ &= \delta(s\,\langle\langle a, b\rangle\rangle - r\,\langle\langle u, v\rangle\rangle + \langle\langle d, s\rangle\rangle - \langle\langle \delta, r\rangle\rangle)\ .\end{aligned}$$

Therefore $\gamma \in I^2(F)$ and $c(\gamma) = (a, b) + (u, v) + (d, s) + (\delta, r) = 0$. Hence $\gamma \in I^3(F)$. □

LEMMA 7.2. $\Gamma(\rho, \psi) \subset I^3(F(\rho, \psi)/F)$.

PROOF. Let $\gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \in \Gamma(\rho, \psi)$. We have $\dim(\gamma_{F(\psi,\rho)})_{an} \le \dim(\rho_{F(\rho)})_{an} + \dim(\psi_{F(\psi)})_{an} + \dim\langle\langle \delta d\rangle\rangle \le 2+2+2 = 6 < 8$. Since $\gamma \in I^3(F)$, the Arason-Pfister Hauptsatz shows that $\gamma_{F(\psi,\rho)}$ is hyperbolic. □

COROLLARY 7.3. *For any $\gamma \in \Gamma(\rho, \psi)$, we have $e^3(\gamma) \in H^3(F(\rho, \psi)/F)$.* □

LEMMA 7.4. *Let $l, k \in F^*$ and let τ be a quadratic form such that $\tau\,\langle\langle k\rangle\rangle \in I^3(F)$. Then $l\tau - \langle\langle k\rangle\rangle\,\tau \equiv lk\tau \pmod{I^4(F)}$.*

PROOF. $l\tau - \langle\langle k\rangle\rangle\,\tau - lk\tau = -\langle\langle l\rangle\rangle\,\langle\langle k\rangle\rangle\,\tau \in \langle\langle l\rangle\rangle\, I^3(F) \subset I^4(F)$. □

LEMMA 7.5. *Let $\gamma \in \Gamma(\rho, \psi)$, $\pi_1 \in P_3(F(\rho)/F)$ and $\pi_2 \in P_3(F(\psi)/F)$. Then there exists $\gamma' \in \Gamma(\rho, \psi)$ such that $\gamma - \pi_1 - \pi_2 \equiv \gamma' \pmod{I^4(F)}$. Moreover, $\gamma + \pi_1 + \pi_2 \equiv \gamma' \pmod{I^4(F)}$.*

PROOF. Let $l_1, l_2 \in F^*$ be such that $\gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle$. By Lemmas 6.1 and 6.2, there exist $k_1, k_2 \in F^*$ such that $\pi_1 = \rho\,\langle\langle k_1\rangle\rangle$, $\pi_2 = \psi\,\langle\langle k_2\rangle\rangle$. By Lemma 7.4, we have

$$\begin{aligned} l_1\rho - \pi_1 &= l_1\rho - \langle\langle k_1\rangle\rangle\,\rho \equiv l_1k_1\rho \pmod{I^4(F)},\\ l_2\psi - \pi_2 &= l_2\psi - \langle\langle k_2\rangle\rangle\,\psi \equiv l_2k_2\psi \pmod{I^4(F)}.\end{aligned}$$

Hence $\gamma - \pi_1 - \pi_2 \equiv l_1k_1\rho + l_2k_2\psi + \langle\langle d\delta\rangle\rangle \pmod{I^4(F)}$. Setting $\gamma' = l_1k_1\rho + l_2k_2\psi + \langle\langle d\delta\rangle\rangle$, we get the required equation $\gamma - \pi_1 - \pi_2 \equiv \gamma' \pmod{I^4(F)}$. The second equation $\gamma + \pi_1 + \pi_2 \equiv \gamma' \pmod{I^4(F)}$ is obvious in view of the congruence $\pi_i \equiv -\pi_i \pmod{I^4(F)}$ (for $i = 1, 2$). □

Corollary 7.6. $\Gamma(\rho,\psi)+I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F)=\Gamma(\rho,\psi)+I^4(F)$.

Proof. It is an obvious consequence of Corollary 6.3 and Lemma 7.5 $\square$

Lemma 7.7. *The following conditions are equivalent:*

(1) $I^3(F(\rho,\psi)/F)\subset I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F)$;
(2) $\Gamma(\rho,\psi)\subset I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F)$;
(3) *there exists* $\gamma\in\Gamma(\rho,\psi)$ *such that* $\gamma\in I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F)$;
(4) $\Gamma(\rho,\psi)$ *contains a hyperbolic form, i.e.* $0\in\Gamma(\rho,\psi)$;
(5) *the quadratic forms* ψ *and* ρ *contain similar* 3*-dimensional subforms;*
(6) $\operatorname{Tors}\operatorname{CH}^2(X_\rho\times X_\psi)=0$;
(7) $H^3(F(\rho,\psi)/F)=H^3(F(\rho)/F)+H^3(F(\psi)/F)$.

Proof. (1)⇒(2). Obvious in view of Lemma 7.2.
(2)⇒(3). Obvious in view of Lemma 7.1.
(3)⇒(4). Let γ be such as in (3). By Corollary 6.3, there exist $\pi_1\in P_3(F(\rho)/F)$ and $\pi_2\in P_3(F(\psi)/F)$ such that $\gamma\in\pi_1+\pi_2+I^4(F)$. Hence $\gamma-\pi_1-\pi_2\in I^4(F)$. By Lemma 7.5, there exists $\gamma'\in\Gamma(\rho,\psi)$ such that $\gamma-\pi_1-\pi_2\equiv\gamma' \pmod{I^4(F)}$. Since $\gamma-\pi_1-\pi_2\in I^4(F)$, we have $\gamma'\in I^4(F)$. By definition of $\Gamma(\rho,\psi)$, $\dim(\gamma')_{an}\le 4+4+2=10<16$. Since $\gamma'\in I^4(F)$, the Arason-Pfister Hauptsatz shows that $\gamma'=0$.
(4)⇒(5). Since $0\in\Gamma(\rho,\psi)$, there exist $l_1,l_2\in F^*$ such that $0=l_1\rho+l_2\psi+\langle\!\langle d\delta\rangle\!\rangle$. Thus $l_1\rho+l_2\psi=-\langle\!\langle d\delta\rangle\!\rangle$. Hence $l_1\rho$ and $l_2\psi$ contain a common subform of the dimension $(\dim(\rho)+\dim(\psi)-\dim\langle\!\langle d\delta\rangle\!\rangle)/2=(4+4-2)/2=3$.
(5)⇒(6). See Corollary 5.10.
(6)⇒(7). See Corollary 2.13.
(7)⇒(1). It is a particular case of Corollary 6.4. $\square$

Proposition 7.8. *For an arbitrary element* $\gamma\in\Gamma(\rho,\psi)$*, one has*
$$H^3(F(\rho,\psi)/F)=H^3(F(\rho)/F)+H^3(F(\psi)/F)+e^3(\gamma)H^0(F)\ .$$

Proof. By Corollary 7.3, the element $e^3(\gamma)$ belongs to $H^3(F(\rho,\psi)/F)$. If $\operatorname{Tors}\operatorname{CH}^2(X_\rho\times X_\psi)=0$ then by Corollary 2.13, we have $H^3(F(\rho,\psi)/F)=H^3(F(\rho)/F)+H^3(F(\psi)/F)$ and the proof is complete. If $\operatorname{Tors}\operatorname{CH}^2(X_\rho\times X_\psi)\neq 0$, Lemma 7.7 shows that $\gamma\notin I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F)$. Hence, by Corollary 6.3, $e^3(\gamma)\notin H^3(F(\rho)/F)+H^3(F(\psi)/F)$. To complete the proof it is sufficient to apply Corollary 2.13 and Theorem 5.7. $\square$

Corollary 7.9.
$$I^3(F(\rho,\psi)/F)\subset I^3(F(\rho)/F)+I^3(F(\psi)/F)+\{\Gamma(\rho,\psi),0\}+I^4(F)\ .$$

Proof. Let $\tau\in I^3(F(\rho,\psi)/F)$. Choose an element $\gamma\in\Gamma(\rho,\psi)$. By Proposition 7.8, either $e^3(\tau)\in H^3(F(\rho)/F)+H^3(F(\psi)/F)$ or $e^3(\tau-\gamma)\in H^3(F(\rho)/F)+H^3(F(\psi)/F)$. It remains to apply Corollary 6.3. $\square$

Proposition 7.10. *Let* $\pi\in I^3(F(\rho,\psi)/F)$*. Then at least one of the following conditions holds*

1) $\pi \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;
2) $\pi \in \Gamma(\rho, \psi) + I^4(F)$.

Proof. Obvious in view of Corollaries 7.9 and 7.6. □

8. Main theorem

Proposition 8.1. *Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ be an anisotropic quadratic form. Let $\psi = \langle -u, -v, uv, \delta\rangle$ and $\rho = \langle -a, -b, ab, d\rangle$. Then:*

1. *The following two conditions are equivalent:*
   (i) $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho, \psi)/F)$,
   (ii) $\phi_{F(\psi)}$ *is isotropic.*
2. *The following two conditions are equivalent:*
   (i) $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$,
   (ii) *there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. Note that $\langle\langle a, b, c\rangle\rangle = \phi - c\rho = \rho - c\phi$.
(1i)⇒(1ii). Let $E = F(\psi)$. If the Pfister form $\langle\langle a, b, c\rangle\rangle_E$ is isotropic, its neighbor $(\langle\langle a, b\rangle\rangle \perp \langle -c\rangle)_E$ is isotropic too. Since $\langle\langle a, b\rangle\rangle \perp \langle -c\rangle \subset \phi$, the form ϕ_E is isotropic. Thus we can suppose that $\langle\langle a, b, c\rangle\rangle_E$ is anisotropic. By the assumption, $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho, \psi)/F) = I^3(E(\rho)/F)$. Hence the anisotropic Pfister form $\langle\langle a, b, c\rangle\rangle_E$ becomes isotropic over the function field of ρ_E. By the Arason-Pfister subform theorem, we have $k\rho_E \subset \langle\langle a, b, c\rangle\rangle_E$ where k is an arbitrary element of $D_E(\rho) \cdot D_E(\langle\langle a, b, c\rangle\rangle)$. Since $(ab)^{-1} \in D_E(\rho)$ and $-abc \in D_E(\langle\langle a, b, c\rangle\rangle)$ we can take $k = (ab)^{-1} \cdot (-abc) = -c$. Thus $-c\rho_E \subset \langle\langle a, b, c\rangle\rangle_E$. Hence $\dim((\langle\langle a, b, c\rangle\rangle \perp c\rho)_E)_{an} \le 8 - 4 = 4$. Since $\langle\langle a, b, c\rangle\rangle + c\rho = \phi$, it follows that $\dim(\phi_E)_{an} \le 4$. Hence $\phi_{F(\psi)} = \phi_E$ is isotropic.
(1ii)⇒(1i). Since $\phi_{F(\psi)}$ and $\rho_{F(\rho)}$ are isotropic, we have $\dim(\phi_{F(\psi)})_{an} \le 4$ and $\dim(\rho_{F(\rho)})_{an} \le 2$. Therefore $\dim(\langle\langle a, b, c\rangle\rangle_{F(\rho,\psi)})_{an} = \dim((\phi - c\rho)_{F(\rho,\psi)})_{an} \le 4 + 2 = 6$. By the Arason-Pfister theorem, $\langle\langle a, b, c\rangle\rangle_{F(\rho,\psi)}$ is hyperbolic. Hence $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho, \psi)/F)$.
(2i)⇒(2ii). By Corollary 6.5, there exist $k_1, k_2 \in F^*$ such that $\langle\langle a, b, k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$, $\langle\langle u, v, k_2\rangle\rangle = \psi\,\langle\langle k_2\rangle\rangle$, and

$$\langle\langle a, b, c\rangle\rangle \equiv \langle\langle a, b, k_1\rangle\rangle + \langle\langle u, v, k_2\rangle\rangle \pmod{I^4(F)}\,.$$

It follows from [**9**, Theorem 4.8] that the Pfister forms $\langle\langle a, b, c\rangle\rangle$, $\langle\langle a, b, k_1\rangle\rangle$, and $\langle\langle u, v, k_2\rangle\rangle$ are linked. Hence there exists $s \in F^*$ such that $s\,\langle\langle u, v, k_2\rangle\rangle = \langle\langle a, b, k_1\rangle\rangle - \langle\langle a, b, c\rangle\rangle$. Since $\langle\langle a, b, k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$ and $\langle\langle a, b, c\rangle\rangle = \rho - c\phi$, we have $s\,\langle\langle u, v, k_2\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle - (\rho - c\phi) = c\phi - k_1\rho$. Therefore $\phi - cs\,\langle\langle u, v, k_2\rangle\rangle = ck_1\rho$. Hence ϕ and $cs\,\langle\langle u, v, k_2\rangle\rangle$ contain a common subform of the dimension

$$\frac{1}{2}(\dim\phi + \dim(sc\,\langle\langle u, v, k_2\rangle\rangle) - \dim(ck_1\rho)) = \frac{1}{2}(6 + 8 - 4) = 5.$$

Let us denote such a form by ϕ_0. By the definition, we have $\phi_0 \subset \phi$. Since $\phi_0 \subset sc\,\langle\langle u, v, k_2\rangle\rangle$, it follows that ϕ_0 is a Pfister neighbor. Since $\langle\langle u, v, k_2\rangle\rangle =$

$\psi\,\langle\!\langle k_2\rangle\!\rangle$, it follows that $\langle\!\langle u,v,k_2\rangle\!\rangle_{F(\psi)}$ is isotropic. Hence the Pfister neighbor $(\phi_0)_{F(\psi)}$ of $\langle\!\langle u,v,k_2\rangle\!\rangle_{F(\psi)}$ is isotropic as well.
(2ii)⇒(2i). Let ϕ_0 be a 5-dimensional Pfister neighbor such that $\phi_0\subset\phi$ and $(\phi_0)_{F(\psi)}$ is isotropic. Let us write ϕ in the form $\phi=\phi_0\perp\langle s_0\rangle$. Since ϕ_0 is a Pfister neighbor, there exists $\pi\in GP_3(F)$ such that $\phi_0\subset\pi$. We can write π in the form $\pi=\phi_0\perp-\langle s_1,s_2,s_3\rangle$. Set $\gamma=\langle s_0,s_1,s_2,s_3\rangle$. We have

$$\gamma=\phi-\pi\equiv\phi=\langle\!\langle a,b,c\rangle\!\rangle+c\rho\equiv c\rho \pmod{I^3(F)}.$$

Since $\dim\gamma=\dim c\rho=4$ it follows from Wadsworth's theorem ([**86**, Theorem 7]) that γ is similar to $c\rho$. Hence there exists $k\in F^*$ such that $\gamma=ck\rho$. We have

$$\langle\!\langle a,b,c\rangle\!\rangle=\rho-c\phi=\rho-c(\gamma+\pi)=\rho-c(ck\rho+\pi)=\langle\!\langle k\rangle\!\rangle\,\rho-c\pi.$$

Now it is sufficient to verify that $\langle\!\langle k\rangle\!\rangle\,\rho\in I^3(F(\rho)/F)$ and $\pi\in I^3(F(\psi)/F)$. We have $\langle\!\langle k\rangle\!\rangle\,\rho=\langle\!\langle a,b,c\rangle\!\rangle+c\pi\in I^3(F)$. Since $\dim(\langle\!\langle k\rangle\!\rangle\,\rho_{F(\rho)})_{an}<8$, the Arason-Pfister Hauptsatz shows that $\langle\!\langle k\rangle\!\rangle\,\rho_{F(\rho)}$ is hyperbolic. Thus $\langle\!\langle k\rangle\!\rangle\,\rho\in I^3(F(\rho)/F)$. Since $\phi_0\subset\pi$ and $(\phi_0)_{F(\psi)}$ is isotropic, $\pi_{F(\psi)}$ is isotropic as well. Since $\pi\in GP_3(F)$, it follows that $\pi_{F(\psi)}$ is hyperbolic. Hence $\pi\in I^3(F(\psi)/F)$. □

COROLLARY 8.2. *Let $\phi=\langle\!\langle a,b\rangle\!\rangle\perp-c\,\langle\!\langle d\rangle\!\rangle$ be an anisotropic quadratic form. Let $\psi=\langle -u,-v,uv,\delta\rangle$ and $\rho=\langle -a,-b,ab,d\rangle$. Suppose that the group $\mathrm{CH}^2(X_\psi\times X_\rho)$ is torsion-free. Then the following conditions are equivalent:*

1. *$\phi_{F(\psi)}$ is isotropic;*
2. *there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0\subset\phi$ and $(\phi_0)_{F(\psi)}$ is isotropic*

PROOF. (1)⇒(2). By Item 1 of Proposition 8.1, we know that $\langle\!\langle a,b,c\rangle\!\rangle\in I^3(F(\rho,\psi)/F)$. Since $\mathrm{Tors}\,\mathrm{CH}^2(X_\psi\times X_\rho)=0$, Corollary 2.13 implies that

$$H^3(F(\rho,\psi)/F)=H^3(F(\rho)/F)+H^3(F(\psi)/F)];.$$

By Corollary 6.4, $I^3(F(\rho,\psi)/F)\subset I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F)$. Applying Proposition 8.1 once again, we are done.
(2)⇒(1). Obvious. □

LEMMA 8.3. *Let ϕ be a 6-dimensional form and ψ be a 4-dimensional form. Suppose that ψ is similar to a subform in ϕ. Then $\operatorname{ind}C_0(\phi)\otimes_F C_0(\psi)=1$.*

PROOF. We can suppose that $\psi\subset\phi$. Hence there exists a 2-dimensional form μ such that $\psi\perp\mu=\phi$. Let E be a field extension of F generated by $\sqrt{d_\pm\phi}$ and $\sqrt{d_\pm\psi}$. Obviously $\phi_E,\psi_E\in I^2(F)$ and $\operatorname{ind}C_0(\phi)\otimes_F C_0(\psi)=\operatorname{ind}C_0(\phi_E)\otimes_E C_0(\psi_E)$. Thus we can reduce our problem to the case where $\phi,\psi\in I^2(F)$. Then $\mu\in I^2(F)$. Since $\dim\mu=2$, the form μ is hyperbolic. Hence $\phi=\psi\perp\mathbb{H}$. Therefore $C_0(\phi)=C_0(\psi)\otimes_F M_2(F)$. Hence $\operatorname{ind}C_0(\phi)\otimes_F C_0(\psi)=1$. □

Corollary 8.4. *Let* $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ *be an anisotropic quadratic form. Let* $\psi = \langle -u, -v, uv, \delta\rangle$ *and* $\rho = \langle -a, -b, ab, d\rangle$.

Suppose that $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) \neq 1$. *Then the following conditions are equivalent:*

(1) $\phi_{F(\psi)}$ *is isotropic and the isotropy is standard;*
(2) *there exits a 5-dimensional Pfister neighbor* ϕ_0 *such that* $\phi_0 \subset \phi$ *and* $(\phi_0)_{F(\psi)}$ *is isotropic;*
(3) $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;
(4) $(a, b, c) \in H^3(F(\rho)/F) + H^3(F(\psi)/F)$.

Proof. (1)$\Rightarrow$(2). Let ϕ and ψ be such as in (1). Let us suppose that the condition (2) is not satisfied. Then by the definition of standard isotropy, ψ is similar to a subform of ϕ. By Lemma 8.3, we have $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$. This contradicts to our assumption.
(2)$\Rightarrow$(1). Obvious.
(3)$\Longleftrightarrow$(4)$\Longleftrightarrow$(1). Follows from Proposition 8.1 and Corollary 6.3. $\square$

Theorem 8.5. *Let* ϕ *be an anisotropic 6-dimensional quadratic form and* ψ *be a 4-dimensional quadratic form with* $d_\pm \psi = d_\pm \phi \neq 1$. *Suppose that* $\phi_{F(\psi)}$ *is isotropic. Then there exits a 5-dimensional Pfister neighbor* ϕ_0 *such that* $\phi_0 \subset \phi$ *and* $(\phi_0)_{F(\psi)}$ *is isotropic.*

Proof. If $\operatorname{ind} C_0(\phi) = 1$ then ϕ is a Pfister neighbor. In this case we can take ϕ_0 to be equal to an arbitrary 5-dimensional subform in ϕ. In the case $\operatorname{ind} C_0(\phi) = 4$, it follows from Theorem 5.5 of Chapter 3 that $\phi_{F(\psi)}$ is anisotropic and we have a contradiction. Thus we can assume that $\operatorname{ind} C_0(\phi) = 2$. Then ϕ is similar to a form of the kind $\langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$. Since $d_\pm \psi = d_\pm \phi$, there exist $u, v \in F^*$ such that ψ is similar to the form $\langle -u, -v, uv, d\rangle$. Replacing ϕ and ψ by similar forms, we can suppose that

$$\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle \text{ and } \psi = \langle -u, -v, uv, d\rangle\,.$$

Let $\rho = \langle -a, -b, ab, d\rangle$. It follows from Theorem 5.1 that $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_\rho) = 0$. Now the result required follows immediately from Corollary 8.2. $\square$

Proposition 8.6. *Let* $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ *and* $\psi = \langle -u, -v, uv, \delta\rangle$ *be anisotropic quadratic forms. Suppose that* $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 4$. *Then the following conditions are equivalent:*

(1) $\phi_{F(\psi)}$ *is isotropic;*
(2) *There is a 5-dimensional subform* $\phi_0 \subset \phi$ *which is a Pfister neighbor and* $(\phi_0)_{F(\psi)}$ *is isotropic.*

Proof. Let $\rho = \langle -a, -b, ab, d\rangle$. Clearly $C_0(\phi) = M_2(F) \otimes_F C_0(\rho)$. Hence $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 4$. It follows from Theorem 5.8 that $\operatorname{Tors} \operatorname{CH}^2(X_\rho \times X_\psi) = 0$. By Corollary 8.2, we are done. $\square$

Proposition 8.7. *Let* $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ *and* $\psi = \langle -u, -v, uv, \delta\rangle$ *be anisotropic quadratic forms with* $\delta \notin F^{*2}$. *Suppose that* $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$. *Then the following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic;*
(2) *Either ψ is similar to a subform in ϕ or there exists a 5-dimensional subform $\phi_0 \subset \phi$ which is a Pfister neighbor and $(\phi_0)_{F(\psi)}$ is isotropic.*

PROOF. (1)⇒(2). Since ϕ is anisotropic, we have $d \notin F^{*2}$. In view of Theorem 8.5 is is sufficient to consider the case $d\delta \notin F^{*2}$. Let $\rho = \langle -a, -b, ab, d\rangle$. Since $C_0(\phi) = M_2(F) \otimes_F C_0(\rho)$, we have $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 1$. Thus all the assumptions of §7 hold. Propositions 7.10 and 8.1 show that at least one of the following conditions holds:

1) $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$,
2) $\langle\langle a, b, c\rangle\rangle \in \Gamma(\rho, \psi) + I^4(F)$.

In the first case, Proposition 8.1 asserts that there exists a 5-dimensional subform $\phi_0 \subset \phi$ which is a Pfister neighbor and $(\phi_0)_{F(\psi)}$ is isotropic.

Thus we can suppose that $\langle\langle a, b, c\rangle\rangle \in \Gamma(\rho, \psi) + I^4(F)$. Let $\gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \in \Gamma(\rho, \psi)$ be such that $\langle\langle a, b, c\rangle\rangle \in \gamma + I^4(F)$. Since $\langle\langle a, b, c\rangle\rangle = \rho - c\phi$, we have

$$l_1\rho - l_1c\phi = l_1 \langle\langle a, b, c\rangle\rangle \equiv \langle\langle a, b, c\rangle\rangle \equiv \gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \pmod{I^4(F)}.$$

Hence $l_1c\phi + l_2\psi + \langle\langle d\delta\rangle\rangle \in I^4(F)$. Since $\dim(l_1c\phi + l_2\psi + \langle\langle d\delta\rangle\rangle)_{an} \le 6 + 4 + 2 = 12 < 16$, the Arason-Pfister Hauptsatz shows that $l_1c\phi + l_2\psi + \langle\langle d\delta\rangle\rangle = 0$. Therefore $\phi = -cl_1l_2\psi - cl_1 \langle\langle d\delta\rangle\rangle$. Since $\dim \phi = 6 = \dim(-cl_1l_2\psi \perp -cl_1 \langle\langle d\delta\rangle\rangle)$, we have $\phi = -cl_1l_2\psi \perp -cl_1 \langle\langle d\delta\rangle\rangle$. Hence ψ is similar to a subform in ϕ.
(2)⇒(1). Obvious. □

Together with results described in Introduction, Theorem 8.5, Propositions 8.6 and 8.7 give rise to the following

THEOREM 8.8. *Let ϕ be an anisotropic quadratic form of dimension ≤ 6 and ψ be such that $\phi_{F(\psi)}$ is isotropic. If the isotropy is non-standard then*

- $\dim \phi = 6$ *and* $\dim \psi = 4$*;*
- $1 \ne d_\pm \phi \ne d_\pm \psi \ne 1$*;*
- $\operatorname{ind} C_0(\phi) = 2$*; and*
- $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 2$. □

9. The case of index 2

Theorem 8.8 implies that if there exists a quadratic form ϕ of dimension ≤ 6 having a non-standard isotropy over the function field of a quadratic form ψ, then there are $a, b, c, d, u, v, \delta \in F^*$ such that $\phi \sim \langle\langle a, b\rangle\rangle \perp -c \langle\langle d\rangle\rangle$, $\psi \sim \langle -u, -v, uv, \delta\rangle$, $d, \delta, d\delta \notin F^{*2}$, and $\operatorname{ind}((a, b) \otimes_F (u, v))_{F(\sqrt{d},\sqrt{\delta})} = 2$.

Set $\rho = \langle -a, -b, ab, d\rangle$. By Corollary 8.2, if $\operatorname{Tors} \mathrm{CH}^2(X_\psi \times X_\rho) = 0$, then the isotropy is standard.

In this section we prove the following

THEOREM 9.1. *Let* $a, b, u, v, d, \delta \in F^{*2}$ *be such that* $d, \delta, d\delta \notin F^{*2}$. *Let* $\rho = \langle -a, -b, ab, d\rangle$ *and* $\psi = \langle -u, -v, uv, \delta\rangle$. *Suppose that* $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 2$. *The following conditions are equivalent:*

1. $\operatorname{Tors} \mathrm{CH}^2(X_\rho \times X_\psi) \neq 0$;
2. *there exists* $c \in F^*$ *such that the quadratic form* $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ *is isotropic over* $F(\psi)$, *but the isotropy is not standard.*

PROOF. (2)⇒(1). Obvious in view of Corollary 8.2.
(1)⇒(2). Since $\operatorname{Tors} \mathrm{CH}^2(X_\rho \times X_\psi) \neq 0$, it follows from Corollary 2.13 that there exists $w \in H^3(F(\rho, \psi)/F)$ such that $w \notin H^3(F(\rho)/F) + H^3(F(\psi)/F)$. Let $\rho_0 = \langle -a, -b, ab\rangle$. It follows from Theorem 5.9 that $\operatorname{ind} C_0(\rho_0) \otimes_F C_0(\psi) \neq \operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 2$. Therefore $\operatorname{ind} C_0(\rho_0) \otimes_F C_0(\psi) = 4$. By Theorem 5.8, we have $\operatorname{Tors} \mathrm{CH}^2(X_{\rho_0} \times X_\psi) = 0$. By Corollary 2.13, we have $H^3(F(\rho_0, \psi)/F) = H^3(F(\rho_0)/F) + H^3(F(\psi)/F)$. Hence

$$w \in H^3(F(\rho, \psi)/F) \subset H^3(F(\rho_0, \psi)/F) = H^3(F(\rho_0)/F) + H^3(F(\psi)/F).$$

Since $H^3(F(\rho_0)/F) = (a, b) \cup H^1(F)$, there exists $c \in F^*$ such that $w - (a, b, c) \in H^3(F(\psi)/F)$, i.e. $w \equiv (a, b, c) \pmod{H^3(F(\psi)/F)}$. By the assumption on w, we see that $(a, b, c) \in H^3(F(\rho, \psi)/F)$ and $(a, b, c) \notin H^3(F(\rho)/F) + H^3(F(\psi)/F)$. Therefore, $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho, \psi)/F$ and

$$\langle\langle a, b, c\rangle\rangle \notin I^3(F(\rho)/F) + I^3(F(\psi)/F) + H^4(F)\ .$$

By Proposition 8.1, the quadratic form $\phi_{F(\psi)}$ is isotropic. By Corollary 8.4, the isotropy is not standard. $\square$

CHAPTER 5

Some new examples in the theory of quadratic forms

We construct a 6-dimensional quadratic form ϕ and a 4-dimensional quadratic form ψ over some field F such that ϕ becomes isotropic over the function field $F(\psi)$ but every proper subform of ϕ is still anisotropic over $F(\psi)$. It is an example of *non-standard isotropy* with respect to some standard conditions of isotropy for 6-dimensional forms over function fields of quadrics, known previously.

Besides of that, we produce an example of 8-dimensional quadratic form ϕ with trivial determinant such that the index of the Clifford invariant of ϕ is 4 but ϕ can not be represented as a sum of two 4-dimensional forms with trivial determinants. Using this, we find a 14-dimensional quadratic form with trivial discriminant and Clifford invariant which which is not similar to a difference of two 3-fold Pfister forms.

Results of this Chapter are obtained in joint work with Oleg Izhboldin.

0. Introduction

Let F be a field of characteristic $\neq 2$. An important problem in the algebraic theory of quadratic forms is to classify pairs of anisotropic quadratic forms ϕ, ψ such that $\phi_{F(\psi)}$ is isotropic, where $F(\psi)$ is the function field of ψ, i.e. the function field of the projective quadric determined by ψ. In the case $\dim \phi \leq 5$, a complete classification is known (see [**15**]). The case $\dim \phi = 6$ was studied in [**16**], [**44**], [**45**], [**49**], and [**54**]. In the case where $\dim \phi = 6$ and $\dim \psi \neq 4$, a complete classification was obtained. It was shown in Chapters 3 and 4, that the same classification is valid for 4-dimensional forms ψ, if the case where $\dim \phi = 6$, $\dim \psi = 4$, $1 \neq \det_{\pm} \phi \neq \det_{\pm} \psi \neq 1$, $\operatorname{ind} C_0(\phi) = 2$, and $\operatorname{ind} C_0(\phi) \otimes C_0(\psi) = 2$ is excluded. Here (see Section 15) we construct in this excepted case an example of ϕ and ψ with non-standard (i.e. not matching the old classification) isotropy of ϕ over $F(\psi)$ (see Theorem 15.2). It is possible to explain what this "non-standard isotropy" does exactly mean without describing the old classification (Lemma 15.3): isotropy of ϕ over $F(\psi)$ is non-standard if and only if the form ϕ is $F(\psi)$-minimal, i.e. no proper subform of ϕ becomes isotropic over $F(\psi)$. A stronger version of Theorem 15.2 states that an example of non-standard isotropy can be obtained starting from an *arbitrary* anisotropic 4-dimensional form ψ (with $\det \psi \neq 1$) over an arbitrary field F_0 by passing to an appropriate extension F of F_0 (see Corollary 15.4).

Let $I(F)$ be the ideal of even-dimensional forms in the Witt ring $W(F)$. Another important problem in the algebraic theory of quadratic forms is to give a classification of low-dimensional quadratic forms belonging to $I^n(F)$ for a fixed $n > 0$. For $n = 2$ and for $n = 3$, this problem was studied by many authors. In [**28**] Jacobson proved that quadratic forms $\phi \in I^2(F)$ of dimension ≤ 6 are uniquely determined up to similarity by the Clifford invariant $c(\phi)$. There exists a good description of 8-dimensional forms $\phi \in I^2(F)$ satisfying the condition $\operatorname{ind} C(\phi) \leq 2$. Namely, these quadratic forms can be written as tensor product of a 2-dimensional subform and a 4-dimensional subform (see e.g. [**41**, Example 9.12]). The case of 8-dimensional quadratic forms $\phi \in I^2(F)$ with $\operatorname{ind} C(\phi) = 4$ is much more complicated. It was an open question if these quadratic forms can be written as $\tau_1 \perp \tau_2$, where τ_1 and τ_2 are 4-dimensional forms with trivial determinant. In Section 13 we construct a counterexample for this question (Corollary 13.8). Nevertheless we find a “weak version” of the decomposition $\phi = \tau_1 \perp \tau_2$. Note that quadratic forms of the type $\tau_1 \perp \tau_2$ can be regarded as Scharlau's transfer $s_{L/F}(\tau)$ in the degenerate case $L = F \times F$. We show that an arbitrary 8-dimensional form $\phi \in I^2(F)$ with $\operatorname{ind} C(\phi) = 4$ can be represented as Scharlau's transfer $s_{L/F}(\tau)$, where L/F is an (étale) quadratic extension and τ is a 4-dimensional L-form with trivial determinant (see Theorem 13.10).

In Section 14 we study quadratic forms $\phi \in I^3(F)$. The structure of ϕ in the case $\dim \phi \leq 12$ was described by Pfister in [**68**, Satz 14 und Zusatz] (see also [**18**]). Our aim is to study 14-dimensional quadratic forms in $I^3(F)$. In [**72**] M. Rost proved that an arbitrary 14-dimensional quadratic form can be represented (up to similarity) as Scharlau's transfer $s_{L/F}(\sqrt{d}\,\tau')$, where $L = F(\sqrt{d})$ and τ' is the pure subform of a 3-fold Pfister form. Note that in the degenerate case $L = F \times F$ we get the decomposition $\phi = k(\tau_1' \perp -\tau_2')$, where τ_1', τ_2' are pure subforms of 3-fold Pfister forms τ_1, τ_2 and $k \in F^*$. It was an open question if any 14-dimensional form $\phi \in I^3(F)$ can be written in the form $\phi = k(\tau_1' \perp -\tau_2')$. It was remarked by D. Hoffmann (1995, Bielefeld, oral communication) that this question is equivalent to the discussed above question on 8-dimensional forms $\phi \in I^2(F)$ with $\operatorname{ind} C(\phi) = 4$. Using the counterexample for 8-dimensional forms, we construct (in Section 14) a counterexample for 14-dimensional forms.

Similar counterexamples of 8-dimensional and 14-dimensional forms in the case of characteristic 0 are independently constructed in [**21**] by using a completely different technique.

Now we explain the structure of the Chapter. It can be divided into two parts: Sections 2–11 and Sections 12–15. All main results listed above are obtained in the second part. In the first part, the necessary preparations are made. The results of Sections 8 and 10 are needed for the counterexamples of 8-and-14-dimensional forms, while the results of Sections 9 and 11 are for the non-standard isotropy. The constructions and proofs of Sections 8, 10 are parallel to that of Sections 9, 11. It is the reason for why we included the results on 8,14-dimensional forms and on the non-standard isotropy in the same

Chapter. However, Sections 8, 10 and Sections 9, 11 are written in an independent manner, so that the reader interested only in one of two groups of results can choose (although, in order to understand more complicated calculations of Section 9, it is better to go through Section 8 first).

Now we explain more precisely the contents of several first preparation Sections.

In Section 2, we show that certain products of (generalized) Severi-Brauer varieties considered as schemes over certain subproducts via the projection can be naturally identified with grassmanians (Corollary 2.4). It was already done in Corollary 6.4 of Chapter 1 and in Proposition 5.3 of Chapter 2. However this time we need more explicit information: namely, we need a description of the vector bundle on the product of the Severi-Brauer varieties corresponding to the tautological vector bundle on the grassmanian under that identification. The answer is given in terms of the canonical vector bundles on the Severi-Brauer varieties.

Let Γ be the grassmanian of "n-planes" in a vector bundle on a variety X. In Section 3, we describe the Grothendieck group of Γ together with the topological filtration on it in terms of the Grothendieck group of X. The general answer (Proposition 3.3) is an easy consequence of the well-known result on the Chow group of a grassmanian. Some additional negligible efforts are made to formulate the result in terms of characteristic classes of the class $[\mathcal{T}]$ of the tautological vector bundle $\mathcal{T}$; classically ([**12**, Proposition 14.6.5]) characteristic classes of $-[\mathcal{T}]$ are used, what is not convenient for practical use. After all this, we apply the general assertion to the case of the grassmanian of 2-planes in a rank 4 vector bundle and get some explicit formulas which are then used in Sections 8 and 9.

In Section 4, we reprove that the pull-back to the generic fiber of a flat morphism is surjective. For what kind of groups? Well, our final goal is the topological filtration i.e., each term of that (Corollary 4.3). We reach the goal starting with the Chow groups (Proposition 4.1) and passing after that to the quotients of the topological filtration (Corollary 4.2). The statement on the Chow groups is not new and even was already used several times in this work. It is a formal consequence of the spectral sequence [**39**, Theorem 3.1]. Here, we give a short direct proof or, better to say, an explanation of the evidence of this fact (Proposition 4.1).

1. Terminology, notation, and backgrounds

1.1. Quadratic forms. By $\phi \perp \psi$, $\phi \simeq \psi$, and $[\phi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F. To simplify notation we write $\phi_1+\phi_2$ instead of $[\phi_1]+[\phi_2]$. The maximal ideal of $W(F)$ generated by the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We denote by $\langle\langle a_1,\dots,a_n\rangle\rangle$ the n-fold Pfister form $\langle 1,-a_1\rangle\otimes\cdots\otimes\langle 1,-a_n\rangle$ and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to an

n-fold Pfister form we denote by $GP_n(F)$. For any field extension L/F, we put $\phi_L = \phi \otimes_F L$.

For a quadratic extension L/F and an L-form ϕ, we denote by $s_{L/F}(\phi)$ Scharlau's transfer [**76**, Ch. 2, §5] corresponding to the F-linear homomorphism $\frac{1}{2}\operatorname{Tr}_{L/F} : L \to F$. In the case where $L = F(\sqrt{d})$, we have $s_{L/F}(\langle 1\rangle) = \langle 1, d\rangle$ and $s_{L/F}(\langle \sqrt{d}\,\rangle) = \langle 1, -1\rangle$.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$.

1.2. Linked forms. We say that quadratic F-forms ϕ and ψ are *linked* if the following equivalent conditions hold:

- there exists a 2-dimensional form μ which is similar to a subform of ϕ and to a subform of ψ,
- there exists a field extension L/F of degree ≤ 2 such that ϕ_L and ψ_L are isotropic,

If ϕ and ψ are forms of dimension ≥ 3, then the condition that ϕ and ψ are linked can be reformulated as follows: there exists a closed point of degree ≤ 2 on the variety $X_\phi \times X_\psi$.

1.3. K-theory and Chow groups. For a smooth algebraic F-variety X, its Grothendieck ring is denoted by $K(X)$. This ring is equipped with the filtration by codimension of support (which respects the multiplication). For a ring (or a group) with filtration A, we denote by G^*A the adjoint graded ring (resp., the adjoint graded group). There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $\mathrm{G}^*K(X)$, its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([**81**, §9]). For a geometrically integral variety of dimension d we set $\mathrm{CH}_i(X) = \mathrm{CH}^{d-i}(X)$ and $\mathrm{G}_iK(X) = \mathrm{G}^{d-i}K(X)$.

1.4. Algebras. Let A be an algebra over a field F. For a field extension E/F (or, more generally, for a unital commutative F-algebra E), we denote by A_E the E-algebra $A \otimes_F E$. For an F-variety X (or, more generally, for an F-scheme X), we denote by A_X the constant X-sheaf of algebras given by A.

In Section 2, the category of commutative unital F-algebras is denoted by F-$\mathfrak{alg}$.

2. Products of Severi-Brauer varieties

Let F be a field and let A be a central simple algebra over F.

Let $n \geq 0$. The generalized Severi-Brauer variety $Y \overset{\text{def}}{=} \mathrm{SB}(n, A)$ of A is characterized as follows: for any $R \in F$-$\mathfrak{alg}$, the set of R-points $Y(R) \overset{\text{def}}{=} \mathrm{Mor}_F(\operatorname{Spec} R, Y)$ of the variety Y consists of the right ideals J of the Azumaya R-algebra $A_R \overset{\text{def}}{=} A \otimes_F R$ having two following properties:

- the injection of A_R-modules $J \hookrightarrow A_R$ splits (in particular, J is projective as the R-module);

- $\operatorname{rk} J = n$, where $\operatorname{rk} J$ is the R-rank of J divided by $\deg A$;

moreover, for any homomorphism $R \to R'$ in the category $F\text{-}\mathfrak{alg}$, the map $Y(R) \to Y(R')$ is given by tensor multiplication $J \mapsto J \otimes_R R'$.

The (usual) Severi-Brauer variety $\mathrm{SB}(A)$ of A is by definition the variety $\mathrm{SB}(1, A)$.

EXAMPLE 2.1. Let A be a quaternion algebra (a, b), where $a, b \in F^*$. The Severi-Brauer variety $\mathrm{SB}(A)$ is isomorphic to the projective conic determined by the quadratic form $\langle 1, -a, -b \rangle$.

EXAMPLE 2.2. Let A be a biquaternion algebra $(a_1, b_1) \otimes (a_2, b_2)$, where a_1, b_1, a_2, $b_2 \in F^*$. The generalized Severi-Brauer variety $\mathrm{SB}(2, A)$ is isomorphic to the projective quadric determined by the Albert form

$$\langle -a_1, -b_1, a_1 b_1, a_2, b_2, -a_2 b_2 \rangle \ .$$

The *canonical vector bundle* $\mathcal{J}$ on the generalized Severi-Brauer variety Y is defined as follows: for any $R \in F\text{-}\mathfrak{alg}$ and a point $J \in Y(R)$, the fiber of $\mathcal{J}$ over J is the R-module J; if $R \to R'$ is a homomorphism in $F\text{-}\mathfrak{alg}$, then the map of the fibers $J \to J'$, where $J' \overset{\mathrm{def}}{=} J \otimes_R R' \in Y(R')$, is defined by the formula $x \mapsto x \otimes 1$.

Since every fiber of $\mathcal{J}$ is right ideal, $\mathcal{J}$ has a structure of right A_Y-module.

PROPOSITION 2.3. *Let $X \overset{\mathrm{def}}{=} \mathrm{SB}(A)$, $Y \overset{\mathrm{def}}{=} \mathrm{SB}(n, A^{\mathrm{op}})$; let $\mathcal{I}$ and $\mathcal{J}$ be the canonical vector bundles on X and Y. The product $X \times Y$, considered over X via the first projection, can be naturally identified (as a scheme over X) with the grassmanian $\mathbb{\Gamma}_n(\mathcal{I})$ of n-planes in $\mathcal{I}$; by this identification, the tautological vector bundle on the grassmanian corresponds to a vector bundle on $X \times Y$ isomorphic to $\mathcal{I} \otimes_{A_{X \times Y}} \mathcal{J}$.*

PROOF. Let $R \in F\text{-}\mathfrak{alg}$ and let I be an R-point of X. To prove the first part of Proposition, it suffices to describe a natural bijection of the fibers over I. The fiber of $X \times Y$ over the point I is the set $Y(R)$. The fiber of $\mathbb{\Gamma}_n(\mathcal{I})$ over the point I is the set of R-submodules N of the R-module I such that the injection $N \hookrightarrow I$ splits and $\operatorname{rk}_R N = n$. For any N like that, the set $\mathrm{Hom}_R(I, N)$ is a right ideal of the R-algebra $\mathrm{End}_R I = A_R^{\mathrm{op}}$ and thus determines an element of $Y(R)$. In this way, we get the natural bijection required.

To describe an isomorphism of the vector bundles (mentioned in the second statement of Proposition), it suffices to give a natural isomorphism of the R-modules $I \otimes_{A_R} \mathrm{Hom}_R(I, N)$ and N. This is given by the rule $x \otimes f \mapsto f(x)$. $\square$

Now, let $A_1, \dots, A_m, A$ be central simple F-algebras such that

$$A = A_1^{\otimes i_1} \otimes \dots \otimes A_m^{\otimes i_m} \ \text{ with certain } \ i_1, \dots, i_m \geq 0 \ .$$

Denote by $X_1, \dots, X_m$ the Severi-Brauer varieties of the algebras $A_1, \dots, A_m$. Put $X \overset{\mathrm{def}}{=} X_1 \times \dots \times X_m$ and $Y \overset{\mathrm{def}}{=} \mathrm{SB}(n, A^{\mathrm{op}})$. For every $j = 1, \dots, m$, let $\mathcal{I}_j$ be the canonical vector bundle on X_j. Put $\mathcal{I} \overset{\mathrm{def}}{=} \mathcal{I}_1^{\otimes i_1} \otimes \dots \otimes \mathcal{I}_m^{\otimes i_m}$; it is a right A_X-module. Let $\mathcal{J}$ be the canonical vector bundle on Y; it is a left A_Y-module.

Corollary 2.4. *In the notation introduced right above, the product $X \times Y$, considered over X via the first projection, can be naturally identified (as a scheme over X) with the grassmanian $\mathbb{\Gamma}_n(\mathcal{I})$; by this identification, the tautological vector bundle on the grassmanian corresponds to a vector bundle on $X \times Y$ isomorphic to $\mathcal{I} \otimes_{A_{X\times Y}} \mathcal{J}$.*

Proof. Put $X' \stackrel{\text{def}}{=} \mathrm{SB}(A)$ and let $\mathcal{I}'$ be the canonical vector bundle on X'.

Let $R \in F\text{-}\mathfrak{alg}$. An element of the set $X(R) = X_1(R) \times \cdots \times X_m(R)$ is a collection $(I_1, \dots, I_m)$ where I_j is a right ideal in $(A_j)_R$. The tensor product $I_1 \otimes_R \cdots \otimes_R I_m$ is a right ideal of A_R. In this way we get a map of sets $X(R) \to X'(R)$ which is natural with respect to R. Let $X \to X'$ be the corresponding morphism of F-varieties.

Consider the cartesian square

$$\begin{array}{ccc} X \times Y & \longrightarrow & X' \times Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' . \end{array}$$

By Proposition, we can identify the product $X' \times Y$ with the grassmanian $\mathbb{\Gamma}_n(\mathcal{I}')$. Since the inverse image of the vector bundle $\mathcal{I}'$ with respect to the morphism $X \to X'$ is $\mathcal{I}$, the variety $X \times Y$ can be therefore identified with $\mathbb{\Gamma}_n(\mathcal{I})$.

Taking the inverse of the vector bundle $\mathcal{I}' \otimes_{A_{X'\times Y}} \mathcal{J}$ with respect to the morphism $X \times Y \to X' \times Y$, we get the vector bundle $\mathcal{I} \otimes_{A_{X\times Y}} \mathcal{J}$. It proves the second statement of Corollary. □

3. The Grothendieck group of a grassmanian

By *ring* we mean a *commutative unital* ring.

Let R be a ring. We consider only the descending filtrations $R^{(i)}$ ($i \in \mathbb{Z}$) on R satisfying the following conditions:

- $R^{(i)} \cdot R^{(j)} \subset R^{(i+j)}$ for all i, j and
- $R^{(0)} = R$.

Let R be a ring with filtration and let M be an R-module. We consider only the descending filtrations $M^{(i)}$ on M satisfying the following conditions:

- $R^{(i)} \cdot M^{(j)} \subset M^{(i+j)}$ for all i, j and
- $M^{(0)} = M$.

Definition 3.1. Let R be a ring with filtration, let M be an R-module, let $e_1, \dots, e_k \in M$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{Z}$. We define the filtration on M induced by the conditions $e_i \in M^{(\alpha_i)}$ for $i = 1, \dots, k$ to be the smallest filtration on M satisfying these conditions.

The following assertion is evident:

LEMMA 3.2. *For every $n \geq 1$, the n-th term $M^{(n)}$ of the filtration on M induced by the conditions $e_i \in M^{(\alpha_i)}$ $(i = 1, \dots, k)$ is determined by the formula*

$$M^{(n)} = \sum_{j=1}^{k} R^{(n-\alpha_j)} \cdot e_j \ .$$

□

We fix the following notation for the rest of this Section: F is an arbitrary field, X is a smooth F-variety, $r \geq n \geq 0$ are integers, $\mathcal{E}$ is a rank r vector bundle over X, $\Gamma \stackrel{\mathrm{def}}{=} \mathbb{\Gamma}_n(\mathcal{E})$ is the grassmanian of n-planes in the vector bundle $\mathcal{E}$, and π is the structure morphism $\Gamma \to X$. We put $m = r - n$.

We denote by $\mathcal{T}$ the *tautological vector bundle* on Γ (also called the *universal vector subbundle* on Γ, see [**12**, §14.6]). The rank of $\mathcal{T}$ equals n.

An (m, n)-*partition* λ is a sequence of integers $(\lambda_1, \dots, \lambda_m)$ of length m satisfying the condition $n \geq \lambda_1 \geq \dots \geq \lambda_m \geq 0$. The *weight* $|\lambda|$ of λ is by definition the sum $\lambda_1 + \dots + \lambda_m$.

Let $s \stackrel{\mathrm{def}}{=} (s_1, s_2, \dots)$ be a sequence of variables. Additionally, we put $s_0 \stackrel{\mathrm{def}}{=} 1$ and $s_i \stackrel{\mathrm{def}}{=} 0$ for all $i < 0$. For an (m, n)-partition λ, the *Schur polynomial* $\Delta_\lambda(s)$ of λ is the determinant of the matrix $(s_{\lambda_i + j - i})_{i,j=1}^{m}$ (see also the definition of [**12**, §14.5]). It is a homogeneous polynomial of weight $|\lambda|$, if every s_i is taken with the weight i.

For every $i \geq 1$, let us substitute $s_i = (-1)^i c_i([\mathcal{T}])$ where $[\mathcal{T}]$ is the class of $\mathcal{T}$ in $K(\Gamma)$ and $c_i \colon K(\Gamma) \to K(\Gamma)$ is the i-th Chern class with values in K (see Definition 2.1 of Chapter 1). For any (m, n)-partition λ, we put $\Delta_\lambda \stackrel{\mathrm{def}}{=} \Delta_\lambda(s)$. Since $c_i([\mathcal{T}]) \in K(\Gamma)^{(i)}$, we have $\Delta_\lambda \in K(\Gamma)^{(|\lambda|)}$.

We consider $K(\Gamma)$ as a $K(X)$-module via the pull-back homomorphism $\pi^* \colon K(X) \to K(\Gamma)$.

PROPOSITION 3.3. *The $K(X)$-module $K(\Gamma)$ is free and the elements $\{\Delta_\lambda\}_\lambda$, where λ runs over all (m, n)-partitions, form its basis. The topological filtration on $K(\Gamma)$ is induced by the conditions $\Delta_\lambda \in K(\Gamma)^{(|\lambda|)}$ (see Definition 3.1).*

PROOF. We have to show that the map

$$\textstyle\bigoplus_\lambda K(X) \xrightarrow{\sum \cdot \Delta_\lambda} K(\Gamma)$$

is an isomorphism of groups with filtrations, where the direct sum is taken over all (m, n)-partitions λ and for every λ, the λ-summand $K(X)$ of the direct sum is considered with the topological filtration shifted by $|\lambda|$.

Since the filtrations are finite, it suffices to show that the induced homomorphism of adjoint graded groups

$$\bigoplus_\lambda \mathrm{G}^{*-|\lambda|} K(X) \xrightarrow{\sum \cdot \bar{\Delta}_\lambda} \mathrm{G}^* K(\Gamma)$$

is an isomorphism, where $\bar{\Delta}_\lambda$ the class of Δ_λ in the quotient $\mathrm{G}^{|\lambda|} K(\Gamma)$. We prove it (see Proposition 3.7) after we have introduced some additional notation and proved some preliminary assertions.

We repeat that we denote by $\bar{\Delta}_\lambda$ the class of Δ_λ in the quotient $\mathrm{G}^{|\lambda|}K(\Gamma)$. Note that $\bar{\Delta}_\lambda$ can be also defined directly in the similar way as Δ_λ by using the the Chern classes with values in G^*K (see Definition 2.8 of Chapter 1) instead of the Chern classes with values in K.

Let us now substitute $s_i = c_i(-[\mathcal{T}]) \in K(\Gamma)$ for every $i \geq 1$. For an (n,m)-partition λ', we put $\nabla_{\lambda'} \stackrel{\mathrm{def}}{=} \Delta_{\lambda'}(s)$. We have $\nabla_{\lambda'} \in K(\Gamma)^{(|\lambda'|)}$ and we denote by $\bar{\nabla}_{\lambda'} \in \mathrm{G}^{|\lambda'|}K(\Gamma)$ the residue class. Note that $\bar{\nabla}_{\lambda'}$ can be also defined directly in the similar way as $\nabla_{\lambda'}$ by using the the Chern classes with values in G^*K. Now, taking the Chern classes with values in CH^*, let us define in the similar way one more element $\nabla^{\mathrm{CH}}_{\lambda'} \in \mathrm{CH}^{|\lambda'|}(\Gamma)$.

Proposition 3.4. *The map*

$$\bigoplus_{\lambda'} \mathrm{CH}^{*-|\lambda'|}(X) \xrightarrow{\sum \cdot \nabla^{\mathrm{CH}}_{\lambda'}} \mathrm{CH}^*(\Gamma),$$

where λ' runs over the set of all (n,m)-partitions, is an isomorphism of graded groups.

Proof. First of all, it is evidently a homomorphism of graded groups because $\nabla^{\mathrm{CH}}_{\lambda'} \in \mathrm{CH}^{|\lambda'|}(\Gamma)$ for every λ'. Thus we only have to show that it is an isomorphism of groups, without taking care for the gradations. This is done in [**12**, Proposition 14.6.5]. $\square$

An (n,m)-partition λ' is called *dual* to an (m,n)-partition λ, if λ'_i (for every $i = 1, \ldots, n$) is equal to the quantity of $\lambda_1, \ldots, \lambda_m$ which are $\geq i$.

Lemma 3.5. *Let λ be an (m,n)-partition and let λ' be the (n,m)-partition dual to λ. Then $\Delta_\lambda = \nabla_{\lambda'}$. In particular, $\bar{\Delta}_\lambda = \bar{\nabla}_{\lambda'}$.*

Proof. The first relation follows from [**12**, Lemma 14.5.1]. After that the second relation is evident (note that $|\lambda| = |\lambda'|$). $\square$

Lemma 3.6 ("Duality theorem"). *Let λ' and λ'' be (n,m)-partitions such that $|\lambda''| + |\lambda'| \leq nm$ and let $\alpha \in \mathrm{G}^*K(X)$. Then*

$$\pi_*\big(\bar{\nabla}_{\lambda'} \cdot \bar{\nabla}_{\lambda''} \cdot \pi^*(\alpha)\big) = \begin{cases} \alpha, & \textit{if } \lambda''_i = m - \lambda'_{n-i+1} \textit{ for all } 1 \leq i \leq n; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Denote by $\tilde{\alpha} \in \mathrm{CH}^*(X)$ an arbitrary preimage of α with respect to the canonical epimorphism $\mathrm{CH}^*(X) \twoheadrightarrow \mathrm{G}^*K(X)$. By the Chow group variant of the duality theorem ([**12**, Proposition 14.6.3]), we have the formula

$$\pi_*\big(\nabla^{\mathrm{CH}}_{\lambda'} \cdot \nabla^{\mathrm{CH}}_{\lambda''} \cdot \pi^*(\tilde{\alpha})\big) = \begin{cases} \tilde{\alpha}, & \text{if } \lambda''_i = m - \lambda'_{n-i+1} \text{ for all } 1 \leq i \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Since the canonical epimorphism $\mathrm{CH}^*(-) \twoheadrightarrow \mathrm{G}^*K(-)$ commutes with push-forwards, pull-backs, and Chern classes, the formula required follows. $\square$

Proposition 3.7. *The map*

$$\bigoplus_\lambda \mathrm{G}^{*-|\lambda|}K(X) \xrightarrow{\sum\cdot\bar\Delta_\lambda} \mathrm{G}^*K(\Gamma)$$

is an isomorphism of graded groups.

Proof. By Lemma 3.5, the homomorphism $\sum\cdot\bar\Delta_\lambda$ coincides with the homomorphism $\sum\cdot\bar\nabla_{\lambda'}$, where λ' runs over all (n,m)-partitions. By Proposition 3.4, the upper arrow in the commutative diagram

$$\begin{tikzcd}
\bigoplus_{\lambda'} \mathrm{CH}^*(X) \arrow[r, "\sum\cdot\nabla^{\mathrm{CH}}_{\lambda'}"] \arrow[d] & \mathrm{CH}^*(\Gamma) \arrow[d] \\
\bigoplus_{\lambda'} \mathrm{G}^*K(X) \arrow[r, "\sum\cdot\bar\nabla_{\lambda'}"] & \mathrm{G}^*K(\Gamma)
\end{tikzcd}$$

is an isomorphism. Therefore the bottom arrow, i.e. the homomorphism $\sum\cdot\bar\nabla_{\lambda'}$, is surjective. So, it remains to prove injectivity.

We prove injectivity of $\sum\cdot\bar\nabla_{\lambda'}$ in exactly the same way as injectivity of $\sum\cdot\nabla^{\mathrm{CH}}_{\lambda'}$ is proved in [**12**] (see the beginning of the proof of [**12**, Proposition 14.6.5]). Suppose that the homomorphism is not injective. Let $\oplus\alpha_{\lambda'}$ be a non-zero element of $\ker(\sum\cdot\bar\nabla_{\lambda'})$. Choose an (n,m)-partition $\tilde\lambda'$ of maximal weight such that $\alpha_{\tilde\lambda'}\neq 0$. Define another (n,m)-partition $\tilde\lambda''$ as follows: $\tilde\lambda''_i \overset{\mathrm{def}}{=} m-\tilde\lambda'_{n-i+1}$ for $i=1,\dots,n$. By Lemma 3.6, we have

$$0=\pi_*\big(\bar\nabla_{\tilde\lambda''}\cdot(\sum_{\lambda'}\cdot\bar\nabla_{\lambda'})(\alpha)\big)=\sum_{\lambda'}\pi_*\big(\bar\nabla_{\tilde\lambda''}\cdot\bar\nabla_{\tilde\lambda'}\cdot\pi^*(\alpha)\big)=\alpha_{\tilde\lambda'}\ ,$$

a contradiction. ☐

The proof of Proposition 3.3 is complete. ☐

Remark 3.8. The assertion that $K(\Gamma)$ is a free $K(X)$-module holds in a more general situation, namely in the case where Γ is a *twisted* grassmanian over X. A proof can be found in [**50**, Theorem 4.4]. However the system of generators which appears there has no "good relation" to the topological filtration and differs from that of Proposition 3.3.

We are especially interested in the case where $m=n=1$ (i.e. in the case of a projective line bundle) and in the case where $m=n=2$.

Corollary 3.9. *Let $\Gamma\to X$ be a projective line bundle and let $\mathcal{T}$ be the tautological vector bundle on Γ. Then $K(\Gamma)$ is a free $K(X)$-module with the basis $1, 1-[\mathcal{T}]$; the topological filtration on $K(\Gamma)$ is induced by the condition $1-[\mathcal{T}]\in K(\Gamma)^{(1)}$.*

Proof. We apply Proposition 3.3 to the particular situation of Corollary. Since now $m=n=1$, we have only two (m,n)-partitions: $\lambda=(0)$ and $\lambda=(1)$. In the first case we get $\Delta_\lambda=1$, in the second case we get $\Delta_\lambda=1-[\mathcal{T}]$. ☐

COROLLARY 3.10. *Let $\Gamma \to X$ be the Grassmanian of 2-planes in a rank 4 vector bundle over X. Denote by $-\eta, \mu \in K(\Gamma)$ respectively the first and the second Chern classes of the tautological vector bundle on Γ. Then $K(\Gamma)$ is a free $K(X)$-module with the basis $1, \eta, \mu, \eta^2, \mu\eta, \mu^2$; moreover, the topological filtration on $K(\Gamma)$ is induced by the conditions $\eta \in K(\Gamma)^{(1)}$; $\eta^2, \mu \in K(\Gamma)^{(2)}$; $\eta\mu \in K(\Gamma)^{(3)}$; $\mu^2 \in K(\Gamma)^{(4)}$.*

PROOF. We apply Proposition 3.3 to the particular situation of Corollary. We are going to compute Δ_λ for every $(2,2)$-partition λ. We use the notation introduced above. In particular, s_0, s_1, s_2 are variables and $s_{-1} = 0 = s_3$.

- For $\lambda = (2,2)$, we have
$$\Delta_\lambda(s) = \det \begin{pmatrix} s_2 & s_3 \\ s_1 & s_2 \end{pmatrix} = s_2^2 - s_1 s_3 = s_2^2\ ; \quad \text{therefore } \Delta_{(2,2)} = \mu^2.$$
- For $\lambda = (2,1)$, we have
$$\Delta_\lambda(s) = \det \begin{pmatrix} s_2 & s_3 \\ s_0 & s_1 \end{pmatrix} = s_2 s_1 - s_0 s_3 = s_2 s_1\ ; \quad \text{therefore } \Delta_{(2,1)} = \mu\eta.$$
- For $\lambda = (2,0)$, we have
$$\Delta_\lambda(s) = \det \begin{pmatrix} s_2 & s_3 \\ s_{-1} & s_0 \end{pmatrix} = s_2 s_0 - s_{-1} s_3 = s_2 s_0\ ; \quad \text{therefore } \Delta_{(2,0)} = \mu.$$
- For $\lambda = (1,1)$, we have
$$\Delta_\lambda(s) = \det \begin{pmatrix} s_1 & s_2 \\ s_0 & s_1 \end{pmatrix} = s_1^2 - s_0 s_2\ ; \quad \text{therefore } \Delta_{(1,1)} = \eta^2 - \mu.$$
- For $\lambda = (1,0)$, we have
$$\Delta_\lambda(s) = \det \begin{pmatrix} s_1 & s_2 \\ s_{-1} & s_0 \end{pmatrix} = s_1 s_0 - s_{-1} s_2 = s_1 s_0\ ; \quad \text{therefore } \Delta_{(1,0)} = \eta.$$
- For $\lambda = (0,0)$, we have
$$\Delta_\lambda(s) = \det \begin{pmatrix} s_0 & s_1 \\ s_{-1} & s_0 \end{pmatrix} = s_0^2 - s_{-1} s_1 = s_0^2\ ; \quad \text{therefore } \Delta_{(0,0)} = 1.$$

To finish the proof, we just replace $\Delta_{(1,1)}$ by $\Delta_{(1,1)} + \Delta_{(2,0)} = \eta^2$. □

4. Pull-back to generic fiber

We fix the following notation for this Section: F is a field, X and Y are irreducible varieties over F, $\pi : X \to Y$ is a flat morphism, θ is the generic point of Y, and $X_\theta \overset{\text{def}}{=} X \times_Y \operatorname{Spec} F(\theta)$ is the fiber of π over θ. We are going to consider the pull-back with respect to the flat morphism of schemes $i\colon X_\theta \to X$.

Note that from the set-theoretical (even topological) point of view, X_θ is really the fiber of π over the point θ (see [**14**, Exercise 3.10 after §3 of Chapter II]). In particular, X_θ is a subset of X.

The group $\mathrm{CH}^*(X)$ is generated by the classes $[x]$ of points $x \in X$. The pull-back homomorphism $i^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(X_\theta)$ is determined by the following rule: if $x \notin X_\theta$ (i.e., if $\pi(x) \neq \theta$), then $i^*([x]) = 0$; if $x \in X_\theta$ (i.e., if $\pi(x) = \theta$), then $i^*([x]) = [x] \in \mathrm{CH}^*(X_\theta)$.

PROPOSITION 4.1. *The pull-back homomorphism $i^* \colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X_\theta)$ is surjective.*

PROOF. Take any generator $\alpha \overset{\text{def}}{=} [x]$ of the group $\mathrm{CH}^*(X_\theta)$, where $x \in X_\theta$. If we consider x as a point of X, we get an element $\beta \overset{\text{def}}{=} [x] \in \mathrm{CH}^*(X)$ such that $i^*(\beta) = \alpha$. $\square$

COROLLARY 4.2. *The pull-back homomorphism $i^* \colon \mathrm{G}^*K(X) \to \mathrm{G}^*K(X_\theta)$ is surjective.*

PROOF. The diagram

$$\begin{array}{ccc} \mathrm{CH}^*(X) & \longrightarrow & \mathrm{CH}^*(X_\theta) \\ \big\downarrow & & \big\downarrow \\ \mathrm{G}^*K(X) & \longrightarrow & \mathrm{G}^*K(X_\theta) \end{array}$$

where the vertical arrows are the canonical epimorphisms (see Section 1), is commutative. Since the map $\mathrm{CH}^*(X) \to \mathrm{CH}^*(X_\theta)$ is surjective (Proposition 4.1) and the map $\mathrm{CH}^*(X_\theta) \to \mathrm{G}^*K(X_\theta)$ is surjective, the map $\mathrm{G}^*K(X) \to \mathrm{G}^*K(X_\theta)$ is surjective as well. $\square$

COROLLARY 4.3. *For any $n \geq 0$, the pull-back homomorphism*

$$i^* \colon K(X)^{(n)} \to K(X_\theta)^{(n)}$$

is surjective.

PROOF. Follows from Corollary 4.2. $\square$

5. Weil transfer via Galois descent

In this Section, L/F is a finite Galois field extension of degree n with the Galois group G. All varieties are assumed to be quasi-projective.

DEFINITION 5.1. Let X be an F-variety. An *L/F-form* of X is an F-variety Y *supplied* with an isomorphism $Y_L \tilde{\to} X_L$. A *morphism* of an L/F-form Y to another L/F-form Y' of the same variety X is a morphism of F-varieties $f \colon Y \to Y'$ such that the diagram of L-morphisms

$$\begin{tikzcd} Y_L \arrow[rr, "f_L"] \arrow[dr] & & Y'_L \arrow[dl] \\ & X_L & \end{tikzcd}$$

commutes.

Let X be an F-variety. The (abstract) group $\mathrm{Aut}(X_L)$ of the automorphisms of the L-variety X_L can be supplied with a structure of G-module in the standard way (see [**79**, §1.1 de Chapitre III]): for $\tau \in G$ and $f \in \mathrm{Aut}(X_L)$ one puts $\tau(f) \stackrel{\mathrm{def}}{=} (\mathrm{id}_X \otimes \tau) \circ f \circ (\mathrm{id}_X \otimes \tau^{-1})$ where $\mathrm{id}_X \otimes \tau$ is the automorphism of the scheme X_L over F given by τ. Denote by $Z^1(G, \mathrm{Aut}(X_L)) = Z^1(L/F, \mathrm{Aut}(X_L))$ the set of 1-cocycles on G with values in $\mathrm{Aut}(X_L)$ ([**79**, §5.1 de Chapitre I]).

Any L/F-form Y of X determines a cocycle $z \in Z^1(L/F, \mathrm{Aut}(X_L))$ ([**79**, §1.3 de Chapitre III]): for any $\tau \in G$, the automorphism $z_\tau \in \mathrm{Aut}(X_L)$ is the composition

$$X_L \tilde{\to} Y_L \xrightarrow{\mathrm{id}_Y \otimes \tau} Y_L \tilde{\to} X_L \xrightarrow{\mathrm{id}_X \otimes \tau^{-1}} X_L \ .$$

Moreover, the rule described above is a 1-1-correspondence between the set of L/F-forms of X (up to the *canonical* isomorphism) and the set

$$Z^1(L/F, \mathrm{Aut}(X_L))$$

(compare to [**79**, 1.3 de Chapitre III]).

Now suppose that $X = \prod_G T$ (the product of n copies of T numbered by the elements of G), where T is a variety over F. We are going to construct a special 1-cocycle $z \in Z^1(L/F, \mathrm{Aut}(X_L))$ in this special situation.

The group S_n of the permutations of the set G (recall that $n = |G|$) can be naturally identified with a subgroup of the group $\mathrm{Aut}\, X_L$: a permutation of the set G corresponds to the automorphism of the product $X_L = \prod_G Y_L$ given by the permutation of the factors. Moreover, this way S_n is a G-submodule of $\mathrm{Aut}\, X_L$ with trivial action of G. In particular, the set $Z^1(G, S_n)$ consists of the group homomorphisms $G \to S_n$.

For any $\tau \in G$ denote by $z_\tau \in S_n$ the left translation by τ, that is the permutation $\sigma \mapsto \tau\sigma$ of the set G. The map $z : G \to S_n$ given by the rule $\tau \mapsto z_\tau$ is a group homomorphism and thus $z \in Z^1(G, S_n)$.

We shall consider z as an element of $Z^1(L/F, \mathrm{Aut}(X_L))$.

Definition 5.2. The following data are fixed: a finite Galois field extension L/F and an F-variety T. The L/F-form (see Definition 5.1) of the variety $X_L \stackrel{\mathrm{def}}{=} \prod_G T_L$ determined by the cocycle $z \in Z^1(L/F, \mathrm{Aut}(X_L))$ constructed above will be denoted by $\mathcal{R}(T)$ or $\mathcal{R}_{L/F}(T)$.

Remark 5.3. The variety $\mathcal{R}_{L/F}(T)$ is the same as the *Weil transfer* (see [**8**, 6.6 de §1 de Chapitre I] and/or [**77**, Chapter 4]) of the L-variety T_L with respect to the extension L/F. Usually, working with varieties over fields, one defines the Weil transfer for any finite separable field extension L/F and a quasi-projective L-variety. However, we are interested here only in the case where the extension L/F is Galois and the L-variety "comes from F". Definition 5.2 can be regarded as an alternative definition of the Weil transfer in this particular situation. It is more convenient for our purposes: the property of $\mathcal{R}(Y)$ we need (see Lemma 5.5 below) becomes evident.

EXAMPLE 5.4. Let us take as L/F a quadratic extension $L = F(\sqrt{d})$ with some $d \in F^*$ and as T the Severi-Brauer variety of a quaternion F-algebra (a, b). Then $\mathcal{R}(T)$ is the quadric determined by the quadratic form

$$\langle -a, -b, ab, d \rangle \ .$$

LEMMA 5.5. *Let L/F be a finite Galois field extension with the Galois group G. Let T be an F-variety. For any $\tau \in G$, the following diagram of isomorphisms commutes*

$$\begin{tikzcd} \mathcal{R}(T)_L \arrow[r, "\mathrm{id}\otimes\tau"] \arrow[d] & \mathcal{R}(T)_L \arrow[d] \\ \prod_G T_L \arrow[r, "(\mathrm{id}\otimes\tau)\circ z_\tau"] & \prod_G T_L \end{tikzcd}$$

PROOF. It is a direct consequence of Definition 5.2. $\square$

6. Galois action on Grothendieck group

In this Section, F is an arbitrary field, L/F is a field extension (e.g. a Galois extension), G is a group of automorphism of L over F (e.g. the Galois group in the case where L/F is a Galois extension), Y is an F-variety.

The group G acts on the Grothendieck group $K(Y_L)$ of the L-variety Y_L. We are interested in a condition on Y which guarantees that the action of G on $K(Y)_L$ is trivial.

LEMMA 6.1. *Suppose that the group $K(Y_L)$ is torsion-free and that the cokernel of the restriction map $\mathrm{res}_{L/F} : K(Y) \to K(Y_L)$ is a torsion group. Then the action of G on $K(Y_L)$ is trivial.*

PROOF. Take any $y \in K(Y_L)$ and any $\sigma \in G$. Since $\mathrm{Coker}(\mathrm{res}_{L/F})$ is a torsion group, some multiple ny of y is in $\mathrm{Im}(\mathrm{res}_{L/F})$, therefore $\sigma(ny) = ny$. Since the group $K(Y_L)$ is torsion-free, it follows that $\sigma(y) = y$. $\square$

Working with homogeneous varieties, we have the first condition of Lemma 6.1 for free: the group $K(Y)$ is natural (with respect to extensions of the base field F) isomorphic to $K(A)$, where A is a separable algebra (i.e. the direct product of simple algebras with centers separable over F) ([**65**, Introduction]). As to the second condition, it is equivalent to the condition that every simple component of A is central over F. We are not interested here in a complete list of homogeneous varieties satisfying this condition. We only notice that the generalized Severi-Brauer varieties are included (see [**69**, Theorem 4.1 of §8] for the case of usual Severi-Brauer varieties and/or [**50**, Theorem 4.4] for the case of generalized Severi-Brauer varieties) as well as their direct products ([**59**, §1.8]). So that we have

COROLLARY 6.2. *Let Y be a product of generalized Severi-Brauer varieties. Then the action of G on $K(Y_L)$ is trivial.* $\square$

Corollary 6.3. *Let L/F be a finite Galois extension, G its Galois group, and Y a product of generalized Severi-Brauer varieties over F. Let us identify $\mathcal{R}(Y)_L$ with $\prod_G Y_L$ (see Definition 5.2). Then for any $\sigma \in G$ the automorphism of $K(\mathcal{R}(Y)_L)$ given by σ corresponds to the automorphism of $K(\prod_G Y_L)$ given by the automorphism of the product induced by the permutation z_σ of the factors, where z_σ is the left translation by σ.*

Proof. By Lemma 5.5, the diagram

$$\begin{array}{ccc} K(\mathcal{R}(Y)_L) & \xrightarrow{\sigma} & K(\mathcal{R}(Y)_L) \\ \downarrow & & \downarrow \\ K(\prod\limits_G Y_L) & \xrightarrow{\sigma\circ z_\sigma} & K(\prod\limits_G Y_L) \end{array}$$

commutes. By Corollary 6.2, σ over the bottom arrow is the identity. $\square$

7. Product of conics

In this Section, the ground field is denoted by l in order to adopt the notation to the situation where the results of this Section will be applied. We fix an algebraic closure $\bar{l}$ of the field l. All algebras and varieties in this Section are algebras and varieties over l. For any variety X, we denote by $\bar{X}$ the $\bar{l}$-variety $X_{\bar{l}}$. For any homogeneous variety X, we identify the ring $K(X)$ with its image in $K(\bar{X})$ under the restriction homomorphism which is injective because the group $K(X)$ is torsion-free ([**65**, Introduction]).

We also need to introduce certain terminology concerning the abelian groups with filtration.

Let A be an abelian group. We consider only the descending filtration $A^{(i)}$ ($i \in \mathbb{Z}$) on A satisfying the condition $A^{(0)} = A$.

Definition 7.1. Let A be an abelian group with filtration and $a \in A$. We define the *codimension* $\operatorname{codim}_A a$ of a in A as

$$\operatorname{codim}_A a \overset{\text{def}}{=} \sup\{i \in \mathbb{Z} |\ a \in A^{(i)}\}\ .$$

Definition 7.2. Let A be an abelian group with filtration. A system of generators $a_1, \dots, a_n$ of A is called *filtering* if for any $i \in \mathbb{Z}$ the term $A^{(i)}$ is generated by a subsystem of $a_1, \dots, a_n$ (and hence, generated by the subsystem of the elements of codimensions $\geq i$). A filtering system is called a *filtering basis* if the abelian group A is free and $a_1, \dots, a_n$ form its basis.

Clearly, to determine a filtration on A, it suffices to give a filtering system of generators with their codimensions. Giving a filtering system of generators, we shall first write down its elements of codimension 0; then, after the ";"-sign, the elements of codimension 1; and so on.

Lemma 7.3. *Let Q be a split quaternion algebra and let Y be its Severi-Brauer variety. Denote by $p \in K(Y)$ the class of a rational point. Then $1; p$ is a filtering basis of $K(Y)$. The multiplication in the ring $K(Y)$ is determined by the formula $p^2 = 0$.*

PROOF. Since the quaternion algebra Q is split, the variety Y is (isomorphic to) a projective line. The statement on the filtering basis follows now e.g. from Corollary 3.9 (note that the tautological vector bundle $\mathcal{T}$ corresponds to the locally free module $\mathcal{O}(-1)$ and that $p = 1 - [\mathcal{T}]$). Since $p^2 \in K(Y)^{(2)}$ and $\dim Y = 1$, it follows that $p^2 = 0$. $\square$

LEMMA 7.4. *Let Q be a quaternion division algebra and let Y be its Severi-Brauer variety. Denote by $p \in K(\bar{Y})$ the class of a rational point. Then $1; 2p$ is a filtering basis of $K(Y)$.*

PROOF. Since there exists a quadratic extension of L splitting Q, the transfer argument shows that $2p \in K(Y)^{(1)}$. Thus $K(Y)$ contains the subgroup of $K(\bar{Y})$ generated by 1 and $2p$; the index of the latter subgroup in $K(\bar{Y})$ is 2. Since there is a natural (with respect to extensions of scalars) isomorphism $K(Y) \simeq K(L) \oplus K(Q)$ ([**69**, Theorem 4.1 of §8]), the index of $K(Y)$ in $K(\bar{Y})$ equals $\operatorname{ind} Q = 2$. Consequently, $K(Y)$ coincides with the subgroup generated by 1 and $2p$.

Now we check the statement on the filtration. We have $K(Y)^{(1)} \subset K(Y) \cap K(\bar{Y})^{(1)}$ and the intersection is evidently generated by $2p$. From the other hand, we have already shown that $2p \in K(Y)^{(1)}$. Thus $K(Y)^{(1)}$ coincides with the subgroup generated by $2p$. $\square$

For two next lemmas, we fix the following notation: $Q_1, \dots, Q_n$ are quaternion algebras; for $i = 1, \dots, n$, let Y_i be the Severi-Brauer variety of Q_i and let $pr_i \colon Y_1 \times \cdots \times Y_n \to Y_i$ be the projection.

LEMMA 7.5. *Suppose that the quaternion algebras $Q_1, \dots, Q_n$ are split. Then the map*

$$pr_1^* \cdots pr_n^* \colon K(Y_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K(Y_n) \to K(Y_1 \times \cdots \times Y_n)$$

is an isomorphism of rings with filtration. In particular,

$$1;\ \{p_i\}_i;\ \{p_i p_j\}_{i<j};\ \dots;\ p_1 \cdots p_n$$

is a filtering basis of $K(Y_1 \times \cdots \times Y_n)$, where $p_i \in K(Y_i)$ are the classes of rational points.

PROOF. Since the quaternion algebras $Q_1, \dots, Q_n$ are split, the varieties $Y_1, \dots, Y_n$ are (isomorphic to) projective lines. $\square$

LEMMA 7.6. *Suppose that the quaternion algebras $Q_1, \dots, Q_n$ are such that the tensor product $Q_1 \otimes_l \cdots \otimes_l Q_n$ is a skewfield. Then the map*

$$pr_1^* \cdots pr_n^* \colon K(Y_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K(Y_n) \to K(Y_1 \times \cdots \times Y_n)$$

is an isomorphism of rings with filtration. In particular,

$$1;\ \{2p_i\}_i;\ \{4p_i p_j\}_{i<j};\ \dots;\ 2^n p_1 \cdots p_n$$

is a filtering basis of $K(Y_1 \times \cdots \times Y_n)$, where $p_i \in K(\bar{Y}_i)$ are the classes of rational points.

PROOF. The map $pr_1^* \cdots pr_n^* \colon K(Y_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K(Y_n) \to K(Y_1 \times \cdots \times Y_n)$ (which is a homomorphism of rings with filtrations) is evidently injective. To check that its image, i.e. the group $K(Y_1) \cdots K(Y_n)$, coincides with $K(Y_1 \times \cdots \times Y_n)$ it suffices to check equality of the indexes (compare to the proof of Lemma 7.4). Since by [**69**, Theorem 4.1 of §8]

$$K(Y_1 \times \cdots \times Y_n) \simeq K\big((F \times Q_1) \otimes_l \cdots \otimes_l (F \times Q_n)\big)$$

and for any $1 \leq i_1 < \cdots < i_k \leq n$ the product $Q_{i_1} \otimes \cdots \otimes Q_{i_k}$, being a division algebra, has the index 2^k, the index of $K(Y_1 \times \cdots \times Y_n)$ in $K(\bar{Y}_1 \times \cdots \times \bar{Y}_n)$ coincides with the index of $K(Y_1) \cdots K(Y_n)$.

Now we check the statement on the filtrations. For any $p \geq 0$, the following inclusions are evident:

$$\begin{aligned}\big(K(Y_1) \cdots K(Y_n)\big)^{(p)} \stackrel{\text{def}}{=} \sum_{i_1 + \cdots + i_n = p} K(Y_1)^{(i_1)} \cdots K(Y_n)^{(i_n)} \subset \\ \subset K(Y_1 \times \cdots \times Y_n)^{(p)} \subset K(Y_1 \times \cdots \times Y_n) \cap K(\bar{Y}_1 \times \cdots \times \bar{Y}_n)^{(p)} .\end{aligned}$$

Since the first term coincides with the last one, we are done. $\square$

Let $\hat{Y} \stackrel{\text{def}}{=} \mathrm{SB}(Q)$ and $\check{Y} \stackrel{\text{def}}{=} \mathrm{SB}(Q^{\mathrm{op}})$, where Q is a quaternion algebra. The canonical vector bundle $\hat{\mathcal{I}}$ on $\hat{Y}$ is a right $Q_{\hat{Y}}$-module while the canonical vector bundle $\check{\mathcal{I}}$ on $\check{Y}$ is a left $Q_{\check{Y}}$-module. Denote by $\mathcal{E}$ the tensor product $\hat{\mathcal{I}} \otimes_{Q_{\hat{Y} \times \check{Y}}} \check{\mathcal{I}}$. It is a vector bundle of rank 1. Let $\hat{p}$ and $\check{p}$ be the classes of rational points on $\bar{\hat{Y}}$ and $\bar{\check{Y}}$.

LEMMA 7.7. *In the notation introduced right above, the class of $\mathcal{E}$ in $K(\hat{Y} \times \check{Y})$ equals $(1 - \hat{p})(1 - \check{p})$.*

PROOF. First note that $4[\mathcal{E}] = [\hat{\mathcal{I}}] \cdot [\check{\mathcal{I}}]$, because $4 = \dim_l Q$. For the rest of the proof we assume that Q is split. The varieties $\hat{Y}$, $\check{Y}$ are (isomorphic to) projective spaces and $[\hat{\mathcal{I}}] = 2[\mathcal{O}_{\hat{Y}}(-1)]$, $[\check{\mathcal{I}}] = 2[\mathcal{O}_{\check{Y}}(-1)]$ ([**69**, §8.4]). Finally, since $\hat{p}$ is the class of a hyperplane, we have $\hat{p} = 1 - [\mathcal{O}_{\hat{Y}}(-1)]$. Analogously, $\check{p} = 1 - [\mathcal{O}_{\check{Y}}(-1)]$. So, we get the formula $4[\mathcal{E}] = 4(1 - \hat{p})(1 - \check{p})$. Since the Grothendieck group is torsion-free, one can divide by 4. $\square$

8. Preliminary calculations I

The goal of this Section is Proposition 8.5.

In this Section, the ground field is denoted by l in order to adopt the notation to the situation where the results of this Section will be applied.

We are going to treat a rather special situation which will occur in Section 10. Let $\hat{Q}_1$ and $\hat{Q}_2$ be quaternion division l-algebras such that the tensor product $\hat{Q} \overset{\text{def}}{=} \hat{Q}_1 \otimes_l \hat{Q}_2$ is a skew-field. For $i = 1, 2$, we put

$$
\begin{array}{llllll}
\check{Q}_i & \overset{\text{def}}{=} \hat{Q}_i^{\,\mathrm{op}}, & \check{Q} & \overset{\text{def}}{=} \hat{Q}^{\,\mathrm{op}} & & \\
\hat{Y}_i & \overset{\text{def}}{=} \mathrm{SB}(\hat{Q}_i), & \check{Y}_i & \overset{\text{def}}{=} \mathrm{SB}(\check{Q}_i), & & \\
\hat{T} & \overset{\text{def}}{=} \mathrm{SB}(2, \check{Q}), & \check{T} & \overset{\text{def}}{=} \mathrm{SB}(2, \hat{Q}), & & \\
X_i & \overset{\text{def}}{=} \hat{Y}_i \times \check{Y}_i, & X & \overset{\text{def}}{=} X_1 \times X_2, & \mathfrak{X} & \overset{\text{def}}{=} X \times \hat{T} \times \check{T}\,.
\end{array}
$$

Let $\hat{\mathcal{I}}_1$ be the canonical vector bundle on $\hat{Y}_1$, $\hat{\mathcal{I}}_2$ the canonical vector bundle on $\hat{Y}_2$, and $\hat{\mathcal{J}}$ the canonical vector bundle on $\hat{T}$. The tensor product $\hat{\mathcal{I}}_1 \otimes \hat{\mathcal{I}}_2$ of the vector bundles over $\mathfrak{X}$ is a right $\hat{Q}_{\mathfrak{X}}$-module, while $\hat{\mathcal{J}}$ is a left $\hat{Q}_{\mathfrak{X}}$-module. Put $\hat{\mathcal{T}} \overset{\text{def}}{=} (\hat{\mathcal{I}}_1 \otimes \hat{\mathcal{I}}_2) \otimes_{\hat{Q}_{\mathfrak{X}}} \hat{\mathcal{J}}$ and denote by $-\hat{\eta}, \hat{\mu} \in K(\mathfrak{X})$ the first and the second Chern classes of $\hat{\mathcal{T}}$.

Let $\check{\mathcal{I}}_1$ be the canonical vector bundle on $\check{Y}_1$, $\check{\mathcal{I}}_2$ the canonical vector bundle on $\check{Y}_2$, and $\check{\mathcal{J}}$ the canonical vector bundle on $\check{T}$. The tensor product $\check{\mathcal{I}}_1 \otimes \check{\mathcal{I}}_2$ of the vector bundles over $\mathfrak{X}$ is a right $\check{Q}_{\mathfrak{X}}$-module, while $\check{\mathcal{J}}$ is a left $\check{Q}_{\mathfrak{X}}$-module. Put $\check{\mathcal{T}} \overset{\text{def}}{=} (\check{\mathcal{I}}_1 \otimes \check{\mathcal{I}}_2) \otimes_{\check{Q}_{\mathfrak{X}}} \check{\mathcal{J}}$ and denote by $-\check{\eta}, \check{\mu} \in K(\mathfrak{X})$ the first and the second Chern classes of $\check{\mathcal{T}}$.

By $\hat{p}_1$, $\hat{p}_2$, $\check{p}_1$, and $\check{p}_2$ we denote the classes of rational points on $\overline{\hat{Y}}_1$, $\overline{\hat{Y}}_2$, $\overline{\check{Y}}_1$, and $\overline{\check{Y}}_2$.

Proposition 8.1. *A filtering basis of the group $K(\mathfrak{X})$ is given by the products of elements of the following table such that from every column at most one element is taken:*

1	$2\hat{p}_1$	$2\hat{p}_2$	$\hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1$	$\hat{p}_2 + \check{p}_2 - \hat{p}_2\check{p}_2$	$\hat{\eta}$	$\check{\eta}$
2					$\hat{\mu}, \hat{\eta}^2$	$\check{\mu}, \check{\eta}^2$
3					$\hat{\mu}\hat{\eta}$	$\check{\mu}\check{\eta}$
4					$\hat{\mu}^2$	$\check{\mu}^2$

The codimension of an element in the table is the number of the line where it is placed; the codimension of an element of the filtering basis is the sum of the codimensions of the factors.

Proof. By Corollary 2.4, the projection $\hat{Y}_1 \times \hat{Y}_2 \times \hat{T} \to \hat{Y}_1 \times \hat{Y}_2$ is a grassmanian. Thus $\mathfrak{X} \to X \times \check{T}$ is a grassmanian as well (this morphism is obtained from the previous one by a base change). More precisely, it is the grassmanian of 2-planes in a rank 4 vector bundle. Moreover, by Corollary 2.4, $\hat{\mathcal{T}}$ is the tautological vector bundle of this grassmanian. Therefore, by Corollary 3.10, $K(\mathfrak{X})$ is a free $K(X \times \check{T})$-module with the basis $1, \hat{\eta}, \hat{\mu}, \hat{\eta}^2, \hat{\mu}\hat{\eta}, \hat{\mu}^2$ and the topological filtration on $K(\mathfrak{X})$ is induced by the conditions $\hat{\eta} \in K(\mathfrak{X})^{(1)}$; $\hat{\eta}^2, \hat{\mu} \in K(\mathfrak{X})^{(2)}$; $\hat{\eta}\hat{\mu} \in K(\mathfrak{X})^{(3)}$; $\hat{\mu}^2 \in K(\mathfrak{X})^{(4)}$.

We have reduced the problem of computation of the group $K(\mathfrak{X})$ with the filtration to the similar problem for $K(X \times \check{T})$.

By Corollary 2.4, the projection $\check{Y}_1 \times \check{Y}_2 \times \check{T} \to \check{Y}_1 \times \check{Y}_2$ is a grassmanian. Thus $X \times \check{T} \to X$ is a grassmanian as well (this morphism is obtained from the previous one by a base change). More precisely, it is (once again) the grassmanian of 2-planes in a rank 4 vector bundle (Corollary 2.4). Moreover, $\check{\mathcal{T}}$ is the tautological vector bundle of this grassmanian. Therefore, by Corollary 3.10, $K(X \times \check{T})$ is a free $K(X)$-module with the basis $1, \check{\eta}, \check{\mu}, \check{\eta}^2, \check{\mu}\check{\eta}, \check{\mu}^2$ and the topological filtration on $K(X \times \check{T})$ is induced by the conditions $\check{\eta} \in K(X \times \check{T})^{(1)}$; $\check{\eta}^2, \check{\mu} \in K(X \times \check{T})^{(2)}$; $\check{\eta}\check{\mu} \in K(X \times \check{T})^{(3)}$; $\check{\mu}^2 \in K(X \times \check{T})^{(4)}$.

We have reduced the problem of computation of the group $K(X \times \check{T})$ with the filtration to the similar problem for $K(X)$.

By Proposition 3.3, the projection $\hat{Y}_2 \times \check{Y}_2 \to \hat{Y}_2$ is a projective line bundle. Thus $X \to X_1 \times \hat{Y}_2$ is a projective line bundle as well (this morphism is obtained from the previous one by a base change). Moreover, according to Lemma 7.7, the class of the tautological vector bundle on this grassmanian equals $(1-\hat{p}_2)(1-\check{p}_2)$. Consequently, by Corollary 3.9, $K(X)$ is a free $K(X_1 \times \hat{Y}_2)$-module with the basis $1, \hat{p}_2 + \check{p}_2 - \hat{p}_2\check{p}_2$; the topological filtration on $K(X)$ is induced by the condition $\hat{p}_2 + \check{p}_2 - \hat{p}_2\check{p}_2 \in K(X)^{(1)}$.

We have reduced the problem of computation of the group $K(X)$ with the filtration to the similar problem for $K(X_1 \times \hat{Y}_2)$.

By Proposition 3.3, the projection $X_1 \to \hat{Y}_1$ is a projective line bundle. Thus $X_1 \times \hat{Y}_2 \to \hat{Y}_1 \times \hat{Y}_2$ is a projective line bundle as well. Moreover, according to Lemma 7.7, the class of the tautological vector bundle on this grassmanian equals $(1-\hat{p}_1)(1-\check{p}_1)$. Consequently, by Corollary 3.9, $K(X_1 \times \hat{Y}_2)$ is a free $K(\hat{T}_1 \times \hat{Y}_2)$-module with the basis $1, \hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1$; the topological filtration on $K(X)$ is induced by the condition $\hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1 \in K(X_1 \times \hat{Y}_2)^{(1)}$.

We have reduced the problem of computation of the group $K(X_1 \times \hat{Y}_2)$ with the filtration to the similar problem for $K(\hat{Y}_1 \times \hat{Y}_2)$.

According to Lemma 7.6, the elements 1; $2\hat{p}_1, 2\hat{p}_2$; $4\hat{p}_1\hat{p}_2$ form a filtering basis of $K(\hat{Y}_1 \times \hat{Y}_2)$. $\square$

A quaternion algebra has a canonical antiautomorphism (the canonical simplectic involution). For $i = 1, 2$, via this isomorphism, we can identify the algebras $\hat{Q}_i$ and $\check{Q}_i$. Taking the product of the antiautomorphisms, we identify the algebras $\hat{Q}$ and $\overset{\vee}{Q}$ as well. Thus the following varieties are identified: $\hat{Y}_1$ and $\overset{\vee}{Y}_1$, $\hat{Y}_2$ and $\overset{\vee}{Y}_2$, $\overset{\vee}{T}$ and $\hat{T}$.

Denote by s the automorphism of

$$\mathfrak{X} = \hat{Y}_1 \times \overset{\vee}{Y}_1 \times \hat{Y}_2 \times \overset{\vee}{Y}_2 \times \hat{T} \times \overset{\vee}{T}$$

given by the permutation of the factors interchanging every $\wedge$-factor with the corresponding $\vee$-factor. The induced ring automorphism of $K(\mathfrak{X})$ will be also denoted by s.

LEMMA 8.2. *Application of s to an element of the filtering basis of $K(\mathfrak{X})$ given in Proposition 8.1 changes every $\wedge$-sign to $\vee$-sign and vise versa.*

PROOF. Clearly, the vector bundles $\overset{\wedge}{\mathcal{T}}$ and $\overset{\vee}{\mathcal{T}}$ are interchanged by s. Thus, the following elements of $K(\mathfrak{X})$ are interchanged by s: $\overset{\wedge}{\eta}$ and $\overset{\vee}{\eta}$; $\overset{\wedge}{\mu}$ and $\overset{\vee}{\mu}$.

We can also consider s as an automorphism of $K(\bar{\mathfrak{X}})$. Clearly, the following elements of $K(\bar{\mathfrak{X}})$ are interchanged by s: $\hat{p}_1$ and $\overset{\vee}{p}_1$; $\hat{p}_2$ and $\overset{\vee}{p}_2$. $\square$

REMARK 8.3. Note that unfortunately s is *not* given by a permutation of the basis of $K(\mathfrak{X})$ (although it is "almost" so): for instance, $s(2\hat{p}_1) = 2\overset{\vee}{p}_1$ is not a basis element while $2\hat{p}_1$ is. If x is a basis element not containing $2\hat{p}_i$ $(i = 1, 2)$ as a factor, then $s(x)$ *is* a basis element.

Denote by L the function field $l(\hat{T} \times \overset{\vee}{T})$ of the variety $\hat{T} \times \overset{\vee}{T}$. We are going to work with the pull-back homomorphism $K(\mathfrak{X}) \to K(X_L)$. First we calculate it in terms of the basis of $K(\mathfrak{X})$. Since it is a homomorphism of $K(X)$-algebras, it suffices to calculate the images of $\overset{\wedge}{\eta}$, $\overset{\wedge}{\mu}$, $\overset{\vee}{\eta}$, and $\overset{\vee}{\mu}$.

LEMMA 8.4. *For the pull-back $K(\mathfrak{X}) \to K(X_L)$, one has*

$$\begin{array}{lll} \overset{\wedge}{\eta} & \mapsto & 2(\hat{p}_1 + \hat{p}_2 - \hat{p}_1\hat{p}_2)\ ; \qquad \overset{\wedge}{\mu} \mapsto 2\hat{p}_1\hat{p}_2 \\ \overset{\vee}{\eta} & \mapsto & 2(\overset{\vee}{p}_1 + \overset{\vee}{p}_2 - \overset{\vee}{p}_1\overset{\vee}{p}_2)\ ; \qquad \overset{\vee}{\mu} \mapsto 2\overset{\vee}{p}_1\overset{\vee}{p}_2 \end{array}$$

PROOF. It suffices to check the statement over an extension of the base field. Thus we may assume that the algebras $\hat{Q}_1$ and $\hat{Q}_2$ are split.

Since $\overset{\wedge}{\mathcal{T}} \overset{\text{def}}{=} (\overset{\wedge}{\mathcal{I}}_1 \otimes \overset{\wedge}{\mathcal{I}}_2) \otimes_{\hat{Q}_{\mathfrak{X}}} \overset{\wedge}{\mathcal{J}}$ and $\dim_F \hat{Q} = 16$, we have $16[\overset{\wedge}{\mathcal{T}}] = [\overset{\wedge}{\mathcal{I}}_1] \cdot [\overset{\wedge}{\mathcal{I}}_2] \cdot [\overset{\wedge}{\mathcal{J}}]$. Applying the pull-back to the right-hand side product, we get $[\overset{\wedge}{\mathcal{I}}_1] \cdot [\overset{\wedge}{\mathcal{I}}_2] \cdot 8$ because the rank of the vector bundle $\overset{\wedge}{\mathcal{J}}$ equals 8. Since $\hat{Y}_1$ and $\hat{Y}_2$ are projective lines, for $i = 1, 2$, we have $[\overset{\wedge}{\mathcal{I}}_i] = 2\overset{\wedge}{\xi}_i$, where $\overset{\wedge}{\xi}_i \overset{\text{def}}{=} [\mathcal{O}_{\hat{Y}_i}(-1)]$.

Since the Chern classes are compatible with the pull-back, it follows that the images of $\overset{\wedge}{\eta}$ and of $\overset{\wedge}{\mu}$ are respectively the first and the second Chern classes of $2\overset{\wedge}{\xi}_1\overset{\wedge}{\xi}_2$.

Let us compute the total Chern class c_t of $2\overset{\wedge}{\xi}_1\overset{\wedge}{\xi}_2$:

$$c_t(2\overset{\wedge}{\xi}_1\overset{\wedge}{\xi}_2) = \big(c_t(\overset{\wedge}{\xi}_1\overset{\wedge}{\xi}_2)\big)^2 = \big(1 + (\overset{\wedge}{\xi}_1\overset{\wedge}{\xi}_2 - 1)t\big)^2\ .$$

Therefore, the first Chern class equals $2(\hat{\xi}_1\hat{\xi}_2 - 1)$ and the second Chern class equals $(\hat{\xi}_1\hat{\xi}_2 - 1)^2$. Substituting $\hat{\xi}_i = 1 - \hat{p}_i$ we get the statement on $\hat{\eta}$ and $\hat{\mu}$ required (remember that by the definition, $\hat{\eta}$ is the first Chern class with *minus*).

The statement on $\check{\eta}$ and $\check{\mu}$ is obtained in the similar way. □

Let $K(\mathfrak{X})^{(4)\,s} \subset K(\mathfrak{X})^{(4)}$ be the subgroup of the s-invariant elements.

Let p be the class of a rational point on $\bar{X}$. Note that $p = \hat{p}_1\check{p}_1\hat{p}_2\check{p}_2 \in K(\bar{X})$.

Proposition 8.5. $2p \notin \operatorname{Im}(K(\mathfrak{X})^{(4)\,s} \to K(X_L))$.

Proof. We are going to work with the following composition:

$$\beta\colon K(\mathfrak{X}) \to K(X_L) \xrightarrow{\operatorname{res}_{\bar{L}/L}} K(X_{\bar{L}}) \twoheadrightarrow K(X_{\bar{L}})/4\cdot K(X_{\bar{L}}),$$

where $\bar{L}$ is an algebraic closure of L. We are going to show that $\beta\big(K(\mathfrak{X})^{(4)^s}\big) = 0$. Since the class of $2p$ in the quotient $K(X_{\bar{L}})/4{\cdot}K(X_{\bar{L}})$ is non-zero (see Lemma 7.5), the affirmation of Proposition will then follow.

A filtering basis of the group $K(\mathfrak{X})$ is given in Proposition 8.1. The elements of this basis having codimensions ≥ 4 form a basis of the term $K(\mathfrak{X})^{(4)}$ (see Definition 7.2). The basis of $K(\mathfrak{X})^{(4)}$ contains the elements

$$\hat{\xi} \overset{\mathrm{def}}{=} (\hat{p}_1{+}\check{p}_1{-}\hat{p}_1\check{p}_1)(\hat{p}_2{+}\check{p}_2{-}\hat{p}_2\check{p}_2)\hat{\mu} \quad\text{and}\quad \check{\xi} \overset{\mathrm{def}}{=} (\hat{p}_1{+}\check{p}_1{-}\hat{p}_1\check{p}_1)(\hat{p}_2{+}\check{p}_2{-}\hat{p}_2\check{p}_2)\check{\mu}\,.$$

Denote by H the subgroup of $K(\mathfrak{X})^{(4)}$ generated by all basis elements except $\hat{\xi}$ and $\check{\xi}$.

Lemma 8.6. *The sum $H + 2K(\mathfrak{X})^{(4)}$ lies in* $\operatorname{Ker}\beta$ *and is s-invariant.*

Proof. Every basis element is a product of the following elements where in the first column the codimension of the element in $K(\mathfrak{X})$ (see Definition 7.1) is given; in the last column the image under β of the element is given (see Lemma 8.4):

(codim $= 1$)	$2\hat{p}_1$	$\mapsto$	$2\hat{p}_1$
(codim $= 1$)	$2\hat{p}_2$	$\mapsto$	$2\hat{p}_2$
(codim $= 1$)	$\hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1$	$\mapsto$	$\hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1$
(codim $= 1$)	$\hat{p}_2 + \check{p}_2 - \hat{p}_1\check{p}_2$	$\mapsto$	$\hat{p}_2 + \check{p}_2 - \hat{p}_2\check{p}_2$
(codim $= 1$)	$\hat{\eta}$	$\mapsto$	$2(\hat{p}_1 + \hat{p}_2 - \hat{p}_1\hat{p}_2)$
(codim $= 1$)	$\check{\eta}$	$\mapsto$	$2(\check{p}_1 + \check{p}_2 - \check{p}_1\check{p}_2)$
(codim $= 2$)	$\hat{\mu}$	$\mapsto$	$2\hat{p}_1\hat{p}_2$
(codim $= 2$)	$\check{\mu}$	$\mapsto$	$2\check{p}_1\check{p}_2$

The image of each element of the table but that of $\hat{p}_1{+}\check{p}_1{-}\hat{p}_1\check{p}_1$ and $\hat{p}_2{+}\check{p}_2{-}\hat{p}_2\check{p}_2$ is divisible by 2. If we like to form a product which is a basic element with a non-zero image under β, we are allowed to take no more than one copy of $\hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1$, no more than one copy of $\hat{p}_2 + \check{p}_2 - \hat{p}_2\check{p}_2$, and no more than one other element of the table. Therefore, the only generators of $K(\mathfrak{X})^{(4)}$, which have a non-zero image with respect to β, are $\overset{\wedge}{\xi}$ and $\overset{\vee}{\xi}$. So, $\beta(H) = 0$. Moreover, since $\beta(\overset{\wedge}{\xi}) = 2p = \beta(\overset{\vee}{\xi})$, it follows that $\beta(2\overset{\wedge}{\xi}) = 0 = \beta(2\overset{\vee}{\xi})$. Thus $H + 2K(\mathfrak{X})^{(4)} \subset \operatorname{Ker}\beta$.

Since $2K(\mathfrak{X})^{(4)}$ is evidently s-invariant, it suffices to check the inclusion

$$s(H) \subset H + 2K(\mathfrak{X})^{(4)} .$$

Take a basis element of H. It is a product of the elements in the table of Proposition 8.1. Therefore, it is either x, either $(2\hat{p}_1)x$, either $(2\hat{p}_2)x$, either $(2\hat{p}_1)(2\hat{p}_2)x$, where x is a basis element of $K(\mathfrak{X})$ not containing $2\hat{p}_i$ $(i = 1, 2)$. Since $s(x)$ is again a basis element of H, we have no problem in the first case.

Set $h_i \overset{\text{def}}{=} \hat{p}_i + \overset{\vee}{p}_i - \hat{p}_i\overset{\vee}{p}_i$. Note that $\overset{\vee}{p}_i = h_i - \hat{p}_i + \hat{p}_i h_i$ (here the relation of Lemma 7.3 $\hat{p}_i^{\,2} = 0$ is used).

In the second and in the third cases, we have (here again $i = 1, 2$):

$$s(2\hat{p}_i x) = 2\overset{\vee}{p}_i s(x) = 2h_i s(x) - 2\hat{p}_i s(x) + 2\hat{p}_i h_i s(x) .$$

The first summand is in $2K(\mathfrak{X})^{(4)}$, the second summand is in H, the third summand is in $K(\mathfrak{X})^{(5)} \subset H$.

Finally, in the fourth case, we have

$$\begin{aligned} s((2\hat{p}_1)(2\hat{p}_2)x) &= (2\overset{\vee}{p}_1)(2\overset{\vee}{p}_2)s(x) = \\ &= \big(2h_1 + (2\hat{p}_1)(h_1 - 1)\big)\big(2h_2 + (2\hat{p}_2)(h_2 - 1)\big)s(x) \equiv \\ &\equiv (2\hat{p}_1)(2\hat{p}_2)(h_1 - 1)(h_2 - 1)s(x) \equiv (2\hat{p}_1)(2\hat{p}_2)s(x) \in H \end{aligned}$$

where the first congruence is modulo $2K(\mathfrak{X})^{(4)}$ and the second congruence is modulo $K(\mathfrak{X})^{(5)} \subset H$. □

Denote by $\widetilde{K(\mathfrak{X})^{(4)}}$ the quotient $K(\mathfrak{X})^{(4)}/(H + 2K(\mathfrak{X})^{(4)})$. According to Lemma 8.6, β determines a homomorphism of $\widetilde{K(\mathfrak{X})^{(4)}}$ and s determines an automorphism of $\widetilde{K(\mathfrak{X})^{(4)}}$. To show that $\beta((K(\mathfrak{X})^{(4)})^s) = 0$ it suffices to show that

$$\beta((\widetilde{K(\mathfrak{X})^{(4)}})^s) = 0 .$$

The group $\widetilde{K(\mathfrak{X})^{(4)}}$ is generated by $\overset{\wedge}{\xi}$ and $\overset{\vee}{\xi}$ subject to the only relations $2\overset{\wedge}{\xi} = 0$, $2\overset{\vee}{\xi} = 0$. These two generators are interchanged by s. Therefore, the group $(\widetilde{K(\mathfrak{X})^{(4)}})^s$ is generated by $\overset{\wedge}{\xi} + \overset{\vee}{\xi}$. Since $\beta(\overset{\wedge}{\xi} + \overset{\vee}{\xi}) = 2p + 2p = 0$, we get the relation required. □

9. Preliminary calculations II

The goal of this Section is Proposition 9.4.

In this Section, the ground field is denoted by l in order to adopt the notation to the situation where the results of this Section will be applied.

We are going to treat a special situation which will occur in Section 11. Let $\hat{Q}_1$ and $\hat{Q}_2$ be quaternion division l-algebras such that the tensor product $\overset{\wedge\wedge}{Q} \overset{\text{def}}{=} \hat{Q}_1 \otimes_l \hat{Q}_2$ is a skew-field. For $i = 1, 2$, we put

$$
\begin{array}{llllll}
\overset{\vee}{Q}_i & \overset{\text{def}}{=} \hat{Q}_i{}^{\text{op}}, & \overset{\vee\vee}{Q} & \overset{\text{def}}{=} \overset{\vee}{Q}_1 \otimes \overset{\vee}{Q}_2 & & \\
\overset{\wedge\vee}{Q} & \overset{\text{def}}{=} \hat{Q}_1 \otimes \overset{\vee}{Q}_2 & \overset{\vee\wedge}{Q} & \overset{\text{def}}{=} \overset{\vee}{Q}_1 \otimes \hat{Q}_2 & & \\
\hat{Y}_i & \overset{\text{def}}{=} \mathrm{SB}(\hat{Q}_i), & \overset{\vee}{Y}_i & \overset{\text{def}}{=} \mathrm{SB}(\overset{\vee}{Q}_i), & & \\
\overset{\wedge\wedge}{T} & \overset{\text{def}}{=} \mathrm{SB}(2, \overset{\vee\vee}{Q}), & \overset{\vee\vee}{T} & \overset{\text{def}}{=} \mathrm{SB}(2, \overset{\wedge\wedge}{Q}), & & \\
\overset{\wedge\vee}{T} & \overset{\text{def}}{=} \mathrm{SB}(2, \overset{\vee\wedge}{Q}), & \overset{\vee\wedge}{T} & \overset{\text{def}}{=} \mathrm{SB}(2, \overset{\wedge\vee}{Q}), & & \\
X_i & \overset{\text{def}}{=} \hat{Y}_i \times \overset{\vee}{Y}_i, & X & \overset{\text{def}}{=} X_1 \times X_2, & \mathfrak{X} & \overset{\text{def}}{=} X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}\,.
\end{array}
$$

Let $\overset{\wedge}{\mathcal{I}}_1$ be the canonical vector bundle on $\hat{Y}_1$, $\overset{\wedge}{\mathcal{I}}_2$ the canonical vector bundle on $\hat{Y}_2$, and $\overset{\wedge\wedge}{\mathcal{J}}$ the canonical vector bundle on $\overset{\wedge\wedge}{T}$. The tensor product $\overset{\wedge}{\mathcal{I}}_1 \otimes \overset{\wedge}{\mathcal{I}}_2$ of the vector bundles over $\mathfrak{X}$ is a right $\overset{\wedge\wedge}{Q}_{\mathfrak{X}}$-module, while $\overset{\wedge\wedge}{\mathcal{J}}$ is a left $\overset{\wedge\wedge}{Q}_{\mathfrak{X}}$-module. Put $\overset{\wedge\wedge}{\mathcal{T}} \overset{\text{def}}{=} (\overset{\wedge}{\mathcal{I}}_1 \otimes \overset{\wedge}{\mathcal{I}}_2) \otimes_{\overset{\wedge\wedge}{Q}_{\mathfrak{X}}} \overset{\wedge\wedge}{\mathcal{J}}$ and denote by $-\overset{\wedge\wedge}{\eta}, \overset{\wedge\wedge}{\mu} \in K(\mathfrak{X})$ the first and the second Chern classes of $\overset{\wedge\wedge}{\mathcal{T}}$.

Let $\overset{\vee}{\mathcal{I}}_1$ be the canonical vector bundle on $\overset{\vee}{Y}_1$, $\overset{\wedge}{\mathcal{I}}_2$ the canonical vector bundle on $\overset{\vee}{Y}_2$, and $\overset{\vee\wedge}{\mathcal{J}}$ the canonical vector bundle on $\overset{\vee\wedge}{T}$. The tensor product $\overset{\vee}{\mathcal{I}}_1 \otimes \overset{\wedge}{\mathcal{I}}_2$ of the vector bundles over $\mathfrak{X}$ is a right $\overset{\vee\wedge}{Q}_{\mathfrak{X}}$-module, while $\overset{\vee\wedge}{\mathcal{J}}$ is a left $\overset{\vee\wedge}{Q}_{\mathfrak{X}}$-module. Put $\overset{\vee\wedge}{\mathcal{T}} \overset{\text{def}}{=} (\overset{\vee}{\mathcal{I}}_1 \otimes \overset{\wedge}{\mathcal{I}}_2) \otimes_{\overset{\vee\wedge}{Q}_{\mathfrak{X}}} \overset{\vee\wedge}{\mathcal{J}}$ and denote by $-\overset{\vee\wedge}{\eta}, \overset{\vee\wedge}{\mu} \in K(\mathfrak{X})$ the first and the second Chern classes of $\overset{\vee\wedge}{\mathcal{T}}$.

We define the vector bundles $\overset{\wedge\vee}{\mathcal{T}}$ and $\overset{\vee\vee}{\mathcal{T}}$ in the similar way:

$$
\overset{\wedge\vee}{\mathcal{T}} \overset{\text{def}}{=} (\overset{\wedge}{\mathcal{I}}_1 \otimes \overset{\vee}{\mathcal{I}}_2) \otimes_{\overset{\wedge\vee}{Q}_{\mathfrak{X}}} \overset{\wedge\vee}{\mathcal{J}}
$$
$$
\overset{\vee\vee}{\mathcal{T}} \overset{\text{def}}{=} (\overset{\vee}{\mathcal{I}}_1 \otimes \overset{\vee}{\mathcal{I}}_2) \otimes_{\overset{\vee\vee}{Q}_{\mathfrak{X}}} \overset{\vee\vee}{\mathcal{J}}
$$

and denote by $-\overset{\wedge\vee}{\eta}, \overset{\wedge\vee}{\mu} \in K(\mathfrak{X})$ the first and the second Chern classes of $\overset{\wedge\vee}{\mathcal{T}}$; by $-\overset{\vee\vee}{\eta}, \overset{\vee\vee}{\mu} \in K(\mathfrak{X})$ the first and the second Chern classes of $\overset{\vee\vee}{\mathcal{T}}$.

By $\hat{p}_1$, $\hat{p}_2$, $\overset{\vee}{p}_1$, and $\overset{\vee}{p}_2$ we denote the classes of rational points on $\overline{\hat{Y}}_1$, $\overline{\hat{Y}}_2$, $\overline{\overset{\vee}{Y}}_1$, and $\overline{\overset{\vee}{Y}}_2$.

Proposition 9.1. *A filtering basis of the group $K(\mathfrak{X})$ is given by the products of elements of the following table such that from every column at most one* *element is taken:*

1	$2\hat{p}_1$	$2\hat{p}_2$	$\hat{p}_1+\check{p}_1-\hat{p}_1\check{p}_1$	$\hat{p}_2+\check{p}_2-\hat{p}_2\check{p}_2$	$\overset{\wedge\wedge}{\eta}$	$\overset{\wedge\vee}{\eta}$	$\overset{\vee\wedge}{\eta}$	$\overset{\vee\vee}{\eta}$
2					$\overset{\wedge\wedge}{\mu}, \overset{\wedge\wedge}{\eta}^2$	$\overset{\wedge\vee}{\mu}, \overset{\wedge\vee}{\eta}^2$	$\overset{\vee\wedge}{\mu}, \overset{\vee\wedge}{\eta}^2$	$\overset{\vee\vee}{\mu}, \overset{\vee\vee}{\eta}^2$
3					$\overset{\wedge\wedge}{\mu}\overset{\wedge\wedge}{\eta}$	$\overset{\wedge\vee}{\mu}\overset{\wedge\vee}{\eta}$	$\overset{\vee\wedge}{\mu}\overset{\vee\wedge}{\eta}$	$\overset{\vee\vee}{\mu}\overset{\vee\vee}{\eta}$
4					$\overset{\wedge\wedge}{\mu}^2$	$\overset{\wedge\vee}{\mu}^2$	$\overset{\vee\wedge}{\mu}^2$	$\overset{\vee\vee}{\mu}^2$

The codimension of an element in the table is the number of the line where it is placed; the codimension of an element of the filtering basis is the sum of the codimensions of the factors.

PROOF. By Corollary 2.4, the projection $\check{Y}_1 \times \hat{Y}_2 \times \overset{\vee\wedge}{T} \to \check{Y}_1 \times \hat{Y}_2$ is a grassmanian. Thus $\mathfrak{X} \to X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T}$ is a grassmanian as well (this morphism is obtained from the previous one by a base change). More precisely, it is the grassmanian of 2-planes in a rank 4 vector bundle. Moreover, by Corollary 2.4, $\overset{\vee\wedge}{\mathcal{T}}$ is the tautological vector bundle of this grassmanian. Therefore, by Corollary 3.10, $K(\mathfrak{X})$ is a free $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T})$-module with the basis $1, \overset{\vee\wedge}{\eta}, \overset{\vee\wedge}{\mu}, \overset{\vee\wedge}{\eta}^2, \overset{\vee\wedge}{\mu}\overset{\vee\wedge}{\eta}, \overset{\vee\wedge}{\mu}^2$ and the topological filtration on $K(\mathfrak{X})$ is induced by the conditions $\overset{\vee\wedge}{\eta} \in K(\mathfrak{X})^{(1)}$; $\overset{\vee\wedge}{\eta}^2, \overset{\vee\wedge}{\mu} \in K(\mathfrak{X})^{(2)}$; $\overset{\vee\wedge}{\eta}\overset{\vee\wedge}{\mu} \in K(\mathfrak{X})^{(3)}$; $\overset{\vee\wedge}{\mu}^2 \in K(\mathfrak{X})^{(4)}$.

We have reduced the problem of computation of the group $K(\mathfrak{X})$ with the filtration to the similar problem for $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T})$.

By Corollary 2.4, the projection $\hat{Y}_1 \times \check{Y}_2 \times \overset{\wedge\vee}{T} \to \hat{Y}_1 \times \check{Y}_2$ is a grassmanian. Thus $X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \to X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T}$ is a grassmanian as well. More precisely, it is the grassmanian of 2-planes in a rank 4 vector bundle. Moreover, $\overset{\wedge\vee}{\mathcal{T}}$ is the tautological vector bundle of this grassmanian (Corollary 2.4). Therefore, by Corollary 3.10, $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T})$ is a free $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T})$-module with the basis $1, \overset{\wedge\vee}{\eta}, \overset{\wedge\vee}{\mu}, \overset{\wedge\vee}{\eta}^2, \overset{\wedge\vee}{\mu}\overset{\wedge\vee}{\eta}, \overset{\wedge\vee}{\mu}^2$ and the topological filtration on $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T})$ is induced by the conditions $\overset{\wedge\vee}{\eta} \in K^{(1)}$; $\overset{\wedge\vee}{\eta}^2, \overset{\wedge\vee}{\mu} \in K^{(2)}$; $\overset{\wedge\vee}{\eta}\overset{\wedge\vee}{\mu} \in K^{(3)}$; $\overset{\wedge\vee}{\mu}^2 \in K^{(4)}$.

We have reduced the problem of computation of the group $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T})$ with the filtration to the similar problem for $K(X \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T})$. A filtering basis for the latter group is given in Proposition 8.1 (note that the notation there is slightly different: namely, $\hat{\eta} = \overset{\wedge\wedge}{\eta}$, $\check{\eta} = \overset{\vee\vee}{\eta}$, $\hat{\mu} = \overset{\wedge\wedge}{\mu}$, and $\check{\mu} = \overset{\vee\vee}{\mu}$. $\square$

A quaternion algebra has a canonical antiautomorphism (the simplectic involution). For $i = 1, 2$, via this isomorphism, we can identify the algebras $\hat{Q}_i$ and $\check{Q}_i$. Taking the product of the antiautomorphisms, we identify $\overset{\wedge\wedge}{Q}$ with $\overset{\vee\vee}{Q}$ and $\overset{\wedge\vee}{Q}$ with $\overset{\vee\wedge}{Q}$. Thus the following varieties are identified: $\hat{Y}_1$ and $\check{Y}_1$, $\hat{Y}_2$ and $\check{Y}_2$, $\overset{\vee\vee}{T}$ and $\overset{\wedge\wedge}{T}$, $\overset{\vee\wedge}{T}$ and $\overset{\wedge\vee}{T}$.

Denote by s_1 the automorphism of

$$\mathfrak{X} = \hat{Y}_1 \times \overset{\vee}{Y}_1 \times \hat{Y}_2 \times \overset{\vee}{Y}_2 \times \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$$

given by the permutation of the factors interchanging $\hat{Y}_1$ with $\overset{\vee}{Y}_1$, $\overset{\wedge\wedge}{T}$ with $\overset{\vee\wedge}{T}$, and $\overset{\wedge\vee}{T}$ with $\overset{\vee\vee}{T}$ (and leaving $\hat{Y}_2$ and $\overset{\vee}{Y}_2$ untouched).

Denote by s_2 the automorphism of $\mathfrak{X}$ given by the permutation of the factors interchanging $\hat{Y}_2$ with $\overset{\vee}{Y}_2$, $\overset{\wedge\wedge}{T}$ with $\overset{\wedge\vee}{T}$, and $\overset{\vee\wedge}{T}$ with $\overset{\vee\vee}{T}$ (and leaving $\hat{Y}_1$ and $\overset{\vee}{Y}_1$ untouched).

The induced ring automorphisms of $K(\mathfrak{X})$ will be also denoted by s_1 and s_2.

LEMMA 9.2. *Application of s_1 to an element of the filtering basis of $K(\mathfrak{X})$ given in Proposition 9.1 changes every "first" $^\wedge$-sign (i.e. a $^\wedge$-sign placed over p_1 or a $^\wedge$-sign placed on the first place over η or over μ) to the $^\vee$-sign and vise versa: every "first" $^\vee$-sign to the $^\wedge$-sign; the "second" $^\wedge$-and-$^\vee$-signs are left untouched. Application of s_2 to an element of the filtering basis changes every "second" $^\wedge$-sign to the $^\vee$-sign and vise versa; the "first" $^\wedge$-and-$^\vee$-signs are left untouched.*

PROOF. We prove only the statement on s_1.

Clearly, s_1 interchanges $\overset{\wedge\wedge}{\mathcal{T}}$ with $\overset{\vee\wedge}{\mathcal{T}}$ and $\overset{\wedge\vee}{\mathcal{T}}$ with $\overset{\vee\vee}{\mathcal{T}}$. Consequently, the following elements of $K(\mathfrak{X})$ are interchanged by s_1: $\overset{\wedge\wedge}{\eta}$ and $\overset{\vee\wedge}{\eta}$; $\overset{\wedge\vee}{\eta}$ and $\overset{\vee\vee}{\eta}$; $\overset{\wedge\wedge}{\mu}$ and $\overset{\vee\wedge}{\mu}$; $\overset{\wedge\vee}{\mu}$ and $\overset{\vee\vee}{\mu}$.

We can also consider s_1 as an automorphism of $K(\bar{\mathfrak{X}})$. Clearly, the elements $\hat{p}_1$ and $\overset{\vee}{p}_1$ of $K(\bar{\mathfrak{X}})$ are interchanged by s while the elements $\hat{p}_2$ and $\overset{\vee}{p}_2$ are left untouched. $\square$

Denote by L the function field of the variety $\overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$. We are going to work with the pull-back homomorphism $K(\mathfrak{X}) \to K(X_L)$. First we calculate it in terms of the basis of $K(\mathfrak{X})$. Since it is a homomorphism of $K(X)$-algebras, it suffices to calculate the images of $\overset{\wedge\wedge}{\eta}$, $\overset{\vee\vee}{\eta}$, $\overset{\wedge\vee}{\eta}$, $\overset{\vee\wedge}{\eta}$, $\overset{\wedge\wedge}{\mu}$, $\overset{\vee\vee}{\mu}$, $\overset{\wedge\vee}{\mu}$, and $\overset{\vee\wedge}{\mu}$.

LEMMA 9.3. *For the pull-back $K(\mathfrak{X}) \to K(X_L)$, one has*

$$\begin{array}{lll@{\qquad\qquad}lll}
\overset{\wedge\wedge}{\eta} & \mapsto & 2(\hat{p}_1 + \hat{p}_2 - \hat{p}_1\hat{p}_2)\ ; & \overset{\wedge\wedge}{\mu} & \mapsto & 2\hat{p}_1\hat{p}_2 \\
\overset{\vee\vee}{\eta} & \mapsto & 2(\overset{\vee}{p}_1 + \overset{\vee}{p}_2 - \overset{\vee}{p}_1\overset{\vee}{p}_2)\ ; & \overset{\vee\vee}{\mu} & \mapsto & 2\overset{\vee}{p}_1\overset{\vee}{p}_2 \\
\overset{\wedge\vee}{\eta} & \mapsto & 2(\hat{p}_1 + \overset{\vee}{p}_2 - \hat{p}_1\overset{\vee}{p}_2)\ ; & \overset{\wedge\vee}{\mu} & \mapsto & 2\hat{p}_1\overset{\vee}{p}_2 \\
\overset{\vee\wedge}{\eta} & \mapsto & 2(\overset{\vee}{p}_1 + \hat{p}_2 - \overset{\vee}{p}_1\hat{p}_2)\ ; & \overset{\vee\wedge}{\mu} & \mapsto & 2\overset{\vee}{p}_1\hat{p}_2
\end{array}$$

PROOF. See the proof of Lemma 8.4. $\square$

Denote by G the subgroup of $\operatorname{Aut} K(\mathfrak{X})$ generated by s_1 and s_2. Let

$$K(\mathfrak{X})^{(3)\,G} \subset K(\mathfrak{X})^{(3)}$$

be the subgroup of the G-invariant elements.

Let p be the class of a rational point on $\bar{X}$. Note that $p = \hat{p}_1\overset{\vee}{p}_1\hat{p}_2\overset{\vee}{p}_2 \in K(\bar{X})$.

Proposition 9.4. $2p \notin \operatorname{Im}(K(\mathfrak{X})^{(3)\,G} \to K(X_L))$.

Proof. We are going to work with the following composition:

$$\beta\colon K(\mathfrak{X}) \to K(X_L) \xrightarrow{\operatorname{res}_{\bar{L}/L}} K(X_{\bar{L}}) \twoheadrightarrow K(X_{\bar{L}})/4\cdot K(X_{\bar{L}}),$$

where $\bar{L}$ is an algebraic closure of L. We are going to show that the residue class of $2p$ is not in $\beta\big((K(\mathfrak{X})^{(3)})^G\big)$.

We introduce some additional notation for elements of $K(\bar{X})$ in order to avoid repetitions of long expressions. For $i = 1, 2$, we put

$$p_i \stackrel{\text{def}}{=} \hat{p}_i\overset{\vee}{p}_i, \quad h_i \stackrel{\text{def}}{=} \hat{p}_i + \overset{\vee}{p}_i - p_i$$

(although it is not essential for the consequent, we remark that: p_i is the class of a rational point on $\bar{X}_i$; h_i is the class of a hyperplane section if we identify $\bar{X}_i$ with the quadric hypersurface in the 3-dimensional projective space via the Segre imbedding). Note that $p = p_1p_2$. We also put $h \stackrel{\text{def}}{=} h_1h_2$.

A filtering basis of the group $K(\mathfrak{X})$ is given in Proposition 9.1. The elements of this basis having codimensions ≥ 3 form a basis of the term $K(\mathfrak{X})^{(3)}$ (see Definition 7.2). Denote by H the subgroup of $K(\mathfrak{X})^{(3)}$ generated by all basis elements except the following ones:

$$2\hat{p}_i h\ ,\quad \overset{*}{\eta} h\ ,\quad \overset{*}{\mu} h\ ,\quad \overset{*}{\mu} h_i$$

where $i = 1, 2$ and $* = {\wedge\wedge}, {\wedge\vee}, {\vee\wedge}, {\vee\vee}$.

Lemma 9.5. *The sum $H + 2K(\mathfrak{X})^{(3)}$ lies in $\operatorname{Ker}\beta$ and is G-invariant.*

Proof. Every basis element is a product of the following elements where in the first column the codimension of the element in $K(\mathfrak{X})$ (see Definition 7.1) is given; in the last column the image under β of the element is given (see Lemma 8.4):

$$\begin{array}{llll}
(\text{codim} = 1) & 2\hat{p}_1 & \mapsto & 2\hat{p}_1 \\
(\text{codim} = 1) & 2\hat{p}_2 & \mapsto & 2\hat{p}_2 \\
(\text{codim} = 1) & h_1 & \mapsto & h_1 \\
(\text{codim} = 1) & h_2 & \mapsto & h_2 \\
(\text{codim} = 1) & \overset{\wedge\wedge}{\eta} & \mapsto & 2(\hat{p}_1 + \hat{p}_2 - \hat{p}_1\hat{p}_2) \\
(\text{codim} = 1) & \overset{\vee\vee}{\eta} & \mapsto & 2(\overset{\vee}{p}_1 + \overset{\vee}{p}_2 - \overset{\vee}{p}_1\overset{\vee}{p}_2) \\
(\text{codim} = 1) & \overset{\wedge\vee}{\eta} & \mapsto & 2(\hat{p}_1 + \overset{\vee}{p}_2 - \hat{p}_1\overset{\vee}{p}_2) \\
(\text{codim} = 1) & \overset{\vee\wedge}{\eta} & \mapsto & 2(\overset{\vee}{p}_1 + \hat{p}_2 - \overset{\vee}{p}_1\hat{p}_2) \\
(\text{codim} = 2) & \overset{\wedge\wedge}{\mu} & \mapsto & 2\hat{p}_1\hat{p}_2 \\
(\text{codim} = 2) & \overset{\vee\vee}{\mu} & \mapsto & 2\overset{\vee}{p}_1\overset{\vee}{p}_2 \\
(\text{codim} = 2) & \overset{\wedge\vee}{\mu} & \mapsto & 2\hat{p}_1\overset{\vee}{p}_2 \\
(\text{codim} = 2) & \overset{\vee\wedge}{\mu} & \mapsto & 2\overset{\vee}{p}_1\hat{p}_2
\end{array}$$

The image of each element of the table but that of h_1 and h_2 is divisible by 2. Easily seen, the basis elements of $K(\mathfrak{X})^{(3)}$, which have a non-zero image with respect to β, are precisely the basic elements excepted in the definition of H. In particular, $\beta(H) = 0$. Moreover, since the image under β of any excepted basis element is divisible by 2, it follows that $H + 2K(\mathfrak{X})^{(3)} \subset \operatorname{Ker}\beta$.

Now we are going to check the G-invariance. We shall check only the affirmation on s_1 (the affirmation on s_2 is checked in the similar way). Since $2K(\mathfrak{X})^{(3)}$ is evidently s_1-invariant, it suffices to check the inclusion

$$s_1(H) \subset H + 2K(\mathfrak{X})^{(3)}\ .$$

Note that the elements $2\hat{p}_1$, h_1, h_2, and h are s_1-invariant.

Take a basis element of H. It is a product of the elements in the table of Proposition 8.1. Therefore, it is either x, either $(2\hat{p}_1)x$, where x is a basis element of $K(\mathfrak{X})$ not containing $2\hat{p}_1$ as a factor. Since $s_1(x)$ is again a basis element of H, we have no problem in the first case.

In the second case, we have $x \in K(\mathfrak{X})^{(2)}$ and

$$s_1(2\hat{p}_1 x) = 2\overset{\vee}{p}_1 s_1(x) = 2h_1 s_1(x) - 2\hat{p}_1 s_1(x) + 2\hat{p}_1 h_1 s_1(x)\ .$$

The first summand is in $2K(\mathfrak{X})^{(3)}$ and the second summand is in H. We are going to show that the third summand is in H.

First suppose that x contains h_1 as a factor. Since h_1 is s_1-invariant, $s_1(x)$ contains h_1 as well. Since $h_1^2 = 2\hat{p}_1\overset{\vee}{p}_1$ and $(\hat{p}_1)^2 = 0$ (Lemma 7.3), we have $2\hat{p}_1 h_1 s_1(x) = 0$ in this case. So, consider the case where x does not contain h_1 as a factor. Since x also does not contain $2\hat{p}_1$, $s_1(x)$ contains neither h_1 nor $2\hat{p}_1$ and so, the product $2\hat{p}_1 h_1 s_1(x)$ is a basis element of $K(\mathfrak{X})^{(4)}$. If it is not in H, then it should be an excepted basis element. However the excepted basis element of the codimension 4 are $h\overset{*}{\mu}$, $* = \wedge\wedge, \wedge\vee, \vee\wedge, \vee\vee$, and $2\hat{p}_1 h_1 s_1(x)$ is not of that kind. Thus $2\hat{p}_1 h_1 s_1(x) \in H$.

$\square$

Denote by $\widetilde{K(\mathfrak{X})^{(3)}}$ the quotient $K(\mathfrak{X})^{(3)}/(H + 2K(\mathfrak{X})^{(3)})$. According to Lemma 9.5, β determines a homomorphism of $\widetilde{K(\mathfrak{X})^{(3)}}$ and the group G acts on $\widetilde{K(\mathfrak{X})^{(3)}}$. To show that $\beta(K(\mathfrak{X})^{(3)\,G}) \not\ni 2p$, it suffices to show that

$$\beta\big((\widetilde{K(\mathfrak{X})^{(3)}})^G\big) \not\ni 2p\ .$$

The group $\widetilde{K(\mathfrak{X})^{(3)}}$ is a vector $\mathbb{Z}/2$-space with the basis given by the classes of the excepted basis elements:

$$2\hat{p}_i h\ ,\quad \overset{*}{\eta} h\ ,\quad \overset{*}{\mu} h\ ,\quad \overset{*}{\mu} h_i\ .$$

It is easy to calculate their images under β:

$$(\dagger)\qquad \begin{gathered} \beta(2\hat{p}_1 h) = 2p_1h_2\ ;\quad \beta(2\hat{p}_2 h) = 2p_2h_1 \\ \beta(\overset{*}{\eta} h) = 2p_1h_2 + 2p_2h_1 + 2p\ ; \beta(\overset{*}{\mu} h) = 2p\ ; \\ \beta(\overset{\bullet\circ}{\mu} h_1) = 2p_1\overset{\circ}{p}_2\ ;\quad \beta(\overset{\bullet\circ}{\mu} h_2) = 2\overset{\bullet}{p}_1 p_2\ ; \end{gathered}$$

where $\bullet, \circ \in \{\vee, \wedge\}$.

Since for $i = 1, 2$

$$s_i(2\hat{p}_i h) = 2\overset{\vee}{p}_i s(h) = 2h_i h - 2\hat{p}_i h + 2\hat{p}_i h_i h = \\ = 2h_i h - 2\hat{p}_i h \equiv 2\hat{p}_i h \pmod{2K(\mathfrak{X})^{(3)}} ,$$

and $s_{3-i}(2\hat{p}_i h) = 2\hat{p}_i h$, the residue classes in $\widetilde{K(\mathfrak{X})^{(3)}}$ of $2\hat{p}_1 h$ and $2\hat{p}_2 h$ are G-invariant. The action of G on $\overset{*}{\eta}h$, $\overset{*}{\mu}h$, and $\overset{*}{\mu}h_i$ is described in Lemma 9.2 (G permutes these generators). Summarizing, one sees that the G-invariant part of $\widetilde{K(\mathfrak{X})^{(3)}}$ is generated by

$$2\hat{p}_1 h,\ \ 2\hat{p}_2 h,\ \ \sum_* \overset{*}{\eta}h,\ \ \sum_* \overset{*}{\mu}h,\ \ \sum_* \overset{*}{\mu}h_1,\ \ \sum_* \overset{*}{\mu}h_2.$$

Using the formulas †, one can easily compute the images of these generators under β:

$$\beta(2\hat{p}_1 h) = 2p_1h_2\ ;\ \ \beta(2\hat{p}_2 h) = 2p_2h_1\ ;$$

$$\beta\left(\sum_* \overset{*}{\eta}h\right) = 4\beta(\overset{\wedge}{\eta}h) = 0\ ;\ \ \beta\left(\sum_* \overset{*}{\mu}h\right) = 4\beta(\overset{\wedge}{\mu}h) = 0\ ;$$

$$\beta\left(\sum_* \overset{*}{\mu}h_1\right) = 4p_1\hat{p}_2 + 4p_1\overset{\vee}{p}_2 = 0\ ;\ \ \beta\left(\sum_* \overset{*}{\mu}h_2\right) = 4\hat{p}_1 p_2 + 4\overset{\vee}{p}_1 p_2 = 0\ .$$

It follows that $\beta\big((\widetilde{K(\mathfrak{X})^{(3)}})^G\big)$ is the subgroup of $K(X_{\bar{L}})/4K(X_{\bar{L}})$ generated by the residue classes of $2p_1h_2 = 2\hat{p}_1\overset{\vee}{p}_1(\hat{p}_2 + \overset{\vee}{p}_2 - \hat{p}_2\overset{\vee}{p}_2)$ and $2p_2h_1 = 2\hat{p}_2\overset{\vee}{p}_2(\hat{p}_1 + \overset{\vee}{p}_1 - \hat{p}_1\overset{\vee}{p}_1)$. Therefore $2p = 2\hat{p}_1\overset{\vee}{p}_1\hat{p}_2\overset{\vee}{p}_2 \notin \beta\big((\widetilde{K(\mathfrak{X})^{(3)}})^G\big)$ (by Lemma 7.5, the group $K(X_{\bar{L}})$ is free with the basis given by the products of $\hat{p}_1$, $\overset{\vee}{p}_1$, $\hat{p}_2$, and $\overset{\vee}{p}_2$). □

10. First basic construction

Let k be a field of characteristic different from 2, containing elements

$$a_1, b_1, a_2, b_2, d \in k^*$$

such that $l \overset{\text{def}}{=} k(\sqrt{d})$ is a field and the biquaternion l-algebra

$$\big((a_1, b_1) \otimes_k (a_2, b_2)\big)_l$$

is a skewfield.

Let T be the generalized Severi-Brauer variety (see Section 2) of rank 2 right ideals in the biquaternion k-algebra $(a_1, b_1) \otimes (a_2, b_2)$. Denote by K the function field of the k-variety $\mathcal{R}(T) = \mathcal{R}_{l/k}(T)$ (see Definition 5.2).

Put $L \overset{\text{def}}{=} K(\sqrt{d})$. Since $\mathcal{R}(T)_l \simeq T_l \times T_l$ (see Section 5), one has

$$\operatorname{ind}\big((a_1, b_1) \otimes (a_2, b_2)\big)_L = 2$$

by the index reduction formula [**7**, Theorem 3].

For $i = 1, 2$, let q_i be the quadratic form $\langle -a_i, -b_i, a_ib_i, d\rangle$ over K.

THEOREM 10.1. *For any odd field extension K'/K, the quadratic forms $(q_1)_{K'}$ and $(q_2)_{K'}$ are non-linked. In particular, the forms q_1 and q_2 themselves are non-linked.*

PROOF. Remember that the quadratic forms q_1 and q_2 are in fact defined over k and denote by X_1 and X_2 the projective quadrics over k determined by q_1 and q_2. Set $X \overset{\text{def}}{=} X_1 \times X_2$. We have to show that the degree of any closed point on the variety X_K is divisible by 4.

Consider the Grothendieck group $K(X_K)$ of the variety X_K supplied with the topological filtration. Let $p \in K(X_{\bar{K}})$ denote the class of a rational point. To show that degree of every closed point on X is divisible by 4, it suffices to show that $2p \notin K(X_K)_{(0)}$, where $K(X_K)_{(0)}$ is the 0-dimensional term of the topological filtration on $K(X_K)$. Since $\dim X = 4$, we have $K(X_K)_{(0)} = K(X_K)^{(4)}$.

The pull-back homomorphism $K(X \times \mathcal{R}(T))^{(4)} \to K(X_K)^{(4)}$, given by the flat morphism of schemes $X_K \to X \times \mathcal{R}(T)$, is surjective by Corollary 4.3. Therefore it suffices to show that $2p$ is not in the image of this homomorphism.

Denote by σ the non-trivial automorphism of l over k. The group $K(X \times \mathcal{R}(T))^{(4)}$ is contained in the σ-invariant part of the group $K(X_l \times \mathcal{R}(T)_l)^{(4)}$. Thus it suffices to show that

$$2p \notin \operatorname{Im}\left(K(X_l \times \mathcal{R}(T)_l)^{(4)\,\sigma} \to K(X_L)\right).$$

For this, we apply Proposition 8.5.

In order to meet the conditions of Proposition 8.5, note that for $i = 1, 2$, one has $X_i \simeq \mathcal{R}(Y_i)$, where $\mathcal{R} = \mathcal{R}_{l/k}$ and Y_i is the Severi-Brauer variety of the quaternion k-algebra (a_i, b_i) (see Example 5.4).

Thus we have $X \times \mathcal{R}(T) \simeq \mathcal{R}(Y_1 \times Y_2 \times T)$. Therefore, we can identify $X_l \times \mathcal{R}(T)_l$ with the product

$$\mathfrak{X} \overset{\text{def}}{=} \hat{Y}_1 \times \hat{Y}_2 \times \hat{T} \times \overset{\vee}{Y}_1 \times \overset{\vee}{Y}_2 \times \overset{\vee}{T}$$

where $\hat{Y}_i$, $\overset{\vee}{Y}_i$ are two copies of $(Y_i)_l$ and $\hat{T}$, $\overset{\vee}{T}$ are two copies of T_l. Moreover, by Corollary 6.3, the automorphism of $K(X_l \times \mathcal{R}(T)_l)$ induced by σ corresponds to the automorphism of $K(\mathfrak{X})$ induced by the permutation of the factors interchanging $\hat{Y}_i$ with $\overset{\vee}{Y}_i$ and $\hat{T}$ with $\overset{\vee}{T}$.

We have met the conditions of Proposition 8.5. Applying it, we get the affirmation required. □

COROLLARY 10.2. *For any field k_0 with $\operatorname{char} k_0 \neq 2$ there exist a field extension K/k_0 and elements $a_1, a_2, b_1, b_2, d \in K^*$ with the following properties:*

- $\operatorname{ind}((a_1, b_1) \otimes (a_2, b_2))_{K(\sqrt{d})} = 2$;
- *for any odd field extension K'/K, the quadratic forms*

  $$q_1 \overset{\text{def}}{=} \langle -a_1, -b_1, a_1b_1, d\rangle \quad \textit{and} \quad q_2 \overset{\text{def}}{=} \langle -a_2, -b_2, a_2b_2, d\rangle$$

  are not linked over K'.

PROOF. Put $k \overset{\text{def}}{=} k_0(a_1, b_1, a_2, b_2, d)$ where a_1, b_1, a_2, b_2, d are indeterminates. Then $l \overset{\text{def}}{=} k(\sqrt{d})$ is a field and the biquaternion l-algebra $\big((a_1, b_1) \otimes (a_2, b_2)\big)_l$ is a skewfield. For the field $K \supset k$ as in Theorem 10.1, all affirmations of Corollary hold. $\square$

11. Second basic construction

Let k be a field of characteristic different from 2, containing elements

$$a_1, b_1, a_2, b_2, d_1, d_2 \in k^*$$

such that $l \overset{\text{def}}{=} k(\sqrt{d_1}, \sqrt{d_2})$ is a field and the biquaternion l-algebra

$$\big((a_1, b_1) \otimes_k (a_2, b_2)\big)_l$$

is a skewfield.

Let T be the generalized Severi-Brauer variety (see Section 2) of rank 2 right ideals in the biquaternion k-algebra $(a_1, b_1) \otimes (a_2, b_2)$. Denote by K the function field of the k-variety $\mathcal{R}(T) = \mathcal{R}_{l/k}(T)$ (see Definition 5.2).

Put $L \overset{\text{def}}{=} K(\sqrt{d_1}, \sqrt{d_2})$. Since $\mathcal{R}(T)_l \simeq T_l^{\times 4}$ (see Section 5), one has

$$\operatorname{ind}\big((a_1, b_1) \otimes (a_2, b_2)\big)_L = 2$$

by the index reduction formula [**7**, Theorem 3].

For $i = 1, 2$, let q_i be the quadratic form $\langle -a_i, -b_i, a_i b_i, d_i\rangle$ over K.

THEOREM 11.1. *Denote by X_1 and X_2 the projective quadric over K determined by q_1 and q_2. The Chow group $\mathrm{CH}^2(X_1 \times X_2)$ has a torsion.*

PROOF. Put $X \overset{\text{def}}{=} X_1 \times X_2$ and consider the Grothendieck group $K(X)$ of the variety X. There is an isomorphism $\mathrm{CH}^2(X) \simeq K(X)^{(2/3)}$ (see [**81**, §9]), where $K(X)^{(2/3)} \overset{\text{def}}{=} K(X)^{(2)}/K(X)^{(3)}$ is the 2-codimensional successive quotient of the topological filtration on $K(X)$. We are going to show that this quotient contains a torsion.

Denote by $p \in K(X_{\bar{K}})$ the class of a rational point. As we did all the time, we identify $K(X)$ with a subgroup of $K(X_{\bar{K}})$ via the restriction homomorphism.

LEMMA 11.2. $2p \in K(X)$.

PROOF. For $i = 1, 2$, denote by $\mathcal{U}_i$ Swan's vector bundle on X_i ([**82**]). It has a structure of right $(Q_i)_{X_i}$-module, where $Q_i \overset{\text{def}}{=} (a_i, b_i)_K$. For the class $[\mathcal{U}_i(2)] \in K(X_i)$ of the 2 (because $2 = \dim X_i$) times twisted Swan's vector bundle, there is a formula ([**32**, Lemma 3.6]): $[\mathcal{U}_i(2)] = 4 + 2h_i + h_i^2$, where h_i is the class of a general hyperplane section of X_i. Lifting to X, we can consider the tensor product $\mathcal{U}_1 \otimes_X \mathcal{U}_2$. It has a structure of right $Q_1 \otimes_K Q_2$-module. Therefore, since $\operatorname{ind} Q_1 \otimes_K Q_2 = 2$, the class $[\mathcal{U}_1(2) \otimes \mathcal{U}_2(2)] = (4 + 2h_1 + h_1^2)(4 + 2h_2 + h_2^2)$ is divisible by 2 in $K(X)$. Thus the product $h_1^2 h_2^2$ is divisible by 2. Since $h_1^2 h_2^2 = 4p$, we are done. $\square$

Since one can find a field extension of K of degree 4 such that the forms q_1 and q_2 become isotropic over this extension, one has $4p \in K(X)^{(4)}$. Therefore, if we manage to show that $2p \notin K(X)^{(3)}$, we get an element of order 2 in the quotient $K(X)/K(X)^{(3)}$, namely the class of $2p$. Since the groups $K(X)^{(0/1)}$ and $K(X)^{(1/2)}$ are torsion-free (see [**74**, Lemme 6.3, (i)] for the statement on $K(X)^{(1/2)} \simeq \mathrm{CH}^1(X)$), it will be a non-trivial torsion element in $K(X)^{(2/3)}$.

So, the last step in the proof of Theorem is the following

LEMMA 11.3. *$2p \notin K(X)^{(3)}$.*

PROOF. Remember that the quadratic forms q_1 and q_2 are in fact defined over k. Let us change the notation and from now on denote by X_1 and X_2 the projective quadrics over k determined by q_1 and q_2. Set $X \stackrel{\mathrm{def}}{=} X_1 \times X_2$. We have to show that $2p \notin K(X_K)^{(3)}$.

The pull-back homomorphism $K(X \times \mathcal{R}(T))^{(3)} \to K(X_K)^{(3)}$, given by the flat morphism of schemes $X_K \to X \times \mathcal{R}(T)$, is surjective by Corollary 4.3. Therefore it suffices to show that $2p$ is not in the image of this homomorphism.

Denote by G the Galois group of the biquadratic field extension l/k. The group $K(X \times \mathcal{R}(T))^{(3)}$ is contained in the G-invariant part of the group $K(X_l \times \mathcal{R}(T)_l)^{(3)}$. Thus it suffices to show that

$$2p \notin \operatorname{Im}\left(K(X_l \times \mathcal{R}(T)_l)^{(3)\,G} \to K(X_L)\right) .$$

For this, we apply Proposition 9.4.

In order to meet the conditions of Proposition 9.4, for $i = 1, 2$, put $l_i \stackrel{\mathrm{def}}{=} k(\sqrt{d_i})$ and denote by σ_i the non-trivial automorphism of l over l_{3-i}. The group G consists of 1, σ_1, σ_2, $\sigma_1\sigma_2$ and is generated by σ_1, σ_2.

Let Y_i be the Severi-Brauer variety of the quaternion k-algebra (a_i, b_i). One has $X_i \simeq R_{l_i/k}(Y_i)$ (see Example 5.4). Therefore, we can identify $(X_i)_l$ with $\hat{Y}_i \times \overset{\vee}{Y}_i$, where $\hat{Y}_i$ and $\overset{\vee}{Y}_i$ are two copies of the variety $(Y_i)_l$; moreover, by Lemma 5.5, the automorphism of $(X_i)_l$ given by σ_i corresponds to the automorphism of $\hat{Y}_i \times \overset{\vee}{Y}_i$ given by σ_i composed with the interchanging of the factors. The automorphism of $(X_i)_l$ given by σ_{3-i} corresponds to the automorphism of $\hat{Y}_i \times \overset{\vee}{Y}_i$ given by σ_{3-i}.

We also can identify $\mathcal{R}(T)_l$ with $\prod_G T_l$. Choosing the following correspondence between the signs $\wedge\!\wedge$, $\vee\!\vee$, $\wedge\!\vee$, $\vee\!\wedge$ and the elements of G:

$$\begin{array}{cccccc}
\wedge\!\wedge & \leftrightarrow & 1 \cdot 1 & = & 1 \\
\vee\!\vee & \leftrightarrow & & & \sigma_1\sigma_2 \\
\wedge\!\vee & \leftrightarrow & 1 \cdot \sigma_2 & = & \sigma_2 \\
\vee\!\wedge & \leftrightarrow & \sigma_1 \cdot 1 & = & \sigma_1
\end{array}$$

we identify $\mathcal{R}(T)_l$ with $\overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$ where $\overset{\wedge\wedge}{T}$, $\overset{\vee\vee}{T}$, $\overset{\wedge\vee}{T}$, $\overset{\vee\wedge}{T}$ are copies of T_l. The automorphism of $\mathcal{R}(T)_l$ given by σ_1 corresponds under this identification to the automorphism of $\overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$ given by σ_1 composed with the interchanging of $\overset{\wedge\wedge}{T}$ with $\overset{\vee\wedge}{T}$ and of $\overset{\vee\vee}{T}$ with $\overset{\wedge\vee}{T}$. Analogously, the automorphism of $\mathcal{R}(T)_l$ given by σ_2 corresponds to the automorphism of $\overset{\wedge\wedge}{T}\times\overset{\vee\vee}{T}\times\overset{\wedge\vee}{T}\times\overset{\vee\wedge}{T}$ given by σ_2 composed with the interchanging of $\overset{\wedge\wedge}{T}$ with $\overset{\wedge\vee}{T}$ and of $\overset{\vee\vee}{T}$ with $\overset{\vee\wedge}{T}$.

Summarizing and passing to the Grothendieck group of the varieties, we get the following commutative diagram (for $i = 1, 2$):

$$\begin{tikzcd}
K(\mathcal{R}_{l_1/k}(Y_1)_l \times \mathcal{R}_{l_2/k}(Y_2)_l \times \mathcal{R}(T)_l) \arrow[r, "\sigma_i"] \arrow[d] & K(\mathcal{R}_{l_1/k}(Y_1)_l \times \mathcal{R}_{l_2/k}(Y_2)_l \times \mathcal{R}(T)_l) \arrow[d] \\
K(\mathfrak{X}) \arrow[r, "\sigma_i \circ s_i"] & K(\mathfrak{X})
\end{tikzcd}$$

where $\mathfrak{X}$ and s_i are as in Proposition 8.5. By Corollary 6.2, σ_i over the bottom arrow is the identity.

We have met the conditions of Proposition 8.5. Applying it, we get the affirmation required. $\square$

Theorem is proved. $\square$

Corollary 11.4. *Let k be a field of characteristic $\neq 2$ and $a, b, u, v, d, \delta \in k^*$. Suppose that $d, \delta, d\delta \notin k^{*2}$ and $((a, b) \otimes (u, v))_{k(\sqrt{d},\sqrt{\delta})}$ is a division algebra. Put $\rho = \langle -a, -b, ab, d\rangle$, $\psi = \langle -u, -v, uv, \delta\rangle$. Then there exists a field extension K/k such that*

$$\operatorname{Tors}\operatorname{CH}^2(X_{\rho_K} \times X_{\psi_K}) \simeq \mathbb{Z}/2\mathbb{Z} \quad \textit{and} \quad \operatorname{ind} C_0(\rho_K) \otimes C_0(\psi_K) = 2 \ .$$

Proof. To come to the situation considered above, we simply put $a_1 = a$, $b_1 = b$, $a_2 = u$, $b_2 = v$, $d_1 = d$, and $d_2 = \delta$, so that $q_1 = \rho$ and $q_2 = \psi$.

Let K be the field extension of k constructed in the beginning of this Section. By Theorem 11.1, the group $\operatorname{Tors}\operatorname{CH}^2(X_{\rho_K} \times X_{\psi_K})$ is non-trivial. From the other hand, by Theorem 5.7 of Chapter 4, the order of this group is at most 2. Therefore $\operatorname{Tors}\operatorname{CH}^2(X_{\rho_K} \times X_{\psi_K}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Finally, let us note that $C_0(\rho) \simeq (a, b)_{k(\sqrt{d})}$ and $C_0(\psi) \simeq (u, v)_{k(\sqrt{\delta})}$. Consequently $\operatorname{ind} C_0(\rho_K) \otimes C_0(\psi_K) = \operatorname{ind}\big((a, b) \otimes (u, v)\big)_L = 2$. $\square$

12. Quadratic forms over complete fields

In this section we need some results concerning the Witt ring over a complete discrete valuation field. We fix the following notation:

- (L, v) is a complete discrete valuation field.
- We set $\mathfrak{O}_L = \{x \in L^* \mid v(x) \geq 0\}$, $\mathfrak{M}_L = \{x \in L \mid v(x) > 0\}$, and $\mathfrak{U}_L = \mathfrak{O}_L - \mathfrak{M}_L = \{x \in L \mid v(x) = 0\}$.
- Residue field $\bar{L}$ is defined as $\mathfrak{O}_L/\mathfrak{M}_L$.
- For any $a \in \mathfrak{O}_L$ we denote by $\bar{a}$ the class of a in $\bar{L} = \mathfrak{O}_L/\mathfrak{M}_L$.

If $a \in \mathfrak{U}_L$, we obviously have $\bar{a} \in \bar{L}^*$. Let π be an element of L such that $v(\pi)$ is odd. Since $L^*/L^{*2} = U_L/U_L^{*2} \times \{1, \pi\}$, an arbitrary quadratic form ϕ over L can be written in the form $\phi = \langle a_1, \ldots, a_k\rangle \perp \pi\langle b_1, \ldots, b_l\rangle$ where

$a_1, \dots, a_k, b_1, \dots, b_l \in \mathfrak{U}_L$. We define quadratic $\bar{L}$-forms $d^1_\pi(\phi)$ and $d^2_\pi(\phi)$ as follows:

$$d^1_\pi(\phi) = \langle \bar{a}_1, \dots, \bar{a}_k \rangle_{an}, \qquad d^2_\pi(\phi) = \left\langle \bar{b}_1, \dots, \bar{b}_l \right\rangle_{an}.$$

REMARK 12.1. 1) Springer's theorem asserts that a quadratic form ϕ and an element $\pi \in L^*$ determine quadratic forms $d^1_\pi(\phi)$ and $d^2_\pi(\phi)$ uniquely up to isomorphism [1]. The maps

$$d^1_\pi,\ d^2_\pi\ :\ \{\text{isometry classes of } L\text{-forms}\} \to \{\text{isometry classes of } \bar{L}\text{-forms}\}$$

give rise to group homomorphisms $W(L) \to W(\bar{L})$, which are called *the first and the second residue class* and denoted by ∂_1 and ∂_2 (see [**48**, §1 of Chapter 6] or [**76**, Definition 2.5 of Chapter 6]).

2) In the case where ϕ is anisotropic, quadratic forms $\langle \bar{a}_1, \dots, \bar{a}_k \rangle$ and $\left\langle \bar{b}_1, \dots, \bar{b}_l \right\rangle$ are anisotropic as well. Thus, in this case

$$d^1_\pi(\phi) = \langle \bar{a}_1, \dots, \bar{a}_k \rangle, \qquad d^2_\pi(\phi) = \left\langle \bar{b}_1, \dots, \bar{b}_l \right\rangle.$$

LEMMA 12.2. *Let ϕ and τ be anisotropic quadratic forms over a complete discrete valuation field (L, v). Let π be an element of L such that $v(\pi)$ is odd. Suppose that $\tau \subset \phi$. Then $d^1_\pi(\tau) \subset d^1_\pi(\phi)$ and $d^2_\pi(\tau) \subset d^2_\pi(\phi)$.*

PROOF. Let γ be such that $\tau \perp \gamma = \phi$. It follows from Remark 12.1 that $d^1_\pi(\tau) \perp d^1_\pi(\gamma) = d^1_\pi(\phi)$ and $d^2_\pi(\tau) \perp d^2_\pi(\gamma) = d^2_\pi(\phi)$. Thus $d^1_\pi(\tau) \subset d^1_\pi(\phi)$ and $d^2_\pi(\tau) \subset d^2_\pi(\phi)$. $\square$

LEMMA 12.3. *Let ϕ_1 and ϕ_2 be anisotropic quadratic k-forms. Let $K = k((t))$, and let $\phi = \phi_1 \perp t\phi_2$ be a quadratic form over K. Let L/K be an odd extension. Suppose that there exists $\tau \in GP_2(L)$ such that $\tau \subset \phi_L$. Then there exists an odd extension l/k of degree $\leq [L : K]$ such that at least one of the following conditions holds:*

- *there exists $\rho \in GP_2(l)$ such that $\rho \subset (\phi_1)_l$.*
- *there exists $\rho \in GP_2(l)$ such that $\rho \subset (\phi_2)_l$.*
- *quadratic forms $(\phi_1)_l$ and $(\phi)_l$ are linked.*

Moreover, we can take $l = \bar{L}$.

PROOF. Since L/K is a finite field extension, L is a complete discrete valuation field. Let v be a valuation on L, and let $l = \bar{L}$ be the residue field of L. We have $[l : k] = [\bar{L} : \bar{K}] \leq [L : K]$. Since L/K is odd, $[l : k]$ is odd too. Besides, the ramification index $e(L/K) = v(t)$ is odd. Thus, d^1_t and d^2_t are well defined. Since $\dim \tau = 4$ and $\det \tau = 1$, it follows that $\dim d^1_t(\tau)$ and $\dim d^2_t(\tau)$ are even, $\dim d^1_t(\tau) + \dim d^2_t(\tau) = 4$, and $\det d^1_t(\tau) \det d^2_t(\tau) = 1$. Thus one of the following conditions holds:

1) $d^1_t(\tau) \in GP_2(\bar{L})$ and $d^2_1(\tau) = 0$,
2) $d^2_t(\tau) \in GP_2(\bar{L})$ and $d^1_t(\tau) = 0$,

[1] In the original version of Springer's theorem, π is an uniformizing element of L. However, we can suppose that π is an arbitrary element such that $v(\pi)$ is odd because there exists a prime element $\pi_L \in L$ such that $\pi \equiv \pi_L$ in L^*/L^{*2}.

3) $\dim d_t^1(\tau) = \dim d_t^2(\tau) = 2$ and $d_t^1(\tau)$ is similar to $d_t^2(\tau)$.

Clearly, $d_t^1(\phi) = (\phi_1)_l$ and $d_t^2(\phi) = (\phi_2)_l$. It follows from Lemma 12.2 that $d_\pi^1(\tau) \subset d_\pi^1(\phi) = (\phi_1)_l$ and $d_\pi^2(\tau) \subset d_\pi^2(\phi) = (\phi_2)_l$. Thus, we are done. □

13. 8-dimensional quadratic forms $\phi \in I^2(F)$

It is an important problem to find a good classification of 8-dimensional quadratic forms $\phi \in I^2(F)$. One of important invariants of ϕ is the Schur index of the Clifford algebra $C(\phi)$. Clearly, $\operatorname{ind} C(\phi)$ is equal to one of the integers: 1, 2, 4, or 8.

If ϕ is a "generic" 8-dimensional form with $\det \phi = 1$, then we have $\operatorname{ind} C(\phi) = 8$. This shows that we cannot say anything specific in the case $\operatorname{ind} C(\phi) = 8$. In the case $\operatorname{ind} C(\phi) = 1$ we have plenty information on the structure of ϕ. Indeed, in this case $c(\phi) = 0$, and hence $\phi \in I^3(F)$. Finally, APH implies that $\phi \in GP_3(F)$. The case $\operatorname{ind} C(\phi) = 2$ is well known too (see for example [**41**, Example 9.12]). Namely, for a quadratic form $\phi \in I^2(F)$ the following two conditions are equivalent: a) $\operatorname{ind} C(\phi) \le 2$; b) ϕ can be written in the form $\phi = \langle\langle a \rangle\rangle\, q$, where $\dim q = 4$.

Thus, the only open case is $\operatorname{ind} C(\phi) = 4$. It is very easy to give examples of quadratic forms ϕ with $\operatorname{ind} C(\phi) \le 4$. If $\phi = \pi_1 \perp \pi_2$ where $\pi_1, \pi_2 \in GP_2(F)$, then $c(\phi) = c(\pi_1) + c(\pi_2)$, and hence $\operatorname{ind} C(\phi) \le 4$. This example gives rise to the following natural

Question 13.1. *Suppose that $\phi \in I^2(F)$ is an 8-dimensional quadratic form with $\operatorname{ind} C(\phi) \le 4$. Do there necessarily exist quadratic forms $\pi_1, \pi_2 \in GP_2(F)$ such that $\phi = \pi_1 \perp \pi_2$?*

In this section we construct a counterexample for this question. We start from the following

Definition 13.2 (cf. [**32**, §7]). Let ϕ be a quadratic form over F.

1) By $S(F)$ we denote the set of quadratic forms over F satisfying the following condition: there exists $\rho \in GP_2(F)$ such that $\rho \subset \phi$.
2) By $S_{\text{odd}}(F)$ we denote the set of quadratic forms over F satisfying the following condition: there exist an odd extension L/F and $\rho \in GP_2(L)$ such that $\rho \subset \phi_L$. In other words,

$$S_{\text{odd}}(F) = \{\phi \mid \text{there exists an odd extension } L/F \text{ such that } \phi_L \in S(L)\}.$$

Clearly, $S(F) \subset S_{\text{odd}}(F)$. We do not know if there exists a field F such that $S(F) \ne S_{\text{odd}}(F)$.[2] Our interest in the set $S_{\text{odd}}(F)$ is motivated by the following

Theorem 13.3 (see [**32**, Theorem 7.3]). *Let ϕ be a quadratic form of dimension ≥ 3. The group $\operatorname{Tors} G_1 K(X_\phi)$ is zero or equal to $\mathbb{Z}/2\mathbb{Z}$. The group*

[2] In [**32**, Remark 7.2], it is remarked that a field F and a 7-dimensional form $\phi \in S_{\text{odd}}(F) \setminus S(F)$ can be constructed. However, recently O. Izhboldin showed that the form ϕ the author had in mind is in fact in $S(F)$.

$\operatorname{Tors} G_1 K(X_\phi)$ *is nontrivial if and only if* ϕ *is anisotropic,* $\dim\phi \geq 5$*, and* $\phi \in S_{\mathrm{odd}}(F)$. □

PROPOSITION 13.4. *Let* $\phi \in I^2(K)$ *be an anisotropic 8-dimensional quadratic form such that* $\operatorname{ind} C(\phi) = 4$*. Then the following conditions are equivalent:*

1) $\phi \in S(K)$*, i.e., there exists* $\rho \in GP_2(K)$ *such that* $\rho \subset \phi$*,*
2) *there exist* $\rho_1, \rho_2 \in GP_2(K)$ *such that* $\phi = \rho_1 \perp \rho_2$*,*
3) ϕ *and* q *are linked, where* q *is an Albert form corresponding to the algebra* $C(\phi)$*.*

PROOF. 1)⇒2). Let ρ' be a complement of ρ in ϕ. We have $\phi = \rho \perp \rho'$. Clearly $\det \rho' = 1$ and $\dim \rho' = 4$. Therefore $\rho' \in GP_2(K)$.

2)⇒3). One can write ρ_1, ρ_2 as follows: $\rho_1 = k_1 \langle\langle a_1, b_1\rangle\rangle$ and $\rho_2 = k_2 \langle\langle a_2, b_2\rangle\rangle$. Then $c(q) = c(\phi) = (a_1, b_1) + (a_2, b_2)$. Therefore, q is similar to the form $\langle -a_1, -b_1, a_1b_1, a_2, b_2, -a_2b_2\rangle$. Obviously, $\phi_{K(\sqrt{a_1})}$ and $q_{K(\sqrt{a_1})}$ are isotropic. Hence ϕ and q are linked.

3)⇒1). Suppose that ϕ and q are linked. Then there exists $s \in K^*$ such that $\phi_{K(\sqrt{s})}$ and $q_{K(\sqrt{s})}$ are isotropic. We claim that $i_W(\phi_{K(\sqrt{s})}) \geq 2$. Suppose at the moment that $i_W(\phi_{K(\sqrt{s})}) = 1$. Then $(\phi_{K(\sqrt{s})})_{an}$ is an anisotropic Albert form. Then $\operatorname{ind} C(\phi_{K(\sqrt{s})}) = 4$. Since $c(q) = c(\phi)$, we see that $\operatorname{ind} C(q_{K(\sqrt{s})}) = 4$. Hence the Albert form $q_{K(\sqrt{s})}$ is anisotropic, a contradiction. Thus $i_W(\phi_{K(\sqrt{s})}) \geq 2$. Hence there exists a 2-dimensional form μ such that $\mu \langle\langle s\rangle\rangle \subset \phi$. To complete the proof it is sufficient to set $\rho = \mu \langle\langle s\rangle\rangle$. □

In this section we construct some new examples of quadratic forms ϕ such that $\phi \notin S_{\mathrm{odd}}(K)$ (and hence $\phi \notin S(K)$). The main tool for our construction is the following

LEMMA 13.5. 1) *Let* ϕ_1 *and* ϕ_2 *be anisotropic* k*-forms such that* $\phi_1, \phi_2 \notin S_{\mathrm{odd}}(k)$*. Denote by* ϕ *the quadratic form* $\phi_1 \perp t\phi_2$ *over* $k((t))$*. Suppose that* $\phi \in S_{\mathrm{odd}}(k((t)))$*. Then there exists a finite odd extension* l/k *such that* $(\phi_1)_l$ *and* $(\phi_2)_l$ *are linked.*

2) *Let* ϕ_1 *and* ϕ_2 *be anisotropic* k*-forms such that* $\phi_1, \phi_2 \notin S(k)$*. Let* $\phi = \phi_1 \perp t\phi_2$ *be a quadratic form over* $k((t))$*. Suppose that* $\phi \in S(k((t)))$*. Then* ϕ_1 *and* ϕ_2 *are linked.*

PROOF. It is an obvious consequence of Lemma 12.3. □

COROLLARY 13.6. *Let* ϕ_1 *and* ϕ_2 *be 4-dimensional* k*-forms not belonging to* $GP_2(k)$*. Suppose that* $(\phi_1)_l$ *and* $(\phi_2)_l$ *are not linked for any odd extension* l/k*. Then the quadratic form* $\phi_1 \perp t\phi_2$ *over* $k((t))$ *does not belong to* $S_{\mathrm{odd}}(k((t)))$*.* □

THEOREM 13.7. *There exist a field* K *and an 8-dimensional quadratic form* $\phi \in I^2(K)$ *such that* $\operatorname{ind} C(\phi) = 4$ *but* $\phi \notin S_{\mathrm{odd}}(K)$*.*

PROOF. Let field k, elements $a_1, a_2, b_1, b_2, d \in k^*$, and 4-dimensional quadratic forms q_1, q_2 be as in Corollary 10.2. We set $K = k((t))$ and

$$\phi = q_1 \perp tq_2 = \langle -a_1, -b_1, a_1b_1, d\rangle \perp t\,\langle -a_2, -b_2, a_2b_2, d\rangle\,.$$

Clearly, $\dim\phi = 4+4 = 8$ and $\det_\pm \phi = 1$. In $W(K)$ we have $\phi = (\langle\langle a_1, b_1\rangle\rangle - \langle\langle d\rangle\rangle) - t(\langle\langle a_2, b_2\rangle\rangle - \langle\langle d\rangle\rangle) = \langle\langle a_1, b_1\rangle\rangle - t\,\langle\langle a_2, b_2\rangle\rangle + \langle\langle d, t\rangle\rangle$. Therefore, $c(\phi) = (a_1, b_1) + (a_2, b_2) + (d, t)$. Applying Tignol's theorem [**83**, Proposition 2.4], we see that

$$\operatorname{ind} C(\phi) = \operatorname{ind}((a_1, b_1) \otimes (a_2, b_2) \otimes (d, t)) = \\ = 2\operatorname{ind}((a_1, b_1) \otimes (a_2, b_2))_{K(\sqrt{d})} = 2 \cdot 2 = 4\,.$$

It follows from Corollary 13.6 that $\phi \notin S_{\mathrm{odd}}(K)$. □

COROLLARY 13.8. *The answer to Question 13.1 is negative.* □

COROLLARY 13.9. *There exist a field K and an 8-dimensional quadratic form $\phi \in I^2(K)$ such that* $\operatorname{Tors} G^i K(X_\phi) = 0$ *for $i \neq 4$ and* $\operatorname{Tors} G^4 K(X_\phi) = \mathbb{Z}/2\mathbb{Z}$.

PROOF. It is an obvious consequence of Theorem 13.7 and [**32**, Theorem 8]. □

THEOREM 13.10. *Let $\phi \in I^2(k)$ be an 8-dimensional quadratic form. Then the following conditions are equivalent:*

1) $\operatorname{ind} C(\phi) \le 4$,
2) *at least one of the following conditions holds:*
   (a) *there exist $\pi_1, \pi_2 \in GP_2(k)$ such that $\phi = \pi_1 \perp \pi_2$,*
   (b) *there exist a field extension l/k of degree 2 and a quadratic form $\tau \in GP_2(l)$ such that $\phi = s_{l/k}(\tau)$.*

PROOF. 1)⇒2). If ϕ is isotropic, we can write ϕ as a sum $\phi = q \perp \langle 1, -1\rangle$, where q is an Albert form. Writing q in the form $q = s\,\langle -a, -b, ab, u, v, -uv, \rangle$, we have $\phi = s\,\langle\langle a, b\rangle\rangle \perp -s\,\langle\langle u, v\rangle\rangle$. Setting $\pi_1 = s\,\langle\langle a, b\rangle\rangle$ and $\pi_2 = -s\,\langle\langle u, v\rangle\rangle$, we are done. Thus we can suppose that ϕ is anisotropic.

Since $\operatorname{ind} C(\phi) \le 4$, there exists an Albert form q such that $c(q) = c(\phi)$. If q is isotropic, then $\operatorname{ind} C(\phi) \le 2$, and hence ϕ can be written in the form $\phi = \langle\langle a\rangle\rangle \otimes \langle b_1, b_2, b_3, b_4\rangle$. Setting $\pi_1 = \langle\langle a\rangle\rangle \otimes \langle b_1, b_2\rangle$ and $\pi_2 = \langle\langle a\rangle\rangle \otimes \langle b_3, b_4\rangle$, we have $\phi = \pi_1 \perp \pi_2$ and $\pi_1, \pi_2 \in GP_2(k)$. Thus in the case where q is isotropic, the proof is complete.

Now, we can suppose that ϕ and q are anisotropic. Let $\rho = \phi \perp tq$ be a quadratic form over $K = k((t))$. Obviously, $\dim\rho = 14$ and $\rho \in I^3(K)$. It follows from [**72**] that there exist $d \in K$ and $\pi \in P_3(K(\sqrt{d}))$ such that $\rho = \phi \perp tq$ is similar to $s_{K(\sqrt{d})/K}(\sqrt{d}\,\pi')$. Let $L = K(\sqrt{d})$.

Since $K^*/K^{*2} = k^*/k^{*2} \times \{1, t\}$, it is sufficient to consider the following two cases:

- $d = a \in k^*$,
- d has the form at with $a \in k^*$.

First, consider the case $d = a \in k^*$. In this case we have $L = l((t))$ with $l = k(\sqrt{a})$. Then an arbitrary L-form γ can be written in the form $\phi_1 \perp t\phi_2$, where ϕ_1 and ϕ_2 are l-forms. We have

$$s_{L/K}(\gamma) = s_{L/K}(\phi_1 \perp t\phi_2) = s_{l/k}(\phi_1) \perp ts_{l/k}(\phi_2).$$

Applying this formula to the case $\gamma = \sqrt{d}\pi'$, we see that $\phi \perp tq$ is similar to $s_{l/k}(\phi_1) \perp ts_{l/k}(\phi_2)$. Hence, one of the k-forms $s_{l/k}(\phi_1)$, $s_{l/k}(\phi_2)$ is similar to ϕ and the other is similar to q. Let i be such that $s_{l/k}(\phi_i) \sim \phi$, and let j be such that $s_{l/k}(\phi_j) \sim q$. Then $\dim\phi_i = 4$ and $\dim\phi_j = 3$. Since $s_{l/k}(\phi_i) \sim \phi$, there exists $r \in k^*$ such that $\phi = r \cdot s_{l/k}(\phi_i) = s_{l/k}(r\phi_i)$. Now it is sufficient to prove that $r\phi_i \in GP_2(l)$. Let $\tilde{\phi_j} = \phi_j \perp \langle \det(\phi_i)\det(\phi_j)\rangle$. Obviously, $\phi_i \perp t\tilde{\phi_j} \in I^2(L)$. Clearly, $\phi_1 \perp t\phi_2$ is similar to $\phi_i \perp t\phi_j$. Therefore π' is similar to $\phi_i \perp t\phi_j$, and hence π is similar to $\phi_i \perp t\tilde{\phi_j}$. Since $\pi \in I^3(l((t)))$, it follows that $\phi_i, \tilde{\phi_j} \in I^2(l)$. Since $\dim\phi_i = 4$, we have $\phi_i \in GP_2(l)$. Thus in the case $d \in k^*$ we are done.

Now, consider the case $d = at$, $a \in k^*$. In this case $L = k((t))(\sqrt{at})$ is a complete discrete valuation field with residue field k and uniformizing element $\sqrt{at}$. Then an arbitrary L-form γ can be written in the form $\phi_1 \perp \sqrt{at}\phi_2$, where ϕ_1 and ϕ_2 are k-forms. We have

$$\begin{aligned} s_{L/K}(\gamma) &= s_{L/K}(\phi_1 \perp \sqrt{at}\phi_2) \\ &= s_{L/K}(\langle 1\rangle) \otimes \phi_1 \perp s_{L/K}(\langle \sqrt{at} \rangle) \otimes \phi_2 \\ &= \langle 1, at\rangle \otimes \phi_1 \perp \langle 1, -1\rangle \otimes \phi_2 \\ &= (\phi_1 \perp \langle 1, -1\rangle \otimes \phi_2) \perp t \cdot a\phi_1. \end{aligned}$$

Applying this formula to the case $\gamma = \sqrt{d}\pi'$, we see that $\phi \perp tq$ is similar to $(\phi_1 \perp \langle 1, -1\rangle \otimes \phi_2) \perp t \cdot a\phi_1$. Therefore one of the forms ϕ, q is similar to $\phi_1 \perp \langle 1, -1\rangle \otimes \phi_2$ and the other is similar to $a\phi_1$. Since ϕ and q are anisotropic, we see that $\dim\phi_2 = 0$. Therefore $\dim(\phi_1 \perp \langle 1, -1\rangle \otimes \phi_2) = \dim a\phi_1$. Hence $\dim\phi = \dim q$, a contradiction.

2)⇒1). In the case where $\phi = \pi_1 \perp \pi_2$ and $\pi_1, \pi_2 \in GP_2(k)$, we have $\operatorname{ind} C(\phi) \le \operatorname{ind} C(\pi_1) \cdot \operatorname{ind} C(\pi_2) \le 2 \cdot 2 = 4$. Now, suppose that there exist a field extension l/k of degree 2 and a quadratic form $\tau \in GP_2(l)$ such that $\phi = s_{l/k}(\tau)$. First of all, we have $\dim\phi = [l : k] \cdot \dim\tau = 8$. Since $\tau \in I^2(l)$, it follows that $\phi = s_{l/k}(\tau) \in I^2(k)$ ([**76**, Corollary 14.9]). Finally, we have $c(\phi) = c(s_{l/k}(\tau)) = \mathrm{Tr}_{l/k}(c(\tau))$. Therefore, $\operatorname{ind} C(\phi) \le 4$. □

Remark 13.11. 1) Setting $l = k \times k$, one can consider Condition 2(a) of Theorem 13.10 as a degenerate case of Condition 2(b).

2) Actually, Theorem 13.10 is an easy consequence of deep Rost's theorem [**72**]. Rost's proof uses numerous results on the algebraic groups. It would be interesting to find a direct proof of Theorem 13.10 in the framework of theory of quadratic forms.

14. 14-dimensional quadratic forms $\phi \in I^3(F)$

In this section we discuss the problem of classification of anisotropic forms $\phi \in I^3(K)$. For anisotropic quadratic forms $\phi \in I^3(K)$, the following results are known: if $\dim \phi < 8$, then ϕ is hyperbolic; if $\dim \phi = 8$, then ϕ is similar to a 3-fold Pfister form; there are no anisotropic 10-dimensional forms belonging to $I^3(K)$; if $\dim \phi = 12$, then there exist a 2-dimensional quadratic form μ and a 6-dimensional Albert form q such that $\phi = \mu \otimes q$. Analyzing these results, one can see that:

- all anisotropic quadratic forms $\phi \in I^3(K)$ of dimension ≤ 12 belongs to $S(K)$,
- any quadratic form $\phi \in I^3(K)$ of dimension ≤ 12 can be represented as a sum $\sum_{i=1}^{k} \rho_i$ with $\rho_i \in GP_3(K)$ and $k \leq 2$.

Here we consider the case $\dim \phi = 14$. It is not difficult to construct a form of dimension 14 belonging to $I^3(K)$. Let τ_1' and τ_2' be pure subforms of 3-fold Pfister forms τ_1 and τ_2. Then for any $k \in K^*$ the quadratic form $\phi = k(\tau_1' \perp -\tau_2')$ has dimension 14 and belongs to $I^3(K)$. This example gives rise to the following

Question 14.1. *Suppose that $\phi \in I^3(K)$ is a 14-dimensional quadratic form. Do there necessarily exist quadratic forms $\tau_1, \tau_2 \in P_3(K)$ and $k \in K^*$ such that $\phi = k(\tau_1' \perp -\tau_2')$?*

We have the following

Proposition 14.2. *Let $\phi \in I^3(K)$ be an anisotropic 14-dimensional form. The following conditions are equivalent:*

1) *$\phi \in S(K)$, i.e., there exists $\rho \in GP_2(K)$ such that $\rho \subset \phi$,*
2) *There exist $\rho_1, \rho_2 \in GP_3(K)$ such that $\phi = \rho_1 + \rho_2$ in $W(K)$,*
3) *There exist $\tau_1, \tau_2 \in P_3(K)$ and $k \in K^*$ such that $\phi = k(\tau_1' \perp -\tau_2')$. Here τ_1' and τ_2' denote pure subforms of Pfister forms τ_1, τ_2,*
4) *There exist $\tau_1, \tau_2 \in P_3(K)$ such that $\phi \equiv \tau_1 + \tau_2 \pmod{I^4(K)}$,*
5) *$e^3(\phi)$ is a sum of two symbols, i.e., there exist $a_1, b_1 c_1, a_2, b_2, c_2 \in K^*$ such that $e^3(\phi) = (a_1, b_1, c_1) + (a_2, b_2, c_2)$.*

Proof. 1)$\Rightarrow$2). Let $s \in K^*$ be such that $\rho_{F(\sqrt{s})}$ is isotropic. Since $\rho \in GP_2(K)$, it follows that $i_W(\phi_{K(\sqrt{s})}) \geq 2$. Therefore $\dim(\phi_{K(\sqrt{s})})_{an} \leq 10$, and hence Pfister's theorem [**68**] implies that $\dim(\phi_{K(\sqrt{s})})_{an} \leq 8$. Thus, $i_W(\phi_{K(\sqrt{s})}) \geq 3$. Hence there exists a 3-dimensional form μ such that $\mu \langle\langle s \rangle\rangle \subset \phi$. We set $\rho_1 = (\mu \perp \langle \det \mu \rangle) \langle\langle s \rangle\rangle$. Clearly, $\rho_1 \in GP_3(K)$. Let $\rho_2 = (\phi \perp -\rho_1)_{an}$. We have $\phi = \rho_1 + \rho_2$ in $W(K)$. It is sufficient to prove that $\rho_2 \in GP_3(K)$. Since $\dim \phi = 14 > 8 = \dim \rho_1$ and $\phi = \rho_1 + \rho_2$, it follows that $\rho_2 \neq 0$. Since $\phi, \rho_1 \in I^3(K)$, it follows that $\rho_2 \in I^3(K)$. Therefore, $\dim \rho_2 \geq 8$. Since ρ_1 and ϕ contain a common 6-dimensional form $\mu \langle\langle s \rangle\rangle$, we have $\dim \rho_2 = \dim(\phi \perp -\rho_1)_{an} \leq 14 + 8 - 2 \cdot 6 = 10$. Since ρ_2 is anisotropic and $\rho_2 \in I^3(K)$, Pfister's theorem implies that $\dim \rho_2 = 8$. Therefore, $\rho_2 \in GP_3(K)$.

2)$\Rightarrow$3). It is a particular case of [**17**, Lemma 3.2] (see also [**9**, Theorem 4.5])

3)$\Rightarrow$4). Since $k(\tau_1' \perp -\tau_2') \equiv \tau_1 + \tau_2 \pmod{I^4(K)}$, we are done.

4)$\Rightarrow$1). Let $L = K(\sqrt{s})$ be a field extension such that $(\rho_2)_L$ is isotropic. We have $\phi_{L(\rho_1)} \equiv (\rho_1 + \rho_2)_{L(\rho_1)} = 0 \pmod{I^4(L(\rho_1))}$. Since $\dim \phi = 14 < 16$, APH implies that $\phi_{L(\rho_1)}$ is hyperbolic. Hence there exists an L-form γ such that $(\phi_L)_{an} = (\rho_1)_L \cdot \gamma$. Hence, $\dim(\phi_L)_{an}$ is divisible by 8. Since $\dim \phi = 14$, it follows that $i_W(\phi_L) \geq (14-8)/2 = 3$. Since $L = K(\sqrt{s})$, there exists a 2-dimensional form μ such that $\langle\langle s\rangle\rangle\, \mu \subset \phi$. Now it is sufficient to set $\rho = \langle\langle s\rangle\rangle\, \mu$.

4)$\Longleftrightarrow$5). It is an easy consequence of bijectivity of $\bar{e}^3 : I^3(K)/I^4(K) \to H^3(K)$. $\square$

THEOREM 14.3. *There exist a field E and a 14-dimensional quadratic form $\tau \in I^3(E)$ such that $\tau \notin S_{\mathrm{odd}}(E)$.*

PROOF. Let K and $\phi \in I^2(K)$ be as in Theorem 13.7. Since $\operatorname{ind} C(\phi) = 4$, there exists an Albert form q such that $c(\phi) = c(q)$. Let $E = K((t))$, and let $\tau = \phi \perp tq$ be a quadratic form over E. Clearly, $\dim \phi = 14$. We have $c(\tau) = c(\phi) + c(q) = 0$. Therefore $\tau \in I^3(E)$. To complete the proof, it suffices to verify that $\tau \notin S_{\mathrm{odd}}(E)$

Suppose at the moment that $\tau \in S_{\mathrm{odd}}(E)$. By Theorem 13.7, we have $\phi \notin S_{\mathrm{odd}}(K)$. Since q is an anisotropic Albert form, it follows that $q \notin S_{\mathrm{odd}}(K)$. Now, it follows from Lemma 13.5 that there exists an odd extension L/K such that ϕ_L and q_L are linked. Proposition 13.4 implies that $\phi_L \in S(L)$. Since L/K is an odd extension, we have $\phi \in S_{\mathrm{odd}}(K)$, a contradiction. $\square$

COROLLARY 14.4. *The answer to Question 14.1 is negative.* $\square$

COROLLARY 14.5. *There exist a field K and a 14-dimensional form $\phi \in I^3(K)$ such that $e^3(\phi)$ cannot be represented as a sum of two symbols.* $\square$

REMARK 14.6. It was proved by D. W. Hoffmann and the first author (independently) that an arbitrary 14-dimensional quadratic form $\phi \in I^3(K)$ can be written in the form $\tau_1 + \tau_2 + \tau_3$ in $W(K)$ where $\tau_1, \tau_2, \tau_3 \in GP_3(K)$ (see for instance [**21**]). In particular, $e^3(\phi)$ can be represented as a sum of 3 symbols.

REMARK 14.7. Let n be an even integer such that $n > 14$. It is not difficult to construct a field E and a quadratic form $\phi \in I^3(E)$ of dimension n such that $\phi \notin S_{\mathrm{odd}}(E)$. The following example shows how to construct a quadratic form $\phi \in I^3(E)$ of dimension $6n$ ($n \geq 4$) so that $\phi \notin S_{\mathrm{odd}}(E)$.

EXAMPLE 14.8. Let $n \geq 4$, and let k_0 be an arbitrary field of characteristic $\neq 2$. Let $k = k_0(X_1, \ldots, X_n, Y_1, \ldots, Y_n, U_1, \ldots, U_n, V_1, \ldots, V_n)$. For any $i = 1, \ldots, n$ we set $A_i = (X_1, Y_1) \otimes_k (U_i, V_i)$ and $q_i = \langle -X_i, -Y_i, X_iY_i, U_i, V_i, -U_iV_i\rangle$. Let $A = A_1 \otimes_k \cdots \otimes_k A_n$ and $K = k(\mathrm{SB}(A))$. Let $1 \leq i < j \leq n$. By the index reduction formula [**78**], we have $\operatorname{ind}(A_i \otimes_k A_j)_K = \min(\operatorname{ind}(A_i \otimes_k A_j), \operatorname{ind}(A_i \otimes_k A_j \otimes_k A)) = \min(4^2, 4^{n-2}) = 16$. Therefore, for any odd extension L/K we have $\operatorname{ind}(A_i \otimes_k A_j)_L = 16$. Then $(q_i)_L$ and $(q_j)_L$ are not linked.

Now we set $E = K((t_1))\dots((t_n))$ and $\phi = t_1(q_1)_E \perp \dots \perp t_n(q_n)_E$. We have $c(\phi) = [(A_1)_E] + \dots + [(A_n)_E] = [A_E] = [(A_{k(\mathrm{SB}(A))})_E] = 0$. Hence $\phi \in I^3(E)$. Applying Lemma 13.5, one can show that $\phi \notin S_{\mathrm{odd}}(E)$.

15. Nonstandard isotropy

Let ϕ and ψ be anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. In the case where $\dim\phi \le 6$ the problem was studied by many authors: the case $\dim\phi \le 4$ was studied by Schapiro in [**75**]; the case $\dim\phi = 5$ was studied by D. W. Hoffmann in [**15**]; for 6-dimensional forms ϕ the problem was studied by D. W. Hoffmann ([**16**]), A. Laghribi ([**44**], [**45**]), D. Leep ([**49**]), A. S. Merkurjev ([**54**]), and in Chapters 3, 4.

In these papers the authors show that under certain conditions on ϕ and ψ the isotropy of ϕ over $F(\psi)$ is standard in a sense. Let us recall the definition of "standard isotropy" given in Chapter 4. [3]

Definition 15.1. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. We say that the isotropy of $\phi_{F(\psi)}$ is *standard*, if at least one of the following conditions holds:

- ψ is similar to a subform in ϕ;
- there exists a subform $\phi_0 \subset \phi$ with the following two properties:
  - the form ϕ_0 is a Pfister neighbor,
  - the form $(\phi_0)_{F(\psi)}$ is isotropic.

Otherwise, we say that the isotropy is *non-standard*.

The main theorem of Chapter 4 asserts that in the case $\dim\phi \le 6$, the isotropy $\phi_{F(\psi)}$ is standard except (possibly) the following case: $\dim\phi = 6$, $\dim\psi = 4$, $1 \ne \det_\pm \phi \ne \det_\pm \psi \ne 1$, and $\operatorname{ind} C_0(\phi) = 2 = \operatorname{ind} C_0(\phi) \otimes_F C_0(\psi)$.

In this section we show that there exist a 6-dimensional quadratic form ϕ and a 4-dimensional quadratic form ψ such that $\phi_{F(\psi)}$ is isotropic, but the isotropy is not standard. More precisely, we prove the following

Theorem 15.2. *Let k be a field of characteristic $\ne 2$, and let $a, b, u, v, d, \delta \in k^*$. Suppose that $d, \delta, d\delta \notin k^{*2}$ and $((a,b) \otimes (u,v))_{k(\sqrt{d},\sqrt{\delta})}$ is a division algebra. Then there exist a field extension K/k and $c \in K^*$ with the following properties:*

1) *Quadratic forms $\phi = \langle\langle a,b\rangle\rangle \perp -c\langle\langle d\rangle\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ are anisotropic, and $\phi_{K(\psi)}$ is isotropic,*
2) *the isotropy $\phi_{K(\psi)}$ is not standard.*

Proof. Let $\rho = \langle -a, -b, ab, d\rangle$. It follows from Corollary 11.4 that there exists a field extension K/k such that $\operatorname{Tors}\mathrm{CH}^2((X_\psi)_K \times (X_\rho)_K) = \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{ind} C_0(\psi_K) \otimes C_0(\rho_K) = 2$. To complete the proof, it is sufficient to apply Theorem 9.1 of Chapter 4. □

[3] If $\dim\phi \le 6$, this definition coincides with definitions given in [**23**] and [**30**]. In this section we consider only the case $\dim\phi \le 6$.

Let ϕ be a K-form and E/F be a field extension. We recall that a quadratic form ϕ is called *E-minimal* [**20**, Definition 1.1] if the following conditions hold:

- ϕ is anisotropic,
- ϕ_E is isotropic,
- $(\phi_0)_E$ is anisotropic for any form $\phi_0 \subset \phi$ with $\dim \phi_0 < \dim \phi$.

LEMMA 15.3. *Let ϕ be a 6-dimensional and ψ a 4-dimensional quadratic forms over K. Suppose that ϕ is anisotropic and $\phi_{F(\psi)}$ is isotropic. Then the following conditions are equivalent:*

1) *the isotropy $\phi_{K(\psi)}$ is not standard,*
2) *ϕ is a $K(\psi)$-minimal form.*

PROOF. 1)$\Rightarrow$2). Suppose at the moment that ϕ is not $F(\psi)$-minimal. Then there exists $\phi_0 \subset \phi$ with $\dim \phi_0 < \dim \phi$ such that $(\phi_0)_{F(\psi)}$ is isotropic. The isotropy $(\phi_0)_{F(\psi)}$ is standard because the dimension of ϕ_0 is ≤ 5. The definition of standard isotropy shows that the isotropy $\phi_{K(\psi)}$ is standard too, a contradiction.

2)$\Rightarrow$1). Suppose that isotropy $\phi_{F(\psi)}$ is standard. Then at least one of the cases of Definition 15.1 holds. First suppose that ψ is similar to a subform of ϕ. Let $\phi_0 \subset \phi$ be such that $\psi \sim \phi_0$. Clearly, $(\phi_0)_{F(\psi)}$ is isotropic and $\dim \phi_0 = 4 < 6 = \dim \phi$. Therefore ϕ is not $F(\psi)$-minimal, a contradiction. Now, consider the second case in Definition 15.1, i.e., suppose that there exists a subform $\phi_0 \subset \phi$ which is a Pfister neighbor such that $(\phi_0)_{F(\psi)}$ is isotropic. If $\dim \phi_0 < \dim \phi$, then ϕ is not a $F(\psi)$-minimal, and we have a contradiction. Now, let $\dim \phi_0 = \dim \phi = 6$. Then $\phi = \phi_0$ is a 6-dimension Pfister neighbor. Since $\phi_{F(\psi)}$ is isotropic, it follows that an arbitrary 5-dimensional subform of ϕ is isotropic over $F(\psi)$. Hence, ϕ is not $F(\psi)$-minimal, a contradiction. $\square$

COROLLARY 15.4. *Let ψ be an anisotropic 4-dimensional quadratic form over k with $\det_{\pm} \psi \neq 1$. Then there exist a field K and a 6-dimensional form ϕ over K such that ϕ is a $K(\psi)$-minimal form.*

PROOF. Replacing ψ by a similar form, we can suppose that ψ has the form $\langle -u, -v, uv, \delta\rangle$. Replacing k by a field of rational functions $k(a, b, d)$, we can suppose that there exist $a, b, d \in k^*$ such that $d, \delta, d\delta \notin k^*$ and $((a, b) \otimes (u, v))_{k(\sqrt{d},\sqrt{\delta})}$ is a division algebra. Let K/k and $c \in K^*$ be as in Theorem 15.2. Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$. Theorem 15.2 implies that $\phi_{K(\psi)}$ is isotropic, but isotropy is not standard. Lemma 15.3 shows that ϕ is a $K(\psi)$-minimal form. $\square$

CHAPTER 6

On the group $H^3(F(\psi, D)/F)$

Let F be a field of characteristic different from 2, ψ a quadratic F-form of dimension ≥ 5, and D a central simple F-algebra of index 8 and exponent 2. We denote by $F(\psi, D)$ the function field of the product $X_\psi \times X_D$, where X_ψ is the projective quadric determined by ψ and where X_D is the Severi-Brauer variety determined by D. We compute the relative Galois cohomology group $H^3(F(\psi, D)/F)$ (with the coefficients $\mathbb{Z}/2$) under the assumption that the index of D goes down when extending the scalars to $F(\psi)$. Using this, we give new, very short proofs for the following results:

– Theorem 7.1, originally proved by Laghribi in [**47**] and
– Theorem 7.2, originally proved by Esnault, Kahn, Levine, and Viehweg in [**10**].

We also generalize the computation of $H^3(F(\psi, D)/F)$ to the case of arbitrary $\operatorname{ind} D$.

Results of this Chapter are obtained in joint work with Oleg Izhboldin.

0. Introduction

Let ψ be a quadratic form and D be an exponent 2 central simple algebra over a field F (always assumed to be of characteristic not 2). Let X_ψ be the projective quadric determined by ψ, X_D the Severi-Brauer variety determined by D, and $F(\psi, D)$ the function field of the product $X_\psi \times X_D$.

A computation of the relative Galois cohomology group

$$H^3(F(\psi, D)/F) \stackrel{\text{def}}{=} \ker\big(H^3(F, \mathbb{Z}/2) \to H^3(F(\psi, D), \mathbb{Z}/2)\big)$$

played a crucial role in obtaining the results of Chapters 3 and 7 concerning the problem of isotropy of quadratic forms over the function fields of quadrics.

The group $H^3(F(\psi, D)/F)$ is closely related to the Chow group $\mathrm{CH}^2(X_\psi \times X_D)$ of 2-codimensional cycles on the product $X_\psi \times X_D$. The main result of this chapter is Theorem 8.1, where the both groups are computed assuming $\dim \psi \geq 5$ and the index of D goes down when extending the scalars to the function field of ψ.

The essential part of the proof is Theorem 6.9 dealing with the case where D is a division algebra of degree 8. This Theorem has two applications in the theory of quadratic forms: a new shorter proof of Theorem 7.1, originally proved by Laghribi ([**47**, Théorème 1]), and a new, shorter, and more elementary proof of Theorem 7.2, originally proved by Esnault, Kahn, Levine, and Viehweg ([**10**, Corollary 9.2]).

An important role in the proof of Theorem 6.9 plays the formula of Proposition 4.4, which is in fact applicable to a wide class of algebraic varieties.

1. Terminology, notation, and backgrounds

1.1. Quadratic forms. Mainly, we use notation of [**48**] and [**76**]. However there is certain slight difference: we denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form

$$\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

and by $P_n(F)$ the set of all n-fold Pfister forms; $GP_n(F)$ is the set of forms similar to a form from $P_n(F)$.

We recall that a quadratic form ψ is called a *(Pfister) neighbor* (of a Pfister form π), if it is similar to a subform in π and $\dim\phi > \frac{1}{2}\dim\pi$. Two quadratic forms ϕ and ϕ^* are *half-neighbors*, if $\dim\phi = \dim\phi^*$ and there exists $s \in F^*$ such that the sum $\phi \perp s\phi^*$ is similar to a Pfister form.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$ and we set $F(\phi) = F(X_\phi)$.

1.2. Generic splitting tower. Let γ be a non-hyperbolic quadratic form over F. Put $F_0 \overset{\text{def}}{=} F$ and $\gamma_0 \overset{\text{def}}{=} \gamma_{an}$. For $i \geq 1$ let $F_i \overset{\text{def}}{=} F_{i-1}(\gamma_{i-1})$ and $\gamma_i \overset{\text{def}}{=} ((\gamma_{i-1})_{F_i})_{an}$. The smallest h such that $\dim\gamma_h \leq 1$ is called the *height* of γ. The sequence $F_0, F_1, \dots, F_h$ is called the *generic splitting tower* of γ ([**40**]). We need some properties of the fields F_s:

Lemma 1.2.1 ([**41**]). *Let M/F be a field extension such that* $\dim(\gamma_M)_{an} = \dim\gamma_s$. *Then the field extension MF_s/M is purely transcendental.*

The following lemma is a consequence of the index reduction formula [**55**].

Lemma 1.2.2 (see [**22**, Theorem 1.6] or [**19**, Proposition 2.1]). *Let ϕ be a quadratic form from $I^2(F)$ with* $\operatorname{ind} C(\phi) \geq 2^r > 1$. *Then there is s with $0 \leq s \leq h(\phi)$ such that* $\dim\phi_s = 2r + 2$ *and* $\operatorname{ind} C(\phi_s) = 2^r$.

Corollary 1.2.3. *Let $\phi \in I^2(F)$ be a quadratic form with* $\operatorname{ind}(C(\phi)) \geq 8$. *Then there is s* $(0 \leq s \leq h(\phi))$ *such that* $\dim\phi_s = 8$ *and* $\operatorname{ind} C(\phi_s) = 8$. □

1.3. Central simple algebras. We are working with finite-dimensional associative algebras over a field. Let D be a central simple F-algebra. We denote by X_D the Severi-Brauer variety of D and by $F(D)$ the function field $F(X_D)$.

For another central simple F-algebra D' and for a quadratic F-form ψ of dimension ≥ 3, we set $F(D', D) \overset{\text{def}}{=} F(X_{D'} \times X_D)$ and $F(\psi, D) \overset{\text{def}}{=} F(X_\psi \times X_D)$.

1.4. Galois cohomology. By $H^*(F)$ we denote the graded ring of Galois cohomology

$$H^*(F, \mathbb{Z}/2\mathbb{Z}) = H^*(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2).$$

For any field extension L/F, we set $H^*(L/F) \overset{\text{def}}{=} \ker(H^*(F) \to H^*(L))$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \mathrm{Br}_2(F)$.

We also work with the cohomology groups $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$, $i = 0, 1, 2$, defined by B. Kahn (see [**29**]). For any field extension L/F, we set

$$H^*(L/F, \mathbb{Q}/\mathbb{Z}(i)) \stackrel{\text{def}}{=} \ker\left(H^*(F, \mathbb{Q}/\mathbb{Z}(i)) \to H^*(L, \mathbb{Q}/\mathbb{Z}(i))\right) .$$

For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with

$$\operatorname{Tors}_2 H^n(F, \mathbb{Q}/\mathbb{Z}(n-1)) .$$

1.5. K-theory and Chow groups. We are mainly working with smooth algebraic varieties over a field, although the smoothness assumption is not always essential.

Let X be a smooth algebraic F-variety. The Grothendieck ring of X is denoted by $K(X)$. This ring is supplied with the filtration "by codimension of support" (which respects the multiplication); the adjoint graded ring is denoted by $G^*K(X)$. There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $G^*K(X)$; its kernel consists only of torsion elements and is trivial in the 0-th, 1-st and 2-nd graded components ([**81**, §9]). In particular we have the following

LEMMA 1.5.1. *The homomorphism $\mathrm{CH}^i(X) \to G^iK(X)$ is bijective if at least one of the following conditions holds:*

- $i = 0$, 1, *or* 2,
- $\mathrm{CH}^i(X)$ *is torsion-free.*

Let X be a variety over F and E/F be a field extension. We denote by $i_{E/F}$ the restriction homomorphism $K(X) \to K(X_E)$. The same notation we use for the restriction homomorphisms $\mathrm{CH}^*(X) \to \mathrm{CH}^*(X_E)$ and $G^*K(X) \to G^*K(X_E)$. We fix a separable closure $\bar{F}$ of the ground field F and denote by $\bar{X}$ the variety $X_{\bar{F}}$. The image of the restriction homomorphism $i_{\bar{F}/F} : G^*K(X) \to G^*K(\bar{X})$ is denoted by $\bar{G}^*K(X)$. The image of the restriction homomorphism $i_{\bar{F}/F} : \mathrm{CH}^*(X) \to \mathrm{CH}^*(\bar{X})$ is denoted by $\bar{\mathrm{CH}}^*(X)$.

For a projective homogeneous variety X, we identify $K(X)$ with a subring of $K(\bar{X})$ via the restriction homomorphism $i_{\bar{F}/F} : K(X) \to K(\bar{X})$ which is injective by [**65**].

We denote by $|S|$ the order of a set S (if S is infinite we set $|S| = \infty$).

2. The group $\operatorname{Tors} G^*K(X)$

LEMMA 2.1. *Let X be a variety over F and E/F be a field extension such that the homomorphism $i_{E/F} : K(X) \to K(X_E)$ is injective and the factor group $K(X_E)/i_{E/F}(K(X))$ is finite. Then*

$$|\ker(G^*K(X) \to G^*K(X_E)| = \frac{|G^*K(X_E)/i_{E/F}(G^*K(X))|}{|K(X_E)/i_{E/F}(K(X))|}$$

PROOF. The proof is the same as the proof of [**33**, Proposition 2]. □

LEMMA 2.2. *Let X be a variety, i be an integer, and E/F be a field extension such that the group $G^iK(X_E)$ is torsion-free. Then*

$$\ker(G^iK(X)\to G^iK(X_E))=\operatorname{Tors}G^iK(X)\ .$$

PROOF. Since $G^iK(X_E)$ is torsion-free, one has

$$\ker(G^iK(X)\to G^iK(X_E))\supset\operatorname{Tors}G^iK(X)\ .$$

On the other hand, transfer (and specialization) arguments show that

$$\ker(G^iK(X)\to G^iK(X_E))\subset\operatorname{Tors}G^iK(X)\ .$$

□

LEMMA 2.3. *Let X be a smooth variety, i be an integer, and E/F be a field extension such that the group $\mathrm{CH}^i(X_E)$ is torsion-free. Then*

- $\mathrm{CH}^i(X_E)\simeq G^iK(X_E)$ *(and hence the group $G^iK(X_E)$ is torsion-free),*
- $\mathrm{CH}^i(X_E)/i_{E/F}(\mathrm{CH}^i(X))\simeq G^iK(X_E)/i_{E/F}(G^iK(X))$.

PROOF. The first assertion is contained in Lemma 1.5.1. The homomorphism $\mathrm{CH}^i(X_E)\to G^iK(X_E)$ induces a homomorphism

$$\mathrm{CH}^i(X_E)/i_{E/F}(\mathrm{CH}^i(X))\to G^iK(X_E)/i_{E/F}(G^iK(X))$$

which is bijective since $\mathrm{CH}^i(X_E)\to G^iK(X_E)$ is bijective and $\mathrm{CH}^i(X)\to G^iK(X)$ is surjective. □

PROPOSITION 2.4. *Suppose that a smooth F-variety X and a field extension E/F satisfy the following three conditions:*

- *the homomorphism $i_{E/F}:K(X)\to K(X_E)$ is injective,*
- *the factorgroup $K(X_E)/i_{E/F}(K(X))$ is finite,*
- *the group $\mathrm{CH}^*(X_E)$ is torsion-free.*

Then

$$|\operatorname{Tors}G^*K(X)|=\frac{|G^*K(X_E)/i_{E/F}(G^*K(X))|}{|K(X_E)/i_{E/F}(K(X))|}=\frac{|\mathrm{CH}^*(X_E)/i_{E/F}(\mathrm{CH}^*K(X))|}{|K(X_E)/i_{E/F}(K(X))|}$$

PROOF. It is an obvious consequence of Lemmas 2.1, 2.2, and 2.3. □

3. Auxiliary lemmas

For an abelian group A we use notation $\mathrm{rk}(A)=\dim_{\mathbb{Q}}(A\otimes_{\mathbb{Z}}\mathbb{Q})$.

LEMMA 3.1. *Let $A_0\subset A$, $B_0\subset B$ be free abelian groups such that* $\mathrm{rk}\,A_0=\mathrm{rk}\,A=r_A$, $\mathrm{rk}\,B_0=\mathrm{rk}\,B=r_B$. *Then*

$$\left|\frac{A\otimes_{\mathbb{Z}}B}{A_0\otimes_{\mathbb{Z}}B_0}\right|=\left|\frac{A}{A_0}\right|^{r_B}\cdot\left|\frac{B}{B_0}\right|^{r_A}.$$

PROOF. One has

$$(A\otimes B)/(A_0\otimes B)\simeq(A/A_0)\otimes B\simeq(A/A_0)\otimes\mathbb{Z}^{r_B}\simeq(A/A_0)^{r_B},$$
$$(A_0\otimes B)/(A_0\otimes B_0)\simeq A_0\otimes(B/B_0)\simeq\mathbb{Z}^{r_A}\otimes(B/B_0)\simeq(B/B_0)^{r_A}\ .$$

Therefore,

$$\left|\frac{A\otimes B}{A_0\otimes B_0}\right| = \left|\frac{A\otimes B}{A_0\otimes B}\right| \cdot \left|\frac{A_0\otimes B}{A_0\otimes B_0}\right| = \left|\frac{A}{A_0}\right|^{r_B} \cdot \left|\frac{B}{B_0}\right|^{r_A}.$$

$\square$

The following lemma is well known.

LEMMA 3.2. *Let A be an abelian group with a finite filtration $A = \mathcal{F}^0A \supset \mathcal{F}^1A \supset \cdots \supset \mathcal{F}^kA = 0$. Let B be a subgroup of A with the filtration $\mathcal{F}^pB = B\cap\mathcal{F}^pA$. Let $G^*A = \bigoplus_{p\geq 0}\mathcal{F}^pA/\mathcal{F}^{p+1}A$ and $G^*B = \bigoplus_{p\geq 0}\mathcal{F}^pB/\mathcal{F}^{p+1}B$. Then*

- $|A/B| = |G^*A/G^*B|$,
- *if A is a finitely generated group then* $\operatorname{rk} G^*A = \operatorname{rk} A$. $\square$

In the following lemma the term "ring" means a *commutative ring with unit.*

LEMMA 3.3. *Let A and B be rings whose additive groups are finitely generated abelian groups. Let I be a nilpotent ideal of A such that $A/I \simeq \mathbb{Z}$. Let R be a subring of $A\otimes_{\mathbb{Z}} B$ and A_R be a subring of A such that $A_R\otimes 1\subset R$. Then the following inequality holds*

$$\left|\frac{A\otimes_{\mathbb{Z}} B}{R}\right| \leq \left|\frac{A}{A_R}\right|^{r_B} \cdot \left|\frac{A\otimes_{\mathbb{Z}} B}{R+(I\otimes_{\mathbb{Z}} B)}\right|^{r_A}$$

where $r_A = \operatorname{rk} A$ and $r_B = \operatorname{rk} B$.

PROOF. Let us denote by B_R the image of R under the following composition $A\otimes B \to (A/I)\otimes B \simeq \mathbb{Z}\otimes B \simeq B$. Obviously,

$$\left|\frac{A\otimes_{\mathbb{Z}} B}{R+(I\otimes_{\mathbb{Z}} B)}\right| = \left|\frac{B}{B_R}\right|.$$

For any $p\geq 0$ we set $\mathcal{F}^pA = \{a\in A \mid \exists m\in\mathbb{N} \text{ such that } ma\in I^p\}$. Clearly, $\operatorname{Tors}(A/\mathcal{F}^pA) = 0$, and so $A/\mathcal{F}^p$ is a free abelian group. Therefore all factor groups $\mathcal{F}^pA/\mathcal{F}^{p+1}A$ $(p = 0,1,\dots)$ are free abelian. Since $A/I\simeq\mathbb{Z}$, it follows that $\mathcal{F}^1A = I$. Thus $A/\mathcal{F}^1A \simeq \mathbb{Z}$. Since I is a nilpotent ideal of A, there exists k such that $I^k = 0$. Then $\mathcal{F}^kA = 0$. Thus the filtration $A = \mathcal{F}^0A \supset \mathcal{F}^1A \supset \mathcal{F}^2A \supset \dots$ is finite and results of Lemma 3.2 can be applied.

Let $\mathcal{F}^pA_R \stackrel{\text{def}}{=} R\cap\mathcal{F}^pA$, $\mathcal{F}^p(A\otimes B) \stackrel{\text{def}}{=} \operatorname{im}(\mathcal{F}^pA\otimes B \to A\otimes B)$, and $\mathcal{F}^pR \stackrel{\text{def}}{=} R\cap\mathcal{F}^p(A\otimes B)$. If K is one of the rings A, A_R, $A\otimes B$, or R, we set $G^pK \stackrel{\text{def}}{=} \mathcal{F}^pK/\mathcal{F}^{p+1}K$ and $G^*K \stackrel{\text{def}}{=} \bigoplus_{p\geq 0}\mathcal{F}^pK/\mathcal{F}^{p+1}K$. Obviously, $\mathcal{F}^pK\cdot\mathcal{F}^qK \subset \mathcal{F}^{p+q}K$ for all p and q. Therefore, $K = \mathcal{F}^0K \supset \mathcal{F}^1K \supset \cdots \supset \mathcal{F}^pK \supset \dots$ is a ring filtration. Hence, the adjoint graded group G^*K has a graded ring structure. Since the additive group of B is free, we have a natural ring isomorphism $G^*A\otimes B \simeq G^*(A\otimes B)$.

Since $A_R\otimes 1\subset R$, we have $G^*A_R\otimes 1 \subset G^*R$. Clearly $G^0(A\otimes B) = (A/I)\otimes B$, and G^0R coincides with the image of the composition $R\to A\otimes B \to (A/I)\otimes B$. By definition of B_R, one has $G^0R = 1_{G^*A}\otimes B_R$ (here

1_{G^*A} denotes the unit of the ring G^*A). Therefore $1_{G^*A}\otimes B_R\subset G^*R$. Since $G^*A_R\otimes 1\subset G^*R$, $1_{G^*A}\otimes B_R\subset G^*R$, and G^*R is a subring of $G^*A\otimes B$, we have $G^*A_R\otimes B_R\subset G^*R$. Therefore $|G^*(A\otimes B)/G^*R|\le|(G^*A\otimes B)/(G^*A_R\otimes B_R)|$. Applying Lemmas 3.1 and 3.2, we have

$$\left|\frac{A\otimes B}{R}\right|=\left|\frac{G^*(A\otimes B)}{G^*R}\right|\le\left|\frac{G^*A\otimes B}{G^*A_R\otimes B_R}\right|=\left|\frac{G^*A}{G^*A_R}\right|^{r_B}\cdot\left|\frac{B}{B_R}\right|^{r_A}=$$
$$=\left|\frac{A}{A_R}\right|^{r_B}\cdot\left|\frac{B}{B_R}\right|^{r_A}=\left|\frac{A}{A_R}\right|^{r_B}\cdot\left|\frac{A\otimes_{\mathbb{Z}}B}{R+(I\otimes_{\mathbb{Z}}B)}\right|^{r_A}.$$

□

4. On the group $\mathrm{CH}^*(X\times Y)$

Let X be a smooth variety. We denote by $\mathcal{F}^p\mathrm{CH}^*(X)$ the group

$$\bigoplus_{i\ge p}\mathrm{CH}^i(X)\ .$$

Let Y be one more smooth variety. For a subgroup A of $\mathrm{CH}^*(X)$ and a subgroup B of $\mathrm{CH}^*(Y)$, we denote by $A\boxtimes B$ the image of the composition $A\otimes B\to\mathrm{CH}^*(X)\otimes\mathrm{CH}^*(Y)\to\mathrm{CH}^*(X\times Y)$.

The following assertion is evident (see also [**39**, §3] and Proposition 4.1 of Chapter 5).

Lemma 4.1. *Let X and Y be smooth varieties over F. Then*

- *the natural homomorphism $\mathrm{CH}^*(X\times Y)\to\mathrm{CH}^*(Y_{F(X)})$ is surjective,*
- *the kernel of the homomorphism $\mathrm{CH}^*(X\times Y)\to\mathrm{CH}^*(Y_{F(X)})$ contains the group $\mathcal{F}^1\mathrm{CH}^*(X)\boxtimes\mathrm{CH}^*(Y)$.*

□

Corollary 4.2. *If the natural homomorphism*

$$\mathrm{CH}^*(X)\otimes\mathrm{CH}^*(Y)\to\mathrm{CH}^*(X\times Y)$$

is bijective and $\mathrm{CH}^*(Y)$ is torsion-free then the homomorphism $\mathrm{CH}^*(X\times Y)\to\mathrm{CH}^*(Y_{F(X)})$ induces an isomorphism

$$\frac{\mathrm{CH}^*(X\times Y)}{\mathcal{F}^1\mathrm{CH}^*(X)\boxtimes\mathrm{CH}^*(Y)}\to\mathrm{CH}^*(Y_{F(X)}).$$

Proof. Since

$$\mathrm{CH}^*(X)\otimes\mathrm{CH}^*(Y)\simeq\mathrm{CH}^*(X\times Y)\ \text{ and }\ \mathrm{CH}^*(X)/\mathcal{F}^1\mathrm{CH}^*(X)\simeq\mathrm{CH}^0(X),$$

the factor group $\mathrm{CH}^*(X\times Y)/(\mathcal{F}^1\mathrm{CH}^*(X)\boxtimes\mathrm{CH}^*(Y))$ is isomorphic to

$$\mathrm{CH}^0(X)\otimes_{\mathbb{Z}}\mathrm{CH}^*(Y)\simeq\mathbb{Z}\otimes_{\mathbb{Z}}\mathrm{CH}^*(Y)\simeq\mathrm{CH}^*(Y)\ .$$

Thus, it is sufficient to prove that the homomorphism $\mathrm{CH}^*(Y)\to\mathrm{CH}^*(Y_{F(X)})$ is injective. It is obvious since $\mathrm{CH}^*(Y)$ is torsion-free. □

Corollary 4.3. *Let X and Y be smooth varieties and E/F be a field extension such that the natural homomorphism $\mathrm{CH}^*(X_E) \otimes \mathrm{CH}^*(Y_E) \to \mathrm{CH}^*(X_E \times Y_E)$ is bijective and $\mathrm{CH}^*(Y_E)$ is torsion-free. Then there exists an isomorphism*

$$\frac{\mathrm{CH}^*(X_E \times Y_E)}{i_{E/F}(\mathrm{CH}^*(X \times Y)) + \mathcal{F}^1\mathrm{CH}^*(X_E) \boxtimes \mathrm{CH}^*(Y_E)} \simeq \frac{\mathrm{CH}^*(Y_{E(X)})}{i_{E(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))}$$

Proof. Obvious in view of Corollary 4.2. □

Proposition 4.4. *Let X and Y be smooth varieties over F and E/F be a field extension such that the following conditions hold*

- *$\mathrm{CH}^*(X_E)$ is a free abelian group of rank r_X,*
- *$\mathrm{CH}^*(Y_E)$ is a free abelian group of rank r_Y,*
- *the canonical homomorphism*
$$\mathrm{CH}^*(X_E) \otimes_{\mathbb{Z}} \mathrm{CH}^*(Y_E) \to \mathrm{CH}^*(X_E \times Y_E)$$
is an isomorphism.

Then

$$\left|\frac{\mathrm{CH}^*(X_E \times Y_E)}{i_{E/F}(\mathrm{CH}^*(X \times Y))}\right| \leq \left|\frac{\mathrm{CH}^*(X_E)}{i_{E/F}(\mathrm{CH}^*(X))}\right|^{r_Y} \cdot \left|\frac{\mathrm{CH}^*(Y_{E(X)})}{i_{E(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))}\right|^{r_X}.$$

Proof. Let $A = \mathrm{CH}^*(X_E)$, $A_R = i_{E/F}(\mathrm{CH}^*(X))$ and

$$I = \bigoplus_{p>0} \mathrm{CH}^p(X_E) = \mathcal{F}^1\mathrm{CH}^*(X_E)\ .$$

Let $B = \mathrm{CH}^*(Y_E)$. By our assumption, we have $\mathrm{CH}^*(X_E \otimes Y_E) \simeq A \otimes_{\mathbb{Z}} B$. We denote by R the image of the composition

$$\mathrm{CH}^*(X \times Y) \to \mathrm{CH}^*(X_E \otimes Y_E) \simeq A \otimes_{\mathbb{Z}} B\ .$$

Clearly, all conditions of Lemma 3.3 hold. Moreover,

$$\left|\frac{\mathrm{CH}^*(X_E \times Y_E)}{i_{E/F}(\mathrm{CH}^*(X \times Y))}\right| = \left|\frac{A \otimes_{\mathbb{Z}} B}{R}\right| \qquad \text{and} \qquad \left|\frac{\mathrm{CH}^*(X_E)}{i_{E/F}(\mathrm{CH}^*(X))}\right| = \left|\frac{A}{A_R}\right|.$$

By Corollary 4.3 we have

$$\left|\frac{A \otimes_{\mathbb{Z}} B}{R + (I \otimes_{\mathbb{Z}} B)}\right| = \left|\frac{\mathrm{CH}^*(Y_{E(X)})}{i_{E(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))}\right|.$$

To complete the prove it suffices to apply Lemma 3.3. □

5. The group $\operatorname{Tors} \mathrm{CH}^2(X_\psi \times X_D)$

Lemma 5.1 (see [**32**, (2.1)]). *Let ψ be a $(2n+1)$-dimensional quadratic form over a separably closed field. Set $X \stackrel{\text{def}}{=} X_\psi$ and $d \stackrel{\text{def}}{=} \dim X = 2n-1$. Then for all $0 \leq p \leq d$ the group $\mathrm{CH}^p(X)$ is canonically isomorphic to $\mathbb{Z}$ (for other p the group $\mathrm{CH}^p(X)$ is trivial). Moreover,*

- *if $0 \leq p < n$, then $\mathrm{CH}^p(X) = \mathbb{Z} \cdot h^p$, where $h \in \mathrm{CH}^1(X)$ denotes the class of a hyperplane section of X;*

- *if* $n \le p \le d$, *then* $\mathrm{CH}^p(X) = \mathbb{Z}\cdot l_{d-p}$, *where* l_{d-p} *denotes the class of a linear subspace in* X *of dimension* $d-p$, *besides* $2l_{d-p} = h^p$.

COROLLARY 5.2. *Let* ψ *be a* $(2n+1)$*-dimensional quadratic form over* F *and let* $X = X_\psi$. *Then*

- $\mathrm{CH}^*(\bar{X})$ *is a free abelian group of rank* $2n$,
- *if* $0 \le p < n$ *then* $|\mathrm{CH}^p(\bar{X})/\bar{\mathrm{CH}}^p(X)| = 1$,
- *if* $n \le p \le 2n-1$ *then* $|\mathrm{CH}^p(\bar{X})/\bar{\mathrm{CH}}^p(X)| \le 2$,
- $|\mathrm{CH}^*(\bar{X})/\bar{\mathrm{CH}}^*(X)| \le 2^n$. □

LEMMA 5.3. *Let* D *be a central simple* F*-algebra of exponent* 2 *and of degree* 8. *Let* $E/L/F$ *be field extensions such that* $\operatorname{ind} D_L = 4$ *and* $\operatorname{ind} D_E = 1$. *Let* $Y = \mathrm{SB}(D)$. *For any* $0 \le p \le \dim Y = 7$, *the group* $\mathrm{CH}^p(Y_E)$ *is canonically isomorphic to* $\mathbb{Z}$. *Moreover, the image of the homomorphism* $i_{E/L} : \mathrm{CH}^p(Y_L) \to \mathrm{CH}^p(Y_E) \simeq \mathbb{Z}$ *contains* 1 *if* $p = 0, 4$*;* 2 *if* $p = 1, 2, 5, 6$*;* 4 *if* $p = 3, 7$.

PROOF. Since $\deg D = 8$ and $\operatorname{ind} D_E = 1$, Y_E is isomorphic to $\mathbb{P}^7_E$. Hence, the group $\mathrm{CH}^p(Y_E) \cong \mathrm{CH}^p(\mathbb{P}^7_E)$ (where $p = 0, \ldots, 7$) is generated by the class h^p of a linear subspace ([**14**]).

The rest part of Lemma is contained in [**34**, Theorem]. For convenience of the reader, we also give a direct construction of the elements required. Let us denote by $\tilde{\mathrm{CH}}^p(Y_L)$ the image of the composition

$$\mathrm{CH}^p(Y_L) \xrightarrow{i_{E/L}} \mathrm{CH}^p(Y_E) \simeq \mathbb{Z}\,\cdot$$

The class of Y_L itself gives $1 \in \tilde{\mathrm{CH}}^0(Y_L)$. Let ξ be the tautological line bundle on the projective space $\mathbb{P}^7_E \simeq Y_E$. Since $\exp D = 2$, the bundle $\xi^{\otimes 2}$ is defined over F and, in particular, over L. Its first Chern class gives $2 \in \tilde{\mathrm{CH}}^1(Y_L)$. Since $\operatorname{ind} D_L = 4$, the bundle $\xi^{\oplus 4}$ is defined over L. Its second Chern class gives $6 \in \tilde{\mathrm{CH}}^2(Y_L)$.[1] Thus $2 \in \tilde{\mathrm{CH}}^2(Y_L)$. The third Chern class of $\xi^{\oplus 4}$ gives $4 \in \tilde{\mathrm{CH}}^3(Y_L)$. The fourth Chern class of $\xi^{\oplus 4}$ gives $1 \in \tilde{\mathrm{CH}}^4(Y_L)$. Finally, taking the product of the cycles constructed in codimensions 1, 2, and 3 with the cycle of codimension 4, one gets the cycles of codimensions 5, 6, and 7 required. □

COROLLARY 5.4. *Under the condition of Lemma 5.3, we have*

$$|\mathrm{CH}^*(Y_E)/i_{E/L}(\mathrm{CH}^*(Y_L))| \le 256\,.$$

PROOF. $\displaystyle\prod_{p=0}^{7} |\mathrm{CH}^p(Y_E)/i_{E/L}(\mathrm{CH}^p(Y_L))| \le 1\cdot 2\cdot 2\cdot 4\cdot 1\cdot 2\cdot 2\cdot 4 = 256\,.$ □

PROPOSITION 5.5. *Let* D *be a central division* F*-algebra of degree* 8 *and exponent* 2. *Let* ψ *be a* 5*-dimensional quadratic* F*-form. Suppose that* $D_{F(\psi)}$ *is not a skewfield. Then* $\operatorname{Tors} G^*K(X_\psi \times X_D) = 0$.

---

[1] In fact, it is enough only to know that the *Grothendieck classes* of the bundles $\xi^{\otimes 2}$ and $\xi^{\oplus 4}$ are in $K(Y_L)$ what can be also seen from the computation of the K-theory.

PROOF. Let $X = X_\psi$ and $Y = X_D$. Corollary 5.2 shows that $\mathrm{CH}^*(X_{\bar F})$ is a free abelian group of rank $r_X = 4$ and $|\mathrm{CH}^*(X_{\bar F})/i_{\bar F/F}(\mathrm{CH}^*(X))| \le 2^2 = 4$.

Since D is a division algebra of degree 8 and $D_{F(\psi)}$ is not division algebra, it follows that $\operatorname{ind} D_{F(X)} = 4$. Applying Corollary 5.4 to the case $L = F(X)$, $E = \bar F(X)$, we have $|\mathrm{CH}^*(Y_{\bar F(X)})/i_{\bar F(X)/F(X)}(\mathrm{CH}^*(Y_{F(X)}))| \le 256$.

Since $Y_{\bar F} = \mathrm{SB}(D_{\bar F}) \simeq \mathbb{P}^7_{\bar F}$, the group $\mathrm{CH}^*(Y_{\bar F})$ is a free abelian of rank $r_Y = 8$ and $\mathrm{CH}^*(X_{\bar F}) \otimes \mathrm{CH}^*(Y_{\bar F}) \simeq \mathrm{CH}^*(X_{\bar F} \times Y_{\bar F})$ (see [**12**, Proposition 14.6.5]). Thus all conditions of Proposition 4.4 hold for X, Y, $E = \bar F$ and we have

$$\left| \frac{\mathrm{CH}^*(X_{\bar F} \times Y_{\bar F})}{i_{\bar F/F}(\mathrm{CH}^*(X \times Y))} \right| \le 4^8 \cdot 256^4 = 2^{48}.$$

Using [**69**, Theorem 4.1 of §8] and [**82**, Theorem 9.1], we get a natural (with respect to extensions of F) isomorphism

$$\begin{aligned} K(X \times Y) &\simeq K\big((F^{\times 3} \times C) \otimes_F (F^{\times 4} \times D^{\times 4})\big) \simeq \\ &\simeq K\big(F^{\times 12} \times C^{\times 4} \times D^{\times 12} \times (C \otimes_F D)^{\times 4}\big) \end{aligned}$$

where $C \overset{\mathrm{def}}{=} C_0(\psi)$ is the even Clifford algebra of ψ. Note that C is a central simple F-algebra of the degree 2^2. Since $D_{F(\psi)}$ is not a skewfield, [**55**, Theorem 1] states that $D \simeq C \otimes_F B$ with some central division F-algebra B. Therefore, $\operatorname{ind} C = \deg C = 2^2$ and $\operatorname{ind} C \otimes D = \operatorname{ind} B = \deg B = 2$. Hence

$$\left| \frac{K(X_{\bar F} \times Y_{\bar F})}{i_{\bar F/F}(K(X \times Y))} \right| = (\operatorname{ind} C)^4 \cdot (\operatorname{ind} D)^{12} \cdot (\operatorname{ind} C \otimes D)^4 = 2^{2\cdot 4 + 3 \cdot 12 + 1 \cdot 4} = 2^{48}\ .$$

Applying Proposition 2.4 to the variety $X \times Y$ and $E = \bar F$, we have

$$|\operatorname{Tors} G^* K(X \times Y)| = \frac{|\mathrm{CH}^*(X_{\bar F} \times Y_{\bar F})/i_{\bar F/F}(\mathrm{CH}^*(X \times Y))|}{|K(X_{\bar F} \times Y_{\bar F})/i_{\bar F/F}(K(X \times Y))|} \le \frac{2^{48}}{2^{48}} = 1\ .$$

Therefore, $\operatorname{Tors} G^* K(X \times Y) = 0$. □

Applying Lemma 1.5.1 we get the following

COROLLARY 5.6. *Under the condition of Proposition 5.5, the group* $\mathrm{CH}^2(X_\psi \times X_D)$ *is torsion-free.* □

6. The group $H^3(F(\psi,D)/F)$

PROPOSITION 6.1 ([**5**, Satz 5.6]). *Let ψ be a quadratic F-form of dimension ≥ 5. The group $H^3(F(\psi)/F)$ is non-trivial iff ψ is a neighbor of an anisotropic 3-Pfister form.*

PROPOSITION 6.2 (see [**66**, Proposition 4.1 and Remark 4.1]). *Let D be a central division F-algebra of exponent 2. Suppose that D is decomposable (in the tensor product of two proper subalgebras). Then*

$$H^3(F(D)/F) = [D] \cup H^1(F)\ .$$

LEMMA 6.3. *If D and D' are Brauer equivalent central simple F-algebras, then the function fields $F(D)$ and $F(D')$ are stably birational equivalent.*

PROOF. Since the algebras $D_{F(D')}$ and $D'_{F(D)}$ are split, the field extensions

$$F(D,D')/F(D') \quad \text{and} \quad F(D,D')/F(D)$$

are pure transcendental. Therefore each of the field extensions $F(D)/F$ and $F(D')/F$ is stably birational equivalent to the extension $F(D,D')/F$. □

COROLLARY 6.4. *Fix a quadratic F-form ψ and integers $i,j \in \mathbb{Z}$. For any central simple F-algebra D, the groups $H^i(F(D)/F)$, $H^i(F(D)/F,\mathbb{Q}/\mathbb{Z}(j))$, $H^i(F(\psi,D)/F)$, $H^i(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(j))$ depend only on the Brauer class of D.* □

LEMMA 6.5. *Let D be a central simple F-algebra of exponent 2 and let ψ be a quadratic F-form. The group $H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2))$ is annihilated by 2.*

PROOF. Let ψ_0 be a 3-dimensional subform of ψ. Clearly,

$$H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2)) \subset H^3(F(\psi_0,D)/F,\mathbb{Q}/\mathbb{Z}(2)) .$$

Therefore, it suffices to show that the latter cohomology group is annihilated by 2. Replacing ψ_0 by the quaternion algebra $C_0(\psi_0)$, we come to a statement covered by [**24**, Lemma A.8]. □

COROLLARY 6.6. *In the conditions of Lemma 6.5, one has*

$$H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi,D)/F) .$$

□

PROPOSITION 6.7. *Let D be an exponent 2 central simple F-algebra and let ψ be a quadratic F-form of dimension ≥ 3. Suppose that* $\operatorname{ind} D_{F(\psi)} < \operatorname{ind} D$. *Then ψ is not a 3-Pfister neighbor and there is an isomorphism*

$$\frac{H^3(F(\psi,D)/F)}{H^3(F(\psi)/F) + [D] \cup H^1(F)} \simeq \operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) .$$

PROOF. By Proposition 2.2 of Chapter 4, there is an isomorphism

$$\frac{H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2))}{H^3(F(\psi)/F,\mathbb{Q}/\mathbb{Z}(2)) + H^3(F(D)/F,\mathbb{Q}/\mathbb{Z}(2))} \simeq$$

$$\simeq \frac{\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D)}{pr_\psi^* \operatorname{Tors}\operatorname{CH}^2(X_\psi) + pr_D^* \operatorname{Tors}\operatorname{CH}^2(X_D)} .$$

By Corollary 6.6, we have $H^3(F(\psi,D)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi,D)/F)$; by Lemma 2.8 of Chapter 4, we have $H^3(F(\psi)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi)/F)$; and by [**24**, Lemma A.8], we have $H^3(F(D)/F,\mathbb{Q}/\mathbb{Z}(2)) = H^3(F(D)/F)$.

Let D' be a division algebra Brauer equivalent to D. By Corollary 6.4, we have $H^3(F(D)/F) = H^3(F(D')/F)$; by Proposition 1.1 of Chapter 2, we have $\operatorname{Tors}\operatorname{CH}^2(X_D) \simeq \operatorname{Tors}\operatorname{CH}^2(X_{D'})$. Since $D'_{F(\psi)}$ is no more a skewfield, there is a homomorphism of F-algebras $C_0(\psi) \to D'$ ([**84**, Théorème 1], see also [**56**, Theorem 2]). Although the algebra $C_0(\psi)$ is not always central simple, it always contains a non-trivial subalgebra central simple over F. Therefore, D' is decomposable, what implies $H^3(F(D')/F) = [D] \cup H^1(F)$ (Proposition 6.2) and $\operatorname{Tors} \mathrm{CH}^2(X_{D'}) = 0$ (Proposition 5.3 of Chapter 1). Finally, the existence of a homomorphism $C_0(\psi) \to D'$ implies that ψ is not a 3-Pfister neighbor; therefore $\operatorname{Tors} \mathrm{CH}^2(X_\psi) = 0$ ([**32**, Theorem 6.1]). $\square$

COROLLARY 6.8. *Let D be a central division F-algebra of degree 8 and exponent 2. Let ψ be a 5-dimensional quadratic F-form. Suppose that $D_{F(\psi)}$ is not a skewfield. Then $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.*

PROOF. It is a direct consequence of Proposition 6.7, Corollary 5.6, and Proposition 6.1. $\square$

THEOREM 6.9. *Let D be a central division F-algebra of degree 8 and exponent 2. Let ψ be a quadratic form of dimension ≥ 5. Suppose that $D_{F(\psi)}$ is not a skewfield. Then $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.*

PROOF. Let ψ_0 be a 5-dimensional subform of ψ. Applying Corollary 6.8, we have $[D] \cup H^1(F) \subset H^3(F(\psi, D)/F) \subset H^3(F(\psi_0, D)/F) = [D] \cup H^1(F)$. Hence $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$. $\square$

COROLLARY 6.10. *Let $\phi \in I^2(F)$ be a 8-dimensional quadratic form such that $\operatorname{ind} C(\phi) = 8$. Let D be a degree 8 central simple algebra such that $c(\phi) = [D]$. Let ψ be a quadratic form of dimension ≥ 5 such that $\phi_{F(\psi)}$ is isotropic. Then*

1) *D is a division algebra;*
2) *$D_{F(\psi)}$ is not a division algebra;*
3) *$H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.* $\square$

7. Corollaries

In this section we demonstrate two applications of Corollary 6.10. Namely, we give very short proofs of the following two theorems:

THEOREM 7.1 ([**47**]). *Let $\phi \in I^2(F)$ be an 8-dimensional quadratic form such that $\operatorname{ind} C(\phi) = 8$. Let ψ be a quadratic form of dimension ≥ 5 such that $\phi_{F(\psi)}$ is isotropic. Then there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$.*

THEOREM 7.2 ([**10**, Corollary 9.3]). *Let $\phi \in I^2(F)$ be a quadratic form such that $\operatorname{ind} C(\phi) \geq 8$. Let A be an algebra such that $c(\phi) = [A]$. Then $\phi_{F(A)} \notin I^4(F(A))$. In particular, $\phi_{F(A)}$ is not hyperbolic. Moreover, if $\dim \phi = 8$ then $\phi_{F(A)}$ is anisotropic.*

We need several lemmas.

LEMMA 7.3. *Let $\phi \in I^2(F)$ be a 8-dimensional quadratic form and let D be an algebra such that $c(\phi) = [D]$. Then $\phi_{F(D)} \in GP_3(F(D))$.*

PROOF. We have $c(\phi_{F(D)}) = c(\phi)_{F(D)} = [D_{F(D)}] = 0$. Hence $\phi_{F(D)} \in I^3(F(D))$. Since $\dim \phi = 8$, we are done. $\square$

LEMMA 7.4. *Let $\phi,\phi^*\in I^2(F)$ be 8-dimensional quadratic forms such that $c(\phi)=c(\phi^*)=[D]$, where D is a 3-quaternion division algebra. Suppose that there is a quadratic form ψ of dimension ≥ 5 such that the forms $\phi_{F(\psi,D)}$ and $\phi^*_{F(\psi,D)}$ are isotropic. Then ϕ and ϕ^* are half-neighbors.*

PROOF. Lemma 7.3 implies that $\phi_{F(\psi,D)},\phi^*_{F(\psi,D)}\in GP_3(F(\psi,D))$. By the assumption of the lemma, $\phi_{F(\psi,D)}$ and $\phi^*_{F(\psi,D)}$ are isotropic. Hence $\phi_{F(\psi,D)}$ and $\phi^*_{F(\psi,D)}$ are hyperbolic. Thus $\phi,\phi^*\in W(F(\psi,D)/F)$.

Let $\tau=\phi\perp\phi^*$. Clearly $\tau\in W(F(\psi,D)/F)$. Since $c(\tau)=c(\phi)+c(\phi^*)=[D]+[D]=0$, we have $\tau\in I^3(F)$. Thus $e^3(\tau)\in H^3(F(\psi,D)/F)$. It follows from Corollary 6.10 that $e^3(\tau)\in[D]\cup H^1(F)$. Hence there exists $s\in F^*$ such that $e^3(\tau)=[D]\cup(s)$. We have $e^3(\tau)=[D]\cup(s)=c(\phi)\cup(s)=e^3(\phi\langle\langle s\rangle\rangle)$. Since $\ker(e^3:I^3(F)\to H^3(F))=I^4(F)$, we have $\tau\equiv\phi\langle\langle s\rangle\rangle \pmod{I^4(F)}$. Therefore $\phi+\phi^*=\tau\equiv\phi\langle\langle s\rangle\rangle=\phi-s\phi \pmod{I^4(F)}$. Hence $\phi^*+s\phi\in I^4(F)$. Hence ϕ and ϕ^* are half-neighbors. □

The following statement was pointed out by Laghribi ([**47**]) as an easy consequence of the index reduction formula [**55**].

LEMMA 7.5. *Let ψ be a quadratic form of dimension ≥ 5 and D be a division 3-quaternion algebra. Suppose that $D_{F(\psi)}$ is not division algebra. Then there exists an 8-dimensional quadratic form $\phi^*\in I^2(F)$ such that $\psi\subset\phi^*$ and $c(\phi^*)=[D]$.* □

PROOF OF THEOREM 7.1. Let D be 3-quaternion algebra such that $c(\phi)=[D]$. Since $\operatorname{ind}C(\phi)=8$, it follows that D is a division algebra. Since $\phi_{F(\psi)}$ is isotropic, $D_{F(\psi)}$ is not a division algebra. It follows from Lemma 7.5 that there exists an 8-dimensional quadratic form $\phi^*\in I^2(F)$ such that $\psi\subset\phi^*$ and $c(\phi^*)=[D]$. Obviously, all conditions of Lemma 7.4 hold. Hence ϕ and ϕ^* are half-neighbors. □

LEMMA 7.6. *Let D be a division 3-quaternion algebra over F. Then there exist a field extension E/F and an 8-dimensional quadratic form $\phi^*\in I^2(E)$ with the following properties:*

(i) *D_E is a division algebra,*
(ii) *$c(\phi^*)=[D_E]$,*
(iii) *$\phi^*_{E(D)}$ is anisotropic.*

PROOF. Let $\phi\in I^2(F)$ be an arbitrary F-form such that $c(\phi)=[D]$. Let $K=F(X,Y,Z)$ and $\gamma=\phi_K\perp\langle\langle X,Y,Z\rangle\rangle$ be a K-form. Let $K=K_0,K_1,\ldots,K_h;\gamma_0,\gamma_1,\ldots,\gamma_h$ be a generic splitting tower of γ.

Since $\gamma\equiv\phi_K \pmod{I^3(K)}$, we have $c(\gamma)=c(\phi_K)=[D_K]$. Since K/F is purely transcendental, $\operatorname{ind}D_K=\operatorname{ind}D=8$. Hence $\operatorname{ind}C(\gamma)=8$. It follows from Corollary 1.2.3 that there exists s such that $\dim\gamma_s=8$ and $\operatorname{ind}C(\gamma_s)=8$. We set $E=E_s$, $\phi^*=\gamma_s$.

We claim that the condition (i)–(iii) of the lemma hold. Since $c(\phi^*)=c(\gamma_E)=c(\phi_E)=[D_E]$, condition (ii) holds. Since $[D_E]=c(\phi^*)=c(\gamma_s)$, we have $\operatorname{ind}D_E=\operatorname{ind}C(\gamma_s)=8$ and thus condition (i) holds.

Now we only need to verify that (iii) holds. Let M_0/F be an arbitrary field extension such that ϕ_{M_0} is hyperbolic. Let $M = M_0(X, Y, Z)$. We have $\gamma_M = \phi_M \perp \langle\langle X, Y, Z\rangle\rangle_M$. Clearly $\langle\langle X, Y, Z\rangle\rangle$ is anisotropic over M. Since ϕ_M is hyperbolic, we have $(\gamma_M)_{an} = \langle\langle X, Y, Z\rangle\rangle_M$ and hence $\dim(\gamma_M)_{an} = 8$. Therefore $\dim(\gamma_M)_{an} = \dim \gamma_s$. By Lemma 1.2.1, we see that the field extension $ME/M = MK_s/M$ is purely transcendental. Hence $\dim(\gamma_{ME})_{an} = \dim(\gamma_M)_{an} = 8$. Since $(\phi^*_{ME})_{an} = (\gamma_{ME})_{an}$, we see that ϕ^*_{ME} is anisotropic. Since ϕ_M is hyperbolic, it follows that $[D_M] = c(\phi_M) = 0$. Hence $[D_{ME}] = 0$ and therefore the field extension $ME(D)/ME$ is purely transcendental. Hence $\phi^*_{ME(D)}$ is anisotropic. Therefore $\phi^*_{E(D)}$ is anisotropic. □

Lemma 7.7. *Let $\phi, \phi^* \in I^2(F)$ be 8-dimensional quadratic forms such that $c(\phi) = c(\phi^*) = [D]$, where D is a 3-quaternion division algebra. Suppose that $\phi^*_{F(D)}$ is anisotropic. Then $\phi_{F(D)}$ is anisotropic.*

Proof. Suppose at the moment that $\phi_{F(D)}$ is isotropic. Then letting $\psi \overset{\text{def}}{=} \phi^*$, we see that all conditions of Lemma 7.4 hold. Hence ϕ and ϕ^* are half-neighbors, i.e., there exists $s \in F^*$ such that $\phi^* + s\phi \in I^4(F)$. Therefore $\phi^*_{F(D)} + s\phi_{F(D)} \in I^4(F(D))$. Since $\phi_{F(D)}$ is hyperbolic, we see that $\phi^*_{F(D)} \in I^4(F(D))$. By the Arason-Pfister Hauptsatz, we see that $\phi^*_{F(D)}$ is hyperbolic. So we get a contradiction to the assumption of the lemma. □

Proposition 7.8. *Let $\phi \in I^2(F)$ be an 8-dimensional quadratic form such that* $\operatorname{ind} C(\phi) = 8$. *Let A be an algebra such that $c(\phi) = [A]$. Then $\phi_{F(A)}$ is anisotropic.*

Proof. Let D be a 3-quaternion algebra such that $c(\phi) = [D]$. Since $\operatorname{ind} C(\phi) = 8$, D is a division algebra. Let E/F and ϕ^* be such that in Lemma 7.6. All conditions of Lemma 7.7 hold for E, ϕ_E, ϕ^*, and D_E. Therefore $\phi_{E(D)}$ is anisotropic. Hence $\phi_{F(D)}$ is anisotropic. Since $[A] = c(\phi) = [D]$, the field extension $F(A)/F$ is stably isomorphic to $F(D)/F$ (Lemma 6.3). Therefore $\phi_{F(A)}$ is anisotropic. □

Proof of Theorem 7.2. Suppose at the moment that $\phi_{F(A)} \in I^4(F(A))$. Since $\operatorname{ind} C(\phi) \geq 8$, it follows that $\dim \phi \geq 8$. By Corollary 1.2.3 there exists a field extension E/F such that $\dim(\phi_E)_{an} = 8$, $\operatorname{ind} C(\phi_E) = 8$. Since $\dim(\phi_E)_{an} = 8$ and $\phi_{E(A)} \in I^4(E(A))$, the Arason-Pfister Hauptsatz shows that $((\phi_E)_{an})_{E(A)}$ is hyperbolic. We get a contradiction to Proposition 7.8. □

8. A generalization

In this section, we generalize Corollary 5.6 and Theorem 6.9 to the case of arbitrary $\operatorname{ind} D$.

Theorem 8.1. *Let D be a central simple F-algebra of exponent 2. Let ψ be a quadratic form of dimension ≥ 5. Suppose that* $\operatorname{ind} D_{F(\psi)} < \operatorname{ind} D$. *Then* $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D) = 0$ *and $H^3(F(\psi, D)/F) = [D] \cup H^1(F)$.*

Proof. By Proposition 6.7, there is a surjection

$$\frac{H^3(F(\psi, D)/F)}{[D] \cup H^1(F)} \twoheadrightarrow \operatorname{Tors} \mathrm{CH}^2(X_\psi \times X_D) .$$

Thus, it suffices to prove the second formula of Theorem.

Proving the second formula, we may assume that $\dim \psi = 5$ (compare to the proof of Theorem 6.9) and D is a division algebra (Corollary 6.4). Under these assumptions, we can write down D as the tensor product $C_0(\psi) \otimes_F B$ (using [**55**, Theorem 1]). In particular, we see that $C_0(\psi)$ is a division algebra, i.e. $\operatorname{ind} C_0(\psi) = \deg C_0(\psi) = 4$.

If $\deg D < 8$, then $D \simeq C_0(\psi)$. In this case, $\psi_{F(D)}$ is a 5-dimensional quadratic form with trivial Clifford algebra; therefore $\psi_{F(D)}$ is isotropic; by this reason, the field extension $F(\psi, D)/F(D)$ is pure transcendental and consequently $H^3(F(\psi, D)/F(D)) = 0$. It follows that

$$H^3(F(\psi, D)/F) = H^3(F(D)/F) = [D] \cup H^1(F) ,$$

where the last equality holds by Proposition 6.2.

If $\deg D > 8$, then $\operatorname{ind} B \geq 4$. Applying the index reduction formula [**78**, Theorem 1.3], we get

$$\operatorname{ind} C_0(\psi)_{F(D)} = \min\{\operatorname{ind} C_0(\psi),\ \operatorname{ind} B\} = 4 .$$

Therefore $\psi_{F(D)}$ is not a 3-Pfister neighbor and by Proposition 6.1 the group $H^3(F(\psi, D)/F(D))$ is trivial. Thus once again

$$H^3(F(\psi, D)/F) = H^3(F(D)/F) = [D] \cup H^1(F) .$$

Finally, if $\deg D = 8$, then we are done by Theorem 6.9 and Proposition 6.7. $\square$

Remark 8.2. A computation of the group $H^3(F(\psi, D)/F)$ in some other cases not covered here is given in Chapters 3 and 7.

CHAPTER 7

Isotropy of 8-dimensional quadratic forms over function fields of quadrics

Let F be a field of characteristic different from 2 and ϕ be an anisotropic 8-dimensional quadratic form over F with trivial determinant. We study the last open cases in the problem of describing the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$.

Results of this Chapter are obtained in joint work with Oleg Izhboldin.

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. More precisely, one studies the question whether the isotropy of ϕ over $F(\psi)$ is standard in a sense.

In this chapter we consider the case where ϕ is an 8-dimensional anisotropic quadratic form with trivial determinant. Necessity of certain sufficient conditions for isotropy of ϕ over $F(\psi)$ was studied by A. Laghribi; we call the isotropy, caused by one of these conditions, *L-standard*:

Definition. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. Besides we suppose that $\dim\phi = 8$ and $\det\phi = 1$. We say that the isotropy of $\phi_{F(\psi)}$ is *L-standard*, if at least one of the following conditions holds:

- there exists a half-neighbor ϕ^* of ϕ such that $\psi \subset \phi^*$;
- there exists a 5-dimensional subform $\phi_0 \subset \phi$ with the following two properties:
  - the form ϕ_0 is a Pfister neighbor,
  - the form $(\phi_0)_{F(\psi)}$ is isotropic.

Otherwise, we say that the isotropy is *non-L-standard*.

In the case when $\dim\psi \geq 5$, the isotropy of $\phi_{F(\psi)}$ is always L-standard ([**46**], [**47**], see also Chapter 6). The main result of this chapter is the following

Theorem. *Let ϕ be an anisotropic 8-dimensional quadratic form with* $\det\phi = 1$ *and ψ be a 4-dimensional quadratic form with* $\det\psi \neq 1$. *Suppose that $\phi_{F(\psi)}$ is isotropic. Then the isotropy of $\phi_{F(\psi)}$ is L-standard except the case* $\operatorname{ind} C(\phi) = \operatorname{ind}(C(\phi)\otimes_F C_0(\psi)) = 4$.

For the exceptional case of the theorem see Corollary 7.4.

For the case where $\dim\psi = 3$ or where $\dim\psi = 4$ and $\det\psi = 1$ see §8.

1. Terminology and notation

1.1. Quadratic forms. Mainly, we use notation of [**48**] and [**76**]. However there is certain slight difference: we denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$. We denote by $P_n(F)$ the set of all n-fold Pfister forms; $GP_n(F)$ is the set of the forms similar to a form from $P_n(F)$. For any field extension L/F, we denote by $P_n(L/F)$ the set of forms from $P_n(F)$ hyperbolic over L; $GP_n(L/F)$ is the set of the forms similar to a form from $P_n(L/F)$.

We recall that a quadratic form ψ is called a *(Pfister) neighbor* (of a Pfister form π), if it is similar to a subform in π and $\dim\phi > \frac{1}{2}\dim\pi$. Two quadratic forms ϕ and ϕ^* are *half-neighbors*, if $\dim\phi = \dim\phi^*$ and there exists $s \in F^*$ such that the sum $\phi \perp s\phi^*$ is similar to a Pfister form.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ if $\dim\phi \geq 3$; $F(\phi) = F(\sqrt{d})$ if $\dim\phi = 2$ and $d = d_\pm\phi \neq 1$; and $F(\phi) = F$ otherwise.

1.2. Algebras. We consider only finite-dimensional F-algebras.

For a central simple F-algebra D, we denote by $\deg(D)$, $[D]$, and $\exp(D)$ respectively the degree of D, the class of D in the Brauer group $\mathrm{Br}(F)$, and the exponent of D, i.e., the order of $[D]$ in the Brauer group.

For a simple F-algebra A, we denote by $\mathrm{ind}(A)$ the Schur index of A. For an algebra B of the form $B = A \times \cdots \times A$ with simple A, we set $\mathrm{ind}\, B = \mathrm{ind}\, A$.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ. By $C_0(\phi)$ we denote the even part of $C(\phi)$. Note that for any quadratic F-form ψ and any central simple F-algebra D, the index of $C_0(\psi) \otimes_F D$ is well-defined.

If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Its class $[C(\phi)]$ in the Brauer group $\mathrm{Br}_2(F)$ is denoted by $c(\phi)$.

Let D be a central simple algebra. We denote by X_D the Severi-Brauer variety of D and by $F(D)$ the function field $F(X_D)$. For another central simple F-algebra D' and for a quadratic F-form ψ of dimension ≥ 3, we set $F(D', D) \overset{\text{def}}{=} F(X_{D'} \times X_D)$ and $F(\psi, D) \overset{\text{def}}{=} F(X_\psi \times X_D)$.

1.3. Cohomology groups. By $H^*(F)$ we denote the graded ring of Galois cohomology $H^*(F, \mathbb{Z}/2\mathbb{Z}) \overset{\text{def}}{=} H^*(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2\mathbb{Z})$.

For $n = 0, 1, 2, 3$, there is a homomorphism $e^n : I^n(F) \to H^n(F)$ which is uniquely determined by the condition $e^n(\langle\langle a_1, \dots, a_n\rangle\rangle) = (a_1, \dots, a_n)$ (see [**5**]). The homomorphism e^n is surjective and $\ker e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [**53**], [**62**], and [**71**]).

We also work with the cohomology groups $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$, $(i = 0, 1, 2)$, defined by B. Kahn (see [**29**]). For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with $\mathrm{Tors}_2\, H^n(F, \mathbb{Q}/\mathbb{Z}(n-1))$.

1.4. K-theory and Chow groups. For a smooth algebraic F-variety X, its Grothendieck ring is denoted by $K(X)$. This ring is supplied with the filtration by codimension of support (also called the topological filtration).

We fix an algebraic closure $\bar{F}$ of the base field F and denote by $\bar{X}$ the $\bar{F}$-variety $X_{\bar{F}}$. If the variety X is projective homogeneous, we identify $K(X)$ with a subring of $K(\bar{X})$ via the restriction homomorphism which is injective by [**65**].

For a ring (or a group) with filtration A, we denote by G^*A the adjoint graded ring (resp., the adjoint graded group).

There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $\mathrm{G}^*K(X)$, its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([**81**, §9]).

Let X_1 and X_2 be two smooth F-varieties. For any $x_1 \in K(X_1)$ and $x_2 \in K(X_2)$, we denote by $x_1 \boxtimes x_2$ the product $pr_1^*(x_1) \cdot pr_2^*(x_2) \in K(X_1 \times X_2)$ where pr_1^* and pr_2^* are the pull-backs with respect to the projections pr_1 and pr_2 of $X_1 \times X_2$ on X_1 and X_2 respectively. For an $\mathcal{O}_{X_1}$-module $\mathcal{F}_1$ and an $\mathcal{O}_{X_2}$-module $\mathcal{F}_2$, we denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the tensor product $pr_1^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} pr_2^*(\mathcal{F}_2)$.

1.5. Relative groups. Let Φ be an arbitrary functor on the category of fields (of characteristic $\neq 2$) with values in the category of abelian groups. For a field extension L/F we use the notation $\Phi(L/F)$ for $\ker(\Phi(F) \to \Phi(L))$. Here is a list of examples that we need in this chapter: $W(L/F)$, $I^n(L/F)$, $H^n(L/F)$, and $H^n(L/F, \mathbb{Q}/\mathbb{Z}(i))$.

2. The groups $H^3(F(\psi, D)/F)$ and $I^3(F(\psi, D)/F)$

Proposition 2.1. *Let D be a central simple F-algebra of exponent 2 and ψ be a quadratic form of dimension ≥ 3. Then there exists a natural isomorphism*

$$\frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)} \simeq \frac{\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D)}{pr_\psi^* \operatorname{Tors}\mathrm{CH}^2(X_\psi) + pr_D^* \operatorname{Tors}\mathrm{CH}^2(X_D)}$$

where pr_ψ^* and pr_D^* are the pull-backs with respect to the projection pr_ψ and pr_D of $X_\psi \times X_D$ to X_ψ and X_D.

Proof. By Proposition 2.2 of Chapter 4, the factor group

$$\frac{H^3(F(\psi, D)/F, \mathbb{Q}/\mathbb{Z}(2))}{H^3(F(\psi)/F, \mathbb{Q}/\mathbb{Z}(2)) + H^3(F(D)/F), \mathbb{Q}/\mathbb{Z}(2))}$$

is isomorphic to

$$\frac{\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D)}{pr_\psi^* \operatorname{Tors}\mathrm{CH}^2(X_\psi) + pr_D^* \operatorname{Tors}\mathrm{CH}^2(X_D)}.$$

Now it is sufficient to apply isomorphisms

$$H^3(F(\psi)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi)/F),$$
$$H^3(F(D)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(D)/F),$$
$$H^3(F(\psi, D)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\psi, D)/F).$$

(see e.g. Lemma 2.8 of Chapter 4, [**24**, Lemma A.8], Corollary 6.6 of Chapter 6). □

Corollary 2.2. *Let D be a biquaternion algebra and ψ be a 4-dimensional quadratic form. Then there exists a natural isomorphism*

$$\frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)} \simeq \operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D).$$

In particular, $2 \cdot \operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) = 0$.

Proof. Since $\dim\psi = 4$, we have $\operatorname{Tors}\operatorname{CH}^2(X_\psi) = 0$ (see e.g. [**32**, Theorem 6.1] or Lemma 2.4 of Chapter 4). By Proposition 5.3 of Chapter 1, we have $\operatorname{Tors}\operatorname{CH}^2(X_D) = 0$. To complete the proof it is sufficient to apply Proposition 2.1. □

Lemma 2.3. *Let ψ be a quadratic form of dimension ≥ 3. Then*
1. *the map $e^3 : P_3(F(\psi)/F) \to H^3(F(\psi)/F)$ is surjective;*
2. *$I^3(F(\psi)/F) + I^4(F) = P_3(F(\psi)/F) + I^4(F)$.*

Proof. 1. Really, we have to verify that the set $H^3(F(\psi)/F)$ consists of symbols. This fact is proved in [**5**, Satz 5.6].
2. Follows from Item 1 and from injectivity of $e^3 : I^3(F)/I^4(F) \to H^3(F)$. □

Lemma 2.4. *Let D be a biquaternion algebra and q be an Albert form of D. Then*
1. *$H^3(F(D)/F) = [D] \cup H^1(F)$;*
2. *$I^3(F(D)/F) + I^4(F) = \{q\,\langle\langle s\rangle\rangle \mid s \in F^*\} + I^4(F)$;*
3. *the map $e^3 : I^3(F(D)/F) \to H^3(F(D)/F)$ is surjective.*

Proof. 1. See [**66**, Corollary 4.5].
2. Obviously $e^3(q\,\langle\langle s\rangle\rangle) = [D] \cup (s)$. Now, it is sufficient to apply Item 1 and injectivity of $e^3 : I^3(F)/I^4(F) \to H^3(F)$.
3. Follows from Items 1 and 2. □

Lemma 2.5. *Let D be a biquaternion algebra and ψ be a quadratic form of dimension ≥ 3. Then the natural homomorphism*

$$\frac{I^3(F(\psi, D)/F) + I^4(F)}{I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)} \to \frac{H^3(F(\psi, D)/F)}{H^3(F(\psi)/F) + H^3(F(D)/F)}$$

is injective.

In particular, the condition $H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$ implies that $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$.

Proof. Obvious consequence of the following facts:
a) $I^3(F)/I^4(F) \to H^3(F)$ is injective;
b) $I^3(F(\psi)/F)) \to H^3(F(\psi)/F)$ is surjective (Lemma 2.3);
c) $I^3(F(D)/F)) \to H^3(F(D)/F)$ is surjective (Lemma 2.4). $\square$

Corollary 2.6. *Let D be a biquaternion algebra and ψ be a quadratic form of dimension ≥ 3 such that* $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) = 0$. *Then*

$$I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)\ .$$

Proof. Follows from Lemma 2.5 and Proposition 2.1. $\square$

3. The group $K(X_\psi \times X_D)$

In this section, ψ is a quadratic F-form of dimension 4, D is a biquaternion F-algebra.

Lemma 3.1. *Consider the tensor product $K(X_\psi) \otimes_{\mathbb{Z}} K(X_D)$ together with the filtration induced by the topological filtrations on $K(X_\psi)$ and $K(X_D)$. The adjoint graded group $G^*(K(X_\psi) \otimes K(X_D))$ is torsion-free.*

Proof. The adjoint graded group $G^*K(X_\psi)$ is torsion-free (see e.g [**32**]). The adjoint graded group $G^*K(X_D)$ is torsion-free as well (see [**34**, Example]). We have a surjection

$$G^*K(X_\psi) \otimes G^*K(X_D) \twoheadrightarrow G^*(K(X_\psi) \otimes K(X_D))\ .$$

The left-hand side term is a finitely generated torsion-free abelian group, i.e. a free abelian group of finite rank. This rank coincides with the rank of the right-hand side term. Therefore, the map is an isomorphism. $\square$

We consider the subgroup $K(X_\psi) \boxtimes K(X_D)$ of $K(X_\psi \times X_D)$ together with the filtration induced by the topological filtration on $K(X_\psi \times X_D)$.

Lemma 3.2. *The homomorphism $K(X_\psi) \otimes K(X_D) \to K(X_\psi) \boxtimes K(X_D)$ is an isomorphism of groups with filtrations.*

Proof. It is a homomorphism of groups respecting the filtrations. First of all let us check that it is an isomorphism of groups, regardless the filtrations. It is evidently an epimorphism. So, we only have to check the injectivity. Since for any extension of the base field F, the restriction homomorphism on the product $K(X_\psi) \otimes K(X_D)$ is injective, it suffices to check the injectivity in a split situation. However, if D splits, then X_D is isomorphic to a projective space; therefore the map $K(X_\psi) \otimes K(X_D) \to K(X_\psi) \boxtimes K(X_D)$ is an isomorphism.

To finish the proof, it is suffices to show that the homomorphism of the adjoint graded groups is injective. Consider the commutative diagram

$$\begin{array}{ccc}
G^*(K(\bar{X}_\psi) \otimes K(\bar{X}_D)) & \longrightarrow & G^*(K(\bar{X}_\psi) \boxtimes K(\bar{X}_D)) \\
\uparrow & & \uparrow \\
G^*(K(X_\psi) \otimes K(X_D)) & \longrightarrow & G^*(K(X_\psi) \boxtimes K(X_D))
\end{array}$$

The left-hand side arrow is injective since the group $G^*(K(X_\psi) \otimes K(X_D))$ is torsion-free (Lemma 3.1) The upper arrow is injective because $\bar{X}_D$ is isomorphic to a projective space. Therefore the bottom arrow is injective too. $\square$

COROLLARY 3.3. *The group $G^*(K(X_\psi) \boxtimes K(X_D))$ is torsion-free.* $\square$

We denote by C the even Clifford algebra $C_0(\psi)$. Let $\mathcal{U}(2)$ be Swan's sheaf $\mathcal{U}$ on the quadric X_ψ ([**82**, §6]), twisted twice. It has a structure of C-module. Let $\mathcal{J}$ be the canonical sheaf on the Severi-Brauer variety X_D ([**69**, §8.4]). It has a structure of D-module.

Put $\mathcal{F} \overset{\text{def}}{=} \mathcal{U}(2) \boxtimes \mathcal{J}$. It is a sheaf on $X_\psi \times X_D$ with a structure of $C \otimes_F D$-module. Denote by f the homomorphism $K(C \otimes D) \to K(X_\psi \times X_D)$ given by the functor of tensor multiplication by $\mathcal{F}$ over $C \otimes D$.

Put $\mathcal{G} \overset{\text{def}}{=} \mathcal{U}(2) \boxtimes \mathcal{J}^{\otimes 3}$. It is a sheaf on $X_\psi \times X_D$ with a structure of $C \otimes_F D^{\otimes 3}$-module. Consider the homomorphism $K(C \otimes D^{\otimes 3}) \to K(X_\psi \times X_D)$ given by the functor of tensor multiplication by $\mathcal{G}$ over $C \otimes D^{\otimes 3}$. Since the algebra $D^{\otimes 2}$ is split, the group $K(C \otimes D^{\otimes 3})$ is canonically isomorphic (via Morita equivalence) to $K(C \otimes D)$. Denote by g the the composition $K(C \otimes D) \xrightarrow{\sim} K(C \otimes D^{\otimes 3}) \to K(X_\psi \times X_D)$.

LEMMA 3.4. 1. *The homomorphism*

$$K(C \otimes D)^{\oplus 2} \to K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)) ,$$

induced by the homomorphisms $f + g$, is surjective.

2. *If $C \otimes D$ is a skewfield, then $K(X_\psi \times X_D) = K(X_\psi) \boxtimes K(X_D)$.*

PROOF. 1. Using Swan's computation of the K-theory for quadrics [**82**, Theorem 9.1] (with $\mathcal{U}(2)$ instead of $\mathcal{U}$) and a generalized Peyre's version [**66**, Proposition 3.1] of Quillen's computation of K-theory for Severi-Brauer schemes [**69**, Theorem 4.1 of §8], we get an isomorphism

$$K(F)^{\oplus 4} \oplus K(C)^{\oplus 2} \oplus K(D)^{\oplus 4} \oplus K(C \otimes D)^{\oplus 2} \simeq K(X_\psi \times X_D)$$

such that the image of $K(F)^{\oplus 4} \oplus K(C)^{\oplus 2} \oplus K(D)^{\oplus 4}$ is contained in $K(X_\psi) \boxtimes K(X_D)$ and the summand $K(C \otimes D)^{\oplus 2}$ is mapped into $K(X_\psi \times X_D)$ via $f + g$. Therefore, $K(C \otimes D)^{\oplus 2} \to K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ is an epimorphism. 2. If the algebra $C \otimes D$ is a skewfield, then its class generates the group $K(C \otimes D)$. The images of this class under f, g are $\mathcal{F}, \mathcal{G} \in K(X_\psi) \boxtimes K(X_D)$. $\square$

COROLLARY 3.5. *If $C \otimes D$ is a skewfield, then the group $G^* K(X_\psi \times X_D)$ is torsion-free.*

PROOF. By Lemma 3.4, $K(X_\psi \times X_D) = K(X_\psi) \boxtimes K(X_D)$. By Corollary 3.3, the group $G^*(K(X_\psi) \boxtimes K(X_D))$ is torsion-free. $\square$

4. The group $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_D)$

THEOREM 4.1. *Let D be a biquaternion algebra and ψ be an anisotropic 4-dimensional quadratic form with $\det \psi \neq 1$. Then the group $\operatorname{CH}^2(X_\psi \times X_D)$ is torsion-free except (possibly) the following two cases:*

(1) $\operatorname{ind} C_0(\psi) \otimes D = \operatorname{ind} D = 4$,
(2) $\operatorname{ind} C_0(\psi) \otimes D = 2$.

PROOF. Set $C \stackrel{\text{def}}{=} C_0(\psi)$ and $s \stackrel{\text{def}}{=} \operatorname{ind}(C \otimes D)$. The possible values of s are 1, 2, 4, 8.

Assume that $s = 8$. Since $\det \psi \neq 1$, it follows that $C \otimes D$ is a skewfield. Therefore, the group $\mathrm{CH}^2(X_\psi \times X_D) \simeq G^2K(X_\psi \times X_D)$ is torsion-free by Corollary 3.5.

Now assume $s = 1$. Then $\operatorname{ind}(C_0(\psi_{F(D)})) = 1$. Therefore the quadratic form $\psi_{F(D)}$ is isotropic. Hence the extension $F(\psi, D)/F(D)$ is purely transcendental and $H^3(F(\psi, D)/F) = H^3(F(D)/F)$. Now, Corollary 2.2 shows that $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) = 0$.

Finally, assume $s = 4$ and $\operatorname{ind} D \neq 4$. Then the biquaternion algebra D is Brauer equivalent to a quaternion F-algebra D'. Since $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D)$ depends only on the Brauer class of D (see e.g. Chapter 6), we may replace D by D'. Let ψ' be a 3-dimensional quadratic F-form such that $C_0(\psi') \simeq D'$. The Severi-Brauer variety $X_{D'}$ is isomorphic to the conic $X_{\psi'}$. Since the tensor product $C_0(\psi) \otimes C_0(\psi')$ has index 4, it is a division algebra. Therefore, by Corollary 4.4 of Chapter 4, the group $\mathrm{CH}^2(X_\psi \times X_{\psi'})$ is torsion-free. □

REMARK 4.2. The assumption ψ is anisotropic is not essential: if ψ is isotropic, then
$$H^3(F(\psi, D)/F) = H^3(F(D)/F)$$
and therefore the group $\mathrm{CH}^2(X_\psi \times X_D)$ is torsion-free as well (by Corollary 2.2).

PROPOSITION 4.3. *Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form with* $\det \psi \neq 1$. *Then the group* $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D)$ *is equal to zero or isomorphic to* $\mathbb{Z}/2$.

PROOF. Since we already know that the torsion in the group $\mathrm{CH}^2(X_\psi \times X_D)$ is annihilated by 2 (Corollary 2.2), it suffices to show that the torsion is cyclic.

Once again we set $C \stackrel{\text{def}}{=} C_0(\psi)$. According to Theorem 4.1, it suffices to consider the case where $\operatorname{ind}(C \otimes D)$ equals 2 or 4.

Consider the quotient $K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ with the filtration induced by the topological filtration on $K(X_\psi \times X_D)$. Since in the exact sequence of the adjoint graded groups
$$0 \to G^*(K(X_\psi) \boxtimes K(X_D)) \to G^*K(X_\psi \times X_D) \to$$
$$\to G^*(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))) \to 0$$
the left-hand side term is torsion-free (Corollary 3.3), we have an injection
$$\operatorname{Tors} G^*K(X_\psi \times X_D) \hookrightarrow G^*(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))) .$$
Since $\mathrm{CH}^2(X_\psi \times X_D) \simeq G^2K(X_\psi \times X_D)$, it suffices to show that the group $G^2(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))$ is cyclic.

Denote by $h \in K(X_\psi)$ the class of a general hyperplane section of X_ψ. Let $\xi \in K(\bar{X}_D)$ be the class of the tautological linear bundle on the projective space $\bar{X}_D$. Note that for any $i \geq 0$, the multiple $(\operatorname{ind} D^{\otimes i}) \cdot \xi^i$ of ξ^i belongs to $K(X_D)$. Thus $\xi^i \in K(X_D)$ for i even and $4\xi^i \in K(X_D)$ for i odd.

The algebra $C \otimes D$ is simple. Therefore, its Grothendieck group is cyclic. By Lemma 3.4, it follows that the quotient $K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D))$ is generated by two elements, namely by $(s/2)x$ and $(s/2)y$, where $x \overset{\text{def}}{=} (4 + 2h + h^2) \boxtimes \xi$, $y \overset{\text{def}}{=} (4 + 2h + h^2) \boxtimes \xi^3$, and $s \overset{\text{def}}{=} \operatorname{ind} C \otimes D$ (we use here the equality $[\mathcal{U}(2)] = 4 + 2h + h^2 \in K(X_\psi)$, [**32**, Lemma 3.6]).

We have a congruence $x \equiv h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2 \pmod{K(X_\psi) \boxtimes K(X_D)}$. The right-hand side element belongs to $K(X_\psi \times X_D)^{(2)}$, because it is in $K(\bar{X}_\psi \times \bar{X}_D)^{(2)}$ and $K(X_\psi \times X_D)^{(2)} = K(\bar{X}_\psi \times \bar{X}_D)^{(2)} \cap K(X_\psi \times X_D)$ (see [**74**, Lemme 6.3, (i)]). Therefore, $(s/2)x \in (K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))^{(2)}$ (take into account that the coefficient $s/2$ is an integer).

We also have another congruence modulo $K(X_\psi) \boxtimes K(X_D)$:

$$y - x \equiv \left(h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2\right) \cdot \left(1 \boxtimes (\xi^2 - 1)\right) .$$

The right-hand side element is in $K(X_\psi \times X_D)^3$ as a product of an element in $K(X_\psi \times X_D)^{(2)}$ and the element $1 \boxtimes (\xi^2 - 1) \in K(X_\psi \times X_D)^{(1)}$. Therefore, $(s/2)(y - x) \in (K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))^{(3)}$ and it follows that the group $G^2(K(X_\psi \times X_D)/(K(X_\psi) \boxtimes K(X_D)))$ is generated by $(s/2)x$. So, in particular, this group is cyclic. □

Remark 4.4. The assumption $\det \psi \neq 1$ is not essential: if $\det \psi = 1$, then $H^3(F(\psi, D)/F) = H^3(F(\psi', D)/F)$ and $H^3(F(\psi)/F) = H^3(F(\psi')/F)$, where ψ' is an arbitrary 3-dimensional subform of ψ (Lemma 5.2 of Chapter 4). Therefore,

$$\mathrm{CH}^2(X_\psi \times X_D) \simeq \mathrm{CH}^2(X_{\psi'} \times X_D) \simeq \mathrm{CH}^2(X_{D'} \times X_D)$$

where D' is the even Clifford algebra of ψ'. The group $\operatorname{Tors} \mathrm{CH}^2(X_{D'} \times X_D)$ is zero or $\mathbb{Z}/2$ according to Theorem 6.1 of Chapter 2.

Theorem 4.5. *Suppose that* $\psi = \langle -x, -y, xy, d\rangle$ *(with* $d \notin F^{*2}$*) and* $D = (x, y) \otimes (u, v)$ *where* $x, y, d, u, v \in F^*$. *Then* $\operatorname{Tors} \mathrm{CH}^2(X_\psi \times X_D) = 0$.

Proof. The even Clifford algebra of the quadratic form ψ is isomorphic to the quaternion algebra $(x, y)_{F(\sqrt{d})}$. Therefore, the tensor product $C_0(\psi) \otimes D$ is Brauer equivalent to the quaternion algebra $(u, v)_{F(\sqrt{d})}$ and in particular has index 2 or 1. In the case, where the index is 1, we are done by Theorem 4.1. Let us assume the index equals to 2. It suffices to show that the element $x \overset{\text{def}}{=} h^2 \boxtimes (\xi - 1) - h \boxtimes (\xi - 1)^2$ is in $K(X_\psi \times X_D)^{(3)}$ (see the proof of Proposition 4.3).

By definition, the element $h \in K(X_\psi)$ is the class of a hyperplane section of the quadric X_ψ. This hyperplane section is the quadric $X_{\psi'}$ determined by a 3-dimensional subform ψ' of ψ. Clearly, x is equal to the image of $x' \overset{\text{def}}{=}$

$h' \boxtimes (\xi - 1) - 1 \boxtimes (\xi - 1)^2 \in K(X_{\psi'} \times X_D)$ under the push-forward homomorphism $K(X_{\psi'} \times X_D) \to K(X_\psi \times X_D)$. Moreover, $x' \in K(X_{\psi'} \times X_D)^{(2)}$, because $x' \in K(\bar{X}_{\psi'} \times \bar{X}_D)^{(2)}$ and $K(X_{\psi'} \times K(X_D))^{(2)} = K(\bar{X}_{\psi'} \times \bar{X}_D)^{(2)} \cap K(X_{\psi'} \times K(X_D))$. Since the codimension of $X_{\psi'} \times X_D$ in $X_\psi \times X_D$ equals to 1, it follows that $x \in K(X_\psi \times X_D)^{(3)}$. □

Remark 4.6. The assumption $d \notin F^{*2}$ is not essential: if $d \in F^{*2}$, then $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) \simeq \operatorname{Tors}\mathrm{CH}^2(X_{C_0(\psi')} \times X_D)$, where $\psi' \stackrel{\text{def}}{=} \langle -x, -y, xy\rangle$; since $C_0(\psi') \simeq (x, y)$, we have $\operatorname{ind} C_0(\psi') \otimes D \le 2$; therefore, by Theorem 6.1 of Chapter 2, the latter group is zero.

5. Standard isotropy in the case $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) = 0$

Definition 5.1. We say that (ϕ, D, q) is a special triple if the following conditions hold:

1) ϕ is an 8-dimensional anisotropic form with $\det \phi = 1$,
2) D is a biquaternion algebra,
3) q is an Albert form,
4) $[D] = c(\phi) = c(q) \in \mathrm{Br}_2(F)$.

In this section we need the following

Theorem 5.2. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let D be an algebra such that $c(\phi) = [D]$. Then $\phi_{F(D)}$ is anisotropic.*

Proof. See [**46**, Théorème 4] and [**10**, Corollary 9.3], see also Theorem 7.2 of Chapter 6. □

Our study of isotropy of 8-dimensional forms over function field of quadrics is based on the following assertion.

Proposition 5.3. *Let (ϕ, D, q) be a special triple and ψ be a quadratic form. Then*

1. *The following two conditions are equivalent:*
   (i) *$\phi + q \in I^3(F(\psi, D)/F)$;*
   (ii) *$\phi_{F(\psi)}$ is isotropic.*
2. *The following two conditions are equivalent:*
   (i) *$\phi + q \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;*
   (ii) *there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. (1i)⇒(1ii). Condition (1i) implies that the quadratic form $(\phi \perp q)_{F(\psi,D)}$ is hyperbolic. Since $q_{F(D)}$ is hyperbolic, it follows that $\phi_{F(\psi,D)}$ is hyperbolic. Let $E \stackrel{\text{def}}{=} F(\psi)$. We see that $\phi_{E(D)}$ is hyperbolic. Theorem 5.2 implies that ϕ_E is isotropic, i.e., condition (1ii) holds.
(1ii)⇒(1i). Suppose that $\phi_{F(\psi)}$ is isotropic. Since $c(\phi) = c(q)$, it follows that $\phi + q \in I^3(F)$. Hence it is sufficient to prove that $\phi_{F(\psi,D)}$ and $q_{F(\psi,D)}$ are hyperbolic. The form $q_{F(\psi,D)}$ is hyperbolic because $q_{F(D)}$ is. Since $c(\phi) = [D]$, we

have $\phi_{F(\psi,D)} \in I^3(F(\psi,D))$. Since $\phi_{F(\psi)}$ is isotropic, we have $\dim(\phi_{F(\psi,D)})_{an} < 8$. The Arason-Pfister Hauptsatz shows that $\phi_{F(\psi,D)}$ is hyperbolic.

(2i)$\Rightarrow$(2ii). By Lemmas 2.3 and 2.4, there exist $\pi \in P_3(F(\psi)/F)$ and $s \in F^*$ such that

$$\phi + q \equiv \pi + q\,\langle\langle s\rangle\rangle \pmod{I^4(F)}.$$

We have $\phi + sq \equiv \pi \pmod{I^4(F)}$. Since $\pi \in P_3(F)$, $\pi_{F(\pi)}$ is hyperbolic. Hence $(\phi + sq)_{F(\pi)} \equiv \pi_{F(\pi)} \equiv 0 \pmod{I^4(F)}$. Since $\dim(\phi + sq) \leq 8 + 4 < 16$, the Arason-Pfister Hauptsatz shows that $(\phi + sq)_{F(\pi)}$ is hyperbolic. Hence there exists a form γ such that $(\phi \perp sq)_{an} = \pi\gamma$. Clearly $0 < 8 - 6 \leq \dim(\phi \perp sq)_{an} \leq 8 + 6 < 16$. This implies that $\dim\gamma = 1$, i.e., there exists $k \in F^*$ such that $\gamma = \langle k\rangle$. Thus $\phi + sq = k\pi$. Therefore $(\phi \perp -k\pi) = -sq$. Hence ϕ and $k\pi$ contain a common subform of dimension

$$\frac{\dim\phi + \dim\pi - \dim q}{2} = \frac{8 + 8 - 6}{2} = 5.$$

Let us denote such a form by ϕ_0. Since $\dim\phi_0 = 5$ and $\phi_0 \subset k\pi$, it follows that ϕ_0 is a Pfister neighbor of π. Since $\pi \in P_3(F(\psi)/F)$, it follows that $(\phi_0)_{F(\psi)}$ is isotropic.

(2ii)$\Rightarrow$(2i). Let ϕ_0 be a 5-dimensional quadratic form such as in (2ii). By the assumption, there exists $\pi \in GP_3(F)$ such that $\phi_0 \subset \pi$. Since $(\phi_0)_{F(\psi)}$ is isotropic, it follows that $\pi \in GP_3(F(\psi)/F)$.

Since $\phi_0 \subset \phi$ and $\phi_0 \subset \pi$, there exist 3-dimensional quadratic forms ρ', ρ'' such that $\phi = \phi_0 \perp \rho'$ and $\pi = \phi_0 \perp \rho''$. We set $\rho = \rho'' \perp -\rho'$. Clearly $\dim\rho = 6$. In the Witt ring $W(F)$ we have $\rho = \rho'' - \rho' = \pi - \phi$. In particular, $\rho \in I^2(F)$. Hence ρ is an Albert form.

We have $c(\rho) = c(\pi) + c(\phi) = 0 + c(\phi) = c(q)$. Hence ρ is similar to q ([**28**, Theorem 3.12]). Let $s \in F^*$ be such that $\rho = sq$. We have $\pi - \phi = \rho = sq$. Hence $\phi = \pi - sq$. Therefore

$$\begin{aligned}\phi + q = \pi + q\,\langle\langle s\rangle\rangle \in GP_3(F(\psi)/F) + [q]\cdot I(F) \subset \\ \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F).\end{aligned}$$

□

COROLLARY 5.4. *Let (ϕ, D, q) be a special triple and ψ be a quadratic form of dimension ≥ 3. Suppose that* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) = 0$. *Then the following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic,*
(2) *there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

PROOF. Obvious in view of Proposition 5.3 and Corollary 2.6. □

6. The group $H^3(F(\psi,D)/F)$ in the case $\operatorname{ind}(C_0(\psi)\otimes_F D) = 2$

In this section we study the group $H^3(F(\psi,D)/F)$ in the case where ψ is a 4-dimensional quadratic form with a non-trivial discriminant, D is a biquaternion division F-algebra, and $\operatorname{ind}(C_0(\psi)\otimes_F D) = 2$.

Let $d \overset{\text{def}}{=} \det\psi$. By our assumption, $d \notin F^{*2}$. Replacing ψ by a similar form, we can suppose $\psi = \langle -a,-b,ab,d\rangle$ with $a,b\in F^*$. Let $L \overset{\text{def}}{=} F(\sqrt{d})$. By our assumption, we have $\operatorname{ind}((a,b)\otimes_F D)_L = 2$. Hence there exists a quaternion F-algebra Q such that $(a,b)_L + [D_L] = [Q_L]$ in $\operatorname{Br}_2(L)$ (see [**42**, Proposition 16.2]). Let us write Q in the form $Q=(r,s)$ with $r,s\in F^*$. We set $\psi' = \langle -r,-s,rs,d\rangle$.

Let q be an Albert form corresponding to D.

Lemma 6.1. *There exist $k,k'\in F^*$ such that $k\psi + k'\psi' + q \in I^3(F)$.*

Proof. Since $(a,b)_L + [D_L] = [Q_L] = (r,s)_L$, it follows that $(a,b)+(r,s)+[D] \in \operatorname{Br}_2(L/F)$. Hence there exist $k\in F^*$ such that $(a,b)+(r,s)+[D] = (d,k)$. Let $k' \overset{\text{def}}{=} -1$ and $\phi \overset{\text{def}}{=} k\psi + k'\psi' + q$. We claim that $\phi\in I^3(F)$. To prove this, it is sufficient to verify that $\phi \in I^2(F)$ and $c(\phi)=0$. We have

$$\begin{aligned}\phi = k\psi + k'\psi' + q = k\langle -a,-b,ab,d\rangle - \langle -r,-s,rs,d\rangle + q =\\ = k(\langle\langle a,b\rangle\rangle - \langle\langle d\rangle\rangle) - (\langle\langle r,s\rangle\rangle - \langle\langle d\rangle\rangle) + q =\\ = k\langle\langle a,b\rangle\rangle + \langle\langle d,k\rangle\rangle - \langle\langle r,s\rangle\rangle + q.\end{aligned}$$

Hence $\phi\in I^2(F)$ and $c(\phi) = (a,b)+(d,k)+(r,s)+c(q) = (a,b)+(d,k)+(r,s)+[D]=0$. $\square$

Definition 6.2. Let D be a biquaternion algebra and ψ be a 4-dimensional quadratic form such that $\det\psi\neq 1$ and $\operatorname{ind}(C_0(\psi)\otimes_F D)=2$. We denote by $\Gamma(\psi,D)$ the set defined as follows

$$\{\gamma\in I^3(F) \mid \text{there exist } k,k',l\in F^* \text{ such that } \gamma = k\psi + k'\psi' + lq\},$$

where q is an Albert form corresponding to D and ψ' is a 4-dimensional quadratic form satisfying the following two properties: $\det\psi' = \det\psi$ and $C_0(\psi')$ is Brauer-equivalent to $C_0(\psi)\otimes_F D$.

Remark 6.3. 1. The set $\Gamma(\psi,D)$ does not depend on the choice of q and ψ': indeed, the condition on q and ψ' determines them uniquely up to similarity.
2. Lemma 6.1 shows the set $\Gamma(\psi,D)$ is not empty.

Lemma 6.4. *$\Gamma(\psi,D)\subset I^3(F(\psi,D)/F)$.*

Proof. Let $\gamma = k\psi + k'\psi' + lq \in \Gamma(\psi,D)$. By the definition of $\Gamma(\psi,D)$, we have $\gamma\in I^3(F)$. Thus it is sufficient to prove that $\gamma_{F(\psi,D)}$ is hyperbolic. We have $\dim(\psi_{F(\psi)})_{an}\le 2$ and $\dim(q_{F(D)})_{an}=0$. Therefore $\dim(\gamma_{F(\psi,D)})_{an} = \dim((k\psi \perp k'\psi' \perp lq)_{F(\psi,D)})_{an} \le 2+4+0 = 6 < 8$. Since $\gamma\in I^3(F)$, the Arason-Pfister Hauptsatz shows that $\gamma_{F(\psi,D)}$ is hyperbolic. $\square$

Lemma 6.5. *Let $\gamma\in\Gamma(\psi,D)$, $\pi\in P_3(F(\psi)/F)$, and $s\in F^*$. Then there exists $\gamma'\in\Gamma(\psi,D)$ such that $\gamma'\equiv \gamma+\pi+q\langle\langle s\rangle\rangle \pmod{I^4(F)}$.*

Proof. Let us write γ in the form $\gamma = k\psi + k'\psi' + lq$. Since $\pi \in P_3(F(\psi)/F)$, there exists $r \in F^*$ such that $\pi = \psi\,\langle\langle r\rangle\rangle$.

We have $k\psi + \pi \equiv k\psi - k\pi = k\psi - k\psi\,\langle\langle r\rangle\rangle = rk\psi \pmod{I^4(F)}$ and $lq + q\,\langle\langle s\rangle\rangle \equiv lq - lq\,\langle\langle s\rangle\rangle = lsq \pmod{I^4(F)}$. Therefore

$$\gamma + \pi + q\,\langle\langle s\rangle\rangle = k\psi + k'\psi' + lq + \pi + q\,\langle\langle s\rangle\rangle \equiv rk\psi + k'\psi' + lsq \pmod{I^4(F)}.$$

Now it is sufficient to set $\gamma' = rk\psi + k'\psi' + lsq$. □

Corollary 6.6.

$$\Gamma(\psi, D) + I^4(F) = \Gamma(\psi, D) + I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)\;.$$

Proof. Obvious in view of Lemmas 6.5, 2.3, and 2.4. □

Lemma 6.7. *The following conditions are equivalent:*

(1) $I^3(F(\psi, D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
(2) $\Gamma(\psi, D) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
(3) *there exists* $\gamma \in \Gamma(\psi, D)$ *such that*

$$\gamma \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F);$$

(4) $\Gamma(\psi, D)$ *contains a hyperbolic form, i.e.* $0 \in \Gamma(\psi, D)$;
(5) *there exist* $x, y, u, v, d \in F^*$ *such that* $\psi \sim \langle -x, -y, xy, d\rangle$ *and* $D \cong (x, y) \otimes_F (u, v)$;
(6) $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_D) = 0$;
(7) $H^3(F(\psi, D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$.

Proof. (1)⇒(2). Obvious in view of Lemma 6.4.
(2)⇒(3). Obvious, because $\Gamma(\psi, D)$ is not empty.
(3)⇒(4). Condition (3) implies that $0 \in \Gamma(\psi, D) + I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$. It follows from Corollary 6.6 that $0 \in \Gamma(\psi, D) + I^4(F)$. Hence there exists $\gamma = k\psi + k'\psi' + lq \in \Gamma(\psi, D)$ such that $\gamma \in I^4(F)$. Since $\dim\gamma = 4 + 4 + 6 = 14 < 16$, the Arason-Pfister Hauptsatz shows that $\gamma = 0$.
(4)⇒(5). Let $\gamma = k\psi + k'\psi' + lq$ be a hyperbolic form. We have $(k\psi \perp lq)_{an} = -k'\psi'_{an}$. Therefore $k\psi$ and $-lq$ contain a common subform of dimension

$$\frac{1}{2}(\dim\psi + \dim q - \dim\psi') = \frac{1}{2}(4 + 6 - 4) = 3.$$

Let us denote such a 3-dimensional form by τ. Let $x, y \in F^*$ be such that $\tau \sim \langle -x, -y, xy\rangle$. Thus $\langle -x, -y, xy\rangle$ is similar to a subform of ψ and similar to a subform of q. Let $d = \det\psi$. Since $\langle -x, -y, xy\rangle$ is similar to a subform of ψ it follows that $\langle -x, -y, xy, d\rangle$ is similar to ψ. Since $\langle -x, -y, xy\rangle$ is similar to a subform of the Albert form q, it follows that there exist $u, v \in F^*$ such that q is similar to $\langle -x, -y, xy, u, v, -uv\rangle$. Then $[D] = c(q) = (x, y) + (u, v)$. Therefore $D \cong (x, y) \otimes_F (u, v)$.
(5)⇒(6). See Theorem 4.5.
(6)⇒(7). See Proposition 2.1.
(7)⇒(1). See Lemma 2.5. □

PROPOSITION 6.8. *Let D be a biquaternion algebra and let ψ be a 4-dimensional quadratic form such that* $\det\psi \neq 1$ *and* $\operatorname{ind}(D\otimes_F C_0(\psi)) = 2$. *Then for any* $\gamma \in \Gamma(\psi, D)$ *one has*

$$H^3(F(\psi,D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F) + e^3(\gamma)H^0(F).$$

PROOF. By Lemma 6.4, the element $e^3(\gamma)$ belongs to $H^3(F(\psi,D)/F)$. If the group $\mathrm{CH}^2(X_\psi \times X_D)$ is torsion-free, then, by Proposition 2.1, we have

$$H^3(F(\psi,D)/F) = H^3(F(\psi)/F) + H^3(F(D)/F)$$

and the proof is complete. If $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) \neq 0$, Lemma 6.7 shows that $\gamma \notin I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$. It follows from Lemma 2.5 that $e^3(\gamma) \notin H^3(F(\psi)/F) + H^3(F(D)/F)$. To complete the proof it is sufficient to apply Proposition 4.3 and Proposition 2.1. □

COROLLARY 6.9.

$$I^3(F(\psi,D)/F) \subset I^3(F(\psi)/F) + I^3(F(D)/F) + \Gamma'(\psi,D) + I^4(F),$$

where $\Gamma'(\psi,D) = \Gamma(\psi,D)\cup\{0\}$.

PROOF. Let $\tau \in I^3(F(\psi,D)/F)$. Choose an element $\gamma \in \Gamma(\psi,D)$. By Proposition 6.8, either the element $e^3(\tau)$ or the element $e^3(\tau-\gamma)$ is in

$$H^3(F(\psi)/F) + H^3(F(D)/F)\ .$$

It remains to apply Lemma 2.5. □

PROPOSITION 6.10. *Let* $\lambda \in I^3(F(\psi,D)/F)$. *Then at least one of the following conditions holds*

1) $\lambda \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
2) $\lambda \in \Gamma(\psi,D) + I^4(F)$.

PROOF. Obvious in view of Corollaries 6.9 and 6.6. □

7. Main theorem

THEOREM 7.1. *Let ϕ be an anisotropic 8-dimensional quadratic form with* $\det\phi = 1$ *and let ψ be a 4-dimensional quadratic form with* $\det\psi \neq 1$. *Suppose that $\phi_{F(\psi)}$ is isotropic and let the case* $\operatorname{ind}C(\phi) = \operatorname{ind}(C(\phi)\otimes_F C_0(\psi)) = 4$ *be excepted. Then the isotropy of $\phi_{F(\psi)}$ is L-standard.*

PROOF. In the case where $\operatorname{ind}C(\phi) = 8$, the theorem is proved in Chapter 3. Thus we can suppose that $\operatorname{ind}C(\phi) \leq 4$. Then there exists a biquaternion algebra D such that $c(\phi) = [D]$. Let q be an Albert form corresponding to D. Clearly (ϕ, D, q) is a special triple .

By Corollary 5.4, we can suppose that $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_D) \neq 0$. Theorem 4.1 asserts that at least one of the following conditions holds:

1) $\operatorname{ind}(D\otimes C_0(\psi)) = \operatorname{ind}D = 4$,
2) $\operatorname{ind}(D\otimes C_0(\psi)) = 2$.

If $\operatorname{ind}(D \otimes C_0(\psi)) = 2$, then all the conditions of Definition 6.2 hold. Thus we have a well-defined set $\Gamma(\psi, D)$.

By Proposition 5.3, we have $\phi + q \in I^3(F(\psi, D)/F)$. By Proposition 6.10, we see that at least one of the following condition holds:

1) $\phi + q \in I^3(F(\psi)/F) + I^3(F(D)/F) + I^4(F)$;
2) $\phi + q \in \Gamma(\psi, D) + I^4(F)$.

In the first case, Proposition 5.3 shows that there exists a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic. This implies that the isotropy of $\phi_{F(\psi)}$ is L-standard.

Therefore, we may assume that $\phi + q \in \Gamma(\psi, D) + I^4(F)$. Thus there exist $k, k', l \in F^*$ (and a 4-dimensional quadratic form ψ') such that

$$\phi + q \equiv k\psi + k'\psi' + lq \pmod{I^4(F)}.$$

Since $k\psi + k'\psi' + lq \in I^3(F)$ it follows that

$$k\psi + k'\psi' + lq \equiv l(k\psi + k'\psi' + lq) \pmod{I^4(F)}.$$

Therefore

$$\phi + q \equiv lk\psi + lk'\psi' + q \pmod{I^4(F)}.$$

Thus $\phi \equiv lk\psi + lk'\psi' \pmod{I^4(F)}$. Let $\phi^* \stackrel{\text{def}}{=} \psi \perp kk'\psi'$. Since $\phi \equiv lk\phi^* \pmod{I^4(F)}$ and $\dim \phi^* = 8$, it follows that ϕ and ϕ^* are half-neighbors. Since $\psi \subset \phi^*$, the isotropy is L-standard and the proof is complete. $\square$

Corollary 7.2. *Let ϕ be an anisotropic 8-dimensional quadratic form with $\det \phi = 1$ and let ψ be a 4-dimensional quadratic form with $\det \psi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic but the isotropy is not L-standard. Then ϕ can be written in the form $\phi = \pi_1 \perp \pi_2$ with $\pi_1, \pi_2 \in GP_2(F)$.*

Proof. Since $\det \phi = 1$, it is sufficient to verify that ϕ contains a 4-dimensional quadratic form with trivial determinant. Suppose at the moment that ϕ contains no 4-dimensional quadratic form with trivial determinant. Then [**25**, Theorem 6.1] implies that there exists a homomorphism of F-algebras, $C_0(\psi) \to C_0(\phi)$. Since $\det \phi = 1$ and $\dim \phi = 8$, there exists a 3-quaternion algebra A such that $C_0(\phi)$ has the form $A \times A$ and $C(\phi) \simeq M_2(A)$. Thus we get a homomorphism $C_0(\psi) \to A$ which is injective because $C_0(\psi)$ is a simple algebra. Then $\operatorname{ind}(C_0(\psi) \otimes_F A) = 2$. Since $M_2(A) \simeq C(\phi)$, we have $\operatorname{ind}(C_0(\psi) \otimes_F C(\phi)) = 2$. Theorem 7.1 implies that isotropy of $\phi_{F(\psi)}$ is L-standard. The contradiction obtained completes the proof. $\square$

It is a natural question if there exists an example of non-L-standard isotropy. One way to find non-L-standard isotropy is based on the following

Lemma 7.3. *Let q be a 6-dimensional quadratic form and let ψ be a 4-dimensional quadratic form over a field k. Suppose that q is a $k(\psi)$-minimal form (see definition in [**20**]). Let $F \stackrel{\text{def}}{=} k((t))$, $\phi \stackrel{\text{def}}{=} q \perp t\langle 1, \det(q)\rangle$. Then the form $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

Proof. The proof of this lemma is absolutely analogous to the proof of Theorem 5.1 of Chapter 3 and we omit it. $\square$

Corollary 7.4. *There exist a field F, an 8-dimensional quadratic form $\phi \in I^2(F)$, and a 4-dimensional F-form ψ with nontrivial determinant such that $\phi_{F(\psi)}$ is isotropic, but the isotropy is not L-standard.*

Proof. By Corollary 15.4 of Chapter 5, there exists a field k, a quadratic form q of dimension 6, and a quadratic form ψ of dimension 4 over a field k such that q is a $k(\psi)$-minimal form. Now the required result follows from Lemma 7.3. $\square$

8. Isotropy over the function field of a conic

In this section we still assume that ϕ is an anisotropic 8-dimensional quadratic form of trivial determinant. We are interested in the question when ϕ is isotropic over the function field of a quadratic form ψ.

For the forms ψ of dimension ≥ 5, the question was studied in [**47**] and [**46**]. The case $\dim\psi = 4$, $d_\pm\psi \neq 1$ is done in the previous section. Thus it suffices to consider only two cases: $\dim\psi = 3$ or $\psi \in GP_2(F)$. Since the function field of a form $\psi \in GP_2(F)$ is stably birational equivalent to the function field of an arbitrary 3-dimensional subform of ψ (see e.g. Lemma 5.2 of Chapter 4), it suffices to handle only one of these cases.

Let us consider the case where $\psi \in GP_2(F)$.

Theorem 8.1. *Let ϕ be an anisotropic 8-dimensional quadratic form (we do not assume* $\det\phi = 1$*) and let $\psi \in GP_2(F)$. Then the form $\phi_{F(\psi)}$ is isotropic if and only if at least one of the following conditions holds:*

a) *there exists a 10-dimensional form ϕ^* such that $\psi \subset \phi^*$ and $s\phi \equiv \phi^*$ $\pmod{I^4(F)}$ for suitable $s \in F^*$;*
b) *there exists a 5-dimensional subform $\phi_0 \subset \phi$ with the following two properties:*
   - *the form ϕ_0 is a Pfister neighbor,*
   - *the form $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. First, suppose that $\phi_{F(\psi)}$ is isotropic. The excellence property of the field extension $F(\psi)/F$ implies that there exist a 6-dimensional form τ and a form γ such that $\phi - \tau = \gamma\psi$ (see [**20**, Proof of Proposition 1.1]). Comparing dimensions, one can see that $1 \leq \dim\gamma \leq 3$. First, consider the case where $\dim\gamma$ is odd (i.e., $\dim\gamma = 1$ or 3). We set $s \stackrel{\text{def}}{=} d_\pm\gamma$. Clearly, $\gamma \equiv \langle s\rangle$ $\pmod{I^2(F)}$. Hence $\gamma\psi \equiv s\psi \pmod{I^4(F)}$. Since $\phi - \tau = \gamma\psi$, it follows that $\phi = \tau + \gamma\psi \equiv \tau + s\cdot\psi \equiv s(s\tau + \psi) \pmod{I^4(F)}$. Setting $\phi^* \stackrel{\text{def}}{=} s\tau \perp \psi$, one can see that condition a) holds. Now, suppose that $\dim\gamma$ is odd, i.e., $\dim\gamma = 2$. Then $\gamma\psi \in GP_3(F)$. We have $\phi - \gamma\psi = \tau$. Therefore, the quadratic forms ϕ and $\gamma\psi$ contain a common subform of dimension $\frac{1}{2}(\dim\phi + \dim\gamma\psi - \dim\tau) = \frac{1}{2}(8 + 8 - 6) = 5$. Let us denote that 5-dimensional form by ϕ_0. Clearly, condition b) holds.

Now, suppose that at least one of conditions a) and b) holds. We have to verify that $\phi_{F(\psi)}$ is isotropic. It is obvious in the case where condition b) holds. Suppose now that condition a) holds. Since $\psi \subset \phi^*$ and $\dim \phi^* = 10$, one has $\dim(\phi^*_{F(\psi)})_{an} \leq 6$. Therefore $\dim((\phi \perp -s\phi^*)_{F(\psi)})_{an} \leq 8 + 6 = 14 < 16$. Since $\phi \perp -s\phi^* \in I^4(F)$, the Arason-Pfister Hauptsatz implies that $\dim((\phi \perp -s\phi^*)_{F(\psi)})_{an}$ is hyperbolic. Hence $(\phi_{F(\psi)})_{an} = (s\phi^*_{F(\psi)})_{an}$. Therefore $\dim(\phi_{F(\psi)})_{an} \leq 6$. Hence $\phi_{F(\psi)}$ is isotropic. $\square$

COROLLARY 8.2. *Let ϕ be an anisotropic 8-dimensional quadratic form with* $\det \phi = 1$ *and let ψ be a 4-dimensional quadratic form with* $\det \psi = 1$ *(or ψ is a 3-dimensional form). Suppose that $\phi_{F(\psi)}$ is isotropic but the isotropy is not L-standard. Then* $\operatorname{ind}(C(\phi) \otimes C_0(\psi)) = 4$.

PROOF. We can assume that $\psi \in GP_2(F)$ (if $\dim \psi = 3$, then we replace ψ by $\psi \perp \langle \det \psi \rangle$) . Let us suppose that $\operatorname{ind}(C(\phi) \otimes C_0(\psi)) \neq 4$. By Theorem 8.1, there exist a 10-dimensional form ϕ^* and an coefficient $s \in F^*$ such that $\psi \subset \phi^*$ and $s\phi \equiv \phi^* \pmod{I^4(F)}$. Then ϕ^* can be written in the form $\phi^* = \psi \perp q$. Clearly, q is an Albert form and $c(q) = c(\phi^*) + c(\psi)$. Therefore $\operatorname{ind} C(q) = \operatorname{ind}(C(\phi^*) \otimes C(\psi)) = \operatorname{ind}(C(\phi) \otimes C_0(\psi)) \neq 4$. Hence, q is isotropic. Thus there exists a 4-dimensional form $\tilde{q}$ such that $\tilde{q}_{an} = q_{an}$. Set $\tilde{\phi}^* \stackrel{\text{def}}{=} \psi \perp \tilde{q}$. Clearly, $\dim \tilde{\phi}^* = 8$, $\psi \subset \tilde{\phi}^*$ and $s\phi \equiv \phi^* \equiv \tilde{\phi}^* \pmod{I^4(F)}$. Therefore, the isotropy $\phi_{F(\psi)}$ is L-standard, a contradiction. $\square$

REMARK 8.3. There are many examples of ϕ and ψ with $\psi \in GP_2(F)$ such that the isotropy of $\phi_{F(\psi)}$ is not L-standard. The condition of Theorem 8.1 can be regarded as a modification of the notion of the L-standard isotropy for the case $\psi \in GP_2(F)$.

CHAPTER 8

A generalization of the Albert-Risman theorems

Let A be a central simple algebra of a prime degree p over a field F and let $B_1, \dots, B_{p-1}$ be central simple F-algebras of degrees $p^{n_1}, \dots, p^{n_{p-1}}$. We show that if every tensor product $A \otimes_F B_i$ has zero divisors, then there exists a field extension E/F of degree $\le p^{n_1+\cdots+n_{p-1}}$ which splits the algebras $B_1, \dots, B_{p-1}$ as well as the algebra A. In the case $p = 2$, this statement was proved 1975 by L. Risman ([**70**]); in the case $p = 2$ and $n_1 = 1$, it is a classical theorem of A. Albert (see [**1**] or [**2**]).

0. Introduction

A well-known theorem of A. Albert states (see [**1**] or [**2**]): if tensor product of two quaternion algebras has zero divisors, then the quaternion algebras can be split by a common extension of the base field of degree ≤ 2.

1975 L. J. Risman gave the following generalization of Albert's theorem ([**70**, Theorem 1]): if tensor product of a degree 2^n (where $n \ge 1$) central simple algebra A and a quaternion algebra B has zero divisors, then A and B possess a common splitting field of degree $\le 2^n$.

Attempts to find a generalization of Risman's theorem to the case of an odd prime p was unsuccessful for a long time. Even worth: 1993 B. Jacob and A. R. Wadsworth ([**27**], see also Chapter 9) have shown that already Albert's theorem has no generalization to the case of *two* degree p algebras. Two degree p central simple algebras A, B with no common splitting field of degree p, they found for every odd prime p, possessed the following property: for any integers $i, j \ge 0$ the index of the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ was $\le p$.

We propose here the following generalization of Risman's theorem:

THEOREM 0.1. *Let A be a central simple algebra of a prime degree p over a field F and let B_1, ..., B_{p-1} be central simple F-algebras of degrees p^{n_1}, ..., $p^{n_{p-1}}$. Set $n \overset{\text{def}}{=} n_1 + \cdots + n_{p-1}$ and suppose that for every $i = 1, \dots, p-1$ the tensor product $A \otimes_F B_i$ has zero divisors. Then there exists a field extension E/F of degree $\le p^n$ which splits all the algebras $A, B_1, \dots, B_{p-1}$.*

In the particular case where $n_1 = \cdots = n_{p-1} = 1$, Theorem 0.1 can be regarded as a generalization of Albert's theorem. For instance, taking $p = 3$ we get the following

EXAMPLE 0.2. Let A, B, C be three degree 3 central simple algebras over a field F. Suppose that each of the two tensor products $A \otimes B$ and $A \otimes C$

has zero divisors. Then there exists a field extension E/F of degree ≤ 9 which splits all the three algebras A, B, C.

In fact, a general method of obtaining results of the type similar to Theorem 0.1 is developed in Chapter 10. However this method allows to control degrees of common splitting fields of algebras only up to a prime to p factor. In order to obtain the announced exact statement, we use here a refinement of that method. It is based on the intersection theory and especially on the theory of *non-negative intersections* developed in [**12**, Chapter 12] (see the proof of Proposition 1.2).

Terminology and notation. Let F be a field. We fix an algebraic closure $\bar{F}$ of F and, for any F-variety X, denote by $\bar{X}$ the $\bar{F}$-variety $X_{\bar{F}}$. Let σ be a cycle on X. We denote by $[\sigma]$ its class in the Chow group $\mathrm{CH}^*(X)$ and by $\bar{\sigma}$ the corresponding cycle on $\bar{X}$. Sometimes, while working on $X \times Y$, where Y is another F-variety, we denote, abusing notation, by σ as well the cycle $\sigma \times Y$.

Degree $\deg(\sigma)$ of a simple 0-dimensional cycle σ is defined to be the degree of its residue field over the base field. Degree of an arbitrary 0-dimensional cycle $\sigma = \sum l_j \sigma_j$, where l_j are integers and σ_j are simple cycles, is defined as $\sum l_j \deg(\sigma_j)$.

A cycle $\sigma = \sum l_j \sigma_j$ (of any dimension) is called *non-negative*, if all the integers l_j are non-negative.

1. Preliminaries

In this section we prove two (independent) statements needed for the next section.

LEMMA 1.1. *For any integers $n, m \geq 0$, let $\phi : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ be the Segre imbedding. Denote by $f \in \mathrm{CH}^1(\mathbb{P}^n)$, $g \in \mathrm{CH}^1(\mathbb{P}^m)$, and $h \in \mathrm{CH}^1(\mathbb{P}^{nm+n+m})$ the classes of hyperplanes. Then*

$$\phi^*(h) = f + g \in \mathrm{CH}^1(\mathbb{P}^n \times \mathbb{P}^m),$$

where $\phi^* \colon \mathrm{CH}^*(\mathbb{P}^{nm+n+m}) \to \mathrm{CH}^*(\mathbb{P}^n \times \mathbb{P}^m)$ is the pull-back homomorphism.

PROOF. Denote by $[x_0 : \dots : x_n]$, $[y_0 : \dots : y_m]$, and $[z_0 : \dots : z_{nm+n+m}]$ the coordinates in $\mathbb{P}^n$, $\mathbb{P}^m$, and $\mathbb{P}^{nm+n+m}$. The Segre embedding ϕ is determined by the rule

$$\phi([x_0 : \dots : x_n] \times [y_0 : \dots : y_m]) = [x_0y_0 : x_0y_1 : \dots : x_ny_{m-1} : x_ny_m] .$$

The intersection of the hyperplane $z_0 = 0$ with $\mathbb{P}^n \times \mathbb{P}^m$ has two transversal components: one of them is determined in $\mathbb{P}^n \times \mathbb{P}^m$ by the equation $x_0 = 0$ and represents f while the other one is determined by the equation $y_0 = 0$ and represents g. $\square$

PROPOSITION 1.2. *Let T be a direct product of Severi-Brauer varieties over a field F. Let σ and σ' be non-negative cycles on T and let $\psi\colon T' \to T$ be a morphism of F-varieties. Then*

1. *there exists a non-negative cycle τ on T such that $[\tau] = [\sigma]\cdot[\sigma'] \in \mathrm{CH}^*(T)$;*
2. *there exists a non-negative cycle τ' on T' such that $\psi^*([\sigma]) = [\tau']$.*

PROOF. We define the cycles τ and τ' à la [**12**]: τ is the intersection product $\sigma \cap \sigma'$ of σ and σ'; τ' is the pull-back $\psi^!(\sigma)$ of the cycle σ with respect to ψ. Note that the cycles τ and τ' are defined *precisely*, not only up to rational equivalence.

Considering τ as a cycle on T and τ' as a cycle on T', we have $[\tau] = [\sigma]\cdot[\sigma'] \in \mathrm{CH}^*(T)$ and $[\tau'] = \psi^*([\sigma]) \in \mathrm{CH}^*(T')$. So, we only have to check that τ and τ' are non-negative. For this it suffices to check that the cycles $\bar\tau$ and $\bar\tau'$ on the $\bar F$-varieties $\bar T$ and $\bar T'$ are non-negative.

Since intersection products commute with flat pull-backs ([**12**, Theorem 6.2]), we still have $\bar\tau = \bar\sigma \cap \bar\sigma'$ and $\bar\tau' = \bar\psi^!(\bar\sigma)$, where $\bar\psi$ is the morphism $\bar T' \to \bar T$ obtained from ψ by the base change. Since the variety $\bar T$ is isomorphic to a direct product of projective spaces, the tangent bundle on $\bar T$ is generated by the global sections (see [**12**, Examples 12.2.1a and 12.2.1c]). Therefore, by [**12**, Corollary 12.2], the cycles $\bar\tau$ and $\bar\tau'$ are non-negative. □

2. The proof

In this section we prove Theorem 0.1.

We denote by $X, Y_1, \dots, Y_{p-1}$, and $T_1, \dots, T_{p-1}$ the Severi-Brauer varieties of the algebras A, $B_1, \dots, B_{p-1}$, and $A\otimes_F B_1, \dots, A\otimes_F B_{p-1}$. We set

$$Y \stackrel{\mathrm{def}}{=} \prod_{i=1}^{p-1} Y_i \ .$$

For every i, tensor product of ideals induces a closed imbedding $\psi_i\colon X \times Y_i \hookrightarrow T_i$ which is a twisted form of the Segre imbedding $\mathbb{P}^{p-1}\times\mathbb{P}^{p^{n_i}-1} \hookrightarrow \mathbb{P}^{p^{n_i+1}-1}$.

Let $f \in \mathrm{CH}^1(\bar X)$, $g_i \in \mathrm{CH}^1(\bar Y_i)$, and $h_i \in \mathrm{CH}^1(\bar T_i)$ be the classes of hyperplanes.

The algebra $A\otimes B_i$ (for every i) has degree p^{n_i+1} and zero divisors, so that its index divides p^{n_i}. Therefore, there exists a simple (and in particular non-negative) p^{n_i}-codimensional cycle σ_i on the variety T_i such that $[\bar\sigma_i] = h_i^{p^{n_i}} \in \mathrm{CH}^{p^{n_i}}(\bar T_i)$ (see [**6**, §3]).

By Item 2 of Proposition 1.2, there exists a non-negative cycle τ_i on $X\times Y_i$ such that $[\tau_i] = \psi_i^*([\sigma_i])$. Since $[\bar\sigma_i] = h_i^{p^{n_i}}$, it follows from Lemma 1.1 that $[\bar\tau_i] = (f+g_i)^{p^{n_i}} \in \mathrm{CH}^{p^{n_i}}(\bar X\times\bar Y_i)$.

Applying Item 1 of Proposition 1.2 to the variety $X\times Y$, we find a non-negative cycle τ on $X\times Y$ such that $[\tau] = [\tau_1]\cdots[\tau_{p-1}] \in \mathrm{CH}^*(X\times Y)$. Note that τ is a cycle of codimension $p^{n_1}+\cdots+p^{n_{p-1}}$ on a variety of dimension

$(p-1)+(p^{n_1}-1)+\cdots+(p^{n_{p-1}}-1) = p^{n_1}+\cdots+p^{n_{p-1}}$, so that it is a 0-dimensional cycle. Moreover,

$$[\bar\tau] = [\bar\tau_1]\cdots[\bar\tau_{p-1}] = (f+g_1)^{p^{n_1}}\cdots(f+g_{p-1})^{p^{n_{p-1}}} = \\ = p^n\cdot(f^{p-1}g_1^{p^{n_1}-1}\cdots g_{p-1}^{p^{n_{p-1}}-1}),$$

where the last equality holds because of the relation $f^p=0$. Since f^{p-1} is the class of a rational point on $\bar X$ and since $g_i^{p^{n_i}-1}$ is the class of a rational point on the variety $\bar Y_i$ for every i, we get $\deg(\tau)=p^n$.

Let τ' be any simple cycle included in τ. Since the cycle τ is non-negative, we have $\deg(\tau')\le p^n$. The residue field E of τ' is a common splitting field of the algebras $A, B_1, \ldots, B_{p-1}$ and $[E:F]=\deg(\tau')\le p^n$. The proof of Theorem 0.1 is complete.

CHAPTER 9

Linked algebras

Two central simple algebras A, B of a prime degree p over a field are called *linked*, if for any integers $i, j \geq 0$ the index of $A^{\otimes i} \otimes B^{\otimes j}$ is at most p. They are called *strongly linked*, if they possess a common splitting field of a finite degree (over the base field) not divisible by p^2.

We show that for any two not strongly linked central simple algebras an extension of the base field can be made such that the algebras become *linked* while still not being *strongly linked*.

0. Introduction

1931 A. Albert proved (see [**1**] or [**2**]): two quaternion division algebras can be split by a common quadratic extension of the base field provided that their tensor product has zero divisors.

Attempts to generalize the theorem of Albert to the case of an odd prime p have led 1987 to counter-examples of J.-P. Tignol and A. R. Wadsworth ([**85**, Proposition 5.1]), who constructed two degree p central division algebras A, B with zero divisors in $A \otimes B$ and without common splitting field extensions of degree p.

Stronger counter-examples was obtained 1993 by B. Jacob and A. R. Wadsworth ([**27**]). Two degree p central division algebras A, B without common splitting field of degree p, they found, possessed the following property: for any integers $i, j \geq 0$ the index of the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ was $\leq p$. It was in fact even shown that any common splitting field of the algebras A, B has degree divisible by p^2 ([**27**, Remark 2]).

In this Chapter we show (see Theorem 1.1) that similar counter-examples can be obtained by an appropriate base extension starting from *any* two degree p central simple algebras A, B provided that degree of every common splitting field of A, B is divisible by p^2 (what is guaranteed if e.g. the tensor product $A \otimes B$ is a division algebra). The proof is essentially different from that of [**27**].

Notation. For a smooth variety X over a field F, we denote by $K(X)$ the Grothendieck group of X; by $K(X)^{(n)}$, where $n \geq 0$, the n-codimensional term of the topological filtration on X (see [**69**, §7] for a definition of the topological filtration); by $\mathrm{CH}^*(X)$ the Chow ring of X graded by codimension of cycles; by $\mathrm{CH}_0(X)$ the 0-dimensional component of the Chow ring, i.e. the component

$\mathrm{CH}^{\dim X}(X)$. For a central simple F-algebra A, we denote by $\mathrm{SB}(A)$ the Severi-Brauer variety of A and by $\mathrm{SB}(r, A)$ (for $r \geq 0$) the generalized Severi-Brauer varieties (also called *generic partial splitting varieties*) of A.

1. The theorem

Throughout this Chapter, A, B are central simple algebras of a prime degree p over a field F. We call them *linked*, if $\mathrm{ind}(A^{\otimes i} \otimes_F A^{\otimes j}) \leq p$ for any $i, j \geq 0$. The algebras A and B are called *strongly linked*, if there exists a finite field extension E/F such that

- the algebras A_E and B_E are split and
- the degree $[E : F]$ is not divisible by p^2.

Clearly, strongly linked algebras are linked. By the theorem of Albert, the inverse is true for $p = 2$. For any odd p, we shall prove the following

THEOREM 1.1. *Let F be a field, p be an odd prime number, and A, B be degree p central simple F-algebras. Suppose that A, B are not strongly linked. Then there exists a field extension $\tilde{F}/F$ such that the algebras $A_{\tilde{F}}, B_{\tilde{F}}$ are linked but still not strongly linked.*

One can take for $\tilde{F}$ the function field $F(T)$ of the following product of generalized Severi-Brauer varieties

$$T \overset{\mathrm{def}}{=} \mathrm{SB}(p, A \otimes B) \times \mathrm{SB}(p, A^{\otimes 2} \otimes B) \times \cdots \times \mathrm{SB}(p, A^{\otimes p-1} \otimes B) .$$

2. Preliminary observations

We set $X \overset{\mathrm{def}}{=} \mathrm{SB}(A) \times \mathrm{SB}(B)$. Let L/F be an arbitrary common splitting field extension of A, B.

LEMMA 2.1. *The algebras A, B are not strongly linked if and only if the image of the restriction homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L)$ is divisible by p^2.*

PROOF. Note that since the algebras A_L, B_L are split, the variety X_L is isomorphic to a product of two projective spaces. Therefore the degree map $\deg : \mathrm{CH}_0(X_L) \to \mathbb{Z}$ is an isomorphism. Also note that the composition $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L) \to \mathbb{Z}$ of the restriction and degree maps is the degree map for $\mathrm{CH}_0(X)$.

If the image of the restriction homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L)$ is not divisible by p^2, then there exists a closed point on X of degree not divisible by p^2. The residue field of this point is a common splitting field for A and B showing that the algebras are strongly linked.

To prove the inverse implication, suppose the image of the restriction homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X_L)$ is divisible by p^2. Then the image of the degree homomorphism $\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$ is divisible by p^2 as well. Let E/F be any finite field extension such that the algebras A_E, B_E are split. The variety X_E has then a rational point. Taking the transfer, we obtain a 0-dimensional cycle of degree $[E:F]$ on X. Consequently, $[E:F]$ is divisible by p^2, i.e. the algebras A, B are not strongly linked. $\square$

LEMMA 2.2. *If the algebras A, B are not strongly linked, then, for any not simultaneously 0 integers $0 \le i, j < p$, the index of tensor product $A^{\otimes i} \otimes B^{\otimes j}$ is divisible by p.*

PROOF. Let i, j be any integers such that $0 \le i, j < p$. Suppose that the tensor product $A^{\otimes i} \otimes B^{\otimes j}$ is split. If $i \ne 0$, then any splitting field of B splits A as well; therefore the algebras A, B possess a common splitting field of degree p in this case, what contradicts to the assumption they are not strongly linked. Thus $i = 0$. The symmetric argument shows that $j = 0$ as well. $\square$

LEMMA 2.3. *Suppose that, for any not simultaneously 0 integers $0 \le i, j < p$, the index of tensor product $A^{\otimes i} \otimes B^{\otimes j}$ is divisible by p. Then the image of the restriction homomorphism $K(X)^{(1)} \to K(X_L)^{(1)}$ is divisible by p.*

PROOF. We have already noticed that since the algebras A_L and B_L are split, the varieties $\mathrm{SB}(A_L)$ and $\mathrm{SB}(B_L)$ are isomorphic to ($(p-1)$-dimensional) projective spaces. Let ξ, η be the Grothendieck classes of the tautological line bundles on $\mathrm{SB}(A_L), \mathrm{SB}(B)_L$ respectively. The group $K(X_L)$ is generated by $\{\xi^i \eta^j\}_{i,j=0}^{p-1}$. Using a generalized Peyre's version [**66**, Proposition 3.1] of Quillen's computation of K-theory for Severi-Brauer schemes [**69**, Theorem 4.1 of §8], one can show that the image of the restriction map $K(X) \to K(X_L)$ is generated by $\{\mathrm{ind}(A^{\otimes i} \otimes B^{\otimes j}) \cdot \xi^i \eta^j\}_{i,j=0}^{p-1}$.

Since the first term of the topological filtration coincides with the kernel of the rank map, the group $K(X_L)^{(1)}$ is generated by $\{\xi^i \eta^j - 1\}_{i,j=0}^{p-1}$ while the image of $K(X)^{(1)} \to K(X_L)^{(1)}$ is generated by $\{\mathrm{ind}(A^{\otimes i} \otimes B^{\otimes j}) \cdot (\xi^i \eta^j - 1)\}_{i,j=0}^{p-1}$.

The assertion required follows now from the assumption on the indexes $\mathrm{ind}(A^{\otimes i} \otimes B^{\otimes j})$. $\square$

COROLLARY 2.4. *In the conditions of Lemma 2.3, for any $n > 0$, the image of the restriction homomorphism $\mathrm{CH}^n(X) \to \mathrm{CH}^n(X_L)$ is divisible by p.*

PROOF. For any $n \ge 0$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^n(X_L) & \longrightarrow & K(X_L)^{(n/n+1)} \\ {\scriptstyle \mathrm{res}_{L/F}} \uparrow & & \uparrow {\scriptstyle \mathrm{res}_{L/F}} \\ \mathrm{CH}^n(X) & \longrightarrow & K(X)^{(n/n+1)} \end{array}$$

where the upper arrow is an isomorphism. Therefore, it suffices to show that for any $n > 0$ the image of the restriction homomorphism $K(X)^{(n)} \to K(X_L)^{(n)}$ is divisible by p. By Lemma 2.3, the image of $K(X)^{(n)} \to K(X_L)^{(1)}$ is divisible by p. Since the quotient $K(X_L)^{(1/n)}$ is torsion-free, we are done. $\square$

3. The proof

In this section we prove Theorem 1.1.

We set $\tilde{F} \stackrel{\text{def}}{=} F(T)$, $\tilde{A} \stackrel{\text{def}}{=} A_{\tilde{F}}$, and $\tilde{B} \stackrel{\text{def}}{=} B_{\tilde{F}}$, where T is the product of generalized Severi-Brauer varieties written down in the end of Section 1. We also set $X \stackrel{\text{def}}{=} \mathrm{SB}(A) \times \mathrm{SB}(B)$ and $\tilde{X} \stackrel{\text{def}}{=} \mathrm{SB}(\tilde{A}) \times \mathrm{SB}(\tilde{B})$.

First of all we note that by the main property of generalized Severi-Brauer varieties one has $\operatorname{ind}(\tilde{A}^{\otimes i} \otimes \tilde{B}) \leq p$ for any $i = 1, \dots, p-1$. Therefore $\operatorname{ind}(\tilde{A}^{\otimes i} \otimes \tilde{B}^{\otimes j}) \leq p$ for any integers $i, j \geq 0$, i.e. the algebras $\tilde{A}, \tilde{B}$ are linked. So we only have to show that they are not strongly linked.

LEMMA 3.1. *For any not simultaneously 0 integers $0 \leq i, j < p$, the index of tensor product $\tilde{A}^{\otimes i} \otimes \tilde{B}^{\otimes j}$ is divisible by p.*

PROOF. First of all, by Lemma 2.2, the assertion holds for the algebras A, B, because they are assumed to be not strongly linked. The assertion on $\tilde{A}, \tilde{B}$ follows now from the index reduction formula for generalized Severi-Brauer varieties ([**61**, Formula 5.11]) (in fact it suffices to apply a simpler statement on triviality of the relative Brauer group for the function field of generalized Severi-Brauer varieties). □

Let L/F be a common splitting field extension for the algebras A, B. Set $\tilde{L} \stackrel{\text{def}}{=} L(T)$. Clearly, $\tilde{L}/\tilde{F}$ is a common splitting field extension for $\tilde{A}, \tilde{B}$.

COROLLARY 3.2. *For any $n > 0$, the image of the restriction homomorphism $\mathrm{CH}^n(\tilde{X}) \to \mathrm{CH}^n(\tilde{X}_{\tilde{L}})$ is divisible by p.*

PROOF. Follows from Lemma 3.1 and Corollary 2.4. □

We consider the graded ring $\mathrm{CH}^*(\tilde{X})$ as a graded $\mathrm{CH}^*(X)$-algebra via the restriction homomorphism $\mathrm{CH}^*(X) \to \mathrm{CH}^*(\tilde{X})$.

PROPOSITION 3.3. *The $\mathrm{CH}^*(X)$-algebra $\mathrm{CH}^*(\tilde{X})$ is generated by its graded components of codimensions $\leq p$.*

PROOF. Since the pull-back $\mathrm{CH}^*(X \times T) \to \mathrm{CH}^*(\tilde{X})$ is an epimorphism of graded $\mathrm{CH}^*(X)$-algebras (see [**39**, Theorem 3.1] or Proposition 4.1 of Chapter 5 for the surjectivity), it suffices to show that the algebra $\mathrm{CH}^*(X \times T)$ is generated by its graded components of codimensions $\leq p$.

Consider the variety $X \times T$ as a scheme over X via the projection. According to Proposition 5.3 of Chapter 2, it is a product (over X) of p-grassmanians. By [**12**, Proposition 14.6.5], $\mathrm{CH}^*(X \times T)$ is therefore generated (as $\mathrm{CH}^*(X)$-algebra) by the Chern classes of the tautological bundles on the grassmanians. Since all these bundles have rank p, they may have non-trivial Chern classes only in codimensions $\leq p$. □

Finally, Theorem 1.1 follows from Lemma 2.1 and the following assertion:

COROLLARY 3.4. *The image of the restriction $\mathrm{CH}_0(\tilde{X}) \to \mathrm{CH}_0(\tilde{X}_{\tilde{L}})$ is divisible by p^2.*

Proof. Note that since $p \neq 2$, we have $p < (p-1)^2 = \dim X$. Thus, by Proposition 3.3, the group $\mathrm{CH}_0(\tilde{X})$ is generated by the image of $\mathrm{CH}_0(X) \to \mathrm{CH}_0(\tilde{X})$ and by the products $\mathrm{CH}^n(\tilde{X}) \cdot \mathrm{CH}^m(\tilde{X})$ with $n, m > 0$.

The image in $\mathrm{CH}_0(\tilde{X}_{\tilde{L}})$ of the first part of the generators is divisible by p^2 since in the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_0(\tilde{X}) & \longrightarrow & \mathrm{CH}_0(\tilde{X}_{\tilde{L}}) \\ \uparrow & & \uparrow \\ \mathrm{CH}_0(X) & \longrightarrow & \mathrm{CH}_0(X_L) \end{array}$$

the image of the bottom arrow is divisible by p^2 (Lemma 2.1).

The image in $\mathrm{CH}_0(\tilde{X}_{\tilde{L}})$ of the second part of the generators is divisible by p^2 since for any $n > 0$ the image of $\mathrm{CH}^n(\tilde{X}) \to \mathrm{CH}^n(\tilde{X}_{\tilde{L}})$ is divisible by p (Corollary 3.2). $\square$

CHAPTER 10

Common splitting fields of division algebras

For every prime number p and every map $\alpha : \mathbb{Z}^n \to \mathbb{Z}$, we find the minimal integer β such that the following assertion holds: any elements $x_1, \dots, x_n$ of the Brauer group $\mathrm{Br}(F)$ of an arbitrary field F, satisfying the conditions $\mathrm{ind}(i_1x_1 + \dots + i_nx_n) = p^{\alpha(i_1,\dots,i_n)}$ for all $i_1, \dots, i_n \in \mathbb{Z}$, possess a finite common splitting field extension E/F with $v_p([E : F]) \le \beta$, where v_p denotes the multiplicity of p.

0. Introduction

Let us fix a prime number p. Let $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}$ be an arbitrary map. We say that α is the *behaviour* of elements $x_1, \dots, x_n$ of the Brauer group $\mathrm{Br}(F)$ of a field F, if for any $i_1, \dots, i_n \in \mathbb{Z}$ the Schur index $\mathrm{ind}(i_1x_1 + \dots + i_nx_n)$ equals $p^{\alpha(i_1,\dots,i_n)}$. We say that α is a *behaviour*, if there exists a field F and elements $x_1, \dots, x_n \in \mathrm{Br}(F)$ with the behaviour α.

Let F be a field and $x \in \mathrm{Br}(F)$. A *splitting field* of x is by definition a field extension E of F such that $x_E = 0 \in \mathrm{Br}(E)$. A *common splitting field* of several elements of $\mathrm{Br}(F)$ is by definition a field which is a splitting field for each of the elements. We consider only (common) splitting fields *finite* over the base field.

Let us fix a behaviour α. In this chapter we determine the minimal integer β such that the following assertion holds (see Theorem 3.1): if some elements in the Brauer group of an arbitrary field F have the behaviour α, then they possess a common splitting field E with $v_p([E : F]) \le \beta$.

Similar questions was already considered in the literature. Here is a list of known results:

1. A classical theorem of Albert (see [**1**] or [**2**]) states: if tensor product of two quaternion division algebras has zero divisors, then the quaternion algebras possess a common splitting field quadratic over the base field.
2. A generalization of Albert's theorem due to Risman ([**70**, Theorem 1]) asserts: if tensor product of a 2-primary division algebra A and a quaternion algebra B has zero divisors, then A and B possess a common splitting field of degree $\deg A$.
3. Jacob and Wadsworth ([**27**], see also Chapter 9) constructed two division algebras of prime degree p over certain field F such that

- $\mathrm{ind}(A^{\otimes i} \otimes_F B^{\otimes j}) \le p$ for any $i, j \ge 0$ and
- the degree of any common splitting field of A and B is divisible by p^2.

4. The following was noticed by M. Rost (unpublished). Three quaternion algebras such that the Brauer class of any tensor product of some of them is represented by a quaternion algebra as well can not be (in general) split by a common quadratic extension of the base field.
5. A generalization of Risman's theorem to the case of odd prime is obtained in Chapter 8.

Note that the theorem presented here does not assert existence of a common splitting field of degree p^β. We even do not know whether such assertion is true in general. However, in certain particular cases (this means, for certain concrete behaviours) the proof can be refined in order to get the stronger result. An example is the theorem of Chapter 8.

Notation. For a smooth variety X over a field F, we denote by $K(X)$ the Grothendieck group of X; by $\Gamma_0 K(X)$ the 0-dimensional term of the gamma-filtration on $K(X)$ (for a definition of the gamma-filtration see [**52**, Definition 8.3] or Definition 2.6 of Chapter 1); by $\mathrm{T}_0 K(X)$ the 0-dimensional term of the topological filtration on X (see [**69**, §7] for a definition of the topological filtration). We fix an algebraic closure $\bar{F}$ of F and denote by $\bar{X}$ the $\bar{F}$-variety $X_{\bar{F}}$.

For any *projective homogeneous* variety X, we identify $K(X)$ with a subgroup in $K(\bar{X})$ via the restriction homomorphism $K(X) \to K(\bar{X})$ which is injective by [**65**].

The order of a finite set S is denoted by $|S|$.

For a central simple F-algebra A, we denote by $\mathrm{SB}(A)$ the Severi-Brauer variety of A and by $\mathrm{SB}(r, A)$ (for $r \geq 0$) the generalized Severi-Brauer varieties of A (also called *generic partial splitting varieties*).

1. "Generic" algebras of given behaviour

For any central simple algebras $A_1, \dots, A_n$ over a field F, we define their behaviour to be the behaviour of their classes in the Brauer group of F.

As in Definition 3.5 of Chapter 2, we say that algebras $A_1, \dots, A_n$ are *disjoint*, if, for any integers $i_1, \dots, i_n \geq 0$, it holds

$$\mathrm{ind}(A^{\otimes i_1} \otimes_F \cdots \otimes_F A^{\otimes i_n}) = \mathrm{ind}(A^{\otimes i_1}) \cdots \mathrm{ind}(A^{\otimes i_n}) .$$

We say that a collection of algebras $\tilde{A}_1, \dots, \tilde{A}_n$ is *"generic"* (compare to Definition 5.4 of Chapter 2), if it can be obtained via the following procedure. We start with some disjoint central simple algebras $A_1, \dots, A_n$ over a field F such that $\mathrm{ind}\, A_j = \exp A_j$ for every $j = 1, \dots, n$. Then we take some central simple algebras $B_1, \dots, B_m$ whose Brauer classes lie in the subgroup of $\mathrm{Br}(F)$ generated by the Brauer classes of $A_1, \dots, A_n$. We denote by Y the direct product $\mathrm{SB}(r_1, A_1) \times \cdots \times \mathrm{SB}(r_m, A_m)$ of generalized Severi-Brauer varieties with some $r_1, \dots, r_m \geq 0$ and we set $\tilde{A}_i \overset{\mathrm{def}}{=} (A_i)_{F(Y)}$ for each $i = 1, \dots, n$.

PROPOSITION 1.1. *For any behaviour $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}$, there exist "generic" division algebras $\tilde{A}_1, \dots, \tilde{A}_n$ (over a suitable field $\tilde{F}$) having the behaviour α.*

We prove the proposition after the following

LEMMA 1.2. *Let A, B be central simple algebras over a field F and let A', B' be central simple algebras over a field F'. Suppose that $\deg B = \deg B'$ and that for any $i \geq 0$ the index of $A' \otimes_{F'} B'^{\otimes i}$ divides the index of $A \otimes_F B^{\otimes i}$. Then, for any $r \geq 0$, the index of $A'_{F'(\mathrm{SB}(r,B'))}$ divides the index of $A_{F(\mathrm{SB}(r,B))}$.*

PROOF. Set $s \overset{\mathrm{def}}{=} \deg B = \deg B'$ and denote by d the greatest common divisor of r and s. By [**60**, Formula 1], one has

$$\operatorname{ind}(A_{F(\mathrm{SB}(r,B))}) = \gcd_{1\leq i\leq s}\left(\frac{d}{\gcd(i,d)}\operatorname{ind}(A\otimes B^{\otimes j})\right).$$

Replacing A by A' and B by B', we get a formula for $\operatorname{ind}(A'_{F'(\mathrm{SB}(r,B'))})$. Since $\operatorname{ind}(A'\otimes B'^{\otimes i})$ divides $\operatorname{ind}(A\otimes B^{\otimes i})$ for any i, we are done. □

PROOF OF PROPOSITION 1.1. We start with disjoint division algebras A_1, $\dots$, A_n over a suitable field F such that for any $j = 1, \dots, n$ one has

$$\deg A_j = \exp A_j = p^{\alpha(0,\dots,0,1,0,\dots,0)}$$

where 1 (in the argument of α) is placed on the j-th position (algebras like that do definitely exist). For every $i_1, \dots, i_n$ with $0 \leq i_j < \deg A_j$, we consider the algebra

$$B_{i_1\dots i_n} \overset{\mathrm{def}}{=} A_1^{\otimes i_1}\otimes\cdots\otimes A_n^{\otimes i_n}$$

and denote by $Y_{i_1\dots i_n}$ the variety $\mathrm{SB}(p^{\alpha(i_1,\dots,i_n)}, B_{i_1\dots i_n})$. We set $Y \overset{\mathrm{def}}{=} \prod Y_{i_1\dots i_n}$ and $\tilde{A}_j \overset{\mathrm{def}}{=} (A_j)_{F(Y)}$ for all $j = 1, \dots, n$. We state $\tilde{A}_1, \dots, \tilde{A}_n$ are "generic" division algebras required.

To show that $\operatorname{ind}(i_1[\tilde{A}_1] + \cdots + i_n[\tilde{A}_n]) = p^{\alpha(i_1,\dots,i_n)}$ for all $i_1, \dots, i_n \in \mathbb{Z}$, it suffices to check that

$$\operatorname{ind}(\tilde{A}_1^{\otimes i_1}\otimes\cdots\otimes\tilde{A}_n^{\otimes i_n}) = p^{\alpha(i_1,\dots,i_n)}$$

for any $i_1, \dots, i_n$ with $0 \leq i_j < \deg A_j$. Since the inequality $\leq$ is evident, it suffices to prove the inverse inequality.

Since α is a behaviour, we can find division algebras $A'_1, \dots, A'_n$ over a field F' having the behaviour α. Clearly, for any $i_1, \dots, i_n$ with $0 \leq i_j < \deg A'_j = \deg A_j$, the index of the algebra $B'_{i_1\dots i_n} \overset{\mathrm{def}}{=} A'^{\otimes i_1}_1 \otimes\cdots\otimes A'^{\otimes i_n}_n$ equals $p^{\alpha(i_1,\dots,i_n)}$ and divides the index of the algebra $B_{i_1\dots i_n}$ (while their degrees coincide). Let us choose some integers $i'_1, \dots, i'_n$ with $0 \leq i'_j \leq \deg A'_j$. By Lemma 1.2, $\operatorname{ind}(B'_{i_1\dots i_n})_{F'(Y'_{i'_1\dots i'_n})}$ divides $\operatorname{ind}(B_{i_1\dots i_n})_{F(Y_{i'_1\dots i'_n})}$, where $Y'_{i'_1\dots i'_n} \overset{\mathrm{def}}{=} \mathrm{SB}(p^{\alpha(i'_1,\dots,i'_n)}, B'_{i'_1\dots i'_n})$. Moreover, the extension $F'(Y'_{i'_1\dots i'_n})/F$ does not in fact affect the index of any F'-algebra, because the variety $Y'_{i'_1\dots i'_n}$ is rational. Therefore, the index of $B'_{i_1\dots i_n}$ itself divides $\operatorname{ind}(B_{i_1\dots i_n})_{F(Y_{i'_1\dots i'_n})}$.

So we see, that if we replace the base field F of the algebras $A_1,\dots,A_n$ by the function field $F(Y_{i'_1\dots i'_n})$, the index of every $B_{i_1\dots i_n}$ is still divisible by $p^{\alpha(i_1,\dots,i_n)}$.

Passing after that to the function field of another variety $Y_{i''_1\dots i''_n}$ and so on, we get in the end the required statement on the indexes. $\square$

2. Definition of β

We fix a prime number p and a behaviour $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}$.

Let us consider a field F and elements $x_1,\dots,x_n \in \mathrm{Br}(F)$ with the behaviour α. Choose division F-algebras representing the elements $x_1,\dots,x_n$ and denote by $X_1,\dots,X_n$ the corresponding Severi-Brauer varieties. Set $X \overset{\text{def}}{=} X_1 \times \dots \times X_n$.

Since $\bar{X}$ is isomorphic to a direct product of projective spaces, $\Gamma_0 K(\bar{X})$ is an infinite cyclic group generated by the class of a rational point. We have $0 \neq \Gamma_0 K(X) \subset \Gamma_0 K(\bar{X}) \simeq \mathbb{Z}$. Therefore, the quotient $\Gamma_0 K(\bar{X})/\Gamma_0 K(X)$ is a finite group.

Definition 2.1. We put $\beta \overset{\text{def}}{=} v_p(|\Gamma_0 K(\bar{X})/\Gamma_0 K(X)|)$.

Lemma 2.2. *The integer β, defined in 2.1, depends only on the prime p and the behaviour α; it does not depend on the choice of the field F and the elements $x_1,\dots,x_n \in \mathrm{Br}(F)$.*

Proof. According to Corollary 2.2 of Chapter 2, the groups $\Gamma_0 K(X)$ and $\Gamma_0 K(\bar{X})$ depend only on p and on the behaviour of $x_1,\dots,x_n$. $\square$

3. The theorem

Theorem 3.1. *For a prime number p and a behaviour $\alpha\colon \mathbb{Z}^n \to \mathbb{Z}$, let β be the integer defined in the previous section. Then*

1. *for any field F, any n elements $x_1,\dots,x_n \in \mathrm{Br}(F)$ with the behaviour α possess a common splitting field E/F satisfying the condition*
$$v_p([E:F]) \le \beta;$$
2. *there exists a field F and elements $x_1,\dots,x_n \in \mathrm{Br}(F)$ with the behaviour α such that any their common splitting field E/F satisfies the condition*
$$v_p([E:F]) \ge \beta\ .$$

We prove the theorem after the following

Lemma 3.2. *Let $A_1,\dots,A_n$ be central simple algebras over a field F and let $X_1,\dots,X_n$ be the corresponding Severi-Brauer varieties. Set $X \overset{\text{def}}{=} X_1 \times \dots \times X_n$ and $\beta' \overset{\text{def}}{=} v_p(|\mathrm{T}_0 K(\bar{X})/\mathrm{T}_0 K(X)|)$. Then*

1. *for any common splitting field E/F of $A_1,\dots,A_n$, it holds*
$$v_p([E:F]) \ge \beta';$$

2. *the algebras* $A_1, \dots, A_n$ *possess a common splitting field* E/F *with*

$$v_p([E:F]) = \beta' .$$

PROOF. For any variety Y, $\mathrm{T}_0K(Y)$ is by definition the subgroup of $K(Y)$ generated by the classes $[y] \in K(Y)$ of the closed points $y \in Y$. If Y is a complete F-variety, the rule $[y] \mapsto \deg(y) \stackrel{\text{def}}{=} [F(y) : F]$, where $F(y)$ is the residue field of y, determines a well-defined homomorphism $\deg\colon T_0K(Y) \to \mathbb{Z}$ (compare to [**14**, Corollary 6.10 of Chapter II]). Note that the composition $\mathrm{T}_0K(Y) \to \mathrm{T}_0K(\bar{Y}) \to \mathbb{Z}$ of the restriction homomorphism with the degree homomorphism for $\bar{Y}$ coincides with the degree homomorphism for Y.

Since $\bar{X}$ is isomorphic to a direct product of projective spaces, the homomorphism $\deg : T_0K(\bar{X}) \to \mathbb{Z}$ is bijective. In particular, since $\mathrm{T}_0K(X)$ is a non-zero subgroup of $\mathrm{T}_0K(\bar{X})$, we see that the quotient $\mathrm{T}_0K(\bar{X})/\mathrm{T}_0K(X)$ is finite.

1. If E is a common splitting field of the algebras $A_1, \dots, A_n$, the variety X_E has a closed rational point. Therefore, there exists a zero-cycle on X of degree $[E : F]$. It follows that the order of the quotient $\mathrm{T}_0K(\bar{X})/\mathrm{T}_0K(X)$ divides $[E : F]$. In particular, $v_p([E : F]) \geq \beta'$.

2. It follows from the definition of β' and the above discussion that there exists a zero-cycle $\sigma = \sum_{i=1}^{r} l_i\sigma_i$ on X (where $l_i \in \mathbb{Z}$ and $\sigma_i \in X$) with $v_p(\deg(\sigma)) = \beta'$. Since

$$\deg(\sigma) \stackrel{\text{def}}{=} \sum_{i=1}^{r} l_i \deg(\sigma_i),$$

one has $v_p(\deg(\sigma_i)) \leq \beta'$ for certain i. Denote by E the residue field of the point σ_i. Since the variety X_E possess a rational point, E is a common splitting field of the algebras $A_1, \dots, A_n$. Therefore, by Item 1, it holds $v_p([E : F]) \geq \beta'$. From the other hand $v_p([E : F]) = v_p(\deg(\sigma_i)) \leq \beta'$. Thus $v_p([E : F]) = \beta'$. □

PROOF OF THEOREM 3.1. 1. Let $x_1, \dots, x_n$ be some elements with the behaviour α in the Brauer group of a field F. Consider the variety X as in Definition 2.1. According to Item 2 of Lemma 3.2, the elements $x_1, \dots, x_n$ possess a common splitting field E/F with $v_p([E : F]) = \beta' \stackrel{\text{def}}{=} v_p(|\mathrm{T}_0K(\bar{X})/\mathrm{T}_0K(X)|)$. From the other hand, $\beta = v_p(\Gamma_0K(\bar{X})/\Gamma_0K(X))$ by Lemma 2.2. Since

$$\mathrm{T}_0K(\bar{X}) = \Gamma_0K(\bar{X}) \;\text{ and }\; \mathrm{T}_0K(X) \supset \Gamma_0K(X)$$

(see [**13**, Theorem 3.9 of Chapter V] for the second relation) the order of the quotient $\mathrm{T}_0K(\bar{X})/\mathrm{T}_0K(X)$ divides the order of the quotient $\Gamma_0K(\bar{X})/\Gamma_0K(X)$. Therefore $\beta' \leq \beta$ and consequently $v_p([E : F]) \leq \beta$.

2. Let $x_1, \dots, x_n$ be the Brauer classes of some "generic" division algebras with the behaviour α (which exist by Proposition 1.1). Let X be the product of the Severi-Brauer varieties of these division algebras. By Item 1 of Lemma 3.2, $v_p([E : F]) \geq \beta'$ for any common splitting field E/F of $x_1, \dots, x_n$. By Theorem 5.5 of Chapter 2, one has $\mathrm{T}_0K(X) = \Gamma_0K(X)$. Therefore $\beta' = \beta$. □

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