ON GENERIC QUADRATIC FORMS IN A POWER OF THE FUNDAMENTAL IDEAL

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Abstract. We work with non-degenerate quadratic forms over fields of characteristic $\neq 2$ and write I for the fundamental ideal in the Witt ring of quadratic forms. For given $n \geq 4$ and $d \geq 2^n + 2^{n-1}$, a construction of a generic d-dimensional quadratic form $q \in I^n$ is not available. To remedy this issue, we introduce a sequence of d-dimensional quadratic forms in I^n, approximating q. It allows us to define and study certain invariants of q including the reduced Chow ring and the indexes of its grassmannians as well as the J-invariant of q. An extension of the results to characteristic 2 is also provided.

Contents

1. Introduction

We work with non-degenerate quadratic forms over fields of characteristic different from 2 and write I for the fundamental ideal in the Witt ring of quadratic forms. For given $n \geq 4$ and $d \geq 2^n + 2^{n-1}$, a construction of a generic d-dimensional quadratic form $q \in I^n$ is not available. To remedy this issue, we introduce in §2 a sequence of d-dimensional quadratic forms in I^n, approximating q. It allows us to define and study certain invariants of q including the *reduced Chow ring* (see §3) and the *indexes* of its *grassmannians* as well as the *J-invariant* of q (see §4).

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In §5, we formulate Conjecture 5.3 on the exact value of the J-invariant of q, derived from some expectations concerning the possible value of the J-invariant on arbitrary quadratic forms. Theorem 6.2 – another of our main highlights – proves this conjecture in some important initial range. One of the steps in the proof is Proposition 6.3, detecting the J-invariant of an anisotropic difference of pure parts of two n-fold Pfister forms.

In §7, we perform a brief study of the index of the highest grassmannian of q. The upper bound on it, provided in Corollary 7.2, generalizes to arbitrary n a classical bound for $n = 3$ available since the seventies.

All results extend to characteristic 2 as explained in §8.

2. Generic sequences

Let F_0 be a field of characteristic $\neq 2$. We work with finite-dimensional non-degenerate quadratic forms over extension fields of F_0. A quadratic form q with some given properties is *generic*, if every quadratic form q' with the same properties is a *specialization* of q meaning that there is a smooth geometrically integral F_0-variety X and a quadratic form Q over X such that q is the generic fiber of Q (and thus a quadratic form over the function field of X) whereas q' is the fiber of Q over a point of X in the base field of q'.

Our interest in generic objects comes from the observation that they are usually easier to study and, at the same time, information on them delivers some information on the arbitrary ones.

Let I be the fundamental ideal in the Witt ring of classes of quadratic forms and let d be a positive even integer. There are two different ways to construct a generic d-dimensional quadratic form q whose class belongs to I^3. One of them, which works "from inside", is based on the fact that the d-dimensional quadratic forms in I^3 over a field $F \supset F_0$ are classified by torsors under the split spin group $\mathrm{Spin}(d)$ over F. Thus a generic q is delivered by a generic $\mathrm{Spin}(d)$-torsor, which can be obtained as the generic fiber of the quotient morphism

$$\mathrm{GL}(N) \to X := \mathrm{GL}(N)/\,\mathrm{Spin}(d)$$

for an embedding of $\mathrm{Spin}(d)$ into a general linear group $\mathrm{GL}(N)$ over F_0, see [14, §5.3]. This generic torsor (and therefore the corresponding generic form q) is defined over the function field $F_0(X)$ of the smooth and geometrically integral quotient variety X.

The second way of getting q works "from outside": we start with a generic d-dimensional quadratic form in I^2, or, more specifically, with the quadratic form

$$q' := \big\langle a_1, \ldots, a_{d-1}, (-1)^{d/2} a_1 \cdots a_{d-1} \big\rangle$$

over the field $F := F_0(a_1, \ldots, a_{d-1})$ of rational functions over F_0 in variables $a_1, \ldots, a_{d-1}$. The form q is then obtained from q' by extending the base field of the latter to the function field $F(\mathrm{SB}(C(q')))$ of the Severi-Brauer variety $X := \mathrm{SB}(C(q'))$ of the Clifford algebra $C(q')$ of q'. So, this second way of getting q is based on the fact that X is a *generic splitting variety* for the Clifford invariant of q'. The value of this invariant is an element in the 2nd Galois cohomology group $H^2(F, \mathbb{Z}/2\mathbb{Z})$. Following [15, Definition 1.8], we say that a smooth F-variety X is a *generic splitting variety* of an element $\alpha \in H^n(F, \mathbb{Z}/2\mathbb{Z})$, provided that for any field extension K/F, $\alpha_K = 0$ if and only if $X(K) \neq \emptyset$.

Now let's fix any $n \geq 4$. Since by [19, Theorem 6.4] and the Arason-Pfister Hauptsatz, for $d < 2^n + 2^{n-1}$, any d-dimensional quadratic form in I^n is Witt-equivalent to a nonzero scalar multiple of an n-fold Pfister form (such a scalar multiple is traditionally called a *general* n-fold Pfister form), it is easy to construct a d-dimensional generic quadratic form in I^n for such d. For $d = 2^n$, for example, one can take $q = a \cdot \langle\langle a_1, \ldots, a_n \rangle\rangle$ over the field of rational functions $F_0(a, a_1, \ldots, a_n)$.

For $d \geq 2^n + 2^{n-1}$ however, a construction of q is not available. Nevertheless, one can produce a sequence of forms in I^n "approximating q". Again, this can be done "from inside" as well as "from outside" of I^n.

The "from inside" way works as follows. One employs that any d-dimensional quadratic form in I^n is Witt-equivalent to an orthogonal sum of several general n-fold Pfister forms. Unlike the case with I^3, we do not know if there exists some N such that N general Pfister forms always suffice. So, instead of just one generic form, we are constructing a sequence of forms, none of which being actually generic, but each next of which becoming "more and more generic" in a sense. Namely, for any $i \geq 1$, we take the orthogonal sum q'_i of, say, $d+i$ generic general n-fold Pfister forms (defined as above, using disjoint sets of variables). To obtain a d-dimensional form out of q'_i, we partially split the latter in a generic way. More precisely, we let the ith form q_i in our final sequence to be a d-dimensional form Witt equivalent to q'_i viewed over the function field of its grassmannian of totally isotropic $((2^n(d+i) - d)/2)$-planes.

The sequence $q_1, q_2, \ldots$ of quadratic forms in I^n, we obtained, has the following property: every d-dimensional quadratic form in I^n is a specialization of the form q_i for some (sufficiently large) i. Moreover, for any i, the form q_i is a specialization of q_{i+1}.

The "from outside" way to get a sequence with the same properties is based on [2, Theorem 0.1], providing a *generic splitting ind-variety* for any element in the Galois cohomology of a field with coefficients in $\mathbb{Z}/2\mathbb{Z}$. More precisely, for any $n \geq 3$, any $F \supset F_0$, and any element $\alpha \in H^n(F, \mathbb{Z}/2\mathbb{Z})$, there is a sequence of smooth geometrically integral closed F-subvarieties $X_1 \subset X_2 \subset \ldots$ such that for any field extension K/F, the element α_K vanishes if and only if $X_i(K) \neq \emptyset$ for some (sufficiently large) i.

For simplicity, let's take $n = 4$. In order to get our sequence $q_1, q_2, \ldots$, we start with a generic d-dimensional quadratic form $q' \in I^3$, consider a splitting ind-variety $X_1 \subset X_2 \subset \ldots$ of the element $\alpha \in H^3(F, \mathbb{Z}/2\mathbb{Z})$ given by the 3d cohomological invariant of q', and we set $q_i := q'_{F(X_i)}$.

This second construction, being very different from the first one, may have advantages for some future applications; but we rely on the first construction in the sequel here.

3. The reduced Chow ring

Let G be the standard split special orthogonal group $\mathrm{SO}(d)$ over the field F_0 and let $P \subset G$ be a standard parabolic subgroup in G. Any d-dimensional quadratic form $q \in I^2$ over a field $F \supset F_0$ yields a G-torsor E_q over F and the F-variety $X_q := E_q/P$. There is a canonical homomorphism of Chow rings $\mathrm{CH}(X_q) \to \mathrm{CH}(X)$ with $X := G/P$, defined, e.g., in [7, §4]. (Since in contrast to [7, §4], the group G here is split, not just quasisplit, the regularity assumption on F/F_0, made in [7, §4], can be omitted.) We define the reduced Chow ring $\bar{\mathrm{CH}}(X_q)$ of X_q as the image of $\mathrm{CH}(X_q)$ in $\mathrm{CH}(X)$; it is isomorphic to the ring

$\mathrm{CH}(X_q)$ modulo its ideal of elements of finite order, but the embedding $\bar{\mathrm{CH}}(X_q) \hookrightarrow \mathrm{CH}(X)$ is also of importance for us. Note that the additive group of $\mathrm{CH}(X)$ is finitely generated; besides, $2^{d/2-1}\,\mathrm{CH}(X) \subset \bar{\mathrm{CH}}(X_q)$ by the transfer argument and because q splits over a finite base field extension of degree dividing this 2-power. It follows that only finitely many subrings in $\mathrm{CH}(X)$ can be of the form $\bar{\mathrm{CH}}(X_q)$ for some q.

By [7, Lemma 4.3], if q' is a specialization of q, then $\bar{\mathrm{CH}}(X_q) \subset \bar{\mathrm{CH}}(X_{q'})$. It follows for a generic sequence $q_1, q_2, \dots$ of d-dimensional quadratic forms in I^n, constructed in §2, that the sequence of the corresponding reduced Chow rings *stabilizes*, i.e.,

$$\bar{\mathrm{CH}}(X_{q_i}) = \bar{\mathrm{CH}}(X_{q_{i+1}}) = \dots$$

for some i. The subring $R \subset \mathrm{CH}(X)$ thus obtained is called the *reduced Chow ring of a generic d-dimensional quadratic form in I^n*. It is characterized by the two following properties:

(1) $R \subset \mathrm{CH}(X_q)$ for any d-dimensional quadratic form $q \in I^n$ over any extension field of F_0;
(2) $R = \mathrm{CH}(X_q)$ for some d-dimensional quadratic form $q \in I^n$ over some extension field of F_0.

Moreover, given the generic sequence $q_1, q_2, \dots$ of d-dimensional quadratic forms in I^n from §2, property (2) holds for $q = q_i$ with some (sufficiently large) i.

4. Invariants derived from the reduced Chow ring

A lot of interesting invariants of a quadratic form q are determined by the reduced Chow ring(s) $\mathrm{CH}(X_q)$. Below we recall definitions for two of them.

The *index* $i(Y)$ of a variety Y is the g.c.d. of degrees of its closed points. The index of X_q is a 2-power which coincides with the index of the subgroup of reduced 0-cycle classes

$$\bar{\mathrm{CH}}_0(X_q) \subset \mathrm{CH}_0(X) = \mathbb{Z}.$$

The value given by the ring R from §3 is the maximum of $i(X_q)$ when q runs over d-dimensional quadratic forms in I^n over extension fields of F_0. In the case of $n = 3$, this maximal value has been extensively studied for arbitrary P in [3], [8], and [6]; for P being the Borel subgroup, the integer $i(X_q)$ coincides with the torsion index of the spin group $\mathrm{Spin}(d)$ and has been computed in [17].

Now let us take a standard parabolic subgroup $P \subset G$ such that $X = G/P$ is a component of the highest grassmannian of the standard split d-dimensional quadratic form over F_0, and let us consider the standard generators $e_1, \dots, e_l$ of the ring $\mathrm{CH}(X)$, defined in [4, §86], where $l := d/2 - 1$. The *J-invariant* of a d-dimensional quadratic form q is originally defined in [20] as the set of those i for which the class of e_i is in the image $\bar{\mathrm{Ch}}(X_q)$ of the composition

$$\mathrm{CH}(X_q) \to \mathrm{CH}(X) \twoheadrightarrow \mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X).$$

(For the sake of convenience, the complement in $\{1, \dots, l\}$ of this set is considered instead in [4].) Note that by [20] (see also [4]), the J-invariant of q determines the entire reduced (modulo 2) Chow ring $\bar{\mathrm{Ch}}(X_q)$.

Example 4.1. Let us fix some $n \geq 2$ and let $q \in I^2$ be a quadratic form of dimension $\geq 2^n$. One has $q \in I^n$ if and only if the J-invariant of q contains all positive integers $< 2^{n-1} - 1$. Indeed, the split form q satisfies the condition on the J-invariant. If $q \in I^n$ is non-split, the *leading form* of q (see [4, §25]) is a general Pfister form of foldness $m \geq n$. The complement of the J-invariant of a general m-fold Pfister form is the singleton $\{2^{m-1} - 1\}$, see, e.g., [4, Example 88.10]. It follows by [4, Corollary 88.6] that $J(q) \supset \{1, \dots, 2^{m-1}-2\} \supset \{1, \dots, 2^{n-1}-2\}$. Conversely, if $q \notin I^n$, then by the *J-filtration conjecture*, proved in [11, Theorem 4.3] (see also [4, Theorem 40.10]), the leading form of q is a general Pfister form of foldness $m < n$ and so $2^{m-1} - 1 \notin J(q)$.

The value of the J-invariant given by R is the smallest (in the sense of inclusions) value of the J-invariant of a d-dimensional quadratic form in I^n over an extension field of F_0. We call it the *J-invariant of a generic d-dimensional quadratic form in I^n* and use the notation J^n_d for it.

5. The J-invariant of a generic quadratic form in I^n

It follows by [4, Corollary 88.6] that $J^n_d = J^n_{d+2} \cap \{1, \dots, l\}$ with, as in §4, $2l + 2 = d$. Therefore J^n_d is just the $\leq l$ part of the union $J^n := \bigcup_d J^n_d$ which we call the *J-invariant of (an infinite-dimensional) generic quadratic form in I^n*. Note that the inclusion $I^n \supset I^{n+1}$ yields the inclusion $J^n \subset J^{n+1}$.

It is known (see, e.g., [17]) that the set J^3 consists of the 2-powers:

$$J^3 = \{1, 2, 4, 8, \dots\}.$$

It would be interesting to determine J^n for $n \geq 4$.

The set J^n satisfies the restriction given by the action of the modulo 2 (reduced) Steenrod algebra on $\mathrm{Ch}(X)$. Namely, by [20, Proposition 5.12] (see also [4, 89.1]), if $i \in J^n$ and the binomial coefficient $\binom{i}{j}$ is odd for some $j \geq 1$, then $i + j \in J^n$ as well. In particular, according to Example 4.1, J^n contains the set S^n *Steenrod-generated* by all $2^m - 1$ with $m \leq n - 2$.

The set S^n has the following nice description:

Lemma 5.1. *The set S^n of integers Steenrod-generated by all $2^m - 1$ with $1 \leq m \leq n - 2$ consists of the positive integers with at most $n - 2$ units in their 2-expansion.*

Proof. By [4, Lemma 78.6], a binomial coefficient $\binom{i}{j}$ is odd if and only if the set of positions of units in the 2-expansion of j is contained in the similar set for i. In particular, the inclusion $2^m - 1 \in S^n$ yields $2^m, \dots, 2^{m+1} - 2 \in S^n$ and so all integers $< 2^{n-1} - 1$ are in S^n.

Let T^n be the set of positive integers with at most $n - 2$ units in the 2-expansion. If $\binom{i}{j}$ is odd, then the number of units in the 2-expansion of $i + j$ is at most the similar number for i. It follows that $S^n \subset T^n$.

Conversely, for $j \in T^n$, we prove that $j \in S^n$ by induction on the number of all digits in the 2-expansion of j. If this number is at most $n - 2$, then $j < 2^{n-1} - 1$ and so $j \in S^n$. If this number is at least $n - 1$, the 2-expansion of i contains at least one zero. Erasing the most left zero, we get an integer $i < j$ which is in S^n by the induction hypothesis. Since the binomial coefficient $\binom{i}{j-i}$ is odd, we conclude that $j = (j - i) + i \in S^n$. □

Remark 5.2. Let q be a quadratic form in I^n. For $i \in J(q)$ with $i < 2^n + 2^{n-2} - 2$, one can show by induction that $e_i \in \bar{\mathrm{CH}}^i(X_q)$. Indeed, a priori $e_i \in \bar{\mathrm{CH}}^i(X_q) + 2\,\mathrm{CH}^i(X)$. But since the two smallest positive integers outside of S^n are $2^{n-1} - 1$ and $2^{n-1} + 2^{n-2} - 1$, any product $f \in \mathrm{CH}^i(X)$ of distinct generators $e_1, e_2, \dots$ contains at most one e_j with $j \notin S^n$. Since $2e_j \in \bar{\mathrm{CH}}(X_q)$, it follows by the induction hypothesis that $2f \in \bar{\mathrm{CH}}(X_q)$.

The positive answer to [20, Question 5.13] implies[1]

Conjecture 5.3. *The set J^n coincides with the set S^n Steenrod-generated by all $2^m - 1$ with $m \le n - 2$.*

Conjecture 5.3 holds for $n = 2$: $J^2 = \emptyset = S^2$, the first equality follows, e.g., from [17, Theorem 3.2]. Since S^3 is the set of 2-powers, Conjecture 5.3 holds for $n = 3$ as well. It is however open for every $n \ge 4$.

Remark 5.4. A priori, the set J^n depends on the initial field F_0. If $F_0 \subset K_0$, then J^n based on F_0 is contained in J^n based on K_0. One can show that J^n is the same for all F_0 containing a given algebraically closed field; in particular, it suffices to prove Conjecture 5.3 for algebraic closures of prime fields. This follows from the observation made in [18, Page 211] that, as a particular case of [16, Corollary 2.3.3], for any smooth scheme X over an algebraically closed field F and any algebraically closed field $K \supset F$, Chow groups of X with finite coefficients map isomorphically onto the Chow groups of X_K.

6. Conjecture 5.3 in the range $< 2^n - 1$

In the range $< 2^n - 1$, Conjecture 5.3 translates as

$$J^n \not\ni 2^n - 2^i - 1 \text{ for } i = n-1, \dots, 1, 0. \tag{6.1}$$

Indeed, all positive integers in this range have at most $n - 1$ units in their 2-expansion and the integers of (6.1) are exactly the ones with $n - 1$ units.

It follows from [5, Theorem 4] that $J^n \not\ni 2^n - 2^i - 1$ for the two initial values $i = n-1, n-2$ of i in (6.1).

Theorem 6.2. *For any n, Conjecture 5.3 holds in the range $< 2^n - 1$.*

Note that through the inclusions $J^n \subset J^{n+1} \subset \dots$, Theorem 6.2 excludes more values from J^n than just the values listed in (6.1).

Proof of Theorem 6.2. By [4, Lemma 82.6], the difference

$$\langle\langle a_1, \dots, a_n \rangle\rangle' \perp - \langle\langle b_1, \dots, b_n \rangle\rangle'$$

of pure parts of two generic n-fold Pfister forms over the field of rational functions

$$F_0(a_1, \dots, a_n, b_1, \dots, b_n)$$

is anisotropic. Therefore, Theorem 6.2 follows from Proposition 6.3 below, computing the J-invariant of an arbitrary anisotropic difference of pure parts of two n-fold Pfister forms. □

[1] An example with the negative answer for a generalization of [20, Question 5.13] has been obtained in [13, §6]. However, no example with the negative answer for the original question is available so far.

Proposition 6.3. *Let q be the difference of pure parts of two n-fold Pfister forms over some field F. If q is anisotropic, then the complement of its J-invariant consists of the integers $2^n - 2^i - 1$, $i = n-1, \dots, 1, 0$.*

Proof. We only need to show that $J(q) \not\ni 2^n - 2^i - 1$ for $i \in \{0, 1, \dots, n-1\}$. Applying the Steenrod operation of the appropriate degree to the element $e_{2^n-2^i-1} \in \mathrm{CH}(X)$, we get the element $e_{2^n-2} \in \mathrm{CH}(X)$. Therefore it is enough to prove that $J(q) \not\ni 2^n - 2$.

Assume the contrary: $J(q) \ni 2^n - 2$. Let Y_q be the projective quadric of q. By Lemma 6.4 below, the subring $\bar{\mathrm{CH}}(X_q)_{F(Y_q)} \subset \mathrm{CH}(X)$ is generated by $\bar{\mathrm{CH}}(X_q) \subset \mathrm{CH}(X)$ and $e_{2^n-2} \in \mathrm{CH}(X)$. Since $2^n - 2 \in J(q)$, we have $e_{2^n-2} \in \bar{\mathrm{CH}}(X_q)$ (see Remark 5.2) and therefore

$$\bar{\mathrm{CH}}(X_q)_{F(Y_q)} = \bar{\mathrm{CH}}(X_q).$$

In particular, the varieties $(X_q)_{F(Y_q)}$ and X_q have the same index. The two n-fold Pfister forms giving q, considered over the field $F(Y_q)$, are 1-*linked* (see [4, §24]) and therefore $i(X_q)_{F(Y_q)} = 2$. So, $i(X_q) = 2$ implying that there is an odd degree field extension K/F such that q_K splits over a quadratic field extension of K. On the other hand, the form q_K is still anisotropic (see [4, Corollary 18.5]), of dimension congruent to 2 modulo 4, and of trivial discriminant; therefore, by [4, Corollary 22.12], it does not split over a quadratic extension. □

The following lemma is an enhanced version of [4, Corollary 88.6]:

Lemma 6.4. *Let q be a quadratic form in I^2 of dimension $d = 2l + 2 \geq 4$ over some field F. Let X_q be a component of the highest grassmannian of q and let Y_q be the projective quadric of q. Then the $\bar{\mathrm{CH}}(X_q)$-algebra $\bar{\mathrm{CH}}(X_q)_{F(Y_q)}$ is generated by e_l.*

Proof. We apply [21, Statement 2.13] to the projection $X_q \times Y_q \to X_q$ and $B \subset \mathrm{CH}(X_q \times Y_q)$ defined as follows. The variety $(Y_q)_{F(X_q)}$ is a split quadric so that the ring $\mathrm{CH}(Y_q)_{F(X_q)}$ is generated by the codimension 1 class of a hyperplane section and a codimension l class of a linear subspace ([4, Proposition 68.1]). We let B be the subring generated by certain lifts of these two generators with respect to the surjective pull-back ring homomorphism

$$\mathrm{CH}(X_q \times Y_q) \to \mathrm{CH}(Y_q)_{F(X_q)}$$

obtained from the morphism of schemes $(Y_q)_{F(X_q)} \to X_q \times Y_q$ given by the generic point of X_q. Namely, we take an arbitrary homogeneous lift of the hyperplane section and we take the Chow class ε of the *incident subvariety* in $X_q \times Y_q$ (c.f. [4, §86]) for the second lift.

As the result of the application of [21, Statement 2.13], we obtain that the $\mathrm{CH}(X_q)$-algebra $\mathrm{CH}(X_q \times Y_q)$ is generated by an element of codimension 1 and the element ε. The pull-back with respect to the morphism $(X_q)_{F(Y_q)} \to X_q \times Y_q$, given by the generic point of Y_q, yields a surjective homomorphism of $\mathrm{CH}(X_q)$-algebras

$$\mathrm{CH}(X_q \times Y_q) \to \mathrm{CH}(X_q)_{F(Y_q)}$$

showing that the $\mathrm{CH}(X_q)$-algebra $\mathrm{CH}(X_q)_{F(Y_q)}$ is generated by certain element of codimension 1 and the image of ε. By [4, Corollary 88.6], the change of field homomorphism $\mathrm{CH}^1(X_q) \to \mathrm{CH}^1(X_q)_{F(Y_q)}$ is surjective for $l \geq 2$ so that the codimension 1 generator can be skipped. Passing to the reduced Chow rings, we get the desired statement because the image of ε in $\mathrm{CH}(X)$ is equal to e_l. □

7. Bounding the index of a generic quadratic form in I^n

Let X_q be a component of the highest grassmannian of a d-dimensional generic quadratic form $q \in I^n$. Like in [4, Proposition 88.11], the index $i(X_q)$ has an upper bound in terms of the complement $\bar{J}^n_d$ of $J^n_d \subset \{1, \dots, d/2-1\}$:

Proposition 7.1. *The index $i(X_q)$ divides $2^{|\bar{J}^n_d|}$.*

Proof. By definition of the J-invariant, for every $i \in J^n_d$, we have $e_i + 2x_i \in \mathrm{CH}^i(X_q)$ for some $x_i \in \mathrm{CH}^i(X)$. By [4, Proposition 86.13], for every $j \in \{1, \dots, d/2-1\}$, we have $2e_j \in \mathrm{CH}^j(X_q)$. The product

$$\prod_{i \in J^n_d,\, j \in \bar{J}^n_d} (e_i + 2x_i)2e_j$$

is a reduced 0-cycle class on X_q of degree congruent to $2^{|\bar{J}^n_d|}$ modulo $2^{|\bar{J}^n_d|+1}$. The product $\prod_{j \in \{1,\dots,d/2-1\}} 2e_j$ is a reduced 0-cycle class on X_q of degree a 2-power. It follows that $i(X_q)$ is a 2-power dividing $2^{|\bar{J}^n_d|}$. □

Since the J-invariant J^n_d contains the $< d/2$ part S^n_d of the set S^n from Lemma 5.1, we get a (weaker but computable) upper bound in terms of the complement $\bar{S}^n_d \supset \bar{J}^n_d$ of $S^n_d \subset \{1, \dots, d/2-1\}$:

Corollary 7.2. $\log_2 i(X_q) \le |\bar{S}^n_d|$. □

Remark 7.3. For $n = 3$, the bound of Proposition 7.1 coincides with the bound of Corollary 7.2 and is the bound of [17, Lemma 3.4], originally obtained in [10].

8. Characteristic 2

All above results extend to the characteristic 2. We briefly discuss the particularities showing up in this extension.

In [4], all the basics for the characteristic 2 case are provided with the exception of the Steenrod operations on the modulo 2 Chow groups, for which our reference is [12].

Instead of the Witt ring of quadratic forms we referred to in characteristic $\neq 2$, in arbitrary characteristic we only have the additive group I of *even-dimensional* non-degenerate quadratic forms, called the *quadratic Witt group*, which is a module over the Witt ring of non-degenerate symmetric bilinear forms. For $n \ge 2$, the subgroup $I^n \subset I$ is defined as the product of I by the $(n-1)$st power of the fundamental ideal of the Witt ring, see [4, §8.B].

The "from inside" way of constructing a generic $q \in I^3$, described in §2, works in characteristic 2 as well. We can skip discussion of the "from outside" way, which requires a modified construction of a generic quadratic form in I^2.

With the correct definition of (general) quadratic n-fold Pfister forms, as in [4, §9.B], their classes still generate I^n. As a result, the "from inside" way of constructing a sequence of forms approximating a generic form in I^n for $n \ge 4$, described in §2, is also valid in arbitrary characteristic. The same is true for the observations on the reduced Chow ring, indexes of grassmannians, and the J-invariant made in §3 and §4. Note that the proof of [4, Theorem 40.10], used in Example 4.1, is based in characteristic 2 on [9] instead of [22] and [11].

After the explanations already made, §6 and §7 go through without any further change everywhere with only one exception: the end of the proof of Proposition 6.3, where we use the fact that an anisotropic quadratic form of dimension congruent to 2 modulo 4 and of trivial discriminant does not split over a quadratic field extension. In characteristic 2, this fact still holds. However the reference [4, Corollary 22.12] used for characteristic $\neq 2$ case covers the case of a separable quadratic field extension only. The reference for the inseparable one is [1].

References

[1] Ahmad, H. On quadratic forms over inseparable quadratic extensions. *Arch. Math. (Basel) 63*, 1 (1994), 23–29.

[2] Demarche, C., and Florence, M. Splitting families in Galois cohomology. *Ann. Sci. Éc. Norm. Supér. (4) 54*, 3 (2021), 779–792.

[3] Devyatov, R. A., Karpenko, N. A., and Merkurjev, A. S. Maximal indexes of flag varieties for spin groups. *Forum Math. Sigma 9* (2021), Paper No. e34, 12.

[4] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[5] Harvey, C. R., and Karpenko, N. A. Quadratic forms in I^n of dimension $2^n + 2^{n-1}$. To appear in *C. R. Math. Acad. Sci. Paris.*

[6] Karpenko, N. A. On special Clifford groups and their characteristic classes. Preprint (19 Jul 2023, 18 pages). Available on author's webpage. To appear in *Ric. Mat.*

[7] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.

[8] Karpenko, N. A., and Merkurjev, A. S. Indexes of generic Grassmannians for spin groups. *Proc. Lond. Math. Soc. (3) 125*, 4 (2022), 825–840.

[9] Kato, K. Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two. *Invent. Math. 66*, 3 (1982), 493–510.

[10] Marlin, R. Anneaux de Chow des groupes algébriques SU(n), Sp(n), SO(n), Spin(n), G_2, F_4; torsion. *C. R. Acad. Sci. Paris Sér. A 279* (1974), 119–122.

[11] Orlov, D., Vishik, A., and Voevodsky, V. An exact sequence for $K_*^M/2$ with applications to quadratic forms. *Ann. of Math. (2) 165*, 1 (2007), 1–13.

[12] Primozic, E. Motivic Steenrod operations in characteristic p. *Forum Math. Sigma 8* (2020), Paper No. e52, 25.

[13] Quéguiner-Mathieu, A., Semenov, N., and Zainoulline, K. The J-invariant, Tits algebras and triality. *J. Pure Appl. Algebra 216*, 12 (2012), 2614–2628.

[14] Serre, J.-P. Cohomological invariants, Witt invariants, and trace forms. In *Cohomological invariants in Galois cohomology*, vol. 28 of *Univ. Lecture Ser.* Amer. Math. Soc., Providence, RI, 2003, pp. 1–100. Notes by Skip Garibaldi.

[15] Suslin, A., and Joukhovitski, S. Norm varieties. *J. Pure Appl. Algebra 206*, 1-2 (2006), 245–276.

[16] Suslin, A. A. Algebraic K-theory of fields. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)* (1987), Amer. Math. Soc., Providence, RI, pp. 222–244.

[17] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.

[18] Totaro, B. *Group cohomology and algebraic cycles*, vol. 204 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2014.

[19] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.

[20] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).
[21] Vishik, A. Fields of u-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.
[22] Voevodsky, V. Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.*, 98 (2003), 59–104.

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