ON GENERIC QUADRATIC FORMS

NIKITA A. KARPENKO

Abstract. Based on Totaro's computation of the Chow ring of classifying spaces for orthogonal groups, we compute the Chow rings of all orthogonal Grassmannians associated with a generic quadratic form of any dimension. This closes the gap between the known particular cases of the quadric and the highest orthogonal Grassmannian. We also relate two different notions of generic quadratic forms.

Contents

1. Introduction

Let k be a field of characteristic different from 2 and let $F_g = k(t_1, \dots, t_n)$ be the field of rational functions over k in variables $t_1, \dots, t_n$ for some $n \geq 2$. We call *generic* the diagonal quadratic form $q_g := \langle t_1, \dots, t_n \rangle$ over F_g. Thus q_g is the n-dimensional quadratic form $F_g^n \to F_g$ on the vector space F_g^n given by the formula

$$q_g \colon (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq n} t_i x_i^2.$$

The Chow ring of the projective quadric defined by q_g has been computed in [8, Corollary 2.2]. The Chow ring of the highest orthogonal Grassmannian of a generic quadratic form has been computed in [18] (see also [20]), but this was done for a different notion of generic, which we call here *standard generic*. As shown in §3, the n-dimensional standard

---

Date: 10 June 2017. Revised: 29 March 2018.

Key words and phrases. Algebraic groups; projective homogeneous varieties; Chow groups. *Mathematical Subject Classification (2010):* 20G15; 14C25.

This work has been supported by a Discovery Grant from the National Science and Engineering Board of Canada.

generic quadratic form q lives over the field of rational functions $F = k(t_{ij})_{1\le i\le j\le n}$ in $n(n+1)/2$ variables t_{ij} and can be defined (in arbitrary characteristic including characteristic 2) by the formula

$$F^n \to F, \ \ (x_1,\dots,x_n) \mapsto \sum_{1\le i\le j\le n} t_{ij}x_ix_j.$$

In the present paper we determine the Chow ring $\mathrm{CH}\,X$ of all orthogonal Grassmannians X associated with the generic and the standard generic quadratic forms. (The characteristic $\neq 2$ assumption is removed in the latter case; the characteristic 2 analog for the first case is provided in §9.) Namely, our Main Theorem (Theorem 6.1, see also Corollary 8.2 and Proposition 9.2) affirms that the ring $\mathrm{CH}\,X$ is generated by the Chern classes of the tautological vector bundle of X. A complete list of relations satisfied by these Chern classes (in general, not only in the generic situation) is provided in Theorem 2.1. All the (well-known) relations that hold over an algebraic closure of the base field actually already hold over the base field itself. This way we obtain a description of the ring $\mathrm{CH}\,X$ in terms of generators and relations. It also follows that the additive group of $\mathrm{CH}\,X$ is torsion-free (see Corollary 6.2).

Proving Main Theorem, we use computation of the Chow ring of classifying spaces for orthogonal groups $O(n)$ performed in [17] as well as in [21] over the field of complex numbers and later in [16] over an arbitrary field of characteristic not 2. We actually need only a piece of this computation which is made in [21] over arbitrary field (of arbitrary characteristic), see Section 5.

Note that the algebraic group $O(n)$ over a field k is not connected if n is even or $\mathrm{char}\,k \neq 2$. In the remaining case (when n is odd and $\mathrm{char}\,k = 2$) the algebraic group $O(n)$ is not smooth. In contrast, the special orthogonal group $O^+(n)$ is always smooth and connected. But since $O(n)$-torsors correspond to all non-degenerate n-dimensional quadratic forms while $O^+(n)$-torsors correspond to quadratic forms of trivial discriminant, it is more appropriate to work with $O(n)$ for the question raised in this paper. On the other hand, since orthogonal Grassmannians depend only on the similarity class of the quadratic form in question and any odd-dimensional quadratic form is similar to that of trivial discriminant, $O(n)$ can be replaced by $O^+(n)$ for odd n.

2. Tautological Chern subring

In this section we consider an arbitrary non-degenerate quadratic form $q : V \to F$ of an arbitrary dimension $n \ge 2$ over an arbitrary field F. (Characteristic 2 is not excluded; non-degenerate quadratic forms are defined as in [5, §7.A]. We require $n \ge 2$ everywhere in the paper because the varieties we are interested in, introduced below, do not occur for $n = 1$.) In particular, V is an n-dimensional F-vector space. We fix an integer $1 \le m \le n/2$ and write X for the orthogonal Grassmannian of isotropic m-planes (i.e., totally isotropic m-dimensional subspaces) in V. Note that the variety X is smooth projective; it is geometrically connected if and only if $m \neq n/2$.

Let $\mathcal{T} = \mathcal{T}_X$ be the tautological (rank-m) vector bundle on X: the fiber of $\mathcal{T}$ over a point of X, given by an isotropic m-plane, is this very m-plane itself. We define the *tautological Chern subring* $\mathrm{CT}\,X$ in the Chow ring $\mathrm{CH}\,X$ as the subring generated by the

Chern classes $c_1(\mathcal{T}),\dots,c_m(\mathcal{T})$. The goal of this section is to determine the ring $\operatorname{CT} X$ by providing a list of defining relations on its generators.

The variety X is a closed subvariety of the usual Grassmannian Γ of all m-planes in V. The Chow ring $\operatorname{CH}\Gamma$ is known to be generated by the Chern classes of the tautological (rank-m) vector bundle on Γ. Therefore the pull-back $\operatorname{CH}\Gamma \to \operatorname{CH} X$ with respect to the closed imbedding $X \hookrightarrow \Gamma$ provides an epimorphism $\operatorname{CH}\Gamma \twoheadrightarrow \operatorname{CT} X$. Since a description of the ring $\operatorname{CH}\Gamma$ by generators and relations is available (see [2, Lemma 1.2] or [6, Example 14.6.6]), we fulfill our goal if we describe the kernel of the epimorphism $\operatorname{CH}\Gamma \twoheadrightarrow \operatorname{CT} X$ in terms of generators of $\operatorname{CH}\Gamma$. For this, it is more convenient to use the generators $c_1,\dots,c_{n-m} \in \operatorname{CH}\Gamma$ given by the Chern classes of $-[\mathcal{T}]$ rather than of $\mathcal{T} = \mathcal{T}_\Gamma$ itself. By $[\mathcal{T}]$ here we mean the class of $\mathcal{T}$ in the Grothendieck ring $K(\Gamma)$. The Chern classes of $-[\mathcal{T}]$ are the Segre classes of $\mathcal{T}$, i.e. the components of the multiplicative inverse to the total Chern class $c(\mathcal{T})$. The tautological vector bundle $\mathcal{T}$ is a subbundle of the trivial (rank-n) vector bundle V and $c_1,\dots,c_{n-m}$ are the Chern classes of the quotient $V/\mathcal{T}$. We define $c_i \in \operatorname{CH}^i(\Gamma)$ for every integer i by setting $c_i := c_i(-[\mathcal{T}]) = c_i(V/\mathcal{T})$. Therefore $c_0 = 1$ and $c_i = 0$ for $i < 0$ as well as for $i > n-m$.

Theorem 2.1. *The kernel of the epimorphism* $\operatorname{CH}\Gamma \twoheadrightarrow \operatorname{CT} X$ *is generated by the elements*

$$c_i^2 - 2c_{i-1}c_{i+1} + 2c_{i-2}c_{i+2} - \dots + (-1)^i 2c_0c_{2i} \quad \text{with } i > n/2 - m \tag{2.2}$$

and c_{n-m}. *The abelian group* $\operatorname{CT} X$ *is free with a basis consisting of the images of the products* $c_1^{\alpha_1}\dots c_{n-m-1}^{\alpha_{n-m-1}}$ *with* $\alpha_1 + \dots + \alpha_{n-m-1} \le m$ *and* $\alpha_i \le 1$ *for* $i > n/2 - m$.

Proof. Let us first check that the elements (2.2) lie in the kernel. The ith element is mapped to the Chern class $c_{2i}(-[\mathcal{T}] - [\mathcal{T}^\vee]) \in \operatorname{CT} X$, where $\mathcal{T} = \mathcal{T}_X$ and $\mathcal{T}^\vee$ is the dual vector bundle. The isomorphism $V/\mathcal{T}^\perp = \mathcal{T}^\vee$, where $\mathcal{T}^\perp$ is the vector bundle given by the orthogonal complement, shows that $-[\mathcal{T}] - [\mathcal{T}^\vee] = -[\mathcal{T}] + [\mathcal{T}^\perp] = [\mathcal{T}^\perp/\mathcal{T}]$. Since the rank of the quotient $\mathcal{T}^\perp/\mathcal{T}$ is $n-2m$ (cf. [5, Proposition 1.5]), its Chern classes vanish in degrees $> n-2m$.

In order to show that c_{n-m} is in the kernel, we proceed similarly to [24, Proof of Proposition 2.1]. One notice that the projective bundle $\mathbb{P}(\mathcal{T})$ over X can be identified with the variety of flags of totally isotropic subspaces in V of dimensions 1 and m. In particular, besides of the projection $\pi : \mathbb{P}(\mathcal{T}) \to X$, we have a projection $\pi_1 : \mathbb{P}(\mathcal{T}) \to X_1$ to the projective quadric X_1 (the orthogonal Grassmannian of 1-planes). Moreover, the tautological line bundle on the projective bundle $\mathbb{P}(\mathcal{T})$ is the pull-back $\pi_1^*(\mathcal{T}_1)$ of the tautological line bundle $\mathcal{T}_1$ on X_1. It follows by [5, §58] or [6, Chapter 3] that $c_i(-[\mathcal{T}]) = \pi_*(\pi_1)^* c_{i+m-1}(-[\mathcal{T}_1])$ for any i. Since $\dim X_1 = n-2$, the Chern class $c_{n-1}(-[\mathcal{T}_1])$ vanishes implying the vanishing of $c_{n-m}(-[\mathcal{T}])$.

In order to show that the kernel is generated by the elements (2.2) and c_{n-m}, we construct additive generators of the quotient C of the ring $\operatorname{CH}\Gamma$ by the ideal generated by the elements (2.2) and c_{n-m}. We recall that the group $\operatorname{CH}\Gamma$ is free, a basis is given by the products $c_1^{\alpha_1}\dots c_{n-m}^{\alpha_{n-m}}$ with $\alpha_1 + \dots + \alpha_{n-m} \le m$. Using the additional relations in C, we can eliminate squares of c_i for $i > n/2 - m$. Indeed, in the quotient of C by the subgroup generated by the products satisfying the additional condition, any element is divisible by an arbitrary 2-power and therefore is 0 since C is finitely generated.

It follows that the group C is generated by the products $c_1^{\alpha_1}\dots c_{n-m-1}^{\alpha_{n-m-1}}$ satisfying the additional condition $\alpha_i \le 1$ for $i > n/2 - m$. It turns out that these are free generators. Moreover, they remain free when we map them to $\mathrm{CT}\, X$ and this finishes the proof of the theorem.

Our products are free in $\mathrm{CT}\, X$ because their images in the $\mathbb{Q}$-vector space $\mathbb{Q}\otimes \mathrm{CH}\,\bar{X}$ are free, where $\bar{X}$ is X over an algebraic closure of F. For odd n this follows from [2, Theorem 2.2(b) and formula (15)] (see Remark 2.3). For even n this follows from [2, Theorem 3.2(b) and formula (40)]. $\square$

Remark 2.3. The paper [2], applied in the above proof, actually deals with the singular cohomology ring instead of the Chow ring. The link is explained by the following two well-known facts: the variety $\bar{X}$ is cellular and the ring $\mathrm{CH}\,\bar{X}$ does not depend on the base field. If the base field is $\mathbb{C}$, then the cycle map from $\mathrm{CH}\,\bar{X}$ to the corresponding singular cohomology ring is an isomorphism, [6, Example 19.1.11(b)].

Remark 2.4. In the case of the highest orthogonal Grassmannian, the ring $\mathrm{CH}\,\bar{X}$ has been described in [23] (see also [5, Proposition 86.16 and Theorem 86.12]).

Remark 2.5. Theorem 2.1 shows that the ring $\mathrm{CT}\, X$ only depends on the integers n and m.

Remark 2.6. For odd n, the ring $\mathrm{CT}\, X$ can be identified with the full Chow ring $\mathrm{CH}\, Y$ of the variety of isotropic m-planes in an $n-1$-dimensional vector space endowed with a non-degenerate alternating bilinear form: there is an isomorphism $\mathrm{CH}\, Y \to \mathrm{CT}\, X$ mapping the Segre classes of the tautological vector bundle on Y to the Segre classes of $\mathcal{T}_X$. (See [2, Theorem 1.2] for a description of the ring $\mathrm{CH}\, Y$ by generators and relations.) This funny observation in the case of the highest orthogonal Grassmannian turned out to be very useful in [22]. We do not use it here.

Our next and ultimate goal is to show that $\mathrm{CT}\, X = \mathrm{CH}\, X$ in the case of generic q. First we need clearness in what is generic. We start with the notion of

3. The standard generic quadratic form

For a field k (of any characteristic) and an integer $n \ge 2$, the standard generic n-dimensional quadratic form is defined as follows.

We consider the orthogonal group $O(n)$ over k and its tautological imbedding into the general linear group $\mathrm{GL}(n)$. The generic fiber of the quotient map

$$\mathrm{GL}(n) \to \mathrm{GL}(n)/O(n)$$

is an $O(n)$-torsor over the function field $F := k(\mathrm{GL}(n)/O(n))$. It determines an n-dimensional quadratic form over F (via the identification of [3, Chapitre III, §5, 2.1]; for the case of smooth $O(n)$ see also [15, (29.28)]) which we call the *standard generic* one.

In order to describe it explicitly, we first recall the interpretation of the quotient variety $\mathrm{GL}(n)/O(n)$ as the variety Q of non-degenerate quadratic forms on the vector space $V := k^n$.[1] The variety of all quadratic forms on V is an affine space (of dimension $n(n+1)/2$) and Q is its open subvariety. The group $\mathrm{GL}(n)$ acts on Q in the evident

[1]This interpretation is mentioned in [21, §15]; however, due to the context, $k = \mathbb{C}$ there.

way. The action is such that for any algebraically closed field $K \supset k$, the abstract group $\mathrm{GL}(n)(K)$ of K-points of $\mathrm{GL}(n)$ acts transitively on the set $Q(K)$ of K-points of Q. Finally, the algebraic group $O(n)$, by its very definition, is the stabilizer of the split quadratic form $q_0 \in Q(k)$, defined by the formulas (7.1) and (7.2).[2] It follows by [3, Proposition 2.1 of Chapter III §3] that Q is the quotient variety $\mathrm{GL}(n)/O(n)$.

For any field extension L/k, an L-point of Q is a non-degenerate quadratic form q on the L-vector space V_L; the fiber of the quotient map $\mathrm{GL}(n) \to Q$ over this point is an $O(n)$-torsor E over L, and q is the quadratic form corresponding to E. In particular, the quadratic form given by the generic fiber of $\mathrm{GL}(n) \to Q$ is defined over the field of rational functions $F = k(t_{ij})_{1\leq i\leq j\leq n}$ (t_{ij} are indeterminates, F/k is purely transcendental of the transcendence degree $n(n+1)/2$) by the formula

$$(x_1, \dots, x_n) \mapsto \sum_{1\leq i\leq j\leq n} t_{ij} x_i x_j.$$

4. Chow rings of classifying spaces

Let F be a field (of arbitrary characteristic) and let G be an affine algebraic group over F, not necessarily smooth. The Chow ring CH_G of the classifying space of G, introduced in [21], is the G-equivariant Chow ring $\mathrm{CH}_G(\operatorname{Spec} F)$. This is a graded ring, the grading is given by codimension of cycles.

The ring CH_G if cofunctorial in G: a homomorphism $G' \to G$ of affine algebraic groups produces a homomorphism of graded rings $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ (see [16, §2]).

By [4, Lemma 4] (see also [10, §3]), if G is a split torus, the homomorphism of graded rings $S(\hat{G}) \to \mathrm{CH}_G$ is an isomorphism, where $\hat{G}$ is the character lattice of G, $S(\hat{G})$ is the symmetric $\mathbb{Z}$-algebra, and a character $\chi \in \hat{G} = S^1(\hat{G})$, viewed as a G-equivariant line bundle over $\operatorname{Spec} F$, is mapped to its first equivariant Chern class in CH^1_G.

We return to the situation where G is an arbitrary affine algebraic group over F:

Proposition 4.1. *Let G' be a closed normal subgroup of G such that the quotient $T := G/G'$ is a split torus. Then the restriction homomorphism $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ is surjective and its kernel is generated by some elements in CH^1_G. More precisely, the kernel is generated by the image of the (additive) homomorphism*

$$\hat{T} = S^1(\hat{T}) = \mathrm{CH}^1_T \to \mathrm{CH}^1_G$$

induced by the quotient homomorphism $G \to T$.

Proof. For any integer i, let us consider a generically free G-representation V possessing an open G-equivariant subset $U \subset V$ such that $\operatorname{codim}_V(V \setminus U) \geq i$ and there are a G-torsor $U \to U/G$ and a G'-torsor $U \to U/G'$. By definition of CH_G (and similarly for G' in place of G), we have a ring homomorphism $\mathrm{CH}_G \to \mathrm{CH}(U/G)$ which is bijective in codimensions $< i$. Moreover, the diagram

$$\begin{array}{ccc} \mathrm{CH}_G & \longrightarrow & \mathrm{CH}_{G'} \\ \downarrow & & \downarrow \\ \mathrm{CH}(U/G) & \longrightarrow & \mathrm{CH}(U/G') \end{array}$$

[2] A quadratic form over k is called *split* if it is isomorphic to q_0.

commutes, where the bottom map is the pull-back homomorphism with respect to the T-torsor $U/G' \to U/G$. Therefore, in order to prove surjectivity of $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ is suffices to prove surjectivity of $\mathrm{CH}(U/G) \to \mathrm{CH}(U/G')$. Moreover, to get the description of the kernel for $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ it suffices to prove the similar description for the kernel of $\mathrm{CH}(U/G) \to \mathrm{CH}(U/G')$, where the homomorphism $\hat{T} \to \mathrm{CH}^1(U/G)$ is the composition $\hat{T} \to \mathrm{CH}^1_G \to \mathrm{CH}^1(U/G)$.

Let us first consider the case of $T = \mathbb{G}_\mathrm{m}$. Let $\mathcal{L}$ be the line bundle $((U/G') \times \mathbb{A}^1)/T$ over U/G. Then U/G' is an open subvariety in $\mathcal{L}$ and its complement is the zero section. By the homotopy invariance and the localization property of Chow groups ([5, Theorem 57.13 and Proposition 57.9]) we have an exact sequence

$$\mathrm{CH}(U/G) \to \mathrm{CH}(U/G) \to \mathrm{CH}(U/G') \to 0,$$

where the first map is the multiplication by the first Chern class of $\mathcal{L}$. This finishes the proof for $T = \mathbb{G}_\mathrm{m}$.

In the general case, we induct on the rank of T. We decompose T as $\mathbb{G}_\mathrm{m} \times T_1$ and define an intermediate subgroup G_1 with $G' \subset G_1 \subset G$ as the kernel of the composition $G \to T \to T_1$. The quotient G/G_1 is then T_1 and the quotient G_1/G' is $\mathbb{G}_\mathrm{m}$. The homomorphism $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ decomposes in the composition $\mathrm{CH}_G \to \mathrm{CH}_{G_1} \to \mathrm{CH}_{G'}$. The surjectivity statement follows because both maps in the composition are surjective by induction. It remains to determine the kernel.

Let $x \in \mathrm{CH}_G$ be an element vanishing in $\mathrm{CH}_{G'}$, then the image of x in CH_{G_1} is the product yx_1 for some $x_1 \in \mathrm{CH}_{G_1}$, where $y \in \mathrm{CH}^1_{G_1}$ is the image of a character of $\mathbb{G}_\mathrm{m}$. Extending the character to T, we get an element $y' \in \mathrm{CH}_G$ lying in the image of $\hat{T} \to \mathrm{CH}^1_G$ and mapped to y. Using the surjectivity of $\mathrm{CH}_G \to \mathrm{CH}_{G_1}$, we find an element $x'_1 \in \mathrm{CH}_G$ mapped to x_1. The difference $x - y'x'_1$ is then in the kernel of $\mathrm{CH}_G \to \mathrm{CH}_{G_1}$ and therefore, by induction, lies in the ideal generated by the image of $\hat{T}$. It follows that x itself lies in the ideal. $\square$

Corollary 4.2. *In the situation of Proposition 4.1, if the ring* $\mathrm{CH}_{G'}$ *is generated by Chern classes (in the sense of* [13, §5]*), then the ring* CH_G *is also generated by Chern classes.*

Proof. For any $i \geq 0$ and any $x \in \mathrm{CH}^i_G$, since the image of x in $\mathrm{CH}^i_{G'}$ is a polynomial in Chern classes, there exists an element $x' \in \mathrm{CH}^i_G$, lying in the Chern subring, such that the difference $x - x'$ vanishes in $\mathrm{CH}_{G'}$. By Proposition 4.1, $x - x'$ belongs to the ideal in CH_G generated by CH^1_G so that we can induct on i. $\square$

Example 4.3. Taking for G a split connected reductive algebraic group and for $G' \subset G$ the semisimple group given by the commutator subgroup of G, we are in the situation of Proposition 4.1: G/G' is a split torus. Therefore Proposition 4.1 describes the relation between the Chow ring of the classifying space of a split reductive group G and that of its semisimple part G'. In particular, by Corollary 4.2, if $\mathrm{CH}_{G'}$ is generated by Chern classes, then CH_G is also generated by Chern classes. This has been proved (by a different method) in [13, Proposition 5.5] in the case of special (split reductive) G, where *special* means that every G-torsor over any field extension of the base field is trivial.

5. Chow rings of classifying spaces for orthogonal groups

The following proposition is a (slightly modified) particular case of [21, Proposition 14.2]. We provide a proof because it is shorter than that of the original statement.

Proposition 5.1. *For any algebraic group G (over any field) and any imbedding of G into a special algebraic group H, the homomorphism $\mathrm{CH}_H \to \mathrm{CH}_G$ is surjective provided that the Chow groups of the quotient H/G over any field extension of the base field are trivial in positive codimensions.*

Proof. As usual, we replace the homomorphism in question by the pull-back homomorphism $\mathrm{CH}(U/H) \to \mathrm{CH}(U/G)$ with respect to the morphism $U/G \to U/H$, where U is an open subvariety in an H-representation, an H-torsor over U/H, and a G-torsor over U/G. Since H is special, every H-torsor is Zariski-locally trivial, [1]. It follows that the fiber of $U/G \to U/H$ over any point $x \in U/H$ is isomorphic to the quotient variety H/G with scalars extended to the residue field of x and therefore has trivial Chow groups in positive codimensions. The statement follows from the spectral sequence of [19, Corollary 8.2] (see also [14, §3]) computing the K-cohomology groups of the total space of the fibration $U/G \to U/H$ in terms of the K-cohomology groups of the base and of the fibers. □

We get the following statement for arbitrary base field of arbitrary characteristic:

Corollary 5.2. *For any $n \geq 2$, the homomorphism $\mathrm{CH}_{\mathrm{GL}(n)} \to \mathrm{CH}_{O(n)}$, given by the tautological imbedding $O(n) \hookrightarrow \mathrm{GL}(n)$, is surjective.*

Proof. As explained in Section 3, the quotient variety $\mathrm{GL}(n)/O(n)$ is identified with the variety Q of n-dimensional non-degenerate quadratic forms. Since Q is an open subvariety in the affine space of all n-dimensional quadratic forms, we have $\mathrm{CH}^{>0}(Q) = 0$ by the homotopy invariance and the localization property of Chow groups. □

6. Main Theorem and its consequences

In this section, k is a field (of any characteristic), n is an integer ≥ 2, F is the function field $k(\mathrm{GL}(n)/O(n))$, E is the standard generic $O(n)$-torsor given by the generic fiber of $\mathrm{GL}(n) \to \mathrm{GL}(n)/O(n)$, and q is the corresponding standard generic quadratic form.

For m with $1 \leq m \leq n/2$, let X be the mth orthogonal Grassmannian of q. We would like to determine the ring $\mathrm{CH}\,X$. The main result is expressed in terms of the tautological (rank-m) vector bundle on X. Its proof will be given in the next section.

Theorem 6.1. *The ring $\mathrm{CH}\,X$ is generated by the Chern classes of the tautological vector bundle.*

Theorem 6.1 claims that $\mathrm{CH}\,X = \mathrm{CT}\,X$ and the ring $\mathrm{CT}\,X$ has been computed in Section 2.

Before proving Theorem 6.1, let us list some consequences. Let Y be any (partial) flag variety of totally isotropic subspaces in q. Let us consider the standard graded epimorphism $\mathrm{CH}\,Y \to GK(Y)$ onto the graded ring associated with the topological filtration (i.e., the filtration by codimension of support) on the Grothendieck ring $K(Y)$.

Corollary 6.2. *The abelian group* $\operatorname{CH} Y$ *is free and, in particular, torsion-free. The ring epimorphism* $\operatorname{CH} Y \to GK(Y)$ *is an isomorphism. The topological filtration on* $K(Y)$ *coincides with the gamma filtration.*

Proof. The variety Y is the variety of flags of totally isotropic subspaces in q of some dimensions $m_1 < \cdots < m_d$. Let X be the orthogonal Grassmannian of m-planes with $m = m_d$. The projection $Y \to X$ is a partial flag variety of subspaces in the tautological vector bundle on X. Therefore, it suffices to prove Corollary 6.2 for X instead of Y.

We have: $\operatorname{CH} X = \operatorname{CT} X$ (Theorem 6.1) and $\operatorname{CT} X$ is a free abelian group(Theorem 2.1).

The kernel of the epimorphism is contained in the torsion subgroup. Since $\operatorname{CH} X$ is torsion-free, the epimorphism is an isomorphism.

Since the Chow ring $\operatorname{CH} X$ is generated by Chern classes, the topological filtration on $K(X)$ coincides with the gamma filtration (see [9, Remark 2.17]). □

7. Proof of Main Theorem

We continue to work over a field k of arbitrary characteristic. We realize the orthogonal group $O(n)$ as the automorphism group of the following split quadratic form $q_0 \colon V \to k$ on the k-vector space $V := k^n$:

$$k^n \ni (x_1, \dots, x_{n/2}, y_{n/2}, \dots, y_1) \mapsto x_1y_1 + x_2y_2 + \cdots + x_{n/2}y_{n/2} \tag{7.1}$$

if n is even and

$$k^n \ni (x_1, \dots, x_{(n-1)/2}, z, y_{(n-1)/2}, \dots, y_1) \mapsto x_1y_1 + x_2y_2 + \cdots + x_{(n-1)/2}y_{(n-1)/2} + z^2 \tag{7.2}$$

if n is odd.

Instead of the mth orthogonal Grassmannian X_0 (for some m with $1 \le m \le n/2$), we consider the variety Y_0 of flags of totally isotropic subspaces in q_0 of dimensions $1, \dots, m$. The group $O(n)$ acts on Y_0 and for any algebraically closed field $K \supset k$ the action of the group $O(n)(K)$ on the set $Y_0(K)$ is transitive. Therefore, by [3, Proposition 2.1 of Chapter III §3], the variety Y_0 is the quotient $O(n)/P$, where P is the stabilizer of the rational point of Y_0 given by the standard flag $V_1 \subset \cdots \subset V_m$ with V_i being the span of the first i vectors in the standard basis of V. (Note that for any m, the variety X_0 is also the quotient of $O(n)$ by the stabilizer of any rational point on X_0; this includes $m = n/2$ even though neither X_0 nor Y_0 are connected in the case: recall that the orthogonal group $O(n)$ is also non-connected for even n.)

Any orthogonal transformation stabilizing this flag also stabilizes the orthogonal complements

$$V_m^\perp = V_{n-m} \subset \cdots \subset V_1^\perp = V_{n-1}.$$

Let $\mathcal{F}$ be the variety of flags of all subspaces in V of dimensions $1, \dots, m, n-m, \dots, n-1$. The group $\operatorname{GL}(n)$ acts on $\mathcal{F}$ and $\mathcal{F} = \operatorname{GL}(n)/S$, where S is the stabilizer of the standard flag $V_1 \subset \cdots \subset V_m \subset V_{n-m} \subset \cdots \subset V_{n-1}$.

Let E be the standard generic $O(n)$-torsor given by the generic fiber of $\operatorname{GL}(n) \to \operatorname{GL}(n)/O(n)$. Let $\mathcal{E}$ be the corresponding $\operatorname{GL}(n)$-torsor obtained via the imbedding

$O(n) \hookrightarrow \mathrm{GL}(n)$. We have a commutative square

$$\begin{array}{ccc} \mathrm{CH}_S & \longrightarrow & \mathrm{CH}(\mathcal{E}/S) \\ \downarrow & & \downarrow \\ \mathrm{CH}_P & \longrightarrow & \mathrm{CH}(E/P) \end{array}$$

with surjective horizontal mappings (cf. [11, Lemma 2.1]).

We claim that the homomorphism $\mathrm{CH}_S \to \mathrm{CH}_P$ is surjective. Admitting the claim for the moment, we conclude that the pull-back homomorphism $\mathrm{CH}(\mathcal{E}/S) \to \mathrm{CH}(E/P) = \mathrm{CH}\,Y$ from the above commutative square is surjective too. Since the group $\mathrm{GL}(n)$ is special, the $\mathrm{GL}(n)$-torsor $\mathcal{E}$ is trivial implying that $\mathcal{E}/S = \mathcal{F}$. We get a surjection $\mathrm{CH}\,\mathcal{F} \twoheadrightarrow \mathrm{CH}\,Y$ implying that the ring $\mathrm{CH}\,Y$ is generated by the Chern classes of the m tautological vector bundles on Y (given by the components of the flags). It follows (see [12, Lemma 4.3]) that $\mathrm{CH}\,X = \mathrm{CT}\,X$.

We finish by proving the claim. The subgroup $S' := \mathbb{G}_{\mathrm{m}}^m \times \mathrm{GL}(n-2m) \times \mathbb{G}_{\mathrm{m}}^m \subset S$ is a Levi subgroup of S, its intersection with $P \subset S$ is $P' := \mathbb{G}_{\mathrm{m}}^m \times O(n-2m)$. The imbedding $P' \hookrightarrow S'$ is the product of the map $\mathbb{G}_{\mathrm{m}}^m \hookrightarrow \mathbb{G}_{\mathrm{m}}^m \times \mathbb{G}_{\mathrm{m}}^m$, $x \mapsto (x, x^{-1})$ and the tautological imbedding $O(n-2m) \hookrightarrow \mathrm{GL}(n-2m)$.

In the commutative square

$$\begin{array}{ccc} \mathrm{CH}_S & \longrightarrow & \mathrm{CH}_{S'} \\ \downarrow & & \downarrow \\ \mathrm{CH}_P & \longrightarrow & \mathrm{CH}_{P'} \end{array}$$

the horizontal maps are isomorphisms, see [13, Proof of Propositrion 6.1]. Therefore, in order to prove the claim, is suffices to prove that the homomorphism $\mathrm{CH}_{S'} \to \mathrm{CH}_{P'}$ is surjective.

In the commutative square

$$\begin{array}{ccc} \mathrm{CH}_{S'} & \longrightarrow & \mathrm{CH}_{\mathrm{GL}(n-2m)} \\ \downarrow & & \downarrow \\ \mathrm{CH}_{P'} & \longrightarrow & \mathrm{CH}_{O(n-2m)} \end{array}$$

the horizontal maps are epimorphisms by Proposition 4.1. The map on the right is an epimorphism by Corollary 5.2. We can now prove the surjectivity of the map on the left in every codimension $i \geq 0$ using induction on i.

For $i = 0$ there is nothing to prove. For $i = 1$, we have a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathbb{G}_{\mathrm{m}}^m \times \mathbb{G}_{\mathrm{m}}^m} & \longrightarrow & \mathrm{CH}^1_{S'} & \xrightarrow{\text{onto}} & \mathrm{CH}^1_{\mathrm{GL}(n-2m)} \\ \downarrow{\scriptstyle \text{onto}} & & \downarrow & & \downarrow{\scriptstyle \text{onto}} \\ \widehat{\mathbb{G}_{\mathrm{m}}^m} & \longrightarrow & \mathrm{CH}^1_{P'} & \xrightarrow{\text{onto}} & \mathrm{CH}^1_{O(n-2m)} \end{array}$$

with a surjection on the left. Since the lower row is exact (by Proposition 4.1),[3] the statement for $i = 1$ follows.

[3] The upper row is also exact but we don't care.

For $i \geq 2$, it suffices to show that any element $x \in \mathrm{CH}^i_{P'}$, vanishing in the group $\mathrm{CH}_{O(n-2m)}$, is in the image of $\mathrm{CH}_{S'}$. Since $x = y_1x_1 + \dots + y_rx_r$ for some $r \geq 0$, some $y_1, \dots, y_r \in \mathrm{CH}^1_{P'}$, and some $x_1, \dots, x_r \in \mathrm{CH}^{i-1}_{P'}$ by Proposition 4.1, we are done.

8. The generic quadratic form in characteristic $\neq 2$

For a field k of characteristic not 2 and an integer $n \geq 2$, we defined in the introduction the generic n-dimensional quadratic form $q_g := \langle t_1, \dots, t_n \rangle$ over the field of rational functions $F_g := k(t_1, \dots, t_n)$, and the the standard generic quadratic form q over the field of rational functions $F := k(t_{ij})_{1 \leq i \leq j \leq n}$. Now we are going to compare q_g with q.

Proposition 8.1. *The field F_g can be k-identified with a subfield in F the way that the field extension F/F_g is purely transcendental and the generic quadratic form q_g with the scalars extended to the field F becomes isomorphic to the standard generic form q.*

Corollary 8.2. *Theorem 6.1 as well as Corollary 6.2 hold for the generic quadratic form in place of the standard generic one.*

Proof. In case of a purely transcendental field extension, the change of field homomorphism for Chow rings is an isomorphism, see [7, Lemma 1.4a]. □

Proof of Proposition 8.1. Let us apply the standard orthogonalization procedure to the standard basis $e_1, \dots, e_n$ of F^n, where the orthogonality refers to the symmetric bilinear form associated with q. This means that we construct an orthogonal basis $e'_1, \dots, e'_n$ by taking for e'_i the sum of e_i and a linear combination of $e_1, \dots, e_{i-1}$, where the coefficients of the linear combination are determined by the condition that e'_i is orthogonal to $e_1, \dots, e_{i-1}$. The procedure works for q because its restriction to the span of $e_1, \dots, e_i$ is non-degenerate for every i.

Then $t_i := q(e'_i)$ equals $t_{ii}+$ a rational function in $t_{11}, \dots, t_{i-1i-1}$ and t_{rs} with $1 \leq r < s \leq n$. It follows that the elements t_{rs} $(1 \leq r < s \leq n)$ and $t_1, \dots, t_n$ all together generate the field F over k and therefore – since their number is the transcendence degree – are algebraically independent over k. In particular, $t_1, \dots, t_n$ are algebraically independent so that the field F_g is identified with the subfield $k(t_1, \dots, t_n) \subset F$. This identification has the required properties. □

9. The generic quadratic form in characteristic 2

In characteristic 2 (actually, in arbitrary characteristic), any non-degenerate quadratic form, depending on the parity of n, is isomorphic to the form

$$[a_1, a_2]\perp \dots \perp[a_{n-1}, a_n] \quad \text{or} \quad [a_1, a_2]\perp \dots \perp[a_{n-2}, a_{n-1}]\perp \langle a_n \rangle\,,$$

where $a_1, \dots, a_n$ are constants from the base field and $a_n \neq 0$ in the case of odd n. The notation $[a_1, a_2]$ stands for the 2-dimensional form $(x_1, x_2) \mapsto a_1x_1^2 + x_1x_2 + a_2x_2^2$. So, the generic n-dimensional quadratic form q_g will be defined as the form

$$[t_1, t_2]\perp \dots \perp[t_{n-1}, t_n] \quad \text{or} \quad [t_1, t_2]\perp \dots \perp[t_{n-2}, t_{n-1}]\perp \langle t_n \rangle \tag{9.1}$$

over the rational function field $F_g := k(t_1, \dots, t_n)$.

Proposition 9.2. *Proposition 8.1 and Corollary 8.2 hold in characteristic 2 as well.*

Proof. We only need to identify the field F_g with a subfield in $F = k(t_{ij})_{1\le i\le j\le n}$ (over k) the way that the field extension F/F_g is purely transcendental and the generic quadratic form q_g with the scalars extended to the field F becomes isomorphic to the standard generic q.

Starting with the standard basis $e_1,\dots,e_n$ of the vector space F^n, we construct a new basis $e'_1,\dots,e'_n$ as follows. For every odd i, the vector e'_i is e_i+ a linear combination of $e_1,\dots,e_{i-1}$ and if $i<n$ then the vector e'_{i+1} is $e_{i+1}+$ a linear combination of $e_1,\dots,e_{i-1}$, where the coefficients of the linear combinations are determined by the condition that the new vectors are orthogonal to each of $e_1,\dots,e_{i-1}$. Additionally, for every even i, we divide the vector e'_i by the non-zero scalar (e'_{i-1},e'_i).

With respect to the new basis, the standard generic quadratic form q has the shape (9.1), where $t_i := q(e'_i)$. For odd i, t_i equals $t_{ii}+$ a rational function in $t_{11},\dots,t_{i-1i-1}$ and t_{rs} with $1\le r<s\le n$. For even i, t_i equals t_{ii}/f_i+ a rational function in $t_{11},\dots,t_{i-2i-2}$ and t_{rs} with $1\le r<s\le n$, where f_i is also a rational function in $t_{11},\dots,t_{i-1i-1}$ and t_{rs} with $1\le r<s\le n$.

It follows that the elements t_{rs} ($1\le r<s\le n$) and $t_1,\dots,t_n$ all together generate the field F over k and therefore are algebraically independent over k. In particular, $t_1,\dots,t_n$ are algebraically independent so that the field F_g is identified with the subfield $k(t_1,\dots,t_n)\subset F$. This identification has the required properties. □

Acknowledgements. I thank Alexander Merkurjev for useful consultations, comments, and suggestions. I thank Raphaël Fino for noticing that the generator c_{n-m} was missing in Theorem 2.1 in a previous version of the paper.

References

[1] *Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications.* Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958.

[2] Buch, A. S., Kresch, A., and Tamvakis, H. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math. 178*, 2 (2009), 345–405.

[3] Demazure, M., and Gabriel, P. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel.

[4] Edidin, D., and Graham, W. Characteristic classes in the Chow ring. *J. Algebraic Geom. 6*, 3 (1997), 431–443.

[5] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2008.

[6] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 1998.

[7] Kahn, B., and Sujatha, R. Motivic cohomology and unramified cohomology of quadrics. *J. Eur. Math. Soc. (JEMS) 2*, 2 (2000), 145–177.

[8] Karpenko, N. A. Chow groups of quadrics and the stabilization conjecture. In *Algebraic K-theory*, vol. 4 of *Adv. Soviet Math.* Amer. Math. Soc., Providence, RI, 1991, pp. 3–8.

[9] Karpenko, N. A. Codimension 2 cycles on Severi-Brauer varieties. *K-Theory 13*, 4 (1998), 305–330.

[10] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.

[11] Karpenko, N. A. Chow groups of some generically twisted flag varieties. *Annals of K-Theory 2*, 2 (2017).

[12] Karpenko, N. A. Chow ring of generically twisted varieties of complete flags. *Adv. Math. 306* (2017), 789–806.

[13] Karpenko, N. A. On generically split generic flag varieties. *Bull. Lond. Math. Soc. 50* (2018), 496–508.

[14] Karpenko, N. A., and Merkurjev, A. S. Chow groups of projective quadrics. *Algebra i Analiz 2*, 3 (1990), 218–235.

[15] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[16] Molina Rojas, L. A., and Vistoli, A. On the Chow rings of classifying spaces for classical groups. *Rend. Sem. Mat. Univ. Padova 116* (2006), 271–298.

[17] Pandharipande, R. Equivariant Chow rings of $\mathrm{O}(k)$, $\mathrm{SO}(2k+1)$, and $\mathrm{SO}(4)$. *J. Reine Angew. Math. 496* (1998), 131–148.

[18] Petrov, V. A. Chow ring of generic maximal orthogonal Grassmannians. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 443* (2016), 147–150.

[19] Rost, M. Chow groups with coefficients. *Doc. Math. 1* (1996), No. 16, 319–393 (electronic).

[20] Smirnov, A., and Vishik, A. Subtle characteristic classes. arXiv:1401.6661 [math.AG] (26 Jan 2014), 31 pages.

[21] Totaro, B. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, vol. 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[22] Totaro, B. The torsion index of the spin groups. *Duke Math. J. 129*, 2 (2005), 249–290.

[23] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).

[24] Vishik, A. Fields of u-invariant 2^r+1. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko