COHOMOLOGY OF RELATIVE CELLULAR SPACES AND ISOTROPIC FLAG VARIETIES

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Abstract. Let A be a separable algebra (with an involution). The varieties of flags of (isotropic) ideals of A are considered and certain decompositions of these varieties in the category of Chow-correspondences are produced. As a consequence, decompositions in various cohomology theories are obtained.

Contents

0. Introduction

Let ϕ be a quadratic form over a field and let ψ be the anisotropic part of ϕ (i.e., an anisotropic quadratic form with the same Witt class as ψ). It was probably first observed by M. Rost that though the precise relationship between the projective quadrics X_ϕ and X_ψ, determined by ϕ and ψ, is rather messy, their relationship in the motivic category is quite clear (see [16]). This relationship allows to compute various cohomological invariants of X_ϕ in terms of the invariants of X_ψ because of the universal property of motives.

Similar result is obtained in [10] for the Severi-Brauer varieties: for a central simple algebra A over a field, the motive of the Severi-Brauer variety X_A determined by A is computed in terms of the motive of X_D, where D is the underlying division algebra (i.e., a central division algebra with the same Brauer class as A).

These two results solve particular cases of the following general problem. Let G be a linear algebraic group (over an arbitrary field) and let X be a projective G-homogeneous variety. One likes to compute the motive of X in terms of motives of some G_{an}-homogeneous varieties, where G_{an} is the semisimple anisotropic kernel of G.

A computation of this type is known in the split case [12]. It is based on the existence of a cellular structure on X. To attack the general situation, a relative analog of this notion is needed. Here, we propose a notion of a *relative cellular space* X (6.1). After that we express the motive of X via the motives of the bases of its cells (6.5). More precisely, this result holds already on the level of the *category of correspondences* (while the motivic category is the pseudo-abelian completion of the category of correspondences). By that reason, we don't work with the motivic category at all.

We apply the theorem on relative cellular spaces to the case of an (isotropic) flag variety X given by a separable algebra (with involution) A and compute the motive of X in terms of the motives of some flag varieties given by the *anisotropic kernel* (10.12, 14.5) of A. In other words, we solve the general problem, formulated above, in the case where the linear algebraic group G is of classical type.

We start the first part with a self-contained discussion of the category of correspondences. Although, similar discussions exist already in the literature (see e.g. [13]), there are at least two reasons to include it here. Usually, only absolutely irreducible varieties (or only varieties with absolutely irreducible components) are allowed. However, it is important for our purposes to be able to work with non-absolutely irreducible varieties as well in order to be able to handle all flag varieties. In fact, even computing the motive of an absolutely irreducible flag variety, one may meet non-absolutely irreducible components in the answer: for instance, if ϕ is an isotropic 4-dimensional quadratic form with non-trivial determinant d, then the quadric X_ϕ is absolutely irreducible, while in the decomposition of its motive the spectrum of the quadratic extension generated by $\sqrt{d}$ occurs.

Another reason to include some back-grounds on correspondences is the following. We need a precise list of properties of a cohomology theory H which guarantee that H determines a functor on the category of correspondences (and therefore a computation of the motive of a variety X gives a computation of $\mathrm{H}(X)$), given in §2, and we don't find a list like that in the literature.

The second part contains a construction of flag varieties via the language of functors of points developed in [5]. Although in the split case it is already done in the literature (see [5, Chap. I] or [8, §§9.7, 9.8] for construction of (flag) varieties of *subspaces*), there is no reference even in the simplest non-split case, the case of Severi-Brauer varieties. Indeed, the definitions of Severi-Brauer varieties, which can be found in the literature, either determine the Severi-Brauer variety X_A of an algebra A by equations in a projective space and are complicated, non-invariant (involving, e.g., the choice of a basis of A as in [1, §1.2]), and difficult to work with; or they are sketched (when, e.g., only the set of rational points of X_A is defined as in [19, §6 of Chap. X]).

We think that the only natural and the most simple way to define and to work with the flag varieties is the way of using the language of functors of points. This way is already being mainly used for working with linear algebraic groups. So, it is natural to expect that homogeneous varieties, being objects closely related to linear algebraic groups, can be also convenient handled by using this language. However, it is of course more complicated with *non-affine* homogeneous varieties as with *affine* algebraic groups. Necessary back-grounds on the language of functors of points (almost without proofs) are given in the beginning of the second part.

Part 1. Cohomology of relative cellular spaces

Conventions.

Let F be a field. By a variety over F or an F-variety, we mean a separated F-scheme of finite type, so no assumptions like "integral" or "reduced" are made. Thus any closed subscheme of an F-variety is an F-variety as well. An F-variety is called *geometrically reduced* (resp. *geometrically irreducible*), if it is reduced (resp. irreducible) over any field extension of F or equivalently over an algebraic closure of F. A variety is called "smooth", if it is "geometrically regular". A variety is called complete, if the structure morphism is proper.

1. Category of correspondences

Let F be a field, $\mathcal{V}$ the category of smooth, complete (or projective) F-varieties. The category of (Chow-)correspondences $\mathcal{CV}$ has the same objects as $\mathcal{V}$, while the set of morphisms $\mathrm{Hom}(X, Y)$ for two varieties X and Y is by definition the Chow group $\mathrm{CH}(X \times Y)$ of algebraic cycles on $X \times Y$ modulo rational equivalence [7]. Here are the properties of the Chow group which are needed for constructing the category of correspondences (we have to apologize for mixing the "data" and "axioms" together):

- for any $X \in \mathcal{V}$, one has an abelian group $\mathrm{CH}(X)$;
- for any $f \in \mathrm{Mor}(X, Y)$, one has
  - the pull-back homomorphism
    $$f^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$$
    which gives a (contravariant) functor $\mathcal{V} \to \mathcal{Ab}$ to the category of abelian groups;
  - the push-forward homomorphism
    $$f_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$$
    which gives a (covariant) functor $\mathcal{V} \to \mathcal{Ab}$;
  - if f is an isomorphism, then the isomorphisms f^* and f_* are mutually inverse;
- (**"compatibility with products"**) for any $f \in \mathrm{Mor}(X, Y)$ and any $Z \in \mathcal{V}$, the composition $pr_Y^* \circ f_*$ coincides with $(f \times id_Z)_* \circ pr_X^*$, where the morphisms in use are shown in the diagram:
  $$\begin{array}{ccc} X \times Z & \xrightarrow{f \times id_Z} & Y \times Z \\ {\scriptstyle pr_X}\big\downarrow & & \big\downarrow{\scriptstyle pr_Y} \\ X & \xrightarrow{f} & Y \end{array}$$
  (pr_X and pr_Y are the projections);
- (**"compatibility with co-products"**) for any $X_i \in \mathcal{V}$, $i = 1, 2$, the abelian group $\mathrm{CH}(X_1 \coprod X_2)$ is a direct sum of $\mathrm{CH}(X_1)$ and $\mathrm{CH}(X_2)$, where the homomorphisms of inclusions and projections are given by

the pull-backs and push-forwards with respect to the imbeddings $X_i \hookrightarrow X_1 \coprod X_2$;

- for any $X \in \mathcal{V}$, the abelian group $\mathrm{CH}(X)$ has a commutative unital ring structure (we shall denote the unit of $\mathrm{CH}(X)$ by 1_X);
- for any $f \in \mathrm{Mor}(X, Y)$
  - the pull-back $f^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ is a ring homomorphism, i.e.

$$f^*(\beta_1 \cdot \beta_2) = f^*(\beta_1) \cdot f^*(\beta_2)$$

    for $\beta_1, \beta_2 \in \mathrm{CH}(Y)$ and $f^*(1_Y) = 1_X$;
  - (**projection formula**) the push-forward $f_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$ is a homomorphism of $\mathrm{CH}(Y)$-modules, i.e.

$$f_*\big(f^*(\beta) \cdot \alpha\big) = \beta \cdot f_*(\alpha)$$

    for $\beta \in \mathrm{CH}(Y)$ and $\alpha \in \mathrm{CH}(X)$.

The Chow group has also a gradation (by codimension of cycles) but we don't consider it yet. Up to §6, CH can be viewed formally as something satisfying the properties listed.

Elements of $\mathrm{Hom}(X, Y)$ are called correspondences.

Definition 1.1. For two correspondences $\alpha \in \mathrm{Hom}(X, Y)$ and $\beta \in \mathrm{Hom}(Y, Z)$ the composition $\beta \circ \alpha \in \mathrm{Hom}(X, Z)$ is defined as

$$(pr_{XZ})_*\Big(pr_{YZ}^*(\beta) \cdot pr_{XY}^*(\alpha)\Big)$$

where the dot stays for the multiplication in the Chow group, asterisks for the pull-backs and push-forward and the morphisms are the projections of the product $X \times Y \times Z$ shown in the diagram:

$$\begin{tikzcd} X \times Y & X \times Y \times Z \arrow[l, "pr_{XY}"'] \arrow[r, "pr_{YZ}"] \arrow[d, "pr_{XZ}"] & Y \times Z \\ & X \times Z & \end{tikzcd}$$

Definition 1.2. Let us define the *graph class* Γ_f of a morphism of varieties $f \in \mathrm{Mor}(X, Y)$ as

$$\Gamma_f = (id_X, f)_*(1_X) \in \mathrm{CH}(X \times Y)$$

where $(id_X, f)\colon X \to X \times Y$ is the morphism given by id_X and f. The graph class Γ_{id_X} for a variety $X \in \mathcal{V}$ will be called the *diagonal class* and denoted by δ_X.

Proposition 1.3. *The category of correspondences $\mathcal{CV}$ is:*
1) really a category; 2) an additive category; 3) additively self-dual.

Proof. 1) Let us check that the diagonal class gives an identity. Take a correspondence $\alpha \in \operatorname{Hom}(X, Y)$ and consider the diagram

$$\begin{tikzcd}
X \arrow[d, "{(id_X, id_X)}"'] & X \times Y \arrow[l, "pr_X"'] \arrow[d, "{(id_X, id_X) \times id_Y}"] & \\
X \times X & X \times X \times Y \arrow[l, "pr_{12}"'] \arrow[r, "pr_{23}"] \arrow[d, "pr_{13}"] & X \times Y \\
& X \times Y &
\end{tikzcd}$$

We have

$$\alpha \circ \delta_X = (pr_{13})_* \Big(pr_{23}^*(\alpha) \cdot \; pr_{12}^*(\delta_X) \Big) \; .$$

Substituting $(id_X, id_X)_*(1_X)$ for δ_X and using the "compatibility with products" for the square from the diagram, we get

$$\alpha \circ \delta_X = (pr_{13})_* \Big(pr_{23}^*(\alpha) \cdot \big((id_X, id_X) \times id_Y \big)_* \circ pr_X^*(1_X) \Big) =$$

$$= (pr_{13})_* \circ \big((id_X, id_X) \times id_Y \big)_* \Big(\big((id_X, id_X) \times id_Y \big)^* \circ pr_{23}^*(\alpha) \cdot pr_X^*(1_X) \Big)$$

where for the last equality the projection formula is used. Since

$$pr_{13} \circ \big((id_X, id_X) \times id_Y \big) = id_{X \times Y} = pr_{23} \circ \big((id_X, id_X) \times id_Y \big)$$

and $pr^*(1_X) = 1_{X \times Y}$, the expression obtained equals α.

The equality $\delta_Y \circ \alpha = \alpha$ can be checked analogously.

To verify that the composition rule is associative, choose some

$$\alpha \in \operatorname{Hom}(T, X), \quad \beta \in \operatorname{Hom}(X, Y), \quad \text{and} \quad \gamma \in \operatorname{Hom}(Y, Z) \; .$$

The composition $\gamma \circ (\beta \circ \alpha)$ will be computed with a help of the following diagram:

$$\begin{tikzcd}
& & T \times Y & & \\
T \times X \times Y \arrow[urr, "pr^{TXY}_{TY}"] \arrow[r, "pr^{TXY}_{XY}"] \arrow[d, "pr^{TXY}_{TX}"'] & X \times Y & & Y \times Z & T \times Y \times Z \arrow[ull, "pr^{TYZ}_{TY}"'] \arrow[l, "pr^{TYZ}_{YZ}"'] \arrow[d, "pr^{TYZ}_{TZ}"] \\
T \times X & & T \times X \times Y \times Z \arrow[ll, "pr_{TX}"'] \arrow[rr, "pr_{TZ}"] \arrow[ull, "pr_{TXY}"] \arrow[ul, "pr_{XY}"'] \arrow[ur, "pr_{YZ}"] \arrow[urr, "pr_{TYZ}"'] & & T \times Z
\end{tikzcd}$$

We have

$$\gamma \circ (\beta \circ \alpha) = (pr^{TYZ}_{TZ})_* \\ \Big((pr^{TYZ}_{YZ})^*(\gamma) \cdot (pr^{TYZ}_{TY})^* \circ (pr^{TXY}_{TY})_* \Big((pr^{TXY}_{XY})^*(\beta) \cdot (pr^{TXY}_{TX})^*(\alpha) \Big) \Big) .$$

Using the "compatibility with products" for the rhombus in the diagram, we replace the composition $(pr^{TYZ}_{TY})^* \circ (pr^{TXY}_{TY})_*$ by $(pr_{TYZ})_* \circ pr^*_{TXY}$. After that, the projection formula applied to the morphism pr_{TYZ} gives

$$\gamma \circ (\beta \circ \alpha) = \\ = (pr^{TYZ}_{TZ})_* \circ (pr_{TYZ})_* \\ \Big(pr^*_{TYZ} \circ (pr^{TYZ}_{YZ})^*(\gamma) \cdot pr^*_{TXY} \Big((pr^{TXY}_{XY})^*(\beta) \cdot (pr^{TXY}_{TX})^*(\alpha) \Big) \Big) = \\ = (pr_{TZ})_* \Big((pr_{YZ})^*(\gamma) \cdot (pr_{XY})^*(\beta) \cdot (pr_{TX})^*(\alpha) \Big) .$$

Computing $(\gamma \circ \beta) \circ \alpha$ in the analogous way, one gets the same expression.

2) For any $X, Y \in \mathcal{CV}$ the set $\mathrm{Hom}(X,Y) = \mathrm{CH}(X \times Y)$ is an abelian group and the composition rule is evidently biadditive.

The zero object is given by $\emptyset$ (it follows from the "compatibility with co-products" that $\mathrm{CH}(\emptyset) = 0$).

For any $X_i \in \mathcal{V}$, $i = 1, 2$, the disjoint union $X_1 \coprod X_2$ is a direct sum of X_1 and X_2 in $\mathcal{CV}$, where the morphisms of inclusions and projections are given by the graph classes of the imbeddings $X_i \hookrightarrow X_1 \coprod X_2$ and their transpositions (use "compatibility with co-products"; see the next paragraph for the definition of transposition).

3) For $X, Y \in \mathcal{V}$ denote by $t \colon X \times Y \to Y \times X$ the morphism of interchanging the factors. Since t is an isomorphism and $t \circ t = id_{X \times Y}$, we have $t_* = t^*$. For $\alpha \in \mathrm{Hom}(X,Y)$, one denotes by α^t (and calls the *transposition* of α) the correspondence

$$t_*(\alpha) = t^*(\alpha) \in \mathrm{Hom}(Y,X) .$$

The identical on the objects (contravariant) functor $\mathcal{CV} \to \mathcal{CV}$ which puts every α to α^t is additive and inverse to itself. □

Proposition 1.4. *The rules*

$$X \in \mathcal{V} \ \mapsto \ X \in \mathcal{CV} , \quad f \in \mathrm{Mor}(X,Y) \ \mapsto \ \Gamma_f \in \mathrm{Hom}(X,Y)$$

determine a (covariant) functor $\mathcal{V} \to \mathcal{CV}$.

Proof. For any $X \in \mathcal{V}$, we have $\Gamma_{id_X} = \delta_X$ by the definition of δ_X (1.2).

Let us check the composition rule $\Gamma_g \circ \Gamma_f = \Gamma_{g \circ f}$ for $f \in \mathrm{Mor}(X,Y)$ and $g \in \mathrm{Mor}(Y,Z)$. Since $\big(id_X, (g \circ f) \big)$ can be decomposed in the composition

$$X \xrightarrow{(id_X, f)} X \times Y \xrightarrow{id_X \times (id_Y, g)} X \times (Y \times Z) \xrightarrow{pr_{XZ}} X \times Z$$

one has

$$\Gamma_{g\circ f} = (pr_{XZ})_* \circ \big(id_X \times (id_Y, g)\big)_* (\Gamma_f) \ .$$

For computation of $\Gamma_g \circ \Gamma_f$ we use the following diagram:

$$\begin{array}{ccccc} & & X\times Y & \xrightarrow{pr_Y} & Y \\ & & {\scriptstyle id_X \times (id_Y, g)}\big\downarrow & & \big\downarrow{\scriptstyle (id_Y, g)} \\ X\times Y & \xleftarrow{pr_{XY}} & X\times Y\times Z & \xrightarrow{pr_{YZ}} & Y\times Z \\ & & {\scriptstyle pr_{XZ}}\big\downarrow & & \\ & & X\times Z & & \end{array}$$

By the "compatibility with products" applied to the square, we have

$$pr_{YZ}^* \circ (id_Y, g)_* = \big(id_X \times (id_Y, g)\big)_* \circ pr_Y^*$$

thereby $\Gamma_g \circ \Gamma_f$ is push-forward with respect to pr_{XZ} of the product

$$\big(id_X \times (id_Y, g)\big)_* \circ pr_Y^*(1_Y) \cdot pr_{XY}^*(\Gamma_f) =$$
$$= \big(id_X \times (id_Y, g)\big)_* \Big(pr_Y^*(1_Y) \cdot \big(id_X \times (id_Y, g)\big)^* \circ pr_{XY}^*(\Gamma_f)\Big)$$

(the projection formula for the morphism $id_X \times (id_Y, g)$ is used). Since

$$pr_{XY} \circ \big(id_X \times (id_Y, g)\big) = id_{X\times Y} \quad \text{and} \quad pr_Y^*(1_Y) = 1_{X\times Y}$$

the product insight of the big delimiters is simply Γ_f. □

2. Geometric cohomology theories

We say that H is a geometric cohomology theory, if

- for any $X \in \mathcal{V}$, one has an abelian group $\mathrm{H}(X)$;
- for any $f \in \mathrm{Mor}(X, Y)$, one has
  - a pull-back homomorphism
    $$f^* \colon \mathrm{H}(Y) \to \mathrm{H}(X)$$
    which gives a (contravariant) functor $\mathcal{V} \to \mathcal{Ab}$;
  - a push-forward homomorphism
    $$f_* \colon \mathrm{H}(X) \to \mathrm{H}(Y)$$
    which gives a (covariant) functor $\mathcal{V} \to \mathcal{Ab}$;
- (**"compatibility with products"**) for any square in $\mathcal{V}$ of the type
  $$\begin{array}{ccc} X\times Y\times Z & \xrightarrow{pr_{YZ}} & Y\times Z \\ {\scriptstyle pr_{XY}}\big\downarrow & & \big\downarrow{\scriptstyle pr^{YZ}} \\ X\times Y & \xrightarrow{pr^{XY}} & Y \end{array}$$
  the compositions $(pr^{YZ})^* \circ pr^{XY}_*$ and $(pr_{YZ})_* \circ pr_{XY}^*$ coincide;
- for any $X \in \mathcal{V}$, the abelian group $\mathrm{H}(X)$ has a structure of a (left) $\mathrm{CH}(X)$-module;

- for any $f \in \mathrm{Mor}(X, Y)$
  - the pull-back $f^* : \mathrm{H}(Y) \to \mathrm{H}(X)$ is a homomorphism of $\mathrm{CH}(Y)$-modules, i.e.
  $$f^*(\beta \cdot y) = f^*(\beta) \cdot f^*(y)$$
  for $\beta \in \mathrm{CH}(Y)$ and $y \in \mathrm{H}(Y)$;
  - (**first projection formula**) the push-forward $f_* : \mathrm{H}(X) \to \mathrm{H}(Y)$ is a homomorphism of $\mathrm{CH}(Y)$-modules, i.e.
  $$f_*\big(f^*(\beta) \cdot x\big) = \beta \cdot f_*(x)$$
  for $\beta \in \mathrm{CH}(Y)$ and $x \in \mathrm{H}(X)$.
  - (**second projection formula**)
  $$f_*\big(\alpha \cdot f^*(y)\big) = f_*(\alpha) \cdot y$$
  for $\alpha \in \mathrm{CH}(X)$ and $y \in \mathrm{H}(Y)$.

Definition 2.1. Let H be a geometric cohomology theory. For any correspondence $\alpha \in \mathrm{Hom}(X, Y)$, we define a group homomorphism $\mathrm{H}(\alpha) : \mathrm{H}(X) \to \mathrm{H}(Y)$ as the composition

$$\mathrm{H}(X) \xrightarrow{pr_X^*} \mathrm{H}(X \times Y) \xrightarrow{\alpha \cdot} \mathrm{H}(X \times Y) \xrightarrow{(pr_Y)_*} \mathrm{H}(Y)$$

where the middle arrow is the multiplication by $\alpha \in \mathrm{CH}(X \times Y)$.

Proposition 2.2. *For any morphism of varieties $f \in \mathrm{Mor}(X, Y)$, one has $\mathrm{H}(\Gamma_f) = f_*$ and $\mathrm{H}(\Gamma_f^t) = f^*$. For any $\alpha \in \mathrm{CH}(X)$ the homomorphism $\mathrm{H}\big((id_X, id_X)_*(\alpha)\big)$ coincides with the multiplication by α.*

Proof. To compute $\mathrm{H}(\Gamma_f)$ we use the following diagram:

$$\begin{tikzcd} & X \arrow[d, "(id_X, f)"] & \\ X & X \times Y \arrow[l, "pr_X"'] \arrow[r, "pr_Y"] & Y \end{tikzcd}$$

For $x \in \mathrm{H}(X)$, one has

$$\mathrm{H}(\Gamma_f)(x) = (pr_Y)_*\Big((id_X, f)_*(1_X) \cdot pr_X^*(x)\Big) =$$
$$= (pr_Y)_* \circ (id_X, f)_*\Big(1_X \cdot (id_X, f)^* \circ pr_X^*(x)\Big)$$

where the latter equality holds by the second projection formula. Since

$$pr_X \circ (id_X, f) = id_X \quad \text{and} \quad pr_Y \circ (id_X, f) = f$$

we get $f_*(x)$.

Computing $\mathrm{H}(\Gamma_f^t)$, we have to use (f, id_X) instead of (id_X, f):

$$\begin{tikzcd} & X \arrow[d, "(f, id_X)"] & \\ Y & Y \times X \arrow[l, "pr_Y"'] \arrow[r, "pr_X"] & X \end{tikzcd}$$

For $y \in \mathrm{H}(Y)$, one has

$$\begin{aligned}\mathrm{H}(\Gamma_f^t)(y) &= (pr_X)_*\Big((f, id_X)_*(1_X) \cdot pr_Y^*(y)\Big) = \\ &= (pr_X)_* \circ (f, id_X)_*\Big(1_X \cdot (f, id_X)^* \circ pr_Y^*(y)\Big) = \\ &= (id_X)_*\big(f^*(y)\big) = f^*(y)\ .\end{aligned}$$

For computation of $\mathrm{H}\left((id_X, id_X)_*(\alpha)\right)$ we use the diagram

$$\begin{tikzcd} & X \arrow[d, "(id_X, id_X)"] & \\ X & X \times X \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & X\ . \end{tikzcd}$$

For $x \in \mathrm{H}(X)$, one has

$$\begin{aligned}\mathrm{H}\left((id_X, id_X)_*(\alpha)\right)(x) &= (pr_2)_*\Big((id_X, id_X)_*(\alpha) \cdot pr_1^*(x)\Big) = \\ &= (pr_2)_* \circ (id_X, id_X)_*\Big(\alpha \cdot (id_X, id_X)^* \circ pr_1^*(x)\Big) = \alpha \cdot x\ .\end{aligned}$$

$\square$

Proposition 2.3. *The rules*

$$\begin{aligned} X \in \mathcal{CV} \ &\mapsto\ \mathrm{H}(X) \in \mathcal{Ab}, \\ \alpha \in \mathrm{Hom}(X, Y) \ &\mapsto\ \mathrm{H}(\alpha) \in \mathrm{Hom}\left(\mathrm{H}(X), \mathrm{H}(Y)\right)\end{aligned}$$

determine an additive functor $\mathrm{H}\colon \mathcal{CV} \to \mathcal{Ab}$.

Proof. First we check that H is a functor. Taking any $X \in \mathcal{V}$ and applying the formula $\mathrm{H}(\Gamma_f) = f_*$ to the particular case $f = id_X$ or the formula

$$\mathrm{H}\left((id_X, id_X)_*(\alpha)\right) = \alpha \cdot$$

to the particular case $\alpha = 1_X$, we see that $\mathrm{H}(\delta_X)$ is the identity.

Let us check the composition rule. Fix $\alpha \in \mathrm{Hom}(X, Y)$, $\beta \in \mathrm{Hom}(Y, Z)$, and $x \in \mathrm{H}(X)$. Here is the diagram of morphisms we shall use:

$$\begin{tikzcd}
 & & Y & & \\
X \times Y \arrow[urr, "pr_Y^{XY}"] \arrow[d, "pr_X^{XY}"'] & & & & Y \times Z \arrow[ull, "pr_Y^{YZ}"'] \arrow[d, "pr_Z^{YZ}"] \\
X & & X \times Y \times Z \arrow[ll, "pr_X"'] \arrow[rr, "pr_Z"] \arrow[ull, "pr_{XY}"'] \arrow[urr, "pr_{YZ}"] \arrow[d, "pr_{XZ}"] & & Z \\
 & & X \times Z \arrow[ull, "pr_X^{XZ}"] \arrow[urr, "pr_Z^{XZ}"'] & &
\end{tikzcd}$$

We have

$$\mathrm{H}(\beta) \circ \mathrm{H}(\alpha)(x) = (pr_Z^{YZ})_* \Big(\beta \cdot (pr_Y^{YZ})^* \circ (pr_Y^{XY})_* \big(\alpha \cdot (pr_X^{XY})^*(x)\big)\Big) .$$

Since the rhombus in the diagram is of the type as in the axiom "compatibility with products", one can replace the composition $(pr_Y^{YZ})^* \circ (pr_Y^{XY})_*$ by $(pr_{YZ})_* \circ (pr_{XY})^*$. After that we use the first projection formula with respect to the morphism pr_{YZ}. It gives

$$(pr_Z^{YZ})_* \circ (pr_{YZ})_* \Big(pr_{YZ}^*(\beta) \cdot pr_{XY}^* \left(\alpha \cdot (pr_X^{XY})^*(x)\right)\Big) =$$
$$= (pr_Z)_* \Big(pr_{YZ}^*(\beta) \cdot pr_{XY}^*(\alpha) \cdot pr_X^*(x)\Big)$$

(besides of functoriality for the pull-back and push-forward, the module property of the pull-back is used here).

From the other hand

$$\begin{aligned}
\mathrm{H}(\beta \circ \alpha)(x) &= (pr_Z^{XZ})_* \Big((pr_{XZ})_* \big(pr_{YZ}^*(\beta) \cdot pr_{XY}^*(\alpha)\big) \cdot (pr_X^{XZ})^*(x)\Big) = \\
&= (pr_Z^{XZ})_* \circ (pr_{XZ})_* \Big(pr_{YZ}^*(\beta) \cdot pr_{XY}^*(\alpha) \cdot pr_{XZ}^* \circ (pr_X^{XZ})^*(x)\Big) = \\
&= (pr_Z)_* \Big(pr_{YZ}^*(\beta) \cdot pr_{XY}^*(\alpha) \cdot pr_X^*(x)\Big)
\end{aligned}$$

(for the first equality we use the definition of $\beta \circ \alpha$ (1.1), for the second one — the second projection formula, for the third one — functorialities). This is the same expression as above.

Thus H is a functor.

For any $X,\ Y \in \mathcal{V}$ the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}\big(\mathrm{H}(X), \mathrm{H}(Y)\big)$$

is evidently a group homomorphism. Thus H is an additive functor. $\square$

Corollary 2.4. *For any $X_i \in \mathcal{V}$, $i = 1, 2$, the abelian group $\mathrm{H}(X_1 \coprod X_2)$ is a direct sum of $\mathrm{H}(X_1)$ and $\mathrm{H}(X_2)$, where the homomorphisms of inclusions and projections are given by the pull-backs and push-forwards with respect to the imbeddings $X_i \hookrightarrow X_1 \coprod X_2$.* $\square$

3. Yoneda's lemma or Manin's identity principle

For $X, Y \in \mathcal{CV}$, one could define the *tensor product* $X \otimes Y$ as the object of $\mathcal{CV}$ given by the (direct) product of varieties $X \times Y \in \mathcal{V}$. However we prefer not to use different signs for "the same thing" and shall write $X \times Y$ instead of $X \otimes Y$ in spite of the fact that it is not the product of X and Y in $\mathcal{CV}$.

Definition 3.1. Let $\alpha \in \mathrm{Hom}(X_1, X_2)$ and $\beta \in \mathrm{Hom}(Y_1, Y_2)$. Consider the projections

$$X_1 \times X_2 \xleftarrow{pr_X} X_1 \times Y_1 \times X_2 \times Y_2 \xrightarrow{pr_Y} Y_1 \times Y_2$$

and define $\alpha \otimes \beta \in \mathrm{Hom}(X_1 \times Y_1, X_2 \times Y_2)$ as

$$\alpha \otimes \beta = pr_X^*(\alpha) \cdot pr_Y^*(\beta)\ .$$

If for any $X \in \mathcal{V}$ we consider $\mathrm{CH}(X)$ as a module over itself, we get a geometric cohomology theory. In particular, for any $\alpha \in \mathrm{Hom}(X, Y)$, a homomorphism

$$\mathrm{CH}(\alpha)\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$$

is defined (2.1).

Lemma 3.2. *For any $\alpha \in \mathrm{Hom}(X, Y)$ and $\beta \in \mathrm{Hom}(Y, Z)$, one has*

$$\mathrm{CH}(\delta_X \otimes \beta)(\alpha) = \beta \circ \alpha\ .$$

Proof. The morphisms in use are shown in the diagram:

$$
\begin{array}{ccccc}
X \times X & & \xleftarrow{(id_X,\, id_X)} & & X \\
{\scriptstyle pr_{13}}\uparrow & & & & \uparrow{\scriptstyle pr_X} \\
X \times Y \times X \times Z & & \xleftarrow{\ f\ } & & X \times Y \times Z \\
& {\scriptstyle pr_{12}} \searrow & & \swarrow {\scriptstyle pr_{XY}} & \\
& & X \times Y & & \\
& {\scriptstyle pr_{24}} \searrow & & \swarrow {\scriptstyle pr_{YZ}} & \\
& & Y \times Z & & \\
& {\scriptstyle pr_{14}} \searrow & & \swarrow {\scriptstyle pr_{XZ}} & \\
& & X \times Z & &
\end{array}
$$

Note that all the three triangles commute. By definition, f is obtained from (id_X, id_X) by the base change with respect to pr_{13}. We have

$$\mathrm{CH}(\delta_X \otimes \beta)(\alpha) = (pr_{14})_* \Big(pr_{13}^*(\delta_X) \cdot pr_{24}^*(\beta) \cdot pr_{12}^*(\alpha) \Big) \ .$$

Substituting $(id_X, id_X)_*(1_X)$ for δ_X, replacing the appeared composition

$$pr_{13}^* \circ (id_X, id_X)_* \quad \text{by} \quad f_* \circ pr_X^*$$

and applying the projection formula with respect to f, we get

$$(pr_{14})_* \circ f_* \Big(pr_X^*(1_X) \cdot f^* \big(pr_{24}^*(\beta) \cdot pr_{12}^*(\alpha) \big) \Big) =$$

$$= (pr_{XZ})_* \Big(pr_{YZ}^*(\beta) \cdot pr_{XY}^*(\alpha) \Big) = \beta \circ \alpha \ .$$

□

Proposition 3.3 (**Manin's identity principle**)**.** *A correspondence*

$$\alpha \in \mathrm{Hom}(X, Y)$$

is an isomorphism (as a morphism in the category $\mathcal{CV}$) if and only if

$$\mathrm{CH}(\delta_T \otimes \alpha) \colon \mathrm{CH}(T \times X) \to \mathrm{CH}(T \times Y)$$

is a group isomorphism for any variety $T \in \mathcal{V}$.

Proof. By the Yoneda's lemma, α is an isomorphism if and only if

$$\alpha \circ \colon \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$$

is an isomorphism for any $T \in \mathcal{CV}$. In our situation we have

$$\mathrm{Hom}(T, X) = \mathrm{CH}(T \times X),\ \mathrm{Hom}(T, Y) = \mathrm{CH}(T \times Y)$$

and by the lemma $\alpha \circ = \mathrm{CH}(\delta_T \otimes \alpha)$. □

4. Gradations

Now it comes the time to remember that for any $X \in \mathcal{V}$ the Chow group $\mathrm{CH}(X)$ has a gradation

$$\mathrm{CH}(X) = \bigoplus_p \mathrm{CH}^p(X)$$

namely the gradation by codimension of cycles. One can also consider the gradation by dimension of cycles

$$\mathrm{CH}(X) = \bigoplus_p \mathrm{CH}_p(X)$$

which can be determined by the rule

$$\mathrm{CH}_p(X) = \mathrm{CH}^{\dim X - p}(X)$$

for an irreducible variety X and if $X \in \mathcal{V}$ is arbitrary then

$$\mathrm{CH}_p(X) = \coprod_k \mathrm{CH}_p(X^k)$$

where X^k are components of X (we use the superscripts for components because we want to reserve the subscripts for a special use). Here are some properties of the gradations:

- the pull-backs respect the gradation by codimension;
- the push-forwards respect the gradation by dimension;
- for every $X \in \mathcal{V}$, the ring structure on $\mathrm{CH}(X)$ respects the gradation by codimension.

Using the gradation on CH one defines a notion of degree for the correspondences. We do it only for irreducible varieties.

Definition 4.1. A correspondence $\alpha \in \mathrm{Hom}(X, Y)$ between irreducible varieties X and Y is called of degree p (notation: $\deg \alpha = p$), if

$$\alpha \in \mathrm{CH}^{\dim X + p}(X \times Y) \ .$$

Suppose that H is a geometric cohomology theory and for every $X \in \mathcal{V}$ the group $\mathrm{H}(X)$ has a gradation

$$\mathrm{H}(X) = \mathrm{H}^*(X) = \bigoplus_p \mathrm{H}^p(X) \ .$$

We shall also refer to this gradation as to the *gradation by codimension*. For an irreducible variety $X \in \mathcal{V}$, we put

$$\mathrm{H}_p(X) = \mathrm{H}^{\dim X - p}(X)$$

and for an arbitrary $X \in \mathcal{V}$, we put

$$\mathrm{H}_p(X) = \coprod_k \mathrm{H}_p(X^k)$$

where X^k are components of X. We refer to this secondary gradation as to the *gradation by dimension*.

We say that H^* is a *graded geometric cohomology theory*, if

- the pull-backs respects the gradation (by codimension);
- the push-forwards respects the gradation by dimension;
- for every $X \in \mathcal{V}$, the structure of $\mathrm{CH}(X)$-module on $\mathrm{H}(X)$ respects the gradations (by codimension).

Lemma 4.2. *Let* H^* *be a graded geometric cohomology theory,* $X, Y \in \mathcal{V}$ *be irreducible varieties, and* $\alpha \in \mathrm{Hom}(X,Y)$ *be a correspondence of degree* p*. Then* $\mathrm{H}(\alpha)\colon \mathrm{H}^*(X) \to \mathrm{H}^*(Y)$ *is a homogeneous homomorphism of degree* p*, i.e.*

$$\mathrm{H}(\alpha)\big(\mathrm{H}^q(X)\big) \subset \mathrm{H}^{p+q}(Y)\ .$$

Proof. Let $x \in \mathrm{H}^q(X)$. Since a pull-back respects the gradation, we have

$$pr_X^*(x) \in \mathrm{H}^q(X \times Y)\ .$$

Further, since $\alpha \in \mathrm{CH}^{\dim X + p}(X \times Y)$ and the multiplication respects the gradation, it holds

$$\alpha \cdot pr_X^*(x) \in \mathrm{H}^{\dim X + p + q}(X \times Y)\ .$$

Although X and Y are assumed to be irreducible, the product $X \times Y$ needs not to be. However, every component of the product is of the same dimension $\dim X + \dim Y$ ([9, Prop. 10.1(d) of Chap. III]), thereby

$$\mathrm{H}^{\dim X + p + q}(X \times Y) = \mathrm{H}_{\dim Y - p - q}(X \times Y)\ .$$

Since a push-forward respects the lower gradation, we obtain

$$\mathrm{H}(\alpha)(x) = (pr_Y)_*\Big(\alpha \cdot pr_X(x)\Big) \in \mathrm{H}_{\dim Y - p - q}(Y) = \mathrm{H}^{p+q}(Y)\ .$$

□

Definition 4.3. Let us call a correspondence $\alpha \in \mathrm{Hom}(X,Y)$ *homogeneous*, if for every component X^k of the variety X and for every component Y^l of Y the component $\alpha^{kl} \in \mathrm{Hom}(X^k, Y^l)$ of α is a correspondence of some degree.

A correspondence $\alpha \in \mathrm{Hom}(X,Y)$ will be called *dichotomous*, if for every l the component $\alpha^{kl} \in \mathrm{Hom}(X^k, Y^l)$ of α is non-zero for at most one value of k.

Lemma 4.4. *Suppose that a correspondence* $\alpha \in \mathrm{Hom}(X,Y)$ *is homogeneous, dichotomous, and an isomorphism in* $\mathcal{CV}$*; let* X^k *and* Y^l *be components of* X *and* Y*, and* $\alpha^{kl} \in \mathrm{Hom}(X^k, Y^l)$ *be the corresponding components of* α*. For every graded geometrical cohomological theory* H^* *and every* k*, one has an isomorphism of graded groups*

$$\big(\mathrm{H}(\alpha^{kl})\big)_l\colon \mathrm{H}^*(X^k) \xrightarrow[\sim]{} \coprod_{l\colon \alpha^{kl} \neq 0} \mathrm{H}^*(Y^l)[\deg \alpha^{kl}]$$

where $\mathrm{H}^*(Y^l)[\deg \alpha^{kl}]$ *denotes the group* $\mathrm{H}^*(Y^l)$ *with the gradation twisted by the integer* $\deg \alpha^{kl}$*.*

Proof. Since α is dichotomous, the homomorphism $\mathrm{H}(\alpha)$ decomposes in the direct sum over k of the homomorphisms

$$\big(\mathrm{H}(\alpha^{kl})\big)_l \colon \mathrm{H}(X^k) \to \coprod_{l\colon \alpha^{kl}\neq 0} \mathrm{H}(Y^l)\ .$$

Since $\mathrm{H}(\alpha)$ is an isomorphism, every summand is an isomorphism as well. Since α is homogeneous, the statement on the gradations follows from (4.2). □

5. Some examples of graded geometric cohomology theories

Chow groups with coefficients

A big source of examples is the theory of Chow groups with coefficients, developed in [17]. We use the terminology and notation of [17]. Let M be a *cycle module* over F [17, def. (2.1)]. For any variety $X \in \mathcal{V}$, the *Chow group* $A^*(X; M)$ *with coefficients in* M is defined [17, §5] and is graded by codimension of cycles. We put

$$\mathrm{H}^*(X) = A^*(X; M)\ .$$

Then H^* is a graded geometric cohomology theory.

For instance, the Quillen's and the Milnor's K-cohomology (see also [11, §2]) occur this way We refer to [17] for a list of further examples of cycle modules.

Notice that if M is $\mathbb{Z}$-graded, one has [17, §5]

$$A^*(X; M) = \coprod_{n\in\mathbb{Z}} A^*(X; M, n)$$

and every component $A^*(X; M, n)$ gives a graded cohomology theory as well.

Higher Chow groups

Since the properties of higher Chow groups $\mathrm{CH}^*(X, n)$ are established in [3] only for quasi-projective varieties X, for this example $\mathcal{V}$ has to be defined as the category of smooth *projective* F-varieties. Fix $n \in \mathbb{Z}$ and put

$$\mathrm{H}^*(X) = \mathrm{CH}^*(X, n)\ .$$

Then H^* is a graded geometric cohomology theory.

Adjoint K-groups

Fix $n \in \mathbb{Z}$. For $X \in \mathcal{V}$, consider the n-th Quillen's K-group $K'_n(X)$ together with the filtration by codimension of support [15, §7]. Let $\mathrm{H}^*(X)$ be the adjoint graded group. Then H^* is a graded geometric cohomology theory.

Étale cohomology

Fix $n, l \in \mathbb{Z}$. For any $p \in \mathbb{Z}$, let $\mathrm{H}^p(X)$ be the étale cohomology group $H^{n+2p}(X, \mu_l^{\otimes p})$ [14], where μ_l is the sheaf of the l-th roots of unity. Then H^* is a graded geometric cohomology theory.

6. Relative cellular spaces

Definition 6.1. A variety $X \in \mathcal{V}$ supplied with the following data:

- with a finite increasing filtration by closed (not necessarily smooth) subvarieties
$$\emptyset = X_{(-1)} \subset X_{(0)} \subset \cdots \subset X_{(n)} = X \ ;$$
- and for every successive difference $X_{(i \setminus i-1)} = X_{(i)} \setminus X_{(i-1)}$ with a vector bundle
$$p_i \colon X_{(i \setminus i-1)} \to Y_i$$
over a variety $Y_i \in \mathcal{V}$

will be called a *(relative) cellular space*. The varieties Y_i will be called the *bases of cells* and the union
$$Y = \coprod_{i=0}^{n} Y_i$$
the *(total) base* of X.

Remark 6.2. One can also say that a variety $X \in \mathcal{V}$ supplied with the filtration is a relative cellular space over Y, if the "adjoint" variety
$$\operatorname{Gr} X = \coprod_{i=0}^{n} X_{(i \setminus i-1)}$$
is a vector bundle over Y. Note that there is a morphism $\operatorname{Gr} X \to X$ given by the (locally closed) imbeddings $X_{(i \setminus i-1)} \hookrightarrow X$.

Remark 6.3. Although in the definition of a cellular space the varieties $X_{(i)}$ are not supposed to be reduced, $\operatorname{Gr} X$ is geometrically reduced (even smooth) as a vector bundle over a variety from $\mathcal{V}$. If one likes, one can introduce the reduced variety structure on every closed subset $X_{(i)} \subset X$; since the adjoint variety will be not changed by this procedure, we shall still have a cellular space over the same base.

Remark 6.4. Up to this §, it was possible to replace the Chow group by any other theory having the properties listed in the very beginning. From now on, we begin to use more specific properties; in particular, proving the theorem below, we work with the Chow groups of non-complete and non-smooth varieties.

In the notation of the definition, the graph of the vector bundle
$$p = \coprod_{i=0}^{n} p_i \colon \operatorname{Gr} X \to Y$$
is a subset of $(\operatorname{Gr} X) \times Y$. Take its closure in $X \times Y$ and denote by π the class of this closure in the Chow group $\operatorname{CH}(X \times Y)$ (by definition, the class in the Chow group of a closed subset is the sum of the classes of its irreducible components; another way of defining is to introduce the reduced variety structure on the closed subset and to use the definition of the class of a subvariety [7, §1.5]; notice that in our case the irreducible components are disjoint, i.e. coincide with the connected components).

Theorem 6.5. *Let X be a cellular space with the total base Y. The correspondence $\pi \in \operatorname{Hom}(X, Y)$ defined above is*

1. *an isomorphism;*
2. *homogeneous and dichotomous (4.3).*

Proof. 1. Instead to deal with π, we shall check that π^t is an isomorphism (that is of course equivalent to the statement on π). For this, it suffices to check that

$$\mathrm{CH}(\delta_T \otimes \pi^t)\colon \mathrm{CH}(T \times Y) \to \mathrm{CH}(T \times X)$$

is an isomorphism for any variety $T \in \mathcal{V}$ (3.3). The variety $T \times X$ has a structure of a cellular space induced from X, namely $(T \times X)_{(i)} = T \times X_{(i)}$ and the cell bases are $T \times Y_i$. Moreover, the correspondence from $\operatorname{Hom}(T \times Y, T \times X)$ we could construct via this cellular space structure coincides with $\delta_T \otimes \pi^t$. Thus it is enough only to verify that

$$\mathrm{CH}(\pi^t)\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$$

is an isomorphism.

Fix some i between 0 and n and consider the exact sequence of Chow groups [7, prop. 1.8]

$$\mathrm{CH}(X_{(i-1)}) \to \mathrm{CH}(X_{(i)}) \to \mathrm{CH}(U_i) \to 0$$

where the first arrow is the push-forward with respect to the closed imbedding $X_{(i-1)} \hookrightarrow X_{(i)}$ and the second arrow is the pull-back with respect to the open imbedding $U_i = X_{(i \setminus i-1)} \hookrightarrow X_{(i)}$. Since $p_i : U_i \to Y_i$ is a vector bundle, the pull-back $p_i^* : \mathrm{CH}(Y_i) \to \mathrm{CH}(U_i)$ is an isomorphism [7, thm. 3.3] and so we obtain an exact sequence

$$\mathrm{CH}(X_{(i-1)}) \to \mathrm{CH}(X_{(i)}) \to \mathrm{CH}(Y_i) \to 0\ .$$

Now we are going to fulfill the following program:

- **a):** to construct a splitting of the epimorphism;
- **b):** to show that the left-hand side arrow is injective;
- **c):** to show that the resulting (obtained by induction on i) isomorphism

$$\mathrm{CH}(Y) = \coprod_i \mathrm{CH}(Y_i) \xrightarrow[\sim]{} \mathrm{CH}(X)$$

coincides with $\mathrm{CH}(\pi^t)$.

a) Denote by $V_i \subset Y_i \times U_i$ the "transposition" of the graph of $p_i\colon U_i \to Y_i$; let Z_i be its closure in $Y_i \times X_{(i)}$ and let

$$Y_i \xleftarrow{pr_{Y_i}} Z_i \xrightarrow{pr_{X_{(i)}}} X_{(i)}$$

be the projections. In the following lemma we consider Z_i as a reduced variety.

Lemma 6.6. *The following square commutes:*

$$\begin{tikzcd}
\mathrm{CH}(X_{(i)}) \arrow[r, "in_{U_i}^*"] & \mathrm{CH}(U_i) \\
\mathrm{CH}(Z_i) \arrow[u, "(pr_{X_{(i)}})_*"] & \mathrm{CH}(Y_i) \arrow[l, "pr_{Y_i}^*"'] \arrow[u, "p_i^*"']
\end{tikzcd}$$

Proof. Note that $pr_{Y_i}^*$ is defined since Y_i is smooth (Z_i might be not) [7, §8.1] while $in_{U_i}^*$ is defined since in_{U_i} is flat (the variety $X_{(i)}$ is not supposed to be regular) [7, thm. 1.7].

Consider the following commutative diagram:

$$\begin{tikzcd}
X_{(i)} & U_i \arrow[l, hook', "in_{U_i}"'] & \\
Z_i \arrow[u, "pr_{X_{(i)}}"] \arrow[dr, "pr_{Y_i}"'] & V_i \arrow[l, hook', "in_{V_i}"'] \arrow[u, "pr_{U_i}"] \arrow[d, "pr^{V_i}"] & U_i \arrow[l, "{(p_i, id_{U_i})}"'] \arrow[ul, "id_{U_i}"'] \arrow[dl, "p_i"] \\
 & Y_i &
\end{tikzcd}$$

Since V_i is closed in $Y_i \times U_i$ as a graph of a morphism of varieties, we have

$$V_i = Z_i \cap (Y_i \times U_i)$$

(first of all, this formula holds on the level of sets; since all varieties involved are reduced (Z_i is so by construction, Y_i is from $\mathcal{V}$, U_i is a vector bundle over Y_i, V_i is isomorphic to U_i) the same formula holds on the level of varieties as well, where $\cap$ means the fiber product of the imbeddings). Thus the square is cartesian and so [7, prop. 1.7]

$$in_{U_i}^* \circ (pr_{X_{(i)}})_* = (pr_{U_i})_* \circ in_{V_i}^*$$

since in_{U_i} is flat (an open imbedding) and $pr_{X_{(i)}}$ proper. For any $\alpha \in \mathrm{CH}(Y_i)$, we have

$$\begin{aligned}
in_{U_i}^* \circ (pr_{X_{(i)}})_* \circ pr_{Y_i}^*(\alpha) &= (pr_{U_i})_* \circ in_{V_i}^* \circ pr_{Y_i}^*(\alpha) = \\
= (pr_{U_i})_* \circ (pr^{V_i})^*(\alpha) &= (pr_{U_i})_* \circ (p_i, id_{U_i})_* \circ (p_i, id_{U_i})^* \circ (pr^{V_i})^*(\alpha) = \\
&= (id_{U_i})_* \circ p_i^*(\alpha) = p_i^*(\alpha)
\end{aligned}$$

(for the third equality notice that the composition $(p_i, id_{U_i})_* \circ (p_i, id_{U_i})^*$ is an identity since $(p_i, id_{U_i})\colon U_i \to V_i$ is an isomorphism). □

Due to the lemma, we get a splitting of the epimorphism

$$\mathrm{CH}(X_{(i)}) \twoheadrightarrow \mathrm{CH}(Y_i)\ ,$$

namely the composition $(pr_{X_{(i)}})_* \circ pr_{Y_i}^*$. So, **a)** is complete.

b) Let us extend the exact sequence of Chow groups to the left by using the K-cohomology groups [17]:

$$H^*(X_{(i)}, K_{*+1}) \to H^*(U_i, K_{*+1}) \to \mathrm{CH}(X_{(i-1)}) \to \mathrm{CH}(X_{(i)}) \to \mathrm{CH}(U_i) \to 0$$

where for a variety T

$$H^*(T, K_{*+1}) = \bigoplus_{l \geq 0} H^l(T, K_{l+1}) \; .$$

The third arrow is injective, if the first arrow is surjective, and it is really surjective because of the following

Lemma 6.7. *The square commutes:*

$$\begin{array}{ccc} H^*(X_{(i)}, K_{*+1}) & \xrightarrow{in^*_{U_i}} & H^*(U_i, K_{*+1}) \\ {\scriptstyle (pr_{X_{(i)}})_*}\Big\uparrow & & \Big\uparrow{\scriptstyle p_i^*} \\ H^*(Z_i, K_{*+1}) & \xleftarrow{pr^*_{Y_i}} & H^*(Y_i, K_{*+1}) \end{array}$$

and p_i^* is an isomorphism.

Proof. Everything goes in the same way as in the proof of the previous lemma. The crucial point is that we have a construction of the pull-back (with "right" properties) for any morphism into a smooth variety [17, §12]; p_i^* is an isomorphism by [17, prop. 8.6]. □

c) The steps **a)** and **b)** together give an isomorphism

$$\coprod_i \mathrm{CH}(Y_i) \xrightarrow[\sim]{} \mathrm{CH}(X)$$

where for every i the corresponding map $\mathrm{CH}(Y_i) \to \mathrm{CH}(X)$ is defined as the composition

$$\mathrm{CH}(Y_i) \xrightarrow{pr^*_{Y_i}} \mathrm{CH}(Z_i) \xrightarrow{(pr_{X_{(i)}})_*} \mathrm{CH}(X_{(i)}) \xrightarrow{(in_{X_{(i)}})_*} \mathrm{CH}(X) \; .$$

To complete the proof of the first statement of the theorem, it remains to show that the composition written up coincides with the homomorphism $\mathrm{CH}((\pi_i)^t)$, where $\pi_i \in \mathrm{CH}(X \times Y_i)$ is the component of $\pi \in \mathrm{CH}(X \times Y)$. Since the element $(\pi_i)^t$ coincides with the class $[Z_i] \in \mathrm{CH}(Y_i \times X)$ of the subvariety $Z_i \subset Y_i \times X$, it suffices to verify the following general fact:

Lemma 6.8. *Let X, $Y \in \mathcal{V}$, $Z \subset Y \times X$ be a closed subvariety (we do not assume that $Z \in \mathcal{V}$ and we do not assume that Z is irreducible) and let*

$$Y \xleftarrow{pr^Z_Y} Z \xrightarrow{pr^Z_X} X$$

be the projections. The composition

$$\mathrm{CH}(Y) \xrightarrow{(pr^Z_Y)^*} \mathrm{CH}(Z) \xrightarrow{(pr^Z_X)_*} \mathrm{CH}(X)$$

coincides with the homomorphism $\mathrm{CH}([Z])$ *(see* [7, §1.5] *for the definition of* $[Z]$*).*

Proof. Consider a diagram:

$$\begin{array}{ccc} & & Y \\ & {}^{pr_Y^Z}\nearrow & \uparrow {}^{pr_Y} \\ Z & \overset{in}{\hookrightarrow} & Y \times X \\ & {}_{pr_X^Z}\searrow & \downarrow {}_{pr_X} \\ & & X \end{array}$$

By the definition of $\mathrm{CH}([Z])$ (2.1), for any $\alpha \in \mathrm{CH}(Y)$, we have

$$\mathrm{CH}([Z])(\alpha) = (pr_X)_*\Big([Z] \cdot pr_Y^*(\alpha)\Big) .$$

Since the multiplication by $[Z]$ on $\mathrm{CH}(Y \times X)$ coincides with the composition $in_* \circ in^*$, we get

$$(pr_X)_* \circ in_* \circ in^* \circ pr_Y^*(\alpha) = (pr_X^Z)_* \circ (pr_Y^Z)^*(\alpha)$$

since both the triangles commutes. □

2. Let Y^l be a component of the total base Y. Since the morphism $p : \operatorname{Gr} X \to Y$ is a vector bundle, the inverse image $T = p^{-1}(Y^l)$ is irreducible and thereafter contained in a component of X, say in X^k. It is clear that for every other $k' \neq k$ the component $\pi^{k'l}$ of π is zero. It means that the correspondence π is dichotomous (4.3).

Moreover, π^{kl} coincides with the class of the closure in $X^k \times Y^l$ of the graph of the morphism $p_{|_T} : T \to Y^l$. Since the graph is irreducible (since T is), the closure is irreducible as well and thus the component π^{kl} is the class of a simple cycle. Thereby π is homogeneous (4.3). The theorem is proven. □

Remark 6.9. Although π and π^t are isomorphisms acting in the mutually inverse directions they are not mutually inverse (in general).

Remark 6.10. In the absolute case, i.e. in the case where $Y_i = \operatorname{Spec} F$ for all i, the proof of the theorem is much more simple. For instance, a splitting for step **a)** is given simply by the pull-back with respect to the structure morphism $X_{(i)} \to \operatorname{Spec} F$.

Corollary 6.11. *Denote by X^k the components of the cellular spaces X, by Y^l the components of its total base and by $\pi^{kl} \in \operatorname{Hom}(X^k, Y^l)$ the components of the correspondence π. In every graded geometric cohomology theory H^*, one has (for any k) an isomorphism of graded groups*

$$\big(\mathrm{H}(\pi^{kl})\big)_l \colon \mathrm{H}^*(X^k) \to \coprod_{l\colon \pi^{kl} \neq 0} \mathrm{H}^*(Y^l)[-r_l]$$

where r_l is the rank of the vector bundle over Y^l given by the cellular structure on X.

Proof. Follows from (4.4), the theorem and an easy observation that $\deg \pi^{kl} = -r_l$. □

7. Operations with relative spaces

Definition 7.1. A variety X supplied with a filtration by closed subvarieties and with a morphism $p : \operatorname{Gr} X \to Y$ is called a *relative space* over Y (so, a relative space is a cellular space if and only if $X, Y \in \mathcal{V}$ and p admits a structure of a vector bundle).

Definition 7.2 (**Product**)**.** Let X and X' be relative spaces over Y and Y' respectively (we use the standard notation for the relative structure data on X and the $'$-notation for the data on X'). The product of varieties $X \times X'$ can be supplied by a structure of a relative space over $Y \times Y'$ as follows:
we use the lexicographic ordering on the set

$$\{0, 1, \dots, n\} \times \{0, 1, \dots, n'\},$$

i.e. $(j, j') < (i, i')$, if $j < i$ or if $j = i$ and $j' < i'$; we put

$$(X \times X')_{(i,i')} = \bigcup_{(j,j') \leq (i,i')} X_{(j)} \times X'_{(j')} = X_{(i-1)} \times X' \cup X_{(i)} \times X'_{(i')} \subset X \times X' \,;$$

thus $\operatorname{Gr}(X \times X') = \operatorname{Gr} X \times \operatorname{Gr} X'$ and we have a morphism

$$p \times p' \colon \operatorname{Gr}(X \times X') \to Y \times Y' \,.$$

The definition can be expanded to the case of several (finitely many) factors in the evident way. A product of cellular spaces is a cellular space as well.

Example 7.3. Let X be a relative space over Y and let $T \in \mathcal{V}$. Then $T \times X$ is a relative space over $T \times Y$. This structure is a particular case of (7.2), if we consider T with the (trivial) structure over itself.

Definition 7.4 (**Composition**)**.** Let X be a relative space over Y and suppose that Y is in turn a relative space over Z. Then X is a relative space over Z in the following way: one takes the refinement of the filtration on X such that $\operatorname{Gr}_{\text{new}} X$ for this new filtration coincides with the inverse image of $\operatorname{Gr} Y \subset Y$ with respect to the morphism $\operatorname{Gr} X \to Y$; the structure morphism is given by the composition

$$\operatorname{Gr}_{\text{new}} X \to \operatorname{Gr} Y \to Z \,.$$

Remark 7.5. The structure (7.2) can be defined via (7.3) and (7.4): if X (resp. X') is a relative space over Y (resp. Y'), then $X \times X'$ is a relative space over $Y \times X'$, which is in turn a relative space over $Y \times Y'$; the composition structure on $X \times X'$ coincides with (7.2).

Definition 7.6 (**Restriction**)**.** Let X be a relative space over Y and let $X' \subset X$, $Y' \subset Y$ be some subvarieties. Suppose that the restriction of the morphism $\operatorname{Gr} X \to Y$ to $X' \cap \operatorname{Gr} X$ is a morphism into Y'. Then X' together with the induced filtration $X'_{(i)} = X' \cap X_{(i)}$ is a relative space over Y'. We refer to this structure on X' as to the structure induced from X.

Part 2. Cohomology of isotropic flag varieties

It seems to be rather convenient to use the language of functors of points (developed in [5]) while working with flag varieties. In §8 we recall some basic notions and facts and fix certain terminology. The terminology used here slightly differs from that of [5]. For instance, we like to preserve the usual meaning for the word "scheme" and will not use this word for functors. To avoid the difficulty that the F-functors (defined below) do not form a category (because the morphisms of an F-functor to another one do not always form a set) we speak only of categories of representable F-functors. A more refined approach, using the notion of *universe*, can be found in [5].

We consider only rings and algebras which are associative and unital; homomorphisms of rings or algebras are supposed to respect 1. If the contrary is not explicitly stated, a module (or vector space) means a *right* module (*right* vector space).

In §8, F is an arbitrary commutative ring; starting from §9, F is a field.

8. Language of functors of points

Let F be a commutative ring (in our applications F will be a field). Covariant functors $F\text{-}\mathfrak{alg} \to \mathfrak{Sets}$ of the category of commutative F-algebras into the category of sets will be called *F-functors*. For any F-scheme (i.e. a scheme over F) X one has an F-functor X, namely

$$R \in F\text{-}\mathfrak{alg} \quad \mapsto \quad X(R) = \mathrm{Mor}_F(\operatorname{Spec} R, X)$$

called the *functor of points* of the F-scheme X. An F-functor isomorphic to the functor of points of an F-scheme (say, X) will be called *representable* (or *represented by* X); the usual categorical sense of the expression "representable functor" means in our terms "F-functor represented by an affine F-scheme" and will be expressed by the words "*affine* F-functor". The evident functor from the category of F-schemes to the category of representable F-functors (a morphism in the latter category is by definition simply a natural transformation of functors) is an equivalence of categories [5, thm. de comparaison on p. 18]. In particular, the category of schemes is equivalent to that of representable $\mathbb{Z}$-functors.

So, any representable F-functor $\mathcal{F}$ determines a unique (up to a canonical isomorphism) F-scheme which is called the *geometric realization* of $\mathcal{F}$ or F-scheme *representing* $\mathcal{F}$ and will be denoted by $\mathcal{F}$ as well.

Example 8.1. Let V be a free F-module of finite rank. The F-functor V with $V(R) = V \otimes_F R$ (defined on the morphisms in the natural way) is called affine space and is represented by the variety "affine space V".

Let $\mathcal{F}$ and $\mathcal{G}$ be F-functors; $\mathcal{G}$ is called a *subfunctor* of $\mathcal{F}$ if $\mathcal{G}(R)$ is a subset of $\mathcal{F}(R)$ and the map $\mathcal{G}(\varphi)\colon \mathcal{G}(R) \to \mathcal{G}(S)$ is the restriction of the map $\mathcal{F}(\varphi)$ for any $R, S \in F\text{-}\mathfrak{alg}$ and any $\varphi \in \mathrm{Hom}_{F\text{-}\mathfrak{alg}}(R, S)$.

The *inverse image* of a subfunctor $\mathcal{G} \subset \mathcal{F}$ with respect to a morphism of F-functors $\mathcal{F}' \to \mathcal{F}$ is by definition the subfunctor $\mathcal{G}' \subset \mathcal{F}'$ with $\mathcal{G}'(R)$ being the inverse image of the subset $\mathcal{G}(R) \subset \mathcal{F}(R)$ with respect to the map $\mathcal{F}'(R) \to \mathcal{F}(R)$.

Let R be a commutative F-algebra. Consider the F-functor $\operatorname{Spec} R$; we have

$$\operatorname{Spec} R(S) = \operatorname{Hom}_{F\text{-}\mathfrak{alg}}(R, S) \text{ for any } S \in F\text{-}\mathfrak{alg}\ .$$

Fixing an ideal $I \subset R$, one can construct two subfunctors of $\operatorname{Spec} R$: for every S the corresponding subsets of $\operatorname{Hom}(R, S)$ are

$$\big\{\varphi \in \operatorname{Hom}(R, S) |\ \varphi(I) \cdot S = S\big\} \text{ and } \big\{\varphi \in \operatorname{Hom}(R, S) |\ \varphi(I) = 0\big\}\ .$$

A subfunctor $\mathcal{G}$ of an F-functor $\mathcal{F}$ is called *open* (resp. *closed*) if for every $R \in F\text{-}\mathfrak{alg}$ and every morphism $\operatorname{Spec} R \to \mathcal{F}$ the inverse image of $\mathcal{G}$ is a subfunctor of $\operatorname{Spec} R$ of the first (resp. second) type for an appropriate ideal $I \subset R$. This definition is of practical use; note the that morphisms $\operatorname{Spec} R \to \mathcal{F}$ are in one-to-one correspondence with the set $\mathcal{F}(R)$. It turns out that every open (resp. closed) subfunctor of a representable F-functor $\mathcal{F}$ is representable (and moreover) by a unique open (resp. closed) subscheme of the F-scheme $\mathcal{F}$.

One says that a family of subfunctors $\{\mathcal{G}_\alpha\}$ of an F-functor $\mathcal{F}$ *covers* $\mathcal{F}$, if

$$\mathcal{F}(R) = \bigcup_\alpha \mathcal{G}_\alpha(R)$$

for any F-algebra R *which is a field*. In the case where $\mathcal{F}$ is representable and every $\mathcal{G}_\alpha$ is open or closed (or locally closed) $\mathcal{F}$ is covered by $\mathcal{G}_\alpha$ if and only if the F-scheme $\mathcal{F}$ is covered by the corresponding subschemes.

In contrast to that, the intersection of a family of subfunctors can be (and is) defined in the simple way:

$$\left(\bigcap_\alpha \mathcal{G}_\alpha\right)(R) = \bigcap_\alpha \mathcal{G}_\alpha(R) \text{ for any } R \in F\text{-}\mathfrak{alg}\ .$$

An F-functor $\mathcal{F}$ is called *local* if for any $R \in F\text{-}\mathfrak{alg}$ and any elements $r_1, \dots, r_l \in R$ generating the unit ideal the sequence of maps of sets

$$\mathcal{F}(R) \to \prod_{i=1}^{l} \mathcal{F}(R_{r_i}) \rightrightarrows \prod_{i,j=1}^{l} \mathcal{F}(R_{r_i r_j})$$

is exact ("exactness" also means injectivity of the first map) or, in other words, if $\mathcal{F}$ is a sheaf in the Zariski topology on $F\text{-}\mathfrak{alg}$. It turns out that an F-functor $\mathcal{F}$ is representable if and only if it is local and admits a covering by open affine subfunctors [5, thm. de comparaison on p. 18] (in this case, the geometric realization of $\mathcal{F}$ is obtained as the direct limit of the schemes $\operatorname{Spec} R$ for all morphisms of F-functors $\operatorname{Spec} R \to \mathcal{F}$, where the limit is taken with respect to the morphisms of schemes $\operatorname{Spec} R \to \operatorname{Spec} R'$ such that the corresponding morphism of F-functors is a morphism over $\mathcal{F}$). So, one can "forget" on schemes and work with the category of such F-functors instead. However we do not do it because we like to use several results and notions from the theory of schemes which is developed better as one of F-functors.

For two F-functors $\mathcal{F}_1$ and $\mathcal{F}_2$ their (direct) product $\mathcal{F}_1 \times \mathcal{F}_2$ is the F-functor with

$$(\mathcal{F}_1 \times \mathcal{F}_2)(R) = \mathcal{F}_1(R) \times \mathcal{F}_2(R) \quad \text{and} \quad (\mathcal{F}_1 \times \mathcal{F}_2)(\varphi) = \mathcal{F}_1(\varphi) \times \mathcal{F}_2(\varphi) \; .$$

For two morphisms of F-functors $\mathcal{F}_1 \to \mathcal{F}$ and $\mathcal{F}_2 \to \mathcal{F}$ the fiber product $\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ over $\mathcal{F}$ is the subfunctor of $\mathcal{F}_1 \times \mathcal{F}_2$ with

$$\big(\mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2\big)(R) = \mathcal{F}_1(R) \times_{\mathcal{F}(R)} \mathcal{F}_2(R)$$

(the intersection of two subfunctors and, more generally, the inverse image of a subfunctor with respect to a morphism of F-functors are examples of fiber products we already met).

Let L be a commutative F-algebra. Since an L-scheme is “the same” as an F-scheme together with a morphism to $\operatorname{Spec} L$ over F the category of (representable) L-functors should be equivalent to the category of (representable) “F-functors over L” whose objects are (representable) F-functors together with a morphism to the F-functor $\operatorname{Spec} L$. It really is: if $\mathcal{F} \to \operatorname{Spec} L$ is an object of the latter category the corresponding L-functor $\mathcal{G}$ is defined directly as follows: $\mathcal{G}(R)$ for $R \in L\text{-}\mathfrak{alg}$ is the inverse image of the structural homomorphism of R with respect to the map of sets

$$\mathcal{F}(R) \to \operatorname{Spec}(L, R) = \operatorname{Hom}_{F\text{-}\mathfrak{alg}}(L, R)$$

where R is considered as an F-algebra in the natural way. Conversely, if $\mathcal{G}$ is an L-functor, one constructs an F-functor $\mathcal{F}$ by putting

$$\mathcal{F}(R) = \coprod_{\operatorname{Hom}_{F\text{-}\mathfrak{alg}}(L,R)} \mathcal{G}(R)$$

and takes the evident morphism $\mathcal{F} \to \operatorname{Spec} L$.

For example, if $\mathcal{F}$ is an F-functor the product $\mathcal{F} \times \operatorname{Spec} L$ is an F-functor over L. The corresponding L-functor is denoted by $\mathcal{F} \otimes_F L$. For any $R \in L\text{-}\mathfrak{alg}$ one has $(\mathcal{F} \otimes_F L)(R) = \mathcal{F}(R)$ where from the right-hand side R is considered as an F-algebra.

Another example is the definition of a fiber. If $f : \mathcal{F}' \to \mathcal{F}$ is a morphism of F-functors and $x \in \mathcal{F}(R)$, where $R \in F\text{-}\mathfrak{alg}$ is an R-point of $\mathcal{F}$, the fiber product of F-functors $\operatorname{Spec} R$ and $\mathcal{F}'$ over $\mathcal{F}$ (where $\operatorname{Spec} R \to \mathcal{F}$ is the morphism defined by $x \in \mathcal{F}(R)$) is an F-functor over R in the natural way. The corresponding R-functor is called the *fiber* of f over x. To obtain its value on $S \in R\text{-}\mathfrak{alg}$, one should take the image of $x \in \mathcal{F}(R)$ in $\mathcal{F}(S)$, where S is considered as an F-algebra, and the inverse image of the result in $\mathcal{F}'(S)$.

The definitions given are evidently related to the corresponding scheme definitions (in the theory of schemes the definition of fiber is mostly applied to the case where x is a *geometric* point, i.e. where R is a field).

It turns out that the language of functors is also suitable for describing sheaves, e.g. sheaves of modules. We discuss it because we have no reference for this matter, however we discuss it shortly since it is only the notion of a vector bundle obtained in the conclusion we do really need for the consequence.

Let $\mathcal{F}$ be a representable F-functor. An algebra $R \in F\text{-}\mathfrak{alg}$ together with a fixed element from $\mathcal{F}(R)$ will be called an $\mathcal{F}$-*algebra*; the category of $\mathcal{F}$-algebras (by definition, a morphism in this category is a homomorphism of the underlying F-algebras respecting the fixed points) will be denoted by $\mathcal{F}\text{-}\mathfrak{alg}$ (note that in the case where $\mathcal{F}$ is the spectrum of an F-algebra L the category $\mathcal{F}\text{-}\mathfrak{alg}$ "coincides" with $L\text{-}\mathfrak{alg}$). Analogously to happened above, considering F-functors over $\mathcal{F}$ is equivalent to considering $\mathcal{F}$-functors, i.e. functors from $\mathcal{F}\text{-}\mathfrak{alg}$ to $\mathfrak{Sets}$. For instance, the category of $\mathcal{F}$-schemes, i.e. schemes over $\mathcal{F}$, is equivalent to the category of representable $\mathcal{F}$-functors. One can give the following definitions:

Definition 8.2. An $\mathcal{F}$-functor $\mathcal{G}$ is called a *sheaf of modules* (over $\mathcal{F}$), if it is local and for every $R \in \mathcal{F}\text{-}\mathfrak{alg}$ the set $\mathcal{G}(R)$ is supplied with a structure of module over the ring R such that for any homomorphism $R \to S$ of $\mathcal{F}$-algebras the map $\mathcal{G}(R) \to \mathcal{G}(S)$ is a homomorphism of R-modules.

Definition 8.3. A sheaf of modules $\mathcal{G}$ over $\mathcal{F}$ is called *quasi-coherent*, if for any homomorphism of $\mathcal{F}$-algebras $R \to S$ the induced homomorphism of S-modules $\mathcal{G}(R) \otimes_R S \to \mathcal{G}(S)$ is an isomorphism.

Definition 8.4. A quasi-coherent sheaf of modules $\mathcal{G}$ over $\mathcal{F}$ is called *coherent*, if for every $R \in \mathcal{F}\text{-}\mathfrak{alg}$ the R-module $\mathcal{G}(R)$ is finitely generated (careful: in the theory of schemes, there are several definitions of quasi-coherentness which coincide only in the noetherian case [9, chap. II, exercise 5.4]; the definition given here is equivalent to [9, def. on p. 111]).

Definition 8.5. A coherent sheaf of modules $\mathcal{G}$ is called *locally free* (or a *vector bundle*), if all modules $\mathcal{G}(R)$ are projective.

Notice that in this language the notions of a locally free sheaf of modules and a vector bundle are not only equivalent, they coincide! The only thing which should be mentioned additionally is an easy observation that a locally free sheaf of modules, defined as above, admits an open covering by affine subfunctors and hence is representable.

Summarizing and decoding, one can give the following

Definition 8.6. A morphism $f : \mathcal{G} \to \mathcal{F}$ of representable F-functors is a *vector bundle*, if for any $R \in F\text{-}\mathfrak{alg}$ and any R-point $x \in \mathcal{F}(R)$ the inverse image $f(R)^{-1}(x)$ of x with respect to the map of sets $f(R) : \mathcal{G}(R) \to \mathcal{F}(R)$ is supplied with a structure of a finitely generated projective R-module such that for any homomorphism of F-algebras $\varphi \colon R \to S$ the map

$$f(R)^{-1}(x) \to f(S)^{-1}(y)$$

where y is the image of $x \in \mathcal{F}(R)$ in $\mathcal{F}(S)$, is a homomorphism of R-modules and the induced homomorphism of S-modules

$$S \otimes_R f(R)^{-1}(x) \to f(S)^{-1}(y)$$

is an isomorphism.

Remark 8.7. We have defined a vector bundle as a morphism with an additional structure. Sometimes in the literature and also here, a morphism (without additional structure) is called vector bundle, if it *admits* a structure of a vector bundle in the above sense.

It is easy to check that the definition of vector bundles given is equivalent to common scheme definitions, e.g. to the (extremely non-invariant) definition from [9, chap. II, exercise 5.18].

9. Grassmanians

From now on F is a field, V a finite-dimensional vector space over F.

Definition 9.1. The (full) *grassmanian* $\mathbb{\Gamma}(V)$ of V is the F-functor defined as follows:

- for any $R \in F\text{-}\mathfrak{alg}$, the set $\mathbb{\Gamma}(V)(R)$ consists of all direct summands of the R-module $V_R = V \otimes_F R$; in other words, $\mathbb{\Gamma}(V)(R)$ consists of all (projective) submodules $N \subset V_R$ such that the quotient V_R/N is projective (as well);
- for any $\varphi \in \operatorname{Hom}_{F\text{-}\mathfrak{alg}}(R, S)$, the map $\mathbb{\Gamma}(V)(R) \to \mathbb{\Gamma}(V)(S)$ is defined with the help of tensoring by S over R:

$$N \subset V_R \quad \mapsto \quad N \otimes_R S \subset V_S\ .$$

Proposition 9.2. *The F-functor $\mathbb{\Gamma}(V)$ is represented by a complete smooth F-variety.*

Proof. First, one checks that the functor $\mathbb{\Gamma}(V)$ is local. Let $R \in F\text{-}\mathfrak{alg}$ and let $r_1, \dots, r_l \in R$ be some elements generating the unit ideal. For any $N \in \mathbb{\Gamma}(V)(R)$, it holds

$$N = V_R \cap \prod_{i=1}^{l} N_{r_i} \quad \subset \prod_{i=1}^{l} V_{R_{r_i}}\ .$$

Thus the first arrow in the sequence

$$\mathbb{\Gamma}(V)(R) \to \prod_{i=1}^{l} \mathbb{\Gamma}(V)(R_{r_i}) \rightrightarrows \prod_{i,j=1}^{l} \mathbb{\Gamma}(V)(R_{r_i r_j})$$

is injective. If for every i we are given an R_{r_i}-module $N_i \in \mathbb{\Gamma}(V)(R_{r_i})$, the condition that $(N_i)_{i=1}^{l}$ "goes to zero to the right" in our sequence means compatibility on the intersections $\operatorname{Spec} R_{r_i} \cap \operatorname{Spec} R_{r_j}$, so we can glue the corresponding sheaves on all $\operatorname{Spec} R_{r_i}$ together to a locally free sheaf on $\operatorname{Spec} R$. Let $N \subset V_R$ be the corresponding (projective) R-module. The quotient V_R/N is projective, since every localization $(V_R/N)_{r_i}$ is. Thus $N \in \mathbb{\Gamma}(V)(R)$.

In order to construct an open affine covering, let us take an epimorphism $p\colon V \twoheadrightarrow U$ and consider the subfunctor $\mathcal{U} \subset \Gamma(V)$ with

$$\mathcal{U}(R) = \left\{N \in \Gamma(V)(R) \mid (p_R)_{|_N}\colon N \to U_R \text{ is an isomorphism}\right\}$$

where $p_R\colon V_R \to U_R$ is the homomorphism of R-modules induced by p. First of all, fixing a splitting of p and setting $U' = \operatorname{Ker} p$, we get a bijection

$$\mathcal{U}(R) \simeq \operatorname{Hom}_R(U_R, U'_R) \simeq \operatorname{Hom}_F(U, U') \otimes_F R$$

for any R. It gives an isomorphism of the F-functors, identifying $\mathcal{U}$ with the affine space $\operatorname{Hom}_F(U, U')$.

To show that the subfunctor $\mathcal{U}$ is open in $\Gamma(V)$, take a morphism $\operatorname{Spec} R \to \Gamma(V)$ and the corresponding R-point $N \in \Gamma(V)(R)$. We have

$$N \in \mathcal{U}(R) \quad \Leftrightarrow \quad p_{R|_N} \text{ is an isomorphism} \quad \Leftrightarrow \quad \operatorname{Coker}(p_{R|_N}) = 0 \text{ and } \operatorname{rk} N \le n$$

where $n = \dim U$. The condition $\operatorname{rk} N \le n$ can be replaced by vanishing of the $(n+1)$-th exterior power $\Lambda^{n+1}N$. Put $P = \operatorname{Coker}(p_{R|_N}) \oplus \Lambda^{n+1}N$. For any $\varphi \in \operatorname{Hom}_{F\text{-}\mathfrak{alg}}(R, S)$, we consequently have

$$N \otimes_R S \in \mathcal{U}(S) \quad \Leftrightarrow \quad P \otimes_R S = 0 \quad \Leftrightarrow \quad \varphi(\operatorname{Ann} P) \cdot S = S$$

where $\operatorname{Ann} P$ stays for the annihilator of P; the latter equivalence holds by

Lemma 9.3. *Let $\varphi\colon R \to S$ be a homomorphism of commutative rings, P a finitely generated R-module. One has $P \otimes_R S = 0$, if and only if the subset $\varphi(\operatorname{Ann} P) \subset S$ generates the unit ideal.*

Proof. If $\varphi(\operatorname{Ann} P)$ generates the unit ideal, the S-module $P \otimes_R S$ is generated by the elements $p \otimes \varphi(r)$ with $r \in \operatorname{Ann} P$ and $p \in P$; since $p \otimes \varphi(r) = p \cdot r \otimes 1 = 0$, it implies that $P \otimes_R S = 0$.

Now suppose that $\varphi(\operatorname{Ann} P) \cdot S \ne S$. Take a maximal ideal $\mathfrak{M} \subset S$ containing $\varphi(\operatorname{Ann} P)$ and put $\mathfrak{p} = \varphi^{-1}(\mathfrak{M})$. Since $\mathfrak{p}$ is a prime ideal containing $\operatorname{Ann} P$ and since P is finitely generated, we have $P_{\mathfrak{p}} \ne 0$. By the Nakoyama lemma (we use again P is finitely generated) $(P/P\mathfrak{p})_{\mathfrak{p}} \ne 0$. Since $(R/\mathfrak{p})_{\mathfrak{p}} \hookrightarrow S/\mathfrak{M}$ is a field extension, the $S/\mathfrak{M}$-module $P \otimes_R S/\mathfrak{M}$ is non-zero as well. Hence $P \otimes_R S \ne 0$. $\square$

We see that the inverse image of the subfunctor $\mathcal{U}$ with respect to a morphism $\operatorname{Spec} R \to \Gamma(V)$ is the open subfunctor of $\operatorname{Spec} R$ defined by the ideal $\operatorname{Ann} P \subset R$. Thus $\mathcal{U}$ is an open subfunctor of the grassmanian.

Lemma 9.4. *Finitely many of the subfunctors $\mathcal{U}$ cover $\Gamma(V)$.*

Proof. Let E be an F-basis of V. By definition, we have to check that the set $\Gamma(V)(L)$ is covered by (finitely many of) the subsets $\mathcal{U}(L)$ for any *field* $L \in F\text{-}\mathfrak{alg}$.

Take an arbitrary $N \in \Gamma(V)(L)$ and consider a maximal subcollection of E such that the L subspace $U'_L \subset V_L$ spanned on this subcollection has the trivial intersection with N. Then $U'_L \oplus N = V_L$ and consequently $N \in \mathcal{U}(L)$ for the subfunctor $\mathcal{U}$ given by $U = V/U'$ and the projection $p\colon V \twoheadrightarrow U$.

Therefore, taking for every subcollection of E the corresponding subfunctor $\mathcal{U}$, we obtain a (finite) covering of $\mathbb{\Gamma}(V)$. $\square$

So, we get a finite open covering by affine spaces. Consequently, the grassmanian is represented by a smooth F-scheme of finite type.

This scheme is proper by the valuative criterion [5, cor. 2.9 and 2.10 on p. 134] or [9, chap. II, thm. 4.7]: for any $R \in F\text{-}\mathfrak{alg}$ which is a (discrete) valuation ring the map

$$\mathbb{\Gamma}(V)(R) \to \mathbb{\Gamma}(V)(\text{quotient field of } R)$$

is evidently bijective (injectivity means separateness, surjectivity means universal closeness). $\square$

Remark 9.5. One can also directly check that $\mathbb{\Gamma}(V)$ is separated: the diagonal is the subfunctor of $\mathbb{\Gamma}(V)^{\times 2}$ with the set of R-points

$$\big\{(N_1, N_2) \in \mathbb{\Gamma}(V)^{\times 2} |\ N_1 \subset N_2 \text{ and } N_2 \subset N_1\big\}$$

and hence it is the intersection of two closed flag subfunctors in $\mathbb{\Gamma}(V)^{\times 2}$ (11.2).

Remark 9.6. We shall construct several closed subfunctors of the products $\mathbb{\Gamma}(V)^{\times m}$, $m \in \mathbb{N}$. They will be automatically represented by complete F-varieties. However we shall need additional attempts to show that some of them will be smooth or at least (geometrically) reduced.

Cellular structure

Let $p : V \twoheadrightarrow W$ be an epimorphism of vector F-spaces. We define an increasing filtration

$$\emptyset = \mathbb{\Gamma}(V)_{(-1)} \subset \mathbb{\Gamma}(V)_{(0)} \subset \cdots \subset \mathbb{\Gamma}(V)_{(\dim W)} = \mathbb{\Gamma}(V)$$

as follows:

$$\mathbb{\Gamma}(V)_{(i)} = \big\{N \in \mathbb{\Gamma}(V) |\ \Lambda^{i+1} p_R(N) = 0\big\}$$

where $p_R \colon V_R \to W_R$ is induced by p and Λ^{i+1} stays for the $(i+1)$-th exterior power.

Lemma 9.7. *Every $\mathbb{\Gamma}(V)_{(i)}$ is a closed subfunctor of $\mathbb{\Gamma}(V)$.*

Proof. It is really a subfunctor clearly. To show that it is closed, take a morphism $\operatorname{Spec} R \to \mathbb{\Gamma}(V)$ and the corresponding R-point $N \in \mathbb{\Gamma}(V)(R)$. Put $M = \Lambda^{i+1} p_R(N)$. It is a submodule of the free module $\Lambda^{i+1} W_R$. By definition we have

$$N \in \mathbb{\Gamma}(V)_{(i)}(R) \quad \Leftrightarrow \quad M = 0\ .$$

Hence, if $\varphi : R \to S$ is a homomorphism of commutative F-algebras and $\varphi^{\Lambda^{i+1} W} : \Lambda^{i+1} W_R \to \Lambda^{i+1} W_S$ the induced homomorphism of R-modules, we have

$$N \otimes_R S \in \mathbb{\Gamma}(V)_{(i)}(S) \quad \Leftrightarrow \quad \varphi^{\Lambda^{i+1} W}(M) = 0\ .$$

The condition from the right-hand side means that $\varphi(r) = 0$ for any coordinate $r \in R$ of an element from M in a fixed basis of $\Lambda^{i+1} W_R$. Thus the inverse image of the subfunctor $\mathbb{\Gamma}(V)_{(i)} \subset \mathbb{\Gamma}(V)$ with respect to the morphism $\operatorname{Spec} R \to \mathbb{\Gamma}(V)$ is the closed subfunctor of $\operatorname{Spec} R$ determined by the ideal consisting of these r. $\square$

Lemma 9.8. *For every $i \geq 0$, the difference $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$ is the subfunctor of $\mathbb{\Gamma}(V)$ with*

$$\mathbb{\Gamma}(V)_{(i\backslash i-1)}(R) = \\ = \left\{ N \in \mathbb{\Gamma}(V)(R) |\ p_R(N) \text{ is a direct summand of } W_R \text{ of rank } i \right\} .$$

Proof. Let $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$ *denotes* the subfunctor of $\mathbb{\Gamma}(V)$ defined as above. We have to show that it is the difference of $\mathbb{\Gamma}(V)_{(i)}$ and $\mathbb{\Gamma}(V)_{(i-1)}$. First of all it is really a subfunctor. Moreover, it is contained in $\mathbb{\Gamma}(V)_{(i)}$ and has no intersection with $\mathbb{\Gamma}(V)_{(i-1)}$. It is also evident that $\mathbb{\Gamma}(V)_{(i-1)}$ and $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$ together cover $\mathbb{\Gamma}(V)_{(i)}$. Hence, we finish the proof, when we show that the F-functor $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$ is local and admits a covering by subfunctors open in $\mathbb{\Gamma}(V)_{(i)}$.

Since $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$ is a subfunctor of a local F-functor, the injectivity of the first arrow in the sequence

$$\mathbb{\Gamma}(V)_{(i\backslash i-1)}(R) \to \prod_{j=1}^{l} \mathbb{\Gamma}(V)_{(i\backslash i-1)}(R_{r_j}) \rightrightarrows \prod_{j,k=1}^{l} \mathbb{\Gamma}(V)_{(i\backslash i-1)}(R_{r_j r_k})$$

needs no proof. Taking an element in the middle term which "vanishes" in the right-hand side term and using locality of $\mathbb{\Gamma}(V)_{(i)}$, we get a module $N \in \mathbb{\Gamma}(V)_{(i)}(R)$ such that the quotient $W_R/p_R(N)$ is locally (in the Zariski topology on $\operatorname{Spec} R$) projective of rank $\dim W - i$. Hence the quotient has these properties globally, i.e. $N \in \mathbb{\Gamma}(V)_{(i\backslash i-1)}(R)$. Thereby the F-functor $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$ is local.

Let $q \colon W \twoheadrightarrow U$ be an epimorphism with $\dim U = i$. Define a subfunctor $\mathcal{U} \subset \mathbb{\Gamma}(V)$ as

$$\mathcal{U}(R) = \left\{ N \in \mathbb{\Gamma}(V) |\ (q_R)\big|_{p_R(N)} : p_R(N) \to U_R \text{ is an isomorphism} \right\} .$$

It is clear that in fact $\mathcal{U} \subset \mathbb{\Gamma}(V)_{(i\backslash i-1)}$ and that all $\mathcal{U}$ (obtained this manner) together cover $\mathbb{\Gamma}(V)_{(i\backslash i-1)}$. To see that $\mathcal{U}$ is open in $\mathbb{\Gamma}(V)_{(i)}$, take an R-point $N \in \mathbb{\Gamma}(V)_{(i)}(R)$. We have

$$N \in \mathcal{U}(R) \quad\Leftrightarrow\quad (q_R)\big|_{p_R(N)} \text{ is an isomorphism} \quad\Leftrightarrow \\ \Leftrightarrow\quad \operatorname{Coker}\left((q_R)\big|_{p_R(N)}\right) = 0 \quad\Leftrightarrow\quad \operatorname{Coker}\left((q_R \circ p_R)\big|_N\right) = 0 .$$

Let us explain why the second equivalence holds: since the R-module U_R is free of rank i and and since $\Lambda^{i+1} p_R(N) = 0$, the homomorphism

$$(q_R)\big|_{p_R(N)} : p_R(N) \to U_R$$

is bijective if and only if it is surjective.

Putting $P = \operatorname{Coker}\big((q_R \circ p_R)_{|_N}\big)$, for any $\varphi \in \operatorname{Hom}_{F\text{-}\mathfrak{alg}}(R, S)$ we consequently have

$$N \otimes_R S \in \mathcal{U}(S) \quad \Leftrightarrow \quad P \otimes_R S = 0 \quad \Leftrightarrow \quad \varphi(\operatorname{Ann} P) \cdot S = S$$

the latter equivalence holds according to (9.3). It shows that the inverse image of the subfunctor $\mathcal{U} \subset \mathbb{\Gamma}(V)_{(i)}$ with respect to the morphism $\operatorname{Spec} R \to \mathbb{\Gamma}(V)_{(i)}$, given by the R-point $N \in \mathbb{\Gamma}(V)_{(i)}(R)$, is the open subfunctor of $\operatorname{Spec} R$ determined by the ideal $\operatorname{Ann} P \subset R$. □

Corollary 9.9. *Put* $\operatorname{Gr} \mathbb{\Gamma}(V) = \coprod \mathbb{\Gamma}(V)_{(i\setminus i-1)}$ *as in (6.2). Then*

$$\operatorname{Gr} \mathbb{\Gamma}(V)(R) = \big\{N \in \mathbb{\Gamma}(V)(R) |\ p_R(N) \textit{ is a direct summand of } W_R\big\}$$

for any $R \in F\text{-}\mathfrak{alg}$.

Proof. Let $\operatorname{Gr} \mathbb{\Gamma}(V)$ *denotes* the subfunctor of $\mathbb{\Gamma}(V)$ with

$$\operatorname{Gr} \mathbb{\Gamma}(V)(R) = \big\{N \in \mathbb{\Gamma}(V)(R) |\ p_R(N) \text{ is a direct summand of } W_R\big\}$$

for any $R \in F\text{-}\mathfrak{alg}$. The direct verification that the scheme $\coprod \mathbb{\Gamma}(V)_{(i\setminus i-1)}$ represents the F-functor $\operatorname{Gr} \mathbb{\Gamma}(V)$ is evident.

One can also argue without passing to the schemes: the F-functor $\operatorname{Gr} \mathbb{\Gamma}(V)$ is local (the same business as in the previous proof), the subfunctors

$$\mathbb{\Gamma}(V)_{(i\setminus i-1)} \subset \operatorname{Gr} \mathbb{\Gamma}(V)$$

are disjoint, cover $\operatorname{Gr} \mathbb{\Gamma}(V)$, and each of them is open (and closed). □

Lemma 9.10. *There is a vector bundle*

$$\operatorname{Gr} \mathbb{\Gamma}(V) \to \mathbb{\Gamma}(W) \times \mathbb{\Gamma}(W'),$$

where $W' = \operatorname{Ker}(p\colon V \to W)$.

Proof. Let $R \in F\text{-}\mathfrak{alg}$, $N \in \mathbb{\Gamma}(V)(R)$. Since in the exact sequence

$$0 \longrightarrow N \cap W'_R \longrightarrow N \xrightarrow{p_R} p_R(N) \longrightarrow 0$$

the module $p_R(N)$ is projective (as a direct summand of the free module W_R), the intersection $N \cap W'_R$ is a direct summand of N. Since N is a direct summand of V_R, the intersection is a direct summand of V_R as well and hence it is also a direct summand of an intermediate module W'_R. Thus there is a map

$$\begin{array}{ccc} \operatorname{Gr} \mathbb{\Gamma}(V)(R) & \to & \mathbb{\Gamma}(W)(R) \times \mathbb{\Gamma}(W')(R) \\ N & \mapsto & \Big(p_R(N),\ N \cap W'_R\Big) \end{array}$$

which determines the morphism of F-functors we are meaning.

To get a structure of vector bundle, we fix a splitting of the epimorphism $p\colon V \to W$, i.e. identify W with a subspace of V complementary to W'. Take any $M \in \mathbb{\Gamma}(W)(R)$ and $M' \in \mathbb{\Gamma}(W')(R)$. The R-modules $N \in \operatorname{Gr} \mathbb{\Gamma}(V)(R)$ such that

$$p_R(N) = M \ \text{ and } \ N \cap W'_R = M'$$

are in one-to-one correspondence with the elements of $\operatorname{Hom}_R(M, W'_R/M')$: for $\varphi \in \operatorname{Hom}_R(M, W'_R/M')$ the corresponding submodule N of $V_R = W_R \oplus W'_R$ is the union of $m + \varphi(m) \subset W_R \oplus W'_R$ for all $m \in M$; for $N \in \operatorname{Gr} \mathbb{\Gamma}(V)(R)$ the corresponding homomorphism $p_R(N) \to W'_R/(N \cap W'_R)$ maps $p_R(n)$ for every $n \in N$ to the class of the W'_R-coordinate of $n \in W_R \oplus W'_R$ modulo $N \cap W'_R$.

Since M and W'_R/M' are finitely generated projective R-modules, the R-module $\operatorname{Hom}_R(M, W'_R/M')$ is finitely generated projective as well. For checking that we get a structure of a vector bundle, it remains to prove that for any homomorphism of commutative F-algebras $R \to S$ the natural homomorphism of S-modules

$$f(M)\colon \operatorname{Hom}_R(M, W'_R/M') \otimes_R S \to \operatorname{Hom}_S(M \otimes_R S, W'_S/M' \otimes_R S)$$

is an isomorphism. Since $f(R)$ is an isomorphism and

$$f(M_1 \oplus M_2) = f(M_1) \oplus f(M_2),$$

$f(M)$ is an isomorphism for any M isomorphic to a direct summand of R^n for some $n \geq 1$, i.e. for any finitely generated projective M. $\square$

Corollary 9.11. *Let $0 \to W' \to V \to W \to 0$ be an exact sequence of vector F-spaces. The grassmanian $\mathbb{\Gamma}(V)$ together with the filtration (9.7) and the vector bundle (9.10) is a relative cellular space over $\mathbb{\Gamma}(W) \times \mathbb{\Gamma}(W')$.* $\square$

Remark 9.12. As seen in the proof of the last lemma, the structure of vector bundle on the morphism $\operatorname{Gr} \mathbb{\Gamma}(V) \to \mathbb{\Gamma}(W) \times \mathbb{\Gamma}(W')$ does depend on the choice of a splitting of the epimorphism $V \twoheadrightarrow W$; nevertheless, the morphism itself as well as the filtration, i.e. all the data, determining the cellular structure, do not.

Corollary 9.13. *In the category of correspondences $\mathcal{CV}$, there is an isomorphism*

$$\mathbb{\Gamma}(V) \simeq \mathbb{\Gamma}(W) \times \mathbb{\Gamma}(W') \,.$$

In particular,

$$\mathrm{H}\left(\mathbb{\Gamma}(V)\right) \simeq \mathrm{H}\left(\mathbb{\Gamma}(W) \times \mathbb{\Gamma}(W')\right)$$

for any geometric cohomology theory H *(§2).* $\square$

Components

Definition 9.14. For every $n \in \mathbb{Z}$, the *n-grassmanian* $\mathbb{\Gamma}_n(V)$ of V is the subfunctor of $\mathbb{\Gamma}(V)$ for which $\mathbb{\Gamma}_n(V)(R)$ ($R \in F\text{-}\mathfrak{alg}$) is the subset in $\mathbb{\Gamma}(V)(R)$ of direct summands having the constant rank n (of course $\mathbb{\Gamma}_n(V)$ is non-empty only if $0 \leq n \leq \dim V$).

Proposition 9.15. *The F-functor $\mathbb{\Gamma}(V)$ is a direct sum of the subfunctors $\mathbb{\Gamma}_n(V)$, $n \in \mathbb{Z}$. The varieties $\mathbb{\Gamma}_n(V)$ are geometrically irreducible.*

Proof. The statement on the direct sum can be proved analogously to (9.9) or by using (9.9) as follows. Take the identity map $p\colon V \to V$. Then

$$\mathbb{\Gamma}(V) = \operatorname{Gr}(V) = \coprod_n \mathbb{\Gamma}(V)_{(n \backslash n-1)} = \coprod_n \mathbb{\Gamma}_n(V)$$

where the first equality holds (for our particular p) by (9.9), the second one — by the definition of Gr, the third one — since $\mathbb{\Gamma}(V)_{(n\backslash n-1)} = \mathbb{\Gamma}_n(V)$ for our particular p.

To show that $\mathbb{\Gamma}_n(V)$ is geometrically irreducible, consider the general linear group $\mathbb{GL}(V)$. For every $R \in F\text{-}\mathfrak{alg}$ the (abstract) group

$$\mathbb{GL}(V)(R) = \mathrm{Aut}_R(V_R)$$

acts on the set $\mathbb{\Gamma}_n(V)(R)$ in the natural way, so that one has a morphism of F-functors

$$\mathbb{GL}(V) \times \mathbb{\Gamma}_n(V) \to \mathbb{\Gamma}_n(V)$$

defining an action of the algebraic group on the n-grassmanian. This action is transitive in the sense that if R is a field, the action "on the level of R-points" is transitive. Since the affine group $\mathbb{GL}(V)$ is (geometrically) irreducible, it follows that the variety $\mathbb{\Gamma}_n(V)$ is geometrically irreducible as well. $\square$

Remark 9.16. The varieties $\mathbb{\Gamma}_n(V)$ are projective: for any $n \in \mathbb{Z}$, the map

$$N \in \mathbb{\Gamma}_n(V)(R) \quad \mapsto \quad \Lambda^n N \in \mathbb{\Gamma}_1(\Lambda^n V)(R)$$

determines a morphism of F-functors, identifying $\mathbb{\Gamma}_n(V)$ with a closed subvariety of the projective space $\mathbb{\Gamma}_1(\Lambda^n V)$.

Corollary 9.17. *In the conditions of (9.11), let* H^* *be a graded geometric cohomology theory (§4). For any* $n \in \mathbb{Z}$*, there is an isomorphism*

$$\mathrm{H}^*\big(\mathbb{\Gamma}_n(V)\big) \simeq \coprod_{i=0}^{n} \mathrm{H}^*\big(\mathbb{\Gamma}_i(W) \times \mathbb{\Gamma}_{n-i}(W')\big)[-i(\dim W' - n + i)] \,.$$

Proof. Follows from (6.11) and that

$$\mathrm{rk}\,\mathrm{Hom}_R(M, W'_R/M') = \mathrm{rk}\, M \cdot \mathrm{rk}(W'_R/M') = i \cdot (\dim_F W' - n + i),$$

if $M \in \mathbb{\Gamma}_i(W)(R)$ and $M' \in \mathbb{\Gamma}_{n-i}(W')(R)$. $\square$

10. Varieties of ideals

Definition 10.1. Let $f\colon V \to V$ be a fixed endomorphism. One defines the subfunctor $\mathbb{\Gamma}^{\mathrm{inv}}(V) \subset \mathbb{\Gamma}(V)$ as follows:

$$\mathbb{\Gamma}^{\mathrm{inv}}(V)(R) = \big\{N \in \mathbb{\Gamma}(V)(R) |\ f_R(N) \subset N\big\}$$

for every $R \in F\text{-}\mathfrak{alg}$, where $f_R\colon V_R \to V_R$ is the endomorphism given by f.

Lemma 10.2. *The subfunctor* $\mathbb{\Gamma}^{\mathrm{inv}}(V) \subset \mathbb{\Gamma}(V)$ *is closed.*

Proof. Take a morphism $\mathrm{Spec}\, R \to \mathbb{\Gamma}(V)$ and let $N \in \mathbb{\Gamma}(V)(R)$ be the corresponding R-point. Choose a splitting s of the projection $p\colon V_R \to V_R/N$ and put

$$M = (s \circ p)\big(f(N)\big) \subset V_R \,.$$

We have

$$N \in \mathbb{\Gamma}^{\mathrm{inv}}(V)(R) \quad \Leftrightarrow \quad M = 0 \,.$$

Hence, if $\varphi : R \to S$ is a homomorphism of commutative F-algebras and $\varphi^V : V_R \to V_S$ the induced homomorphism of R-modules, we have

$$N \otimes_R S \in \mathbb{\Gamma}^{\mathrm{inv}}(V)(S) \quad \Leftrightarrow \quad \varphi^V(M) = 0 \ .$$

The condition $\varphi^V(M) = 0$ means that $\varphi(r) = 0$ for any coordinate $r \in R$ of an element from M in a fixed basis of V_R. Thus the inverse image of the subfunctor $\mathbb{\Gamma}^{\mathrm{inv}}(V) \subset \mathbb{\Gamma}(V)$ with respect to the morphism $\operatorname{Spec} R \to \mathbb{\Gamma}(V)$ is the closed subfunctor of $\operatorname{Spec} R$ determined by the ideal of R consisting of all these r. □

Let A be an arbitrary F-algebra finite-dimensional over F, V a finitely generated A-module (then V is finite-dimensional as a vector space over F).

Definition 10.3. We define a subfunctor $\mathbb{\Gamma}^A(V) \subset \mathbb{\Gamma}(V)$ as follows:

$$\mathbb{\Gamma}^A(V)(R) = \big\{ N \in \mathbb{\Gamma}(V)(R) | \ N \text{ is an } A_R\text{-submodule of } V_R \big\}$$

for any $R \in F$-$\mathfrak{alg}$. The F-functor $\mathbb{\Gamma}^A(A)$, where A is considered as a right module over itself, will be called the *variety of (right) ideals* of A and denoted shortly by $\mathbb{\Gamma}^A$.

Corollary 10.4. *The subfunctor $\mathbb{\Gamma}^A(V) \subset \mathbb{\Gamma}(V)$ is closed (and thereby represented by a complete F-variety).*

Proof. Multiplication by any $a \in A$ is an F-endomorphism of V, so it determines a closed subfunctor (10.2). Since $N \in \mathbb{\Gamma}(V)(R)$ is an A_R-submodule of V_R if and only if N is stable under multiplication by every $a \in A$, the intersection of all of these closed subfunctors gives $\mathbb{\Gamma}^A(V)$. □

Lemma 10.5. *Let A_1 and A_2 be two arbitrary finite-dimensional F-algebras. Then $\mathbb{\Gamma}^{A_1 \times A_2} \simeq \mathbb{\Gamma}^{A_1} \times \mathbb{\Gamma}^{A_2}$.*

Proof. For every $R \in F$-$\mathfrak{alg}$ we still have

$$(A_1 \times A_2)_R \simeq (A_1)_R \times (A_2)_R \ ;$$

thus $\mathbb{\Gamma}^{A_1 \times A_2}(R) \simeq \mathbb{\Gamma}^{A_1}(R) \times \mathbb{\Gamma}^{A_2}(R)$. □

Definition 10.6. A finite-dimensional F-algebra A is called *separable*, if it is semisimple after extending scalars to any field extension of F. An equivalent definition: A is separable, if it is semisimple and the center of every simple component of A is a (finite) separable extension of F [4, exercise 2 from §71]. Another equivalent definition: A is separable, if there exists a separable extension E/F such that the algebra A_E is isomorphic to a direct product of matrix algebras over E.

Let A be a separable F-algebra, V a finitely generated A-module.

Lemma 10.7 (Morita equivalence). *Put $B = \operatorname{End}_A V$. Then $\mathbb{\Gamma}^A(V) \simeq \mathbb{\Gamma}^B$.*

Proof. By definition, the set $\mathbb{\Gamma}^A(V)(R)$ for any $R \in F\text{-}\mathfrak{alg}$ consists of the A_R-submodules N of V_R such that the exact sequence

$$(*) \qquad 0 \to N \to V_R \to V_R/N \to 0$$

splits over R. However, since A_R is a generalized Azumaya R-algebra and the R-module V_R/N is finitely presented, the sequence $(*)$ splits over A_R as well [20, prop. 3.1]; so, the set $\mathbb{\Gamma}^A(V)(R)$ consists of the direct summands of the A_R-module V_R.

For any $R \in F\text{-}\mathfrak{alg}$ it holds: $B_R = B \otimes_F R$ is a generalized Azumaya R-algebra isomorphic to $\operatorname{End}_{A_R} V_R$ and V_R has also the structure of a left B_R-module. Since V_R is finitely generated projective (even free) over R it is also finitely generated projective over A_R and over B_R [20, cor. 3.2] and hence it is a *progenerator* in the category of A_R-modules [6, prop. and def. 4.3]. Thus by the Morita theory [6, thm. 4.29] the functor

$$\begin{array}{ccc} A_R\text{-}\mathfrak{mod} & \to & B_R\text{-}\mathfrak{mod} \\ N & \mapsto & \operatorname{Hom}_{A_R}(V_R, N) \end{array}$$

is an equivalence of the category of (right) A_R-modules and the category of (right) B_R-modules, where the abelian group $\operatorname{Hom}_{A_R}(V_R, N)$ is considered as a B_R-module in the natural way. The inverse equivalence is defined as follows:

$$\begin{array}{ccc} B_R\text{-}\mathfrak{mod} & \to & A_R\text{-}\mathfrak{mod} \\ M & \mapsto & \operatorname{Hom}_{B_R}(V_R^*, M) \end{array}$$

where $V_R^* = \operatorname{Hom}_{A_R}(V_R, A_R) \in B\text{-}\mathfrak{mod}$. The equivalences are additive (also inclusions preserving) and $V_R \in A_R\text{-}\mathfrak{mod}$ corresponds to $B_R \in B_R\text{-}\mathfrak{mod}$. Thus, restricting to the direct summands of V_R and, on the other hand, to the direct summands (ideals) of B_R, we get a bijection

$$\mathbb{\Gamma}^A(V)(R) \simeq \mathbb{\Gamma}^B(R) .$$

It determines an isomorphism of F-functors since

$$\operatorname{Hom}_{A_R}(V_R, N) \otimes_R S \simeq \operatorname{Hom}_{A_S}(V_S, N \otimes_R S)$$

for any $R \to S \in \operatorname{Hom}_{F\text{-}\mathfrak{alg}}(R, S)$. □

Corollary 10.8. *The variety $\mathbb{\Gamma}^A(V)$ is smooth.*

Proof. Put $B = \operatorname{End}_A V$ and let $\bar{F}$ be an algebraic closure of F. Since B is a separable F-algebra, $B_{\bar{F}}$ is isomorphic to a product

$$\operatorname{End}_{\bar{F}} W_1 \times \cdots \times \operatorname{End}_{\bar{F}} W_n$$

for some vector $\bar{F}$-spaces $W_1, \ldots, W_n$. We have

$$\begin{aligned} &\mathbb{\Gamma}^A(V) \simeq \mathbb{\Gamma}^B && \text{by (10.7) ;} \\ &\mathbb{\Gamma}^{B_{\bar{F}}} \simeq \mathbb{\Gamma}^{\operatorname{End} W_1} \times \cdots \times \mathbb{\Gamma}^{\operatorname{End} W_n} && \text{by (10.5) .} \end{aligned}$$

Thus $\mathbb{\Gamma}^A(V)_{\bar{F}}$ is isomorphic to the smooth variety $\mathbb{\Gamma}(W_1) \times \cdots \times \mathbb{\Gamma}(W_n)$. □

Theorem 10.9. *Let A be a separable F-algebra and let*

$$0 \to W' \to V \to W \to 0$$

be an exact sequence of finitely generated A-modules. The variety $\mathbb{\Gamma}^A(V)$ is a relative cellular space over $\mathbb{\Gamma}^A(W) \times \mathbb{\Gamma}^A(W')$.

Proof. We state that the cellular structure on $\mathbb{\Gamma}(V)$ (9.11) induces a cellular structure on $\mathbb{\Gamma}^A(V) \subset \mathbb{\Gamma}(V)$ in the sense of (7.6). We have to verify that the restriction to

$$\operatorname{Gr}\mathbb{\Gamma}^A(V) = \mathbb{\Gamma}^A(V) \cap \operatorname{Gr}\mathbb{\Gamma}(V)$$

of the vector bundle

$$\operatorname{Gr}\mathbb{\Gamma}(V) \to \mathbb{\Gamma}(W) \times \mathbb{\Gamma}(W')$$

is a vector bundle over $\mathbb{\Gamma}^A(W) \times \mathbb{\Gamma}^A(W')$.

To obtain a vector bundle structure, one has to fix an A-splitting of the epimorphism $V \twoheadrightarrow W$, i.e. identify W with an A-submodule of V complementary to W'. Take any $M \in \mathbb{\Gamma}^A(W)(R)$ and $M' \in \mathbb{\Gamma}^A(W')(R)$. The modules $N \in \operatorname{Gr}\mathbb{\Gamma}^A(V)(R)$ such that

$$p_R(N) = M \quad \text{and} \quad N \cap W'_R = M'$$

are in one-to-one correspondence with the elements of the finitely generated projective R-module $\operatorname{Hom}_{A_R}(M, W'_R/M')$ (compare with the proof of (9.10)). For any homomorphism of commutative F-algebras $R \to S$ the natural homomorphism of S-modules

$$\operatorname{Hom}_{A_R}(M, W'_R/M') \otimes_R S \to \operatorname{Hom}_{A_S}(M \otimes_R S, W'_S/M' \otimes_R S)$$

is an isomorphism. Thus we are done. □

Corollary 10.10. *In the category of correspondences $\mathcal{CV}$, there is an isomorphism*

$$\mathbb{\Gamma}^A(V) \simeq \mathbb{\Gamma}^A(W) \times \mathbb{\Gamma}^A(W') \,.$$

In particular,

$$\mathrm{H}\left(\mathbb{\Gamma}^A(V)\right) \simeq \mathrm{H}\left(\mathbb{\Gamma}^A(W) \times \mathbb{\Gamma}^A(W')\right)$$

for any geometric cohomology theory H. □

Corollary 10.11. *Let $\mathrm{M}_n(A)$ stays for the algebra of $n \times n$-matrices over A. Then $\mathbb{\Gamma}^{\mathrm{M}_n(A)} \simeq \mathbb{\Gamma}^{A \times n}$ in $\mathcal{CV}$.*

Proof. Since $\mathrm{M}_n(A) = \operatorname{End}_A(A^n)$, we have $\mathbb{\Gamma}^{\mathrm{M}_n(A)} \simeq \mathbb{\Gamma}^A(A^n)$. Using the exact sequence

$$0 \to A^{n-1} \to A^n \to A \to 0$$

and induction by n, we get an isomorphism $\mathbb{\Gamma}^A(A^n) \simeq (\mathbb{\Gamma}^A)^{\times n}$ in $\mathcal{CV}$. Now apply (10.5). □

Definition 10.12. Every separable algebra A is isomorphic to a product

$$\mathrm{M}_{n_1}(D_1) \times \cdots \times \mathrm{M}_{n_m}(D_m)$$

with separable *division* algebras $D_1, \ldots, D_m$. The separable algebra

$$D_1^{\times n_1} \times \cdots \times D_m^{\times n_m}$$

will be called the *anisotropic kernel* of A and denoted by A_{an} (the definition of A_{an} is non-canonical; however the isomorphism class of A_{an} is unique).

Theorem 10.13. *For any separable algebra A, in the category $\mathcal{CV}$ there is an isomorphism $\mathbb{\Gamma}^A \simeq \mathbb{\Gamma}^{A_{\mathrm{an}}}$.* □

Example 10.14. We consider a situation which occurs in §15. Suppose that we are given a decomposition $V = V_1 \oplus V_2 \oplus V_3$ into a sum of A-modules. Using the exact sequences

$$0 \to V_2 \oplus V_3 \to V \to V_1 \to 0 \quad \text{and} \quad 0 \to V_2 \to V_2 \oplus V_3 \to V_3 \to 0$$

and composing the relative structures (7.4) we turn $\mathbb{\Gamma}^A(V)$ into a relative (cellular) space over $\mathbb{\Gamma}^A(V_1) \times \mathbb{\Gamma}^A(V_2) \times \mathbb{\Gamma}^A(V_3)$: for any $R \in F\text{-}\mathfrak{alg}$ the set $\mathrm{Gr}\,\mathbb{\Gamma}^A(V)(R)$ consists of $N \in \mathbb{\Gamma}^A(V)(R)$ such that the projection of N to $(V_1)_R$ is in $\mathbb{\Gamma}^A(V_1)(R)$ and the projection of $N \cap (V_2 \oplus V_3)_R$ to $(V_3)_R$ is in $\mathbb{\Gamma}^A(V_3)_R$ (then $N \cap (V_2)_R \in \mathbb{\Gamma}^A(V_2)(R)$ automatically).

Fix a triple

$$(N_1, N_2, N_3) \in \Big(\mathbb{\Gamma}^A(V_1) \times \mathbb{\Gamma}^A(V_2) \times \mathbb{\Gamma}^A(V_3)\Big)(R)$$

and A_R-modules N'_j such that $(V_j)_R = N_j \oplus N'_j$ for $j = 2, 3$. The set of $N \in \mathrm{Gr}\,\mathbb{\Gamma}^A(V)(R)$ lying over the fixed triple is in one-to-one correspondence with

$$\mathrm{Hom}_{A_R}(N_1, N'_2) \oplus \mathrm{Hom}_{A_R}(N_1, N'_3) \oplus \mathrm{Hom}_{A_R}(N_3, N'_2)\ .$$

For an element $f_{12} \oplus f_{13} \oplus f_{32}$ of this sum, the corresponding A_R-module $N \subset V$ equals

$$N = \big\{ n_1 + n_2 + f_{12}(n_1) + f_{32}(n_3) + n_3 + f_{13}(n_1) \mid n_1 \in N_1, n_2 \in N_2, n_3 \in N_3 \big\}\ .$$

Components

We compute the components of the variety $\mathbb{\Gamma}^A(V)$ only for a central simple algebra A. So, let A be a finite-dimensional central simple F-algebra, V a finitely generated A-module.

We denote by $\deg A$ (*degree* of A) the square root of $\dim_F A$ and by $\mathrm{rk}_A V$ (*rank* of V over A) the integer $\dim_F V / \deg A$.

The decomposition of the grassmanian into the sum of its components (9.15) produces the decomposition

$$\mathbb{\Gamma}^A(V) = \coprod_n \mathbb{\Gamma}^A(V) \cap \mathbb{\Gamma}_n(V)\ .$$

Lemma 10.15. *The intersection $\mathbb{\Gamma}^A(V) \cap \mathbb{\Gamma}_n(V)$ is empty whenever the integer n is not divisible by $\deg A$.*

Proof. Take any $R \in F\text{-}\mathfrak{alg}$ and $N \in \mathbb{\Gamma}_n(V)(R)$. Let $R \to L$ be a homomorphism of F-algebras and L be a field. If the R-module N is an A_R-module, then $N \otimes_R L$ is a module over the central simple algebra A_L; thus $n = \dim_L N$ is divisible by $\deg A_L = \deg A$. □

Hence it is natural to give the following

Definition 10.16. We put

$$\mathbb{\Gamma}_n^A(V) = \mathbb{\Gamma}^A(V) \cap \mathbb{\Gamma}_{n \cdot \deg A}(V) \quad \subset \mathbb{\Gamma}(V) .$$

The analogously defined F-functors $\mathbb{\Gamma}_n^A$ are called the *generalized Severi-Brauer varieties* of A [2]; $\mathbb{\Gamma}_1^A$ is the *Severi-Brauer variety* of A [1].

Lemma 10.17 (**Morita equivalence**). *Put* $B = \operatorname{End}_A V$. *Then* $\mathbb{\Gamma}_n^A(V) \simeq \mathbb{\Gamma}_n^B$ *for every* $n \in \mathbb{Z}$.

Proof. We state that the restrictions of the mutually inverse isomorphisms between $\mathbb{\Gamma}^A(V)$ and $\mathbb{\Gamma}^B$, constructed in the proof of (10.7), are isomorphisms between $\mathbb{\Gamma}_n^A(V)$ and $\mathbb{\Gamma}_n^B$. Let us check that for every $R \in F\text{-}\mathfrak{alg}$ the image of $\mathbb{\Gamma}_n^A(V)$ is contained in $\mathbb{\Gamma}_n^B$:

The condition $N \in \mathbb{\Gamma}_n^A(V)(R)$ means that N has the constant rank $n \cdot \deg A$ as an R-module. Thus for every commutative R-algebra L, which is a field, the A_L-module $N \otimes_R L$ is of the rank n. Consequently

$$\dim_L \operatorname{Hom}_{A_L}(V_L, N \otimes_R L) = \operatorname{rk}_{A_L} V_L \cdot \dim_L(N \otimes_R L) = \operatorname{rk}_A V \cdot n = \deg B \cdot n$$

and hence $\operatorname{Hom}_{A_R}(V_R, N) \in \mathbb{\Gamma}_n^B(R)$. □

Corollary 10.18. *The varieties* $\mathbb{\Gamma}_n^A(V)$ *are geometrically irreducible;* $\mathbb{\Gamma}^A(V)$ *is their direct sum.*

Proof. Let us show that $\mathbb{\Gamma}_n^A(V)_{\bar{F}}$ is irreducible, where $\bar{F}$ is an algebraic closure of F. Put $B = \operatorname{End}_A V$. Since $B_{\bar{F}}$ splits, there is a vector space W over $\bar{F}$ such that $B_{\bar{F}} \simeq \operatorname{End}_{\bar{F}} W$. We have

$$\mathbb{\Gamma}_n^A(V)_{\bar{F}} \simeq (\mathbb{\Gamma}_n^B)_{\bar{F}} \simeq \mathbb{\Gamma}_n^{B_{\bar{F}}} \simeq \mathbb{\Gamma}_n(W) .$$

Now use (9.15). □

Corollary 10.19. *In the conditions of (10.9), assume that the* F*-algebra* A *is central simple. Let* H^* *be a graded geometric cohomology theory. For any* $n \in \mathbb{Z}$*, there is an isomorphism*

$$\mathrm{H}^*\left(\mathbb{\Gamma}_n^A(V)\right) \simeq \coprod_{i=0}^{n} \mathrm{H}^*\left(\mathbb{\Gamma}_i^A(W) \times \mathbb{\Gamma}_{n-i}^A(W')\right)[-i(\operatorname{rk}_A W' - n + i)] .$$

Proof. Follows from (6.11) and that

$$\operatorname{rk}_R \operatorname{Hom}_{A_R}(M, W'_R/M') = \operatorname{rk}_{A_R} M \cdot \operatorname{rk}_{A_R}(W'_R/M') = i \cdot (\operatorname{rk}_A W' - n + i)$$

if $M \in \mathbb{\Gamma}_i^A(W)(R)$ and $M' \in \mathbb{\Gamma}_{n-i}^A(W')(R)$ (compare with (9.17). □

Example 10.20 (Decomposition of Severi-Brauer varieties).

$$\mathrm{H}^*(\mathbb{\Gamma}_1^{\mathrm{M}_n(A)}) \simeq \coprod_{i=0}^{n} \mathrm{H}^*(\mathbb{\Gamma}_1^A)[-i \cdot \deg A] \ .$$

This decomposition was obtained in [10].

11. Varieties of flags of subspaces

We return to the situation of the §9: V is again simply a finite-dimensional vector space over a field F.

Definition 11.1. For any $m \in \mathbb{N}$, one defines a subfunctor $\mathbf{\Phi}_m(V) \subset \mathbb{\Gamma}(V)^{\times m}$, called the *variety of m-flags (of subspaces)* as follows:

$$\mathbf{\Phi}_m(V)(R) = \left\{(N_1, \dots, N_m) \in \mathbb{\Gamma}(V)(R)^{\times m} | \ N_1 \subset \cdots \subset N_m\right\} .$$

Note that $\mathbf{\Phi}_1(V) = \mathbb{\Gamma}(V)$.

Lemma 11.2. *The subfunctor of m-flags $\mathbf{\Phi}_m(V) \subset \mathbb{\Gamma}(V)^{\times m}$ is closed (and hence represented by a complete F-variety).*

Proof. If $m > 2$ then

$$\mathbf{\Phi}_m(V) = \bigcap_{i=0}^{m-2} \mathbb{\Gamma}(V)^{\times i} \times \mathbf{\Phi}_2(V) \times \mathbb{\Gamma}(V)^{\times (m-2-i)} \quad \subset \mathbb{\Gamma}(V)^{\times m} \ .$$

Thus it is enough only to check that $\mathbf{\Phi}_2(V)$ is closed in $\mathbb{\Gamma}(V)^{\times 2}$.

Take $N_1, N_2 \in \mathbb{\Gamma}(V)(R)$ for some $R \in F\text{-}\mathfrak{alg}$ and denote by s a splitting of the surjection $p\colon V_R \twoheadrightarrow V_R/N_2$. Put $M = (s \circ p)(N_1) \subset V_R$. We have

$$(N_1, N_2) \in \mathbf{\Phi}_2(V)(R) \ \Leftrightarrow \ N_1 \subset N_2 \ \Leftrightarrow \ M = 0 \ .$$

Thus for any homomorphism of commutative F-algebras, we have

$$(N_1 \otimes_R S, N_2 \otimes_R S) \in \mathbf{\Phi}_2(V)(S) \ \Leftrightarrow \ \varphi^V(M) = 0$$

where $\varphi^V : V_R \to V_S$ is the map induced by φ. The end of the proof is standard (see the proofs of (9.7) or (10.2)). $\square$

Proposition 11.3. *For every $m \in \mathbb{N}$, the variety $\mathbf{\Phi}_m(V)$ is smooth.*

Proof. Let $U_1 \subset \cdots \subset U_m$ and $U'_1 \supset \cdots \supset U'_m$ be two chains of subspaces in V such that $U'_i \oplus U_i = V$ for every i. Take the epimorphisms $p_i\colon V \twoheadrightarrow V/U'_i \simeq U_i$ and consider the open subfunctors $\mathcal{U}_i \subset \mathbb{\Gamma}(V)$ defined as in the proof of (9.2). It is clear that the F-functor $\mathbf{\Phi}_m(V)$ is covered by the open subfunctors of the type

$$\mathcal{U} = \mathbf{\Phi}_m(V) \cap \prod_{i=1}^{m} \mathcal{U}_i \ .$$

We are going to show that $\mathcal{U}$ is an affine space.

For any $R \in F\text{-}\mathfrak{alg}$, the set $\prod \mathcal{U}_i(R)$ is identified with

$$\prod_{i=1}^{m} \mathrm{Hom}_R\left((U_i)_R, (U'_i)_R\right) \ .$$

An element $(f_i)_{i=1}^m$ from the latter product corresponds to an element from $\mathbf{\Phi}_m(V)(R)$ if and only if the following diagram commutes:

$$\begin{array}{ccccccc}
(U'_1)_R & \hookleftarrow & (U'_2)_R & \hookleftarrow & \dots & \hookleftarrow & (U'_m)_R \\
\downarrow f_1 & & \downarrow f_2 & & & & \downarrow f_m \\
(U_1)_R & \twoheadleftarrow & (U_2)_R & \twoheadleftarrow & \dots & \twoheadleftarrow & (U_m)_R
\end{array}$$

In this way we get a structure of an R-module on the set $\mathcal{U}(R)$. Moreover, $\mathcal{U}(R) \simeq \mathcal{U}(F) \otimes_F R$. Thus $\mathcal{U}$ is isomorphic to the affine space on the vector space $\mathcal{U}(F)$. □

Proposition 11.4. *Let $0 \to W' \to V \to W \to 0$ be an exact sequence of vector F-spaces. The variety $\mathbf{\Phi}_m(V)$ is a relative cellular space over $\mathbf{\Phi}_m(W) \times \mathbf{\Phi}_m(W')$.*

Proof. Let us consider $\mathbb{\Gamma}(V)^{\times m}$ as a cellular space over $\mathbb{\Gamma}(W)^{\times m} \times \mathbb{\Gamma}(W')^{\times m}$ using the product (7.2) of the cellular structures (9.11). We state that $\mathbf{\Phi}_m(V) \subset \mathbb{\Gamma}(V)^{\times m}$ is a cellular subspace in the sense of (7.6). To see that the morphism

$$\operatorname{Gr} \mathbf{\Phi}_m(V) \to \mathbf{\Phi}_m(W) \times \mathbf{\Phi}_m(W')$$

is a vector bundle, identify W with a complementary to W' subspace of V and choose an R-point

$$(M_1, \dots, M_m; M'_1, \dots, M'_m) \in \Big(\mathbf{\Phi}_m(W) \times \mathbf{\Phi}_m(W')\Big)(R) \ .$$

The set of R-points of $\operatorname{Gr} \mathbf{\Phi}_m(V)$ lying over the fixed point is in one-to-one correspondence with the elements $(f_i)_{i=1}^m$ of the product

$$\prod_{j=1}^m \operatorname{Hom}_R(M_j, W'_R/M'_j)$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
M_1 & \hookrightarrow & M_2 & \hookrightarrow & \dots & \hookrightarrow & M_m \\
\downarrow f_1 & & \downarrow f_2 & & & & \downarrow f_m \\
W'_R/M'_1 & \twoheadrightarrow & W'_R/M'_2 & \twoheadrightarrow & \dots & \twoheadrightarrow & W'_R/M'_m
\end{array}$$

For every j we can find an R-module M''_j such that $M_{j-1} \oplus M''_j = M_j$. Our set is an R-module isomorphic to

$$\prod_{j=1}^m \operatorname{Hom}_R(M''_j, W'_R/M'_j) \ .$$

□

Corollary 11.5. *For every $m \in \mathbb{N}$, in the category of correspondences $\mathcal{CV}$, there is an isomorphism*

$$\mathbf{\Phi}_m(V) \simeq \mathbf{\Phi}_m(W) \times \mathbf{\Phi}_m(W') \ .$$

In particular,

$$\mathrm{H}\left(\mathbf{\Phi}_m(V)\right) \simeq \mathrm{H}\left(\mathbf{\Phi}_m(W) \times \mathbf{\Phi}_m(W')\right)$$

for any geometric cohomology theory H. □

Components

Definition 11.6. For every sequence of integers $n_1, \dots, n_m$ the intersection

$$\mathbf{\Phi}_m(V) \cap \prod_{i=1}^{m} \mathbb{\Gamma}_{n_i}(V) \quad \subset \mathbb{\Gamma}(V)^{\times m}$$

is called the variety of $(n_1, \dots, n_m)$-flags and denoted by $\mathbf{\Phi}_{(n_1,\dots,n_m)}(V)$. Of course it is non-empty only if $0 \le n_1 \le \cdots \le n_m \le \dim V$; if $n_i = n_{i+1}$ for some i then

$$\mathbf{\Phi}_{(n_1,\dots,n_m)}(V) = \mathbf{\Phi}_{(n_1,\dots,n_i,n_{i+2},\dots,n_m)}(V)\ .$$

Proposition 11.7. *The varieties $\mathbf{\Phi}_{(n_1,\dots,n_m)}(V)$ are geometrically irreducible; $\mathbf{\Phi}_m(V)$ is their direct sum.*

Proof. Since $\mathbb{\Gamma}(V)$ is a direct sum of $\prod \mathbb{\Gamma}_{n_i}(V)$, the latter statement is clear.

Since the algebraic group $\mathbb{GL}(V)$ acts transitively on every $\mathbf{\Phi}_{(n_1,\dots,n_m)}(V)$ (compare with the proof of (9.15)), these varieties are geometrically irreducible. □

Corollary 11.8. *In the conditions of (11.4), let H^* be a graded geometric cohomology theory. For any sequence of integers $n_1, \dots, n_m$, there is an isomorphism*

$$\begin{aligned} &\mathrm{H}^*\left(\mathbf{\Phi}_{(n_1,\dots,n_m)}(V)\right) \simeq \\ &\simeq \coprod_{i_1,\dots,i_m} \mathrm{H}^*\left(\mathbf{\Phi}_{(i_1,\dots,i_m)}(W) \times \mathbf{\Phi}_{(n_1-i_1,\dots,n_m-i_m)}(W')\right) \\ &\quad [-i_1(d-n_1+i_1) - (i_2-i_1)(d-n_2+i_2) - \cdots - (i_m-i_{m-1})(d-n_m+i_m)] \end{aligned}$$

where $d = \dim W'$.

Proof. Follows from (6.11) and that (we use notation of (11.4))

$$\begin{aligned} &\operatorname{rk} \prod_{j=1}^{m} \operatorname{Hom}_R(M''_j, W'_R/M'_j) = \\ &= i_1(d-n_1+i_1) + (i_2-i_1)(d-n_2+i_2) + \cdots + (i_m-i_{m-1})(d-n_m+i_m), \end{aligned}$$

if $(M_j)_{j=1}^m \in \mathbf{\Phi}_{(i_1,\dots,i_m)}(W)(R)$ and $(M'_j)_{j=1}^m \in \mathbf{\Phi}_{(n_1-i_1,\dots,n_m-i_m)}(W')(R)$. □

12. Varieties of flags of ideals

Again, A stays for an arbitrary finite-dimensional F-algebra; V is a finitely generated A-module.

Definition 12.1. For any $m \in \mathbb{N}$, we put

$$\mathbf{\Phi}_m^A(V) = \mathbf{\Phi}_m(V) \cap \mathbb{\Gamma}^A(V)^{\times m} \quad \subset \mathbb{\Gamma}(V)^{\times m}\ .$$

The shorten notation $\mathbf{\Phi}_m^A$ in the case $V = A$ is used. The F-functor $\mathbf{\Phi}_m^A$ (as well as its geometric realization) is called the *variety of m-flags of (right) ideals* (of A).

Lemma 12.2. *Let A_1 and A_2 be two arbitrary finite-dimensional F-algebras. Then $\mathbf{\Phi}_m^{A_1 \times A_2} \simeq \mathbf{\Phi}_m^{A_1} \times \mathbf{\Phi}_m^{A_2}$ for any $m \in \mathbb{N}$.*

Proof. Evident. □

For the further consideration we restrict ourself to the case where the algebra A is separable.

Lemma 12.3 (Morita equivalence). *Put $B = \operatorname{End}_A V$. Then $\mathbf{\Phi}^A_m(V) \simeq \mathbf{\Phi}^B_m$ for any $m \in \mathbb{N}$.*

Proof. In the proof of (10.7), mutually inverse isomorphisms of $\mathbb{I}\Gamma^A(V)$ and $\mathbb{I}\Gamma^B$ are described. They induce mutually inverse isomorphisms of $\mathbf{\Phi}^A_m(V) \subset \mathbb{I}\Gamma^A(V)^{\times m}$ and $\mathbf{\Phi}^B_m \subset (\mathbb{I}\Gamma^B)^{\times m}$. □

Corollary 12.4. *For every $m \in \mathbb{N}$, the variety $\mathbf{\Phi}^A_m(V)$ is smooth.*

Proof. Put $B = \operatorname{End}_A V$ and let $\bar{F}$ be an algebraic closure of F. Since B is a separable F-algebra, $B_{\bar{F}}$ is isomorphic to a product

$$\operatorname{End}_{\bar{F}} W_1 \times \cdots \times \operatorname{End}_{\bar{F}} W_n$$

for some vector $\bar{F}$-spaces $W_1, \ldots, W_n$. We have

$$\mathbf{\Phi}^A_m(V) \simeq \mathbf{\Phi}^B_m \qquad \text{by (12.3)};$$
$$\mathbf{\Phi}^{B_{\bar{F}}}_m \simeq \mathbf{\Phi}^{\operatorname{End} W_1}_m \times \cdots \times \mathbf{\Phi}^{\operatorname{End} W_n}_m \qquad \text{by (12.2)}.$$

Thus $\mathbf{\Phi}^A_m(V)_{\bar{F}}$ is isomorphic to the smooth variety $\mathbf{\Phi}_m(W_1) \times \cdots \times \mathbf{\Phi}_m(W_n)$. □

Theorem 12.5. *Let A be a separable F-algebra and let*

$$0 \to W' \to V \to W \to 0$$

be an exact sequence of finitely generated A-modules. For every $m \in \mathbb{N}$, the variety $\mathbf{\Phi}^A_m(V)$ is a relative cellular space over $\mathbf{\Phi}^A_m(W) \times \mathbf{\Phi}^A_m(W')$.

Proof. The proof is standard — compare with (10.9) or (11.4). □

Corollary 12.6. *In the category of correspondences $\mathcal{CV}$, there is an isomorphism*

$$\mathbf{\Phi}^A_m(V) \simeq \mathbf{\Phi}^A_m(W) \times \mathbf{\Phi}^A_m(W')\ .$$

In particular,

$$\mathrm{H}\left(\mathbf{\Phi}^A_m(V)\right) \simeq \mathrm{H}\left(\mathbf{\Phi}^A_m(W) \times \mathbf{\Phi}^A_m(W')\right)$$

for any geometric cohomology theory H. □

Theorem 12.7. *For any separable algebra A, in the category $\mathcal{CV}$ there is an isomorphism $\mathbf{\Phi}^A_m \simeq \mathbf{\Phi}^{A_{\mathrm{an}}}_m$ (for any $m \in \mathbb{N}$), where A_{an} is the anisotropic kernel of A (10.12).* □

Components

We compute the components of the varieties $\mathbf{\Phi}^A_m(V)$ only for a central simple algebra A. So, let A be a finite-dimensional central simple F-algebra, V a finitely generated A-module.

Definition 12.8. For every sequence of integers $(n_1, \dots, n_m)$, we put

$$\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V) = \mathbf{\Phi}_m(V) \cap \prod_{i=1}^m \Gamma^A_{n_i}(V) = \mathbf{\Phi}^A_m(V) \cap \mathbf{\Phi}_{(n_1 \deg A, \dots, n_m \deg A)}(V) .$$

The shorten notation $\mathbf{\Phi}^A_m$ in the case $V = A$ is used.

Lemma 12.9 (Morita equivalence). *Put $B = \operatorname{End}_A V$. Then*

$$\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V) \simeq \mathbf{\Phi}^B_{(n_1,\dots,n_m)}$$

for any sequence of integers $(n_1, \dots, n_m)$. □

Corollary 12.10. *The varieties $\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V)$ are geometrically irreducible and $\mathbf{\Phi}^A_m(V)$ is their direct sum.*

Proof. The latter statement is evident. Let us show that $\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V)_{\bar F}$ is irreducible, where $\bar F$ is an algebraic closure of F. Put $B = \operatorname{End}_A V$. Since $B_{\bar F}$ splits, there is a vector space W over $\bar F$ such that $B_{\bar F} \simeq \operatorname{End}_{\bar F} W$. We have

$$\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V)_{\bar F} \simeq \left(\mathbf{\Phi}^B_{(n_1,\dots,n_m)}\right)_{\bar F} \simeq \mathbf{\Phi}^{B_{\bar F}}_{(n_1,\dots,n_m)} \simeq \mathbf{\Phi}_{(n_1,\dots,n_m)}(W) .$$

Now use (11.7). □

Corollary 12.11. *In the conditions of (12.5), assume that the F-algebra A is central simple. Let H^* be a graded geometric cohomology theory. For any sequence of integers $(n_1, \dots, n_m)$, there is an isomorphism*

$$\mathrm{H}^*\left(\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V)\right) \simeq$$
$$\simeq \coprod_{i_1,\dots,i_m} \mathrm{H}^*\left(\mathbf{\Phi}^A_{(i_1,\dots,i_m)}(W) \times \mathbf{\Phi}^A_{(n_1-i_1,\dots,n_m-i_m)}(W')\right)$$
$$[-i_1(d-n_1+i_1) - (i_2-i_1)(d-n_2+i_2) - \dots - (i_m-i_{m-1})(d-n_m+i_m)]$$

where $d = \operatorname{rk}_A W'$. □

13. Varieties of isotropic subspaces

We return to the situation when V is simply a finite-dimensional vector F-space.

Definition 13.1. Let $h\colon V \times V \to W$ be a fixed F-bilinear map to a finite-dimensional vector F-space W. One defines the *subfunctor* $\Gamma(V,h) \subset \Gamma(V)$ *of* (more precisely — *totally*) *isotropic subspaces* as follows:

$$\Gamma(V,h)(R) = \left\{N \in \Gamma(V)(R) \mid h(N,N) = 0\right\} .$$

Lemma 13.2. *The subfunctor $\Gamma(V,h) \subset \Gamma(V)$ is closed (and hence represented by a complete F-variety).*

Proof. Let $N \in \Gamma(V)(R)$, $R \in F$-$\mathfrak{alg}$. Put $M = h(N,N) \subset W_R$. We have

$$N \in \Gamma(V,h)(R) \quad \Leftrightarrow \quad M = 0 .$$

Hence, if $\varphi\colon R \to S$ is a homomorphism of F-algebras and $\varphi^W : W_R \to W_S$ is the induced homomorphism of R-modules, we have

$$N \otimes_R S \in \mathbb{\Gamma}(V,h)(S) \quad \Leftrightarrow \quad \varphi^W(M) = 0 \ .$$

The end of the proof is standard (compare with (9.7) or (10.2)). $\square$

From now on, the characteristic of the field F is assumed to differ from 2. Suppose that the bilinear form h on V is:

- non-degenerate and
- symmetric or skew-symmetric.

Proposition 13.3. *Under the assumptions made above, the variety $\mathbb{\Gamma}(V,h)$ is smooth.*

Proof. We have to show that $\mathbb{\Gamma}(V,h)_{\bar F}$ is regular, where $\bar F$ is an algebraic closure of F. Since $\mathbb{\Gamma}(V,h)_{\bar F} \simeq \mathbb{\Gamma}(V_{\bar F}, h_{\bar F})$ we may simply assume that F itself is algebraically closed. Under this assumption, we produce a covering by open subvarieties isomorphic to affine spaces.

For any vector F-space U, the hyperbolic space on U, denoted by $\mathbb{H}(U)$, is by definition the direct sum $U \oplus U^*$ of U and the dual space $U^* = \mathrm{Hom}_F(U,F)$ supplied with the (context depending) symmetric or skew-symmetric bilinear form, determined by the conditions: $(U,U) = 0 = (U^*,U^*)$ and $(u^*,u) = u^*(u)$ for any $u^* \in U^*$, $u \in U$.

For every orthogonal decomposition of V of the type

$$V \simeq \mathbb{H}(U) \perp W$$

consider the projection $p : V \twoheadrightarrow U$ and the corresponding open subfunctor $\mathcal{U} \subset \mathbb{\Gamma}(V)$ (see the proof of (9.2)).

Lemma 13.4. *The subfunctors $\mathcal{U} \cap \mathbb{\Gamma}(V,h)$ cover $\mathbb{\Gamma}(V,h)$.*

Proof. Since F is algebraically closed, it suffices to check the covering on the level of F only. For every $U \in \mathbb{\Gamma}(V,h)(F)$, there exists a decomposition $V \simeq \mathbb{H}(U) \perp W$. If $\mathcal{U}$ is the corresponding open subfunctor, then $U \in \mathcal{U}(F)$. $\square$

Lemma 13.5. *The variety $\mathcal{U} \cap \mathbb{\Gamma}(V,h)$ is isomorphic to an affine space (the assumption that F is algebraically closed is not needed here).*

Proof. We know already that $\mathcal{U}$ itself is an affine space (see the proof of (9.2)). Namely, for any $R \in F\text{-}\mathfrak{alg}$, we identify the set $\mathcal{U}(R)$ with

$$\mathrm{Hom}_R(U_R, U_R^* \oplus W_R) = \mathrm{Bil}(U_R) \oplus \mathrm{Hom}_R(U_R, W_R)$$

where $\mathrm{Bil}(U_R)$ denotes the R-module of R-bilinear forms on U_R. A pair

$$\Big(b \in \mathrm{Bil}(U_R),\ f \in \mathrm{Hom}_R(U_R, W_R)\Big)$$

corresponds to an element from $\mathbb{\Gamma}(V,h)(R)$ if and only if the submodule

$$\big\{u + b(u,\cdot) + f(u) |\ u \in U\big\} \ \subset U \oplus U^* \oplus W = V$$

is totally isotropic, i.e. for any $u_1, u_2 \in U$ the scalar product

$$h\Big(u_1 + b(u_1,\cdot) + f(u_1),\ u_2 + b(u_2,\cdot) + f(u_2)\Big) =$$
$$= b(u_1,u_2) + \lambda b(u_2,u_1) + h\big(f(u_1), f(u_2)\big) \tag{$*$}$$

is zero, where $\lambda = 1$ (resp. $\lambda = -1$) in the symmetric (resp. skew-symmetric) case.

Notice, that since $\operatorname{char} F \neq 2$, the R-module $\operatorname{Bil}(U_R)$ decomposes in the direct sum $\operatorname{Bil}^1(U_R) \oplus \operatorname{Bil}^{-1}(U_R)$ of the submodules of the symmetric and the skew-symmetric forms: for any $b \in \operatorname{Bil}(U_R)$, its symmetric and skew-symmetric components are

$$(x,y) \mapsto \frac{b(x,y)+b(y,x)}{2} \quad \text{and} \quad (x,y) \mapsto \frac{b(x,y)-b(y,x)}{2}\ .$$

The condition $(*) = 0$ means that the λ-symmetric component of b is uniquely determined by f. However, there is no restriction on the choice of the $-\lambda$-symmetric component. Thus for any R, we obtain a bijection

$$\big(\mathcal{U} \cap \mathbb{\Gamma}(V,h)\big)(R) \simeq \operatorname{Bil}^{-\lambda}(U_R) \oplus \operatorname{Hom}(U_R, W_R)$$

giving an isomorphism with an affine space desired (note that $\mathcal{U} \cap \mathbb{\Gamma}(V,h)$ is (in general) not a linear subspace of the affine space $\mathcal{U}$). $\square$

With this lemma, we completed the proof of the proposition. $\square$

A cellular structure is not considered in this § because it will be obtained at once in a more general situation in §15.

Components

Definition 13.6. For every $n \in \mathbb{Z}$, we put

$$\mathbb{\Gamma}_n(V,h) = \mathbb{\Gamma}(V,h) \cap \mathbb{\Gamma}_n(V) \quad \subset \mathbb{\Gamma}(V)$$

(of course, $\mathbb{\Gamma}_n(V,h)$ is non-empty only if $0 \le n \le \dim V/2$).

Proposition 13.7. *The varieties $\mathbb{\Gamma}_n(V,h)$ are geometrically irreducible excluding the symmetric case with $n = \dim V/2$; $\mathbb{\Gamma}(V,h)$ is their direct sum. In the excluded case, the variety $\mathbb{\Gamma}_n(V,h)$ either is irreducible, or consists of two isomorphic (geometrically irreducible) components.*

Proof. Set $G = \mathbb{SO}(V,h)$ in the symmetric and $G = \mathbb{Sp}(V,h)$ in the skew-symmetric case. The affine algebraic group G is irreducible. If the exceptional case is excluded, it acts on $\mathbb{\Gamma}_n(V,h)$ transitively. Thus $\mathbb{\Gamma}_n(V,h)$ is absolutely irreducible.

In the exceptional case, the algebraic group $\mathbb{O}(V,h)$ acts transitively on $\mathbb{\Gamma}_n(V,h)$. Since $\mathbb{SO}(V,h)$ is the connected component of $\mathbb{O}(V,h)$ and $[\mathbb{O}(V,h) : \mathbb{SO}(V,h)] = 2$, the statement on the exceptional case follows. $\square$

14. Involutions and hermitian forms

Let A be a ring. An *involution* on A is an anti-automorphism of order ≤ 2. An isomorphism of rings with involutions is an isomorphism of the rings commuting with the involutions.

For two rings with involutions (A_1, σ_1) and (A_2, σ_2), the product $(A_1, \sigma_1) \times (A_2, \sigma_2)$ is the ring $A_1 \times A_2$ with the involution $\sigma_1 \times \sigma_2$.

Let σ be an involution on A and suppose that the ring A is *semisimple* [4, def. (24.5)]. Since σ acts on the set of simple components of A, every component is either invariant or interchanged with another one. Thus (A, σ) is isomorphic to a product of (already indecomposable) factors of the following two types:

- a simple ring [4, def. (25.14)] with an involution;
- $B \times B^{\mathrm{op}}$, where B is a simple ring and B^{op} is its opposite ring, with the *switch* involution τ:
$$\tau(b, b^{\mathrm{op}}) = \tau(b^{\mathrm{op}}, b) \text{ for } b \in B,\ b^{\mathrm{op}} \in B^{\mathrm{op}}.$$

Let (A, σ) be a ring with involution and let V', V be A-modules. A *sesquilinear mapping* h on $V' \times V$ is a biadditive map $h \colon V' \times V \to A$ such that
$$h(v'a', va) = \sigma(a') \cdot h(a', a) \cdot a \text{ for any } v' \in V',\ v \in V,\ a', a \in A.$$
The set of sesquilinear mappings on $V' \times V$ is denoted by $\mathrm{Sesq}(V', V)$. It is a module over the ring of σ-invariant elements of the center of A, isomorphic to $\mathrm{Hom}_A(V', V^*)$, where $V^* = \mathrm{Hom}_A(V, A)$ is considered as a right A-module (using the involution).

In the case $V' = V$, we obtain the notion of a *sesquilinear form* on V. The set of sesquilinear forms on V will be denoted simply by $\mathrm{Sesq}(V)$.

If h is a sesquilinear form on V, then σh, defined as
$$(\sigma h)(v', v) = \sigma\big(h(v, v')\big)$$
is once again a sesquilinear form on V. Put $\lambda = 1$ or $\lambda = -1$. A sesquilinear form h on V is called *λ-hermitian*, if $h = \lambda \cdot (\sigma h)$; one also says *hermitian* for 1-hermitian and *skew-hermitian* for (-1)-hermitian. The set of λ-hermitian forms on V is denoted by $\mathrm{Herm}^\lambda(V)$; it is a module over the ring of σ-invariant elements of the center of A.

Example 14.1. For $V = A$ put $h(a', a) = \sigma(a') \cdot a$ for any $a', a \in A$. Then h is a hermitian form on V.

Example 14.2 (Hyperbolic space). Let (A, σ) be a ring with involution, U an A-module, $\lambda = \pm 1$. The λ-hermitian *hyperbolic space on U*, denoted as $\mathbb{H}^\lambda(U)$ or simply $\mathbb{H}(U)$, is the direct sum $U \oplus U^*$ supplied with the λ-hermitian form $\mathbb{h}$, determined by the conditions
$$\mathbb{h}(U, U) = 0 = \mathbb{h}(U^*, U^*) \text{ and } \mathbb{h}(u^*, u) = u^*(u) \text{ for any } u^* \in U^*,\ u \in U.$$
If $U = U_1 \oplus U_2$, then $\mathbb{H}(U) = \mathbb{H}(U_1) \perp \mathbb{H}(U_2)$.

Suppose that $2 \in A^\times$. Any $h \in \mathrm{Sesq}(V)$ can be decomposed in the sum

$$h = \frac{h + \sigma h}{2} + \frac{h - \sigma h}{2} \ .$$

The first (resp. second) summand is a hermitian (resp. skew-hermitian) form on V called hermitian (resp. skew-hermitian) component of h. Since a simultaneously hermitian and skew-hermitian form is necessarily 0, we have a decomposition in a direct sum (of modules over the ring of σ-invariant elements of the center of A)

$$\mathrm{Sesq}(V) = \mathrm{Herm}^1(V) \oplus \mathrm{Herm}^{-1}(V) \ .$$

A λ-hermitian form on V is called *non-degenerate*, if the induced homomorphism of A-modules $\hat{h}\colon V \to V^*$ is an isomorphism. In this case, there is a unique involution σ_h on the ring $\mathrm{End}_A V$ satisfying the condition

$$h\big(v', f(v)\big) = h\big(\sigma_h(f)(v'), v\big) \ \text{ for any } v', v \in V, \ f \in \mathrm{End}_A V.$$

It is called the *adjoint* involution (with respect to h) and can be constructed as follows:

$$\sigma_h(f) = \hat{h}^{-1} \circ f^* \circ \hat{h} \quad \text{for any } f \in \mathrm{End}_A V,$$

where f^* is the endomorphism of V^* induced by f.

Proposition 14.3 ([18, cor. 9.2 of chap. 7]). *Let A be a skew-field with an involution, V a finite-dimensional vector A-space and h a non-degenerate λ-hermitian form on V. There exists an orthogonal decomposition*

$$(V, h) = \mathbb{H}(U) \perp (W, h)$$

with anisotropic $W \subset V$ (moreover, U and (W, h) are unique up to isomorphisms).

We specialize our consideration to the case where A is a finite-dimensional semisimple algebra over a field F and $\mathrm{char}\, F \neq 2$. By an involution on A we always mean an involution of the ring A which is F-linear or in other words which is trivial on F.

Lemma 14.4 ([18, discussion after thm. 7.4 of chap.8]). *Suppose that A is a simple F-algebra, V a finitely generated A-module. Every involution on the F-algebra $\mathrm{End}_A V$ is adjoint with respect to an appropriate chosen involution on A and a non-degenerate λ-hermitian form on V.*

Definition 14.5. For any semisimple F-algebra with an involution (A, σ), we define an F-algebra with an involution $(A, \sigma)_{\mathrm{an}}$, called the *anisotropic kernel* of (A, σ), by the following rules:

- if (A, σ) decomposes, $(A, \sigma)_{\mathrm{an}}$ is the product of the anisotropic kernels of the indecomposable components of (A, σ);
- if $(A, \sigma) \simeq \big(\mathrm{M}_n(D) \times \mathrm{M}_n(D)^{\mathrm{op}}, \mathrm{switch}\big)$, where D is a division algebra, we put

$$(A, \sigma)_{\mathrm{an}} = (D \times D^{\mathrm{op}}, \mathrm{switch})^{\times n} \ ;$$

- let D be a division F-algebra with an involution, V a vector space over D with a λ-hermitian form h and let
$$(V,h) = \mathbb{H}(U) \perp (W,h)$$
be a decomposition as in (14.3); if $A \simeq \operatorname{End}_D V$ and if σ is the adjoint involution on A, then
$$(A,\sigma)_{\mathrm{an}} = (D \times D^{\mathrm{op}}, \mathrm{switch})^{\times \dim_D U} \times (\operatorname{End}_D W, \sigma)\ .$$

The anisotropic kernel of a semisimple F-algebra with an involution is once again a semisimple F-algebra with an involution, defined up to an isomorphism.

Remark 14.6. The anisotropic kernel can be defined in a more intrinsic way as follows. One proves an analogy of (14.3) for the case of a semisimple F-algebra A and applies it to the case $V = A$ and h as in (14.1) in order to get a decomposition
$$V = \mathbb{H}(U_1) \perp \ldots \perp \mathbb{H}(U_n) \perp W$$
with anisotropic W and simple $U_1, \ldots, U_n$. The anisotropic kernel of (A,σ) is then the algebra
$$\operatorname{End}_A U_1 \times (\operatorname{End}_A U_1)^{\mathrm{op}} \times \cdots \times \operatorname{End}_A U_n \times (\operatorname{End}_A U_n)^{\mathrm{op}} \times \operatorname{End}_A W$$
with the restriction of σ in the capacity of the involution.

15. Varieties of isotropic ideals

Definition 15.1. For a finite-dimensional F-algebra A witn an involution σ and a finitely generated A-module V with a λ-hermitian form h, we put
$$\mathbb{I}\Gamma^A(V,h) = \mathbb{I}\Gamma^A(V) \cap \mathbb{I}\Gamma(V,h) \quad \subset \mathbb{I}\Gamma(V)\ .$$
In the case where $V = A$ and where h is as in (14.1), we use the notation $\mathbb{I}\Gamma^{A,\sigma}$ for $\mathbb{I}\Gamma^A(V,h)$ and call it the *variety of (right) (totally) isotropic ideals* of (A,σ).

Lemma 15.2. *For two finite-dimensional F-algebras with involutions (A_1,σ_1) and (A_2,σ_2), it holds $\mathbb{I}\Gamma^{A_1\times A_2,\sigma_1\times\sigma_2} \simeq \mathbb{I}\Gamma^{A_1,\sigma_1} \times \mathbb{I}\Gamma^{A_2,\sigma_2}$.* □

Definition 15.3. For a ring A and a right (resp. left) ideal $I \subset A$, we denote by $\operatorname{Ann} I$ the left (resp. right) annihilator of I. It is a left (resp. right) ideal of A.

Lemma 15.4. *Let A be a generalized Azumaya algebra (over a commutative ring R), $I \subset A$ a right or a left ideal, which is a direct summand of A. Then:*
- $\operatorname{Ann} I$ *is a direct summand of A;*
- $\operatorname{Ann}\operatorname{Ann} I = I$.

Proof. Since a left ideal in A is a right ideal in A^{op}, it suffices to consider the case of a right ideal I only.

Let us identify A with $\operatorname{End}_A A$. Then a right ideal I is identified with $\operatorname{Hom}_A(A,I)$. Hence $\operatorname{Ann} I = \operatorname{Hom}_A(A/I, A)$ and it becomes evident that $\operatorname{Ann} I$ is a direct summand of A in the case where I is.

Suppose that R is a field. Using the identification obtained, we can count the dimensions and see that $\dim_R I + \dim_R \operatorname{Ann} I = \dim_R A$.

For an arbitrary R, we evidently have $I \subset \operatorname{Ann}\operatorname{Ann} I$. By the dimension formula of the previous paragraph, for any homomorphism of R into a field L, it holds $(\operatorname{Ann}\operatorname{Ann} I/I) \otimes L = 0$. Thus $I = \operatorname{Ann}\operatorname{Ann} I$ (note that $\operatorname{Ann}\operatorname{Ann} I$ is a finitely generated R-module as a direct summand of A). □

Lemma 15.5. *Let A be a separable F-algebra. Set $B = A \times A^{op}$ and let τ be the switch involution on B. Then $\mathbb{I}\Gamma^{B,\tau} \simeq \mathbf{\Phi}_2^A$.*

Proof. Take any $R \in F\text{-}\mathfrak{alg}$. A right ideal of $B_R = A_R \times A_R^{\mathrm{op}}$ is a product $I \times I'$, where I is a right ideal, I' is a *left* ideal of A_R. An ideal $I \times I'$ is totally isotropic if and only if $I' \cdot I = 0$, i.e. $I \subset \operatorname{Ann} I'$. Since $\operatorname{Ann} I'$ is a direct summand of A_R by the previous lemma, we obtain a map

$$I \times I' \in \mathbb{I}\Gamma^{B,\tau}(R) \quad\mapsto\quad (I, \operatorname{Ann} I') \in \mathbf{\Phi}_2^A(R)$$

determining a morphism of the F-functors. The inverse morphism is given by the map

$$(I \subset J) \in \mathbf{\Phi}_2^A(R) \quad\mapsto\quad I \times \operatorname{Ann} J \in \mathbb{I}\Gamma^{B,\tau}(R)$$

and is really inverse by the previous lemma once again. □

For the rest of the §, we fix a separable algebra A with an involution σ and a finitely generated A-module V with a non-degenerate λ-hermitian form h (where $\lambda = 1$ or $\lambda = -1$).

Lemma 15.6 (Morita equivalence). *Put $B = \operatorname{End}_A V$ and let τ be the adjoint involution on B. Then $\mathbb{I}\Gamma^A(V, h) \simeq \mathbb{I}\Gamma^{B,\tau}$.*

Proof. By the definition of τ, we have

$$h\big(v,\ f(v')\big) = h\big(\tau(f)(v),\ v'\big)$$

for any $v, v' \in V$ and $f \in B$. For an arbitrary $R \in F\text{-}\mathfrak{alg}$, consider the given by Morita theory map (see the proof of (10.7))

$$\begin{array}{ccc} \mathbb{I}\Gamma^A(V)(R) & \to & \mathbb{I}\Gamma^B(R) \\ N & \mapsto & \operatorname{Hom}_{A_R}(V_R, N) \end{array}$$

Suppose that N is totally isotropic. We like to show that its image is totally isotropic as well. Take any $f, f' \in \operatorname{Hom}_{A_R}(V_R, N)$. We have

$$0 = h\Big(f(v), f'(v')\Big) = h\Big(\big(\tau(f') \circ f\big)(v),\ v'\Big) \text{ for any } v, v' \in V_R\,.$$

Since h is non-degenerate, it follows that $\big(\tau(f') \circ f\big)(v) = 0$ for any $v \in V_R$, i.e. $\tau(f') \circ f = 0 \in B_R$. Thus the image of N is really a totally isotropic ideal of B_R.

Since the inverse map

$$\mathbb{I}\Gamma^A(V)(R) \leftarrow \mathbb{I}\Gamma^B(R)$$

is defined in an analogous manner (see the proof of (10.7)), a similar verification shows that it preserves the property of being totally isotropic as well. Thus we obtain mutually inverse isomorphisms of $\mathbb{\Gamma}^A(V,h)$ and $\mathbb{\Gamma}^{B,\tau}$. $\square$

Corollary 15.7. *The variety $\mathbb{\Gamma}^A(V,h)$ is smooth.*

Proof. It is enough to handle only the variety $\mathbb{\Gamma}^{A,\sigma}$ (15.6), under the assumption that F is algebraically closed and (A,σ) is indecomposable (15.2). Then there is a vector F-space W such that either $A \simeq \operatorname{End}_F W$ or $A \simeq \operatorname{End} W \times \operatorname{End} W^*$ and σ is the switch in the second case.

Consider the first case. Depending on the type of σ, there is a symmetric or skew-symmetric non-degenerate bilinear form h on W such that σ is the adjoint involution with respect to h (and the identity involution on F). Thus $\mathbb{\Gamma}^{A,\sigma} \simeq \mathbb{\Gamma}(W,h)$ by (15.6) and is smooth by (13.3).

In the second case, $\mathbb{\Gamma}^{A,\sigma} \simeq \mathbf{\Phi}_2(W)$ by (15.5) and is smooth by (11.3). $\square$

Theorem 15.8. *Let A be a separable F-algebra with an involution, V a finitely generated A-module with a non-degenerate λ-hermitian form h and*

$$V = \mathbb{H}(U) \perp W$$

an orthogonal decomposition. The variety $\mathbb{\Gamma}^A(V,h)$ is a relative cellular space over $\mathbf{\Phi}_2^A(U) \times \mathbb{\Gamma}^A(W,h)$.

Proof. Using the decomposition of the A-module $V = U \oplus U^* \oplus W$, we obtain a relative structure on $\mathbb{\Gamma}^A(V)$ over $\mathbb{\Gamma}^A(U) \times \mathbb{\Gamma}^A(U^*) \times \mathbb{\Gamma}^A(W)$ as in (10.14). Restricting the morphism

$$\operatorname{Gr} \mathbb{\Gamma}^A(V) \to \mathbb{\Gamma}^A(U) \times \mathbb{\Gamma}^A(U^*) \times \mathbb{\Gamma}^A(W)$$

to $\operatorname{Gr} \mathbb{\Gamma}^A(V,h) = \mathbb{\Gamma}^A(V,h) \cap \operatorname{Gr} \mathbb{\Gamma}^A(V)$, we get a morphism into $\mathbf{\Phi}_2^A(U) \times \mathbb{\Gamma}^A(W,h)$. Indeed, let $R \in F\text{-}\mathfrak{alg}$ and let

$$(N_1, (U_R/N_2)^*, N_3) \in \Big(\mathbb{\Gamma}^A(U) \times \mathbb{\Gamma}^A(U^*) \times \mathbb{\Gamma}^A(W)\Big)(R)$$

be the image of some $N \in \operatorname{Gr} \mathbb{\Gamma}^A(V)(R)$. If N is totally isotropic, then N_3 is totally isotropic as well; also the submodule $N_1 \oplus (U_R/N_2)^*$ is totally isotropic, that is $N_1 \subset N_2$.

We are going to produce a structure of a vector bundle on the morphism

$$\operatorname{Gr} \mathbb{\Gamma}^A(V,h) \to \mathbf{\Phi}_2^A(U) \times \mathbb{\Gamma}^A(W,h)$$

constructed right now.

Fix $R \in F\text{-}\mathfrak{alg}$, $(N_1,N_2) \in \mathbf{\Phi}_2^A(U)(R)$, $N_3 \in \mathbb{\Gamma}^A(W,h)(R)$, a splitting of the inclusion $N_2 \hookrightarrow U_R$ and an A_R-module N_3' such that $N_3 \oplus N_3' = W$. The elements $N \in \operatorname{Gr} \mathbb{\Gamma}^A(V,h)(R)$, lying over the fixed R-point, are in one-to-one correspondence with

$$\begin{aligned} &\operatorname{Hom}_{A_R}(N_1, N_2^*) \oplus \operatorname{Hom}_{A_R}(N_1, N_3') \oplus \operatorname{Hom}(N_3, N_2^*) = \\ &= \operatorname{Sesq}(N_1,N_2) \oplus \operatorname{Hom}_{A_R}(N_1, N_3') \oplus \operatorname{Sesq}(N_3,N_2) \end{aligned}$$

(compare to (10.14)). An element $h_{12} \oplus f_{13} \oplus h_{32}$ of the latter set corresponds to

$$N = \big\{ n_1 + n_2^* + h_{12}(n_1, \cdot) + h_{32}(n_3, \cdot) + n_3 + f_{13}(n_1) \in V_R | \\ n_1 \in N_1, n_2^* \in (U_R/N_2)^*, n_3 \in N_3 \big\} .$$

Let us compute the hermitian product of two elements from this N:

$$h\Big(n_1 + n_2^* + h_{12}(n_1, \cdot) + h_{32}(n_3, \cdot) + n_3 + f_{13}(n_1), \\ m_1 + m_2^* + h_{12}(m_1, \cdot) + h_{32}(m_3, \cdot) + m_3 + f_{13}(m_1) \Big) =$$

$$(*) \quad = h_{12}(n_1, m_1) + \lambda\sigma\big(h_{12}(m_1, n_1)\big) + h_{32}(n_3, m_1) + \lambda\sigma\big(h_{32})(m_3, n_1)\big) + \\ + h\big(n_3, f_{13}(m_1)\big) + \lambda\sigma\Big(h\big(m_3, f_{13}(n_1)\big)\Big) + h\big(f_{13}(n_1), f_{13}(m_1)\big)$$

where $n_1, m_1 \in N_1$, $n_2^*, m_2^* \in (U_R/N_2)^*$, $n_3, m_3 \in N_3$.

Suppose that $(*) = 0$ for any n_1, n_3, m_1, m_3. Taking $n_3 = 0 = m_3$, we get a condition

(i)

$$h_{12}(n_1, m_1) + \lambda\sigma\big(h_{12}(m_1, n_1)\big) = -h\big(f_{13}(n_1), f_{13}(m_1)\big) \text{ for any } n_1, m_1 \in N_1 .$$

Thus the rest of $(*)$

$$h_{32}(n_3, m_1) + \lambda\sigma\big(h_{32}(m_3, n_1)\big) + h\big(n_3, f_{13}(m_1)\big) + \lambda\sigma\Big(h\big(m_3, f_{13}(n_1)\big)\Big)$$

is zero as well. Taking $m_3 = 0 = n_1$, we get a condition

$$\text{(ii)} \qquad h_{32}(n_3, m_1) = -h\big(n_3, f_{13}(m_1)\big) \text{ for any } n_3 \in N_3 \text{ and } m_1 \in N_1$$

Conversely, the conditions (i) and (ii) together imply $(*)$.

The condition (ii) means that the restriction of $h_{32} \in \mathrm{Sesq}(N_3, N_2)$ to $N_3 \times N_1$ is uniquely determined by f_{13}. The condition (i) means that the λ-hermitian component of the restriction of the sesquilinear map $h_{12} \in \mathrm{Sesq}(N_1, N_2)$ to the subset $N_1 \times N_1$ is uniquely determined by f_{13} as well. Thus we get a bijection with the following set:

$$\mathrm{Sesq}(N_1, N_1') \oplus \mathrm{Herm}^{-\lambda}(N_1) \oplus \mathrm{Hom}_{A_R}(N_1, N_3') \oplus \mathrm{Sesq}(N_3, N_1')$$

where N_1' is a fixed A_R-module such that $N_2 = N_1 \oplus N_1'$. It gives a structure of a vector bundle, since the sum written down is a finitely generated projective R-module compatible with tensor products by commutative R-algebras. □

Corollary 15.9. *In the category of correspondences $\mathcal{CV}$, there is an isomorphism*

$$\mathbb{\Gamma}^A(V, h) \simeq \mathbf{\Phi}_2^A(U) \times \mathbb{\Gamma}^A(W, h) .$$

In particular,

$$\mathrm{H}\left(\mathbb{\Gamma}^A(V)\right) \simeq \mathrm{H}\left(\mathbf{\Phi}_2^A(U) \times \mathbb{\Gamma}^A(W, h)\right)$$

for any geometric cohomology theory H. □

Theorem 15.10. *For any separable algebra A with an involution σ, in the category $\mathcal{CV}$ there is an isomorphism $\mathbb{\Gamma}^{A,\sigma} \simeq \mathbb{\Gamma}^{(A,\sigma)_{\mathrm{an}}}$, where $(A,\sigma)_{\mathrm{an}}$ is the anisotropic kernel (14.5)* □

Components

Let A be a central simple F-algebra supplied with an involution σ (called an involution *of the first kind* in the literature). We put

$$t = \begin{cases} 1 & \text{in the case of orthogonal } \sigma; \\ -1 & \text{in the case of symplectic } \sigma. \end{cases}$$

As before, V is a finitely generated A-module, supplied with a non-degenerate λ-hermitian form h (where $\lambda = \pm 1$).

Definition 15.11. For any $n \in \mathbb{Z}$, we put

$$\mathbb{\Gamma}_n^A(V,h) = \mathbb{\Gamma}^A(V,h) \cap \mathbb{\Gamma}_n^A(V) \quad \subset \mathbb{\Gamma}^A(V)\ .$$

Analogously, the F-functors $\mathbb{\Gamma}_n^{A,\sigma}$ are defined. The variety $\mathbb{\Gamma}_1^{A,\sigma}$ in the orthogonal case was called the *involution variety* and studied in [21].

Lemma 15.12 (Morita equivalence). *Put $B = \mathrm{End}_A V$ and let τ be the adjoint involution on B. Then $\mathbb{\Gamma}_n^A(V,h) \simeq \mathbb{\Gamma}_n^{B,\tau}$.*

Proof. Using (10.7), we identify $\mathbb{\Gamma}^A(V)$ with $\mathbb{\Gamma}^B$. By (10.17), the subfunctor $\mathbb{\Gamma}_n^A(V)$ is identified with $\mathbb{\Gamma}_n^B$; by (15.6), the subfunctor $\mathbb{\Gamma}^A(V,h)$ is identified with $\mathbb{\Gamma}^{B,\tau}$. Thus the intersections are identified with each other. □

Corollary 15.13. *The varieties $\mathbb{\Gamma}_n^A(V,h)$ are geometrically irreducible excluding the case where $t\lambda = 1$ and $n = \mathrm{rk}_A V/2$; $\mathbb{\Gamma}^A(V,h)$ is their direct sum. In the excluded case, the variety $\mathbb{\Gamma}_n^A(V,h)$ either is irreducible, or consists of two isomorphic (geometrically irreducible) components.*

Proof. The last statement is straightforward. Let us prove the rest.

By the lemma, it suffices to handle the variety $\mathbb{\Gamma}_n^{B,\tau}$ only. We also may assume that F is algebraically closed. Then $B \simeq \mathrm{End}_F W$ for a vector F-space W and the involution on $\mathrm{End}_F W$ is adjoint with respect to a $t\lambda$-symmetric non-degenerate bilinear form h on W. Thus $\mathbb{\Gamma}_n^{B,\tau} \simeq \mathbb{\Gamma}_n(W,h)$ by (15.12). The variety $\mathbb{\Gamma}_n(W,h)$ is either irreducible or consists of two (geometrically irreducible) components by (13.7). □

Corollary 15.14. *In the conditions of (15.8), assume that the F-algebra A is central simple and the involution σ is of the type t. Let H^* be a graded geometric cohomology theory. For any $n \in \mathbb{Z}$, there is an isomorphism*

$$\mathrm{H}^*\left(\mathbb{\Gamma}_n^A(V,h)\right) \simeq \coprod_{i+(\mathrm{rk}\,U-j)+k=n} \mathrm{H}^*\left(\mathbf{\Phi}_{(i,j)}^A(U) \times \mathbb{\Gamma}_k^A(W,h)\right)[-r_{ijk}]$$

where $r_{ijk} = i(j-i) + i(i-t\lambda)/2 + i(\mathrm{rk}_A W - k) + k(j-i)$.

Proof. We only have to check that the twisting numbers r_{ijk} are correct. In the notation of the proof of (15.8), suppose that

$$(N_1, N_2) \in \mathbf{\Phi}^A_{(i,j)}(U)(R) \quad \text{and} \quad N_3 \in \mathbb{I}\Gamma^A_k(W,h)(R)\ .$$

Then

$$\begin{aligned} r_{ijk} = &\operatorname{rk}_R \operatorname{Sesq}(N_1, N_1') + \operatorname{rk}_R \operatorname{Herm}^{-\lambda}(N_1) + \\ &+ \operatorname{rk}_R \operatorname{Hom}_{A_R}(N_1, N_3') + \operatorname{rk}_R \operatorname{Sesq}(N_3, N_1')\ . \end{aligned}$$

Computing these ranks, we may assume that R is a field. To avoid additional notation, we simply assume that $R = F$.

Lemma 15.15. *Let A be a central simple F-algebra with an involution of type t, N' and N finitely generated A-modules. Then*

(i) $$\dim_F \operatorname{Sesq}(N', N) = \operatorname{rk}_A N' \cdot \operatorname{rk}_A N\ ;$$

(ii) $$\dim_F \operatorname{Herm}^\lambda(N) = \frac{1}{2} \operatorname{rk}_A N(\operatorname{rk}_A N + t\lambda)\ .$$

Proof. We have already observed that $\dim_F \operatorname{Hom}_A(N', N) = \operatorname{rk}_A N' \cdot \operatorname{rk}_A N$. Since $\operatorname{Sesq}(N', N) \simeq \operatorname{Hom}_A(N', N^*)$, the equality (i) needs no proof.

The equality (ii) is evidently fulfilled for $N = A$: the F-space $\operatorname{Herm}^\lambda(A)$ is isomorphic to the space of λ-symmetric elements of A; its dimension equals $\deg A(\deg A + t\lambda)/2$ by one of possible definitions of an orthogonal (resp. symplectic) involution [18, def. 7.6 of chap. 8].

For an arbitrary A-module N and any $n \in \mathbb{N}$, notice that

$$\operatorname{Herm}^\lambda(N^n) \simeq \operatorname{Sesq}(N)^{n(n-1)/2} \oplus \operatorname{Herm}^\lambda(N)^n$$

and therefore (ii) is fulfilled for N if and only if it holds for N^n. This ends the proof since $N^n \simeq A^m$ for some $n, m \in \mathbb{N}$. □

Since $\operatorname{rk}_A N_1 = i$, $\operatorname{rk}_A N_1' = j - i$, $\operatorname{rk}_A N_3 = k$ and $\operatorname{rk}_A N_3' = \operatorname{rk}_A W - k$, we get

$$r_{ijk} = i(j-i) + i(i - t\lambda)/2 + i(\operatorname{rk}_A W - k) + k(j-i)\ .$$

□

16. Varieties of flags of isotropic ideals

Definition 16.1. For a finite-dimensional F-algebra A witn an involution σ and a finitely generated A-module V with a λ-hermitian form h, for any $m \in \mathbb{N}$, we put

$$\mathbf{\Phi}^A_m(V,h) = \mathbf{\Phi}^A_m(V) \cap \mathbb{I}\Gamma^A(V,h)^{\times m} \quad \subset \mathbb{I}\Gamma^A(V)^{\times m}\ .$$

In the case where $V = A$ and where h is as in (14.1), we use the notation $\mathbf{\Phi}^{A,\sigma}_m$ for $\mathbf{\Phi}^A_m(V,h)$ and call it the *variety of m-flags of (right) (totally) isotropic ideals* of (A, σ).

Lemma 16.2. *For two finite-dimensional F-algebras with involutions (A_1, σ_1) and (A_2, σ_2), for any $m \in \mathbb{N}$, it holds*

$$\mathbf{\Phi}_m^{A_1 \times A_2, \sigma_1 \times \sigma_2} \simeq \mathbf{\Phi}_m^{A_1, \sigma_1} \times \mathbf{\Phi}_m^{A_2, \sigma_2} .$$

$\square$

Lemma 16.3. *Let A be a separable F-algebra. Set $B = A \times A^{op}$ and let τ be the switch involution on B. Then $\mathbf{\Phi}_m^{B,\tau} \simeq \mathbf{\Phi}_{2m}^{A}$ for any $m \in \mathbb{N}$.*

Proof. The mutually inverse isomorphisms of the F-functors are defined as follows (compare with the proof of (15.5)). For any $R \in F\text{-}\mathfrak{alg}$, an element

$$(I_1 \times I'_1 \subset \cdots \subset I_m \times I'_m) \in \mathbf{\Phi}_m^{B,\tau}(R)$$

is identified with

$$(I_1 \subset \cdots \subset I_m \subset \operatorname{Ann} I'_m \subset \cdots \subset \operatorname{Ann} I'_1) \in \mathbf{\Phi}_{2m}^{A}(R) ;$$

conversely, an element

$$(I_1 \subset \cdots \subset I_m \subset J_m \subset \cdots \subset J_1) \in \mathbf{\Phi}_{2m}^{A}(R)$$

is identified with

$$(I_1 \times \operatorname{Ann} J_1 \subset \cdots \subset I_m \times \operatorname{Ann} J_m) \in \mathbf{\Phi}_m^{B,\tau}(R) .$$

$\square$

For the rest of §, we fix a separable algebra A with an involution σ and a finitely generated A-module V with a non-degenerate λ-hermitian form h (as always $\lambda = 1$ or $\lambda = -1$).

Lemma 16.4 (Morita equivalence). *Put $B = \operatorname{End}_A V$ and let τ be the adjoint involution on B. Then $\mathbf{\Phi}_m^{A}(V, h) \simeq \mathbf{\Phi}_m^{B,\tau}$ for any $m \in \mathbb{N}$.* $\square$

Corollary 16.5. *The varieties $\mathbf{\Phi}_m^{A}(V, h)$ are smooth.*

Proof. It is enough to handle only the variety $\mathbf{\Phi}_m^{A,\sigma}$ (16.4), under the assumption that F is algebraically closed and (A, σ) is indecomposable (16.2). Then there is a vector F-space W such that either $A \simeq \operatorname{End}_F W$ or $A \simeq \operatorname{End} W \times \operatorname{End} W^*$ and σ is the switch in the second case.

Consider the first case. Depending on the type of σ, there is a symmetric or skew-symmetric non-degenerate bilinear form h on W such that σ is the adjoint involution with respect to h (and the identity involution on F). Thus $\mathbf{\Phi}_m^{A,\sigma} \simeq \mathbf{\Phi}_m^{F}(W, h)$ by (16.4). The latter variety is smooth, since it can be covered by affine spaces (compare to (11.3) and (13.3)).

In the second case, $\mathbf{\Phi}_m^{A,\sigma} \simeq \mathbf{\Phi}_{2m}(W)$ by (16.3) and is smooth by (11.3). $\square$

Theorem 16.6. *Let A be a separable F-algebra with an involution, V a finitely generated A-module with a non-degenerate λ-hermitian form h and*

$$V = \mathbb{H}(U) \perp W$$

an orthogonal decomposition. For any $m \in \mathbb{N}$, the variety $\mathbf{\Phi}_m^{A}(V, h)$ is a relative cellular space over $\mathbf{\Phi}_{2m}^{A}(U) \times \mathbf{\Phi}_m^{A}(W, h)$. $\square$

Corollary 16.7. *In the category of correspondences $\mathcal{CV}$, for any $m \in \mathbb{N}$, there is an isomorphism*

$$\mathbf{\Phi}^A_m(V,h) \simeq \mathbf{\Phi}^A_{2m}(U) \times \mathbf{\Phi}^A_m(W,h) \ .$$

In particular,

$$\mathrm{H}\left(\mathbf{\Phi}^A_m(V,h)\right) \simeq \mathrm{H}\left(\mathbf{\Phi}^A_{2m}(U) \times \mathbf{\Phi}^A_m(W,h)\right)$$

for any geometric cohomology theory H. □

Theorem 16.8. *For any separable algebra A with an involution σ, for any $m \in \mathbb{N}$, in the category $\mathcal{CV}$ there is an isomorphism*

$$\mathbf{\Phi}^{A,\sigma}_m \simeq \mathbf{\Phi}^{(A,\sigma)_{\mathrm{an}}}_m$$

where $(A,\sigma)_{\mathrm{an}}$ is the anisotropic kernel of (A,σ) (14.5). □

Components

Let A be a central simple F-algebra supplied with an involution σ of type t, V a finitely generated A-module supplied with a non-degenerate λ-hermitian form h.

Definition 16.9. For any sequence of integers $(n_1,\dots,n_m)$, we put

$$\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V,h) = \mathbf{\Phi}^A_m(V,h) \cap \mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V) \quad \subset \mathbb{\Gamma}^A(V)^{\times m} \ .$$

Analogously, the F-functors $\mathbf{\Phi}^{A,\sigma}_{(n_1,\dots,n_m)}$ are defined.

Lemma 16.10 (Morita equivalence). *Put $B = \operatorname{End}_A V$ and let τ be the adjoint involution on B. Then $\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V,h) \simeq \mathbf{\Phi}^{B,\tau}_{(n_1,\dots,n_m)}$.* □

Corollary 16.11. *The varieties $\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V,h)$ are geometrically irreducible excluding the case where $t\lambda = 1$ and $n_m = \operatorname{rk}_A V/2$; $\mathbf{\Phi}^A_m(V,h)$ is their direct sum. In the excluded case, the variety $\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V,h)$ either is irreducible, or consists of two isomorphic (geometrically irreducible) components.*

Proof. The last statement is straightforward. Let us prove the rest.

By the lemma, it suffices to handle the variety $\mathbf{\Phi}^{B,\tau}_{(n_1,\dots,n_m)}$ only. We also may assume that F is algebraically closed. Then $B \simeq \operatorname{End}_F W$ for a vector F-space W and the involution on $\operatorname{End}_F W$ is adjoint with respect to a $t\lambda$-symmetric non-degenerate bilinear form h on W. Thus

$$\mathbf{\Phi}^{A,\sigma}_{(n_1,\dots,n_m)} \simeq \mathbf{\Phi}^F_{(n_1,\dots,n_m)}(W,h)$$

by (16.10). Put

$$G = \begin{cases} \mathbb{SO}(W,h), & \text{if } t\lambda = 1; \\ \mathbb{S}\mathrm{p}(W,h), & \text{if } t\lambda = -1. \end{cases}$$

If the exceptional case is excluded, the algebraic group G acts transitively on $\mathbf{\Phi}^F_{(n_1,\dots,n_m)}(W,h)$, whence geometrically irreducibility.

In the exceptional case, the group $\mathbb{O}(W,h)$ acts transitively on the variety $\mathbf{\Phi}^F_{(n_1,\dots,n_m)}(W,h)$. Since $\mathbb{SO}(W,h)$ is the connected component of $\mathbb{O}(W,h)$ and $[\mathbb{O}(W,h):\mathbb{SO}(W,h)]=2$, the statement on the exceptional case follows. $\square$

Corollary 16.12. *In the conditions of (16.6), assume that the F-algebra A is central simple. Let H^* be a graded geometric cohomology theory. For any sequence of integers $(n_1,\dots,n_m)$, there is an isomorphism*

$$\begin{aligned}\mathrm{H}^*\left(\mathbf{\Phi}^A_{(n_1,\dots,n_m)}(V,h)\right) &\simeq \\ &\simeq \coprod_{i_1+(\operatorname{rk}U-j_1)+k_1=n_1} \cdots \coprod_{i_m+(\operatorname{rk}U-j_m)+k_m=n_m} \\ &\qquad \mathrm{H}^*\left(\mathbf{\Phi}^A_{(i_1,\dots,i_m,j_m,\dots,j_1)}(U)\times \mathbf{\Phi}^A_{(k_1,\dots,k_m)}(W,h)\right)[\dots]\end{aligned}$$

(a computation of the twisting numbers is omitted). $\square$

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