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Some new examples in the theory of quadratic forms

Oleg T. Izhboldin[1], **Nikita A. Karpenko**[2]

[1] Universität Bielefeld, Fakultät für Mathematik, Postfach 10 01 31, D-33501 Bielefeld, Germany (e-mail: oleg@mathematik.uni-bielefeld.de)
[2] Westfälische Wilhelms-Universität, Mathematisches Institut, Einsteinstraße 62, D-48149 Münster, Germany (e-mail: karpenk@math.uni-muenster.de)

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Abstract. We construct a 6-dimensional anisotropic quadratic form ϕ and a 4-dimensional quadratic form ψ over some field F such that ϕ becomes isotropic over the function field $F(\psi)$ but every proper subform of ϕ is still anisotropic over $F(\psi)$. It is an example of *non-standard isotropy* with respect to some standard conditions of isotropy for 6-dimensional forms over function fields of quadrics, known previously. Besides of that, we produce an 8-dimensional quadratic form ϕ with trivial determinant such that the index of the Clifford invariant of ϕ is 4 but ϕ can not be represented as a sum of two 4-dimensional forms with trivial determinants. Using this, we find a 14-dimensional quadratic form with trivial discriminant and Clifford invariant, which is not similar to a difference of two 3-fold Pfister forms. The proofs are based on computations of the topological filtration on the Grothendieck group of certain projective homogeneous varieties. To do these computations, we develop several methods, covering a wide class of varieties and being, to our mind, of independent interest.

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1. Introduction

Let F be a field of characteristic $\neq 2$. An important problem in the algebraic theory of quadratic forms is to classify the pairs of anisotropic quadratic forms ϕ, ψ over F such that $\phi_{F(\psi)}$ is isotropic, where $F(\psi)$ is the function field of ψ, i.e. the function field of the projective quadric determined by ψ. In the case $\dim \phi \leq 5$, a complete classification is known (see [11]). The case $\dim \phi = 6$ was studied in [12], [31], [32], [34], and [37]. In the case where $\dim \phi = 6$ and $\dim \psi \neq 4$, a complete classification was obtained. After that, it was shown in [18] and [19], that the same classification is valid for 4-dimensional forms ψ, if the case where $\dim \phi = 6$, $\dim \psi = 4$, $1 \neq \det_{\pm} \phi \neq \det_{\pm} \psi \neq 1$, $\operatorname{ind} C_0(\phi) = 2$, and $\operatorname{ind} C_0(\phi) \otimes C_0(\psi) = 2$ is excluded. Here (see Sect. 18) we construct in this excepted case an example of ϕ and ψ with the non-standard (i.e. not matching the old classification) isotropy of ϕ over $F(\psi)$ (see Theorem 18.2). It is possible to explain what this "non-standard isotropy" does exactly mean without describing the old classification (Lemma 18.3): isotropy of a form ϕ over $F(\psi)$ is non-standard if and only if the form ϕ is $F(\psi)$-minimal, i.e. no proper subform of ϕ becomes isotropic over $F(\psi)$. A stronger version of Theorem 18.2 states that an example of the non-standard isotropy can be obtained starting from an *arbitrary* anisotropic 4-dimensional form ψ (with $\det \psi \neq 1$) over an arbitrary field F_0 by passing to an appropriate extension F of F_0 (see Corollary 18.4).

Let $I(F)$ be the ideal of even-dimensional forms in the Witt ring $W(F)$ of the quadratic forms over F. Another important problem in the algebraic theory of quadratic forms is to give a classification of low-dimensional forms belonging to $I^n(F)$ for a fixed $n > 0$. For $n = 2$ and for $n = 3$, this problem was studied by many authors. In [20] N. Jacobson proved that the quadratic forms $\phi \in I^2(F)$ of dimension ≤ 6 are uniquely determined up to similarity by the Clifford invariant $c(\phi)$. There exists a good description of 8-dimensional forms $\phi \in I^2(F)$ satisfying the condition $\operatorname{ind} C(\phi) \leq 2$. Namely, these quadratic forms can be written as tensor product of a 2-dimensional subform and a 4-dimensional subform (see e.g. [28, Ex. 9.12]). The case of 8-dimensional quadratic forms $\phi \in I^2(F)$ with $\operatorname{ind} C(\phi) = 4$ is much more complicated. It was an open question if these quadratic forms can be written as $\tau_1 \perp \tau_2$, where τ_1 and τ_2 are 4-dimensional forms with trivial determinant. In Sect. 16 we construct a counterexample for this question (Corollary 16.8). Nevertheless we find a "weak version" of the decomposition $\phi = \tau_1 \perp \tau_2$. Note that quadratic forms of the type $\tau_1 \perp \tau_2$ can be regarded as Scharlau's transfer $s_{L/F}(\tau)$ in the degenerate case $L = F \times F$. We show that an arbitrary 8-dimensional form $\phi \in I^2(F)$ with $\operatorname{ind} C(\phi) = 4$ can be represented as Scharlau's transfer $s_{L/F}(\tau)$, where L/F is an (étale) quadratic extension and τ is a 4-dimensional L-form with trivial determinant (see Theorem 16.10).

In Sect. 17 we study the quadratic forms $\phi \in I^3(F)$. The structure of ϕ in the case $\dim \phi \leq 12$ was described by Pfister in [40, Satz 14 und Zusatz] (see also [14]). Our aim is to study the 14-dimensional quadratic forms in $I^3(F)$. In [42] M. Rost proved that an arbitrary 14-dimensional quadratic form can be represented (up to similarity) as Scharlau's transfer $s_{L/F}(\sqrt{d}\,\tau')$, where $L = F(\sqrt{d})$ and τ' is the pure subform of a 3-fold

Pfister form. Note that in the degenerate case $L = F \times F$ we get the decomposition $\phi = k(\tau_1' \perp -\tau_2')$, where τ_1', τ_2' are pure subforms of some 3-fold Pfister forms τ_1, τ_2 and $k \in F^*$. It was an open question if any 14-dimensional form $\phi \in I^3(F)$ can be written in the form $\phi = k(\tau_1' \perp -\tau_2')$. It was remarked by D. Hoffmann (1995, Bielefeld, oral communication) that this question is equivalent to the discussed above question on 8-dimensional forms $\phi \in I^2(F)$ with $\operatorname{ind} C(\phi) = 4$. Using the counterexample for 8-dimensional forms, we construct (in Sect. 17) a counterexample for 14-dimensional forms.

Similar counterexamples of 8-dimensional and 14-dimensional forms in the case of characteristic 0 are independently constructed in [16] by using completely different techniques.

Our methods are based on the computation of the topological filtration on the Grothendieck group for certain projective homogeneous varieties. There are numerous works on K-theory of particular projective homogeneous varieties (Quillen [41], Swan [51], Levine-Srinivas-Weyman [35], Tao [52], and others) and a general work of Panin [39], where the K-theory is computed. However none of them does not consider the question about the topological filtration on the K-theory. In Part I we develop a machinery which makes possible to compute the topological filtration on K_0 for a wide class of homogeneous varieties (see Corollaries 9.6, 10.6, and 11.4 which are, in fact, not about the homogeneous varieties only). So, as to the future applications, we consider Part I as the most interesting part of the article and Part II as an example of an application of Part I to two known problems in the theory of quadratic forms.

2. Plan of works

The paper consists of two Parts. All main results, mentioned in the Introduction, are obtained in Part II. However their proofs are based on the results of Part I: the example of non-standard isotropy is based on Theorem 14.1; the examples of the 8-dimensional and 14-dimensional quadratic forms are based on Theorem 13.1. Although these two groups of our main results are rather far from each other, Theorems 13.1 and 14.1 are quite similar. In fact, they both are about an upper bound for the codimension (with respect to the topological filtration) of a certain element (namely, the doubled rational point class) in the Grothendieck group of certain varieties (see the proof of Theorem 13.1 and the statement of Lemma 14.3).

Moreover, the varieties are quite similar (we mean the variety X_K in the proof of Theorem 13.1 and the variety $X_1 \times X_2$ in Theorem 14.1). They both are of the form $X_{F(\mathcal{R}(T))}$ with certain F-varieties X and T, where $\mathcal{R}(T)$ is the Weil transfer of the L-variety T_L with respect to a Galois (in fact, quadratic or biquadratic) field extension L/F. Note that the varieties are not chosen by chance: in certain sense, they are generic models of the situation we need.

In Sects. 4–11 of Part I we build up a technology for computing the topological filtration on the Grothendieck group $K(X_{F(\mathcal{R}(T))})$. Now we are going to explain the purpose of each Section.

The techniques shown in Sect. 5 allows one to move the problem from the variety $X_{F(\mathcal{R}(T))}$ to the F-variety $X \times \mathcal{R}(T)$.

Necessary back-grounds on the Weil transfer are given in Sect. 6.

Since $\mathcal{R}(T)_L$ is just the product of several copies of T_L, our current variety $X \times \mathcal{R}(T)$ looks much simpler over L. Clearly, the Galois group G of the field extension L/F acts on $K(X_L \times \mathcal{R}(T)_L)$ and there is an inclusion

$$K(X \times \mathcal{R}(T)) \subset K(X_L \times \mathcal{R}(T)_L)^G$$

respecting the filtrations.[1] Let us consider the filtration on the right-hand side group as an upper bound for the filtration on the left. It turns out that this upper bound is good enough for the success. By that reason we forget about $K(X \times \mathcal{R}(T))$ and work further with $K(X_L \times \mathcal{R}(T)_L)^G$ instead.

In Sect. 7 we obtain some quite obvious general assertion on the Galois action on the Grothendieck groups.

After that we come to the problem: how the topological filtration for a variety of the type $X \times T$ can be computed. Note that in our case X is a product of Severi-Brauer varieties and T is a product of generalized Severi-Brauer varieties. In Sect. 4 we study the structure of such $X \times T$ as schemes over X. We describe a situation where $X \times T$ turns out to be isomorphic to a grassmanian bundle over X. We obtain also certain additional information in this situation: namely, a description of the tautological vector bundle on the grassmanian bundle as a vector bundle on $X \times T$.

Our next problem looks as follows: given a grassmanian bundle $\Gamma \to X$ and knowledge of the topological filtration for X, how can we find the topological filtration for Γ? To describe the answer, we develop in Sect. 8 the language of *filtered bases* of filtered modules. In Sect. 10 we prove a general assertion (Proposition 10.3) which answers our question immediately in the particular case of a projective bundle. In Sect. 11, using the same method, we give an answer for an arbitrary grassmanian bundle.

The last step we need is a computation of the topological filtration (and of the Grothendieck group itself) for the variety X. For this, we develop in Sect. 9 a method of computation of the topological filtration for products of varieties in the so called *disjoint* case. In fact, our variety X is defined as the direct product of two quadric surfaces $X_1 \times X_2$, which are disjoint;

[1] We consider the filtration on $K(X_L \times \mathcal{R}(T)_L)^G$ induced by the topological filtration on $K(X_L \times \mathcal{R}(T)_L)$.

using this (and Swan's computation [51] of $K(X_i)$), one can compute the topological filtration for X via Sect. 9. However the way chosen in the paper differs from that one and is even simpler (e.g. because it does not use [51]): it makes use of the fact that each X_i is a projective line bundle over a conic Y_i and the conics Y_1, Y_2 are disjoint. It is also important that this way we obtain a filtered basis of $K(X)$ in the terms which are more suitable for the further purposes.

So, after we have shown how everything can be computed, we do some specific computation in Sect. 12. After that almost all is done to proof the basic Theorems in Sects. 13 and 14.

In the conclusion, we like to make certain additional remarks on the contents of some Sections.

In Sect. 5, we prove that the pull-back to the generic fiber of a flat morphism is surjective. For what kind of groups? Well, our final goal is the topological filtration, i.e. each term of that (Corollary 5.3). We reach the goal starting from the Chow groups (Proposition 5.1) and passing after that to the successive quotients of the topological filtration (Corollary 5.2). The statement on the Chow groups is not new; it is a formal consequence of the spectral sequence [27, Th. 3.1]. What we give here is a short direct proof or, better to say, an explanation of the evidence of this fact (Proposition 5.1).

The Weil transfer (also known as Weil restriction or Weil corestriction and under several other names) is a common, well-known, frequently used tool. However, we don't have any reference for some of its basic properties. By that reason, Sect. 6 is included. We consider only the situation of a Galois field extension (since we need only it). This allows to *define* the Weil transfer via Galois descent. With this definition, the properties we need become straight-forward.

In Sect. 4, we show that certain products of (generalized) Severi-Brauer varieties considered as schemes over certain subproducts via the projection can be naturally identified with grassmanians bundles (Corollary 4.4). Similar assertions were already proved in [24, Cor. 6.4] and in [25, Prop. 5.3]. However this time we need more explicit information: namely, we need a description of the vector bundle on the product of the Severi-Brauer varieties corresponding to the tautological vector bundle on the grassmanian bundle under that identification; the answer is given in terms of the tautological vector bundles on the Severi-Brauer varieties. Also notice that the basic statement of this section (Item 1 of Proposition 4.3) has a more general form, which clarify the things happening.

3. Terminology, notation, and backgrounds

3.1. Quadratic forms

By $\phi \perp \psi$ and $\phi \simeq \psi$ we denote respectively orthogonal sum of forms and isometry of forms. Sometimes ϕ denotes also the class of ϕ in the Witt ring $W(F)$ of the field F, e.g. in expressions like $\phi + \psi$; we apologize for this abusing of notation.

The maximal ideal of $W(F)$ generated by the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \dots \otimes \langle 1, -a_n\rangle$ and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to an n-fold Pfister form we denote by $GP_n(F)$. For any field extension L/F, we put $\phi_L = \phi \otimes_F L$.

For a quadratic extension L/F and an L-form ϕ, we denote by $s_{L/F}(\phi)$ the Scharlau's transfer [46, Sect. 5 of Chap. 2] corresponding to the F-linear homomorphism $\frac{1}{2}\operatorname{Tr}_{L/F} : L \to F$. In the case where $L = F(\sqrt{d})$, we have $s_{L/F}(\langle 1\rangle) = \langle 1, d\rangle$ and $s_{L/F}(\langle \sqrt{d} \rangle) = \langle 1, -1\rangle$.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$.

3.2. Linked forms

We say that quadratic F-forms ϕ and ψ are *linked* if the following equivalent conditions hold:

– there exists a 2-dimensional form μ which is similar to a subform of ϕ and to a subform of ψ,
– there exists a field extension L/F of degree ≤ 2 such that ϕ_L and ψ_L are isotropic,

If ϕ and ψ are forms of dimension ≥ 3, then the condition that ϕ and ψ are linked can be reformulated as follows: there exists a closed point of degree ≤ 2 on the variety $X_\phi \times X_\psi$.

3.3. K-theory and Chow groups

For a smooth algebraic F-variety X, its Grothendieck ring is denoted by $K(X)$. This ring is equipped with the filtration by codimension of support (which respects the multiplication); its n-th term (the term of codimension n other speaking) is denoted by $K(X)^{(n)}$.

For a ring (or a group) with filtration A, we denote by G^*A the adjoint graded ring (resp., the adjoint graded group). There is a canonical surjective homomorphism of the graded Chow ring $\mathrm{CH}^*(X)$ onto $G^*K(X)$, its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([50, Sect. 9]). For a geometrically integral variety of dimension d we set $\mathrm{CH}_i(X) = \mathrm{CH}^{d-i}(X)$ and $G_iK(X) = G^{d-i}K(X)$.

Very often, we identify $K(X)$ with a subgroup of $K(X_E)$, where X is an F-variety and E/F is a field extension such that the restriction homomorphism $K(X) \to K(X_E)$ is injective.

Let X_1, X_2 be F-varieties and $x_i \in K(X_i)$ for $i = 1, 2$. In expressions like x_1x_2 or $x_1 + x_2$ we consider x_1, x_2 as elements of $K(X_1 \times X_2)$ with the help of the pull-backs under the projections (so that the expressions become a sense).

3.4. Algebras

Let A be an algebra over a field F. For a field extension E/F (or, more generally, for a unital commutative F-algebra E), we denote by A_E the E-algebra $A \otimes_F E$. For an F-variety X (or, more generally, for an F-scheme X), we denote by A_X the constant X-sheaf of algebras given by A.

In Sect. 4, the category of commutative unital F-algebras is denoted by F-$\mathfrak{alg}$.

Part 1. Basic constructions

4. Products of Severi-Brauer varieties

Let F be a field and let A be a central simple algebra over F.

Let $n \geq 0$. The generalized Severi-Brauer variety $Y \stackrel{\mathrm{def}}{=} \mathrm{SB}(n, A)$ of A is characterized as follows (cf. [26]): for any $R \in F$-$\mathfrak{alg}$, the set of R-points $Y(R) \stackrel{\mathrm{def}}{=} \mathrm{Mor}_F(\mathrm{Spec}\, R, Y)$ of the variety Y consists of the right ideals J of the Azumaya R-algebra $A_R \stackrel{\mathrm{def}}{=} A \otimes_F R$ having two following properties:

- the injection of A_R-modules $J \hookrightarrow A_R$ splits (in particular, J is projective as an R-module);
- the R-module J has the constant rank $n \cdot \deg A$;

moreover, for any homomorphism $R \to R'$ in the category F-$\mathfrak{alg}$, the map $Y(R) \to Y(R')$ is given by the tensor multiplication $J \mapsto J \otimes_R R'$.

The (usual) Severi-Brauer variety $\mathrm{SB}(A)$ of A is by definition the variety $\mathrm{SB}(1, A)$.

Example 4.1. Let A be a quaternion algebra (a, b), where $a, b \in F^*$ (we suppose that $\mathrm{char}\, F \neq 2$ in this Example). The Severi-Brauer variety $\mathrm{SB}(A)$

is isomorphic to the projective conic determined by the quadratic form $\langle 1, -a, -b\rangle$.

Example 4.2. Let A be a biquaternion algebra $(a_1, b_1) \otimes (a_2, b_2)$, where a_1, b_1, a_2, $b_2 \in F^*$ (we suppose that $\operatorname{char} F \neq 2$ in this Example). The generalized Severi-Brauer variety $\mathrm{SB}(2, A)$ is isomorphic to the projective quadric determined by the Albert form $\langle -a_1, -b_1, a_1b_1, a_2, b_2, -a_2b_2\rangle$.

The *tautological* (also called *canonical*) vector bundle $\mathcal{J}$ on the generalized Severi-Brauer variety $Y = \mathrm{SB}(n, A)$ is defined as follows: for any $R \in F\text{-}\mathfrak{alg}$ and an R-point $J \in Y(R)$, the fiber of $\mathcal{J}$ over J is the R-module J; if $R \to R'$ is a homomorphism in $F\text{-}\mathfrak{alg}$, then the map of the fibers $J \to J'$, where $J' \stackrel{\mathrm{def}}{=} J \otimes_R R' \in Y(R')$, is defined by the formula $x \mapsto x \otimes 1$.

Since every fiber of $\mathcal{J}$ is a right ideal, $\mathcal{J}$ has a structure of right A_Y-module.

Proposition 4.3. *Let A be a central simple F-algebra. Let X be an F-scheme endowed with a right A_X-module $\mathcal{M}$ which is a locally free $\mathcal{O}_X$-module of rank* $\deg A$. *Then*

1. *the product $X \times \mathrm{SB}(n, A^{\mathrm{op}})$, considered over X via the first projection, can be naturally identified (as a scheme over X) with the grassmanian bundle $\varGamma_n(\mathcal{M})$ of n-planes in $\mathcal{M}$;*
2. *under this identification, the tautological vector bundle on the grassmanian bundle corresponds to the vector bundle $\mathcal{M} \otimes_A \mathcal{J}$ on $X \times \mathrm{SB}(n, A^{\mathrm{op}})$, where $\mathcal{J}$ denotes the tautological vector bundle on* $\mathrm{SB}(n, A^{\mathrm{op}})$.

Proof. Let $Y \stackrel{\mathrm{def}}{=} \mathrm{SB}(n, A^{\mathrm{op}})$. Let $R \in F\text{-}\mathfrak{alg}$ and let x be an R-point of X. To prove the first statement of the Proposition, it suffices to describe a natural bijection of the fibers over x. The fiber of $X \times Y$ over the point x is the set $Y(R)$. The fiber of $\varGamma_n(\mathcal{M})$ over the point x is the set of R-submodules N of the R-module $\mathcal{M}_x$ such that the injection $N \hookrightarrow \mathcal{M}_x$ splits and $\operatorname{rk}_R N = n$. For any N like that, the set $J = \{a \in A_R \mid \mathcal{M}_x \cdot a \subset N\}$ is a left ideal of the R-algebra A_R (i.e. a right ideal of A_R^{op}), determining an element of $Y(R)$. This way, we get the natural bijection required.

To describe an isomorphism of the vector bundles (for the second statement of Proposition), it suffices to give a natural isomorphism of the R-modules $\mathcal{M}_x \otimes_{A_R} J$ and N. This is given by the rule $x \otimes a \mapsto x \cdot a$. $\square$

Now we consider a special situation, where Proposition 4.3 can be applied. Suppose that

$$A \stackrel{\mathrm{def}}{=} A_1^{\otimes i_1} \otimes \ldots \otimes A_m^{\otimes i_m},$$

where $A_1, \ldots, A_m$ are some central simple F-algebras and $i_1, \ldots, i_m$ are some non-negative integers. Let $X_1, \ldots, X_m$ be the Severi-Brauer varieties of the algebras $A_1, \ldots, A_m$. Put $X \stackrel{\text{def}}{=} S \times X_1 \times \ldots \times X_m$, where S is an F-variety. For every $j = 1, \ldots, m$, denote by $\mathcal{I}_j$ the tautological vector bundle on X_j. Put

$$\mathcal{M} \stackrel{\text{def}}{=} \mathcal{O}_S \otimes \mathcal{I}_1^{\otimes i_1} \otimes \ldots \otimes \mathcal{I}_m^{\otimes i_m};$$

it is a right A_X-module which is a locally free $\mathcal{O}_X$-module of rank $\deg A$. Finally, let $Y \stackrel{\text{def}}{=} \mathrm{SB}(n, A^{\mathrm{op}})$ and let $\mathcal{J}$ be the tautological vector bundle on Y; it is a left A_Y-module. Applying Proposition 4.3, we get the following

Corollary 4.4. *In the notation introduced right above, the product $X \times Y$, considered over X via the first projection, can be naturally identified (as a scheme over X) with the grassmanian bundle $\Gamma_n(\mathcal{M})$; under this identification, the tautological vector bundle on the grassmanian bundle corresponds to the vector bundle $\mathcal{M} \otimes_A \mathcal{J}$ on $X \times Y$.* $\square$

Since the projective space bundle $\mathbb{P}(\mathcal{M})$ is (by definition) $\Gamma_1(\mathcal{M})$, we get the following

Corollary 4.5. *Let A be a central simple F-algebra and let S be an F-variety. Set $X \stackrel{\text{def}}{=} S \times \mathrm{SB}(A)$ and $Y \stackrel{\text{def}}{=} \mathrm{SB}(A^{\mathrm{op}})$. Denote by $\mathcal{I}$ the tautological vector bundle on $\mathrm{SB}(A)$ and by $\mathcal{J}$ the tautological vector bundle on Y. Set $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{O}_S \otimes \mathcal{I}$.*

Then the product $X \times Y$, considered over X via the first projection, can be naturally identified (as a scheme over X) with the projective space bundle $\mathbb{P}(\mathcal{M})$; under this identification, the tautological vector bundle on $\mathbb{P}(\mathcal{M})$ corresponds to the vector bundle $\mathcal{M} \otimes_A \mathcal{J}$ on $X \times Y$. $\square$

5. Pull-back to generic fiber

We fix the following notation for this section: F is a field, Y and T are irreducible F-varieties, $\pi\colon Y \to T$ is a flat morphism, θ is the generic point of T, and $Y_\theta \stackrel{\text{def}}{=} Y \times_T \operatorname{Spec} F(\theta)$ is the generic fiber of π, i.e. the fiber of π over θ. We are going to consider the pull-back i^* with respect to the flat morphism of schemes $i\colon Y_\theta \to Y$.

Note that from the set-theoretical (even topological) point of view, Y_θ is really the fiber of π over the point θ (see [10, Exercise 3.10 after Sect. 3 of Chap. II]). In particular, Y_θ is a subset of Y.

The group $\mathrm{CH}^*(Y)$ is generated by the classes $[y]$ of points $y \in Y$. The pull-back homomorphism $i^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y_\theta)$ is determined by the following rule: if $y \notin Y_\theta$ (i.e., if $\pi(y) \neq \theta$), then $i^*([y]) = 0$; if $y \in Y_\theta$ (i.e., if $\pi(y) = \theta$), then $i^*([y]) = [y] \in \mathrm{CH}^*(Y_\theta)$.

Proposition 5.1. *The pull-back homomorphism* $i^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y_\theta)$ *is surjective.*

Proof. Take any generator $\alpha \overset{\text{def}}{=} [y]$ of the group $\mathrm{CH}^*(Y_\theta)$, where $y \in Y_\theta$. If we consider y as a point of Y, we get an element $\beta \overset{\text{def}}{=} [y] \in \mathrm{CH}^*(Y)$ such that $i^*(\beta) = \alpha$. □

Now we pass from the Chow group to the Grothendieck group.

Corollary 5.2. *The pull-back homomorphism* $i^* : G^*K(Y) \to G^*K(Y_\theta)$ *is surjective.*

Proof. The diagram

$$\begin{array}{ccc} \mathrm{CH}^*(Y) & \to & \mathrm{CH}^*(Y_\theta) \\ \downarrow & & \downarrow \\ G^*K(Y) & \to & G^*K(Y_\theta) \end{array}$$

where the vertical arrows are the canonical epimorphisms (see Sect. 3.3), is commutative. Since the map $\mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y_\theta)$ is surjective (Proposition 5.1) and the map $\mathrm{CH}^*(Y_\theta) \to G^*K(Y_\theta)$ is surjective, the map $G^*K(Y) \to G^*K(Y_\theta)$ is surjective as well. □

Corollary 5.3. *For any* $n \geq 0$*, the pull-back homomorphism* $i^* : K(Y)^{(n)} \to K(Y_\theta)^{(n)}$ *is surjective.*

Proof. Follows from Corollary 5.2. □

Example 5.4. We shall apply Corollary 5.3 only to the particular situation, where $Y = X \times T$ (with certain F-varieties X and T) and $\pi : Y = X \times T \to T$ is the projection. Thus $Y_\theta = X_{F(T)}$ and i^* will be a homomorphism of $K(X \times T)$ onto $K(X_{F(T)})$. Note that i^* is a homomorphism of $K(X)$-algebras if we consider $K(X \times T)$ and $K(X_{F(T)})$ as $K(X)$-algebras in the natural way (i.e. via the pull-backs). We shall use the notation f for the homomorphism i^* in the situation like described here.

6. Weil transfer via Galois descent

In this section, L/F is a finite Galois field extension with the Galois group G. All varieties here are assumed to be quasi-projective.

Definition 6.1. Let X be an F-variety. An L/F-*form* of X is an F-variety Y *supplied* with an L-isomorphism $Y_L \tilde{\to} X_L$. A *morphism* of an L/F-form Y to another L/F-form Y' of the same variety X is a morphism of F-varieties $f : Y \to Y'$ such that the diagram of L-morphisms

$$\begin{array}{ccc} Y_L & \xrightarrow{f_L} & Y'_L \\ & \searrow \quad \swarrow & \\ & X_L & \end{array}$$

commutes (note that the set of morphisms of forms $Y \to Y'$ contains at most 1 element; in particular, an isomorphism of two L/F-forms of X is always canonical).

Let X be an F-variety. The (abstract) group $\mathrm{Aut}_L(X_L)$ of the automorphisms of the L-variety X_L can be supplied with a structure of G-module in the standard way (see [48, Sect. 1.1 de Chap. III]): for $\tau \in G$ and $f \in \mathrm{Aut}_L(X_L)$ one puts $\tau(f) \stackrel{\mathrm{def}}{=} (\mathrm{id}_X \otimes \tau) \circ f \circ (\mathrm{id}_X \otimes \tau^{-1})$ where $\mathrm{id}_X \otimes \tau$ is the automorphism of the scheme X_L over F given by τ. Denote by $Z^1(L/F, \mathrm{Aut}_L(X_L)) = Z^1(G, \mathrm{Aut}_L(X_L))$ the set of 1-cocycles on G with values in $\mathrm{Aut}_L(X_L)$ ([48, Sect. 5.1 de Chap. I]).

Any L/F-form Y of X determines a cocycle $z \in Z^1(L/F, \mathrm{Aut}_L(X_L))$ ([48, 1.3 de Chap. III]): for any $\tau \in G$, the automorphism $z_\tau \in \mathrm{Aut}_L(X_L)$ is the composition

$$X_L \tilde{\to} Y_L \xrightarrow{\mathrm{id}_Y \otimes \tau} Y_L \tilde{\to} X_L \xrightarrow{\mathrm{id}_X \otimes \tau^{-1}} X_L\ .$$

Moreover, the rule described above is a 1-1-correspondence between the set of L/F-forms of X (up to the *canonical* isomorphism) and the set

$$Z^1(L/F, \mathrm{Aut}_L(X_L))$$

(see [3, Prop. 2.6]).

Now suppose that $X = \prod_G T$ (the product of $|G|$ copies of T numbered by the elements of G), where T is a variety over F. We are going to construct a special 1-cocycle $z \in Z^1(L/F, \mathrm{Aut}_L(X_L))$ in this situation.

For any $\tau \in G$, consider the left translation by τ, that is the permutation $\sigma \mapsto \tau\sigma$ of the set G, and denote by $z_\tau \in \mathrm{Aut}_L(X_L)$ the automorphism of the product $X_L = \prod_G Y_L$ given by the corresponding permutation of factors. The map $z\colon G \to \mathrm{Aut}_L(X_L)$, $\tau \mapsto z_\tau$ is a 1-cocycle.

Definition 6.2. The following data are fixed: a finite Galois field extension L/F and an F-variety T. The L/F-form (see Definition 6.1) of the variety $X \stackrel{\mathrm{def}}{=} \prod_G T$ determined by the cocycle $z \in Z^1(L/F, \mathrm{Aut}_L(X_L))$ constructed above will be denoted by $\mathcal{R}(T)$ or $\mathcal{R}_{L/F}(T)$.

Remark 6.3. The variety $\mathcal{R}_{L/F}(T)$ is the same as the *Weil transfer* (see [3, Sect. 2.8] and/or [4, 6.6 de Sect. 1 de Chap. I] and/or [47, Chap. 4]) of the L-variety T_L with respect to the extension L/F. Usually, working with varieties over fields, one defines the Weil transfer for any finite separable field extension L/F and a quasi-projective L-variety. However, we are interested here only in the case where the extension L/F is Galois and the L-variety "comes from F". Definition 6.2 can be regarded as an alternative definition of the Weil transfer in this particular situation. It is more convenient for our purposes: the property of $\mathcal{R}(Y)$ we need (see Lemma 6.5 below) becomes evident.

Example 6.4. Let us take as L/F a quadratic extension $L = F(\sqrt{d})$ with some $d \in F^*$ and as T the Severi-Brauer variety of a quaternion F-algebra (a, b) (we suppose that $\operatorname{char} F \neq 2$ in this Example). Then $\mathcal{R}(T)$ is isomorphic to the projective quadric hypersurface, determined by the quadratic form

$$\langle -a, -b, ab, d\rangle .$$

Lemma 6.5. *Let L/F be a finite Galois field extension with the Galois group G. Let T be an F-variety. For any $\tau \in G$, the following diagram of isomorphisms commutes*

$$\begin{array}{ccc} \mathcal{R}(T)_L & \xrightarrow{\mathrm{id}\otimes\tau} & \mathcal{R}(T)_L \\ \downarrow & & \downarrow \\ \prod_G T_L & \xrightarrow{(\mathrm{id}\otimes\tau)\circ z_\tau} & \prod_G T_L \end{array}$$

Proof. It is a direct consequence of Definition 6.2. □

7. Galois action on Grothendieck group

In this section, F is an arbitrary field, L/F is a field extension (e.g., a Galois field extension), G is a group of automorphism of L over F (e.g., the Galois group in the case where L/F is a Galois extension), Y is an F-variety.

The group G acts on the Grothendieck group $K(Y_L)$ of the variety Y_L. We are interested in a condition on Y which guarantees that the action of G on $K(Y_L)$ is trivial.

Lemma 7.1. *Suppose that the group $K(Y_L)$ is torsion-free and that the cokernel of the restriction map $\operatorname{res}_{L/F} : K(Y) \to K(Y_L)$ is a torsion group. Then the action of G on $K(Y_L)$ is trivial.*

Proof. Take any $y \in K(Y_L)$ and any $\sigma \in G$. Since $\operatorname{Coker}(\operatorname{res}_{L/F})$ is a torsion group, some multiple ny of y is in $\operatorname{Im}(\operatorname{res}_{L/F})$, therefore $\sigma(ny) = ny$. Since the group $K(Y_L)$ is torsion-free, it follows that $\sigma(y) = y$. □

Working with homogeneous varieties, we have the first condition of Lemma 7.1 for free: the group $K(Y)$ is natural (with respect to extensions of the base field F) isomorphic to $K(A)$, where A is a separable algebra (i.e. a direct product of simple algebras with centers separable over F) ([39, Introduction]). The second condition holds for Y and for all extensions L/F if and only if every simple component of A is central over F. We do not need here the complete list of such varieties. We only notice that the generalized Severi-Brauer varieties are included[2] as well as their direct products[3]. So that we have

Corollary 7.2. *Let Y be a product of generalized Severi-Brauer varieties. Then the action of G on $K(Y_L)$ is trivial.* □

Corollary 7.3. *Let L/F be a finite Galois extension, G be its Galois group, and Y be the product of some generalized Severi-Brauer varieties over F. Let us identify $\mathcal{R}(Y)_L$ with $\prod_G Y_L$ (see Definition 6.2). Then, for any $\sigma \in G$, the automorphism of $K(\mathcal{R}(Y)_L)$, given by σ, corresponds to the automorphism of $K(\prod_G Y_L)$, given by the automorphism of the product, induced by the permutation z_σ of the factors, where z_σ is the left translation by σ.*

Proof. By Lemma 6.5, the diagram

$$\begin{array}{ccc} K(\mathcal{R}(Y)_L) & \xrightarrow{\sigma} & K(\mathcal{R}(Y)_L) \\ \downarrow & & \\ K(\prod\limits_G Y_L) & \xrightarrow{\sigma \circ z_\sigma} & K(\prod\limits_G Y_L) \end{array}$$

commutes. By Corollary 7.2, σ over the bottom arrow is the identity. □

8. Filtered rings, modules, and bases

In this section we introduce some terminology concerning filtrations on abstract rings and modules. This terminology will be then applied (in the further Sections) to the Grothendieck rings of varieties.

A commutative unital non-zero ring R is called *filtered*, if it is supplied with a finite filtration $R^{(n)}$ ($n \in \mathbb{Z}$), satisfying the following conditions:

- $R^{(n)} \cdot R^{(m)} \subset R^{(n+m)}$ for all n, m and
- $R^{(0)} = R$

(note that the filtration of a filtered ring is automatically descending and $R^{(1)} \neq R$).

Let R be a filtered ring. An R-module M is called *filtered*, if it is supplied with a finite filtration $M^{(n)}$ ($n \in \mathbb{Z}$), satisfying the following conditions:

- $R^{(n)} \cdot M^{(m)} \subset M^{(n+m)}$ for all n, m and

[2] This follows from [41, Th. 4.1 of Sect. 8] in the particular case of usual Severi-Brauer varieties and [35, Th. 4.4] in the general case of generalized Severi-Brauer varieties.

[3] It is a consequence of the following assertion: if A_1 is the separable algebra for a homogeneous variety Y_1 and A_2 is the separable algebra for a homogeneous variety Y_2, then $A_1 \otimes_F A_2$ is the separable algebra for $Y_1 \times Y_2$, see [38, Sect. 1.8].

– $M^{(0)} = M$

(note that the filtration of a filtered module is automatically descending).

Example 8.1. Let $X \to Y$ be a morphism of smooth varieties. The Gorthendieck ring $K(Y)$ supplied with the topological filtration $K(Y)^{(n)}$ is a filtered ring. Considering $K(X)$ as a $K(Y)$-module via the pull-back homomorphism $K(Y) \to K(X)$, we get an example of a filtered module (where $K(X)$ is supplied with the topological filtration as well).

Let M be a filtered module over a filtered ring R. The codimension $\operatorname{codim} x$ of an element $x \in M$ is defined as

$$\operatorname{codim} x = \operatorname{codim}_M x \stackrel{\text{def}}{=} \sup\{n \in \mathbb{Z} \mid\ x \in M^{(n)}\} .$$

Let R be a filtered ring and let M be a (free and finitely generated) filtered R-module. An R-basis $e_1, \dots, e_k$ of M is called *filtered,* if for any $n \in \mathbb{Z}$ one has

$$M^{(n)} = \sum_{j=1}^{k} R^{(n-n_j)} \cdot e_j \ ,$$

where $n_j \stackrel{\text{def}}{=} \operatorname{codim} e_j$.

Clearly, a filtered module is uniquely determined by its filtered basis and the codimensions of the basis elements.

Supplying $\mathbb{Z}$ with the trivial filtration

$$\mathbb{Z}^{(n)} \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } n > 0; \\ \mathbb{Z}, & \text{if } n \leq 0, \end{cases}$$

we get a filtered ring. This way we transfer definitions from above to the case of abelian groups, obtaining the notions of a *filtered (abelian) group* and a *filtered basis* of it.

Example 8.2 (cf. Lemmas 9.8 and 9.7 and Corollary 9.6). Let $Y_1, \dots, Y_n$ be projective lines. Denote by p_i the class of a rational point on Y_i. Then the elements of the form $p_1^{\varepsilon_1} \cdots p_n^{\varepsilon_n}$, where $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$, constitute a filtered basis of the filtered group $K(Y_1 \times \ldots \times Y_n)$; besides

$$\operatorname{codim}(p_1^{\varepsilon_1} \cdots p_n^{\varepsilon_n}) = \varepsilon_1 + \ldots + \varepsilon_n \ .$$

For a non-zero element u of a filtered module M, we denote by $\bar{u}$ the class of u in $G^{\operatorname{codim} u} M \subset G^* M$.

The following three assertions are evident:

Lemma 8.3. *let R be a filtered ring, M be a filtered R-module, and*

$$u_1, \dots, u_k \in M$$

be non-zero elements such that G^*M is a free G^*R-module with the basis $(\bar{u}_1, \dots, \bar{u}_k)$. Then $(u_1, \dots, u_k)$ is a filtered basis of the R-module M. $\square$

Corollary 8.4. *A finitely generated filtered group A possess a filtered basis if and only if the group G^*A is torsion-free.* $\square$

Lemma 8.5. *Let R be a filtered ring, let S be a filtered ring which is a filtered R-module, and let M be a filtered S-module. If $(e_1, \dots, e_n)$ is a filtered basis of S over R and $(f_1, \dots, f_m)$ is a filtered basis of M over S, then the collection $(e_i f_j)$ (where $i = 1, \dots, n$ and $j = 1, \dots, m$) is a filtered basis of M over R and* $\operatorname{codim}_M(e_i f_j) = \operatorname{codim}_S e_i + \operatorname{codim}_M f_j$.
$\square$

9. Filtered bases for products of disjoint varieties

Let $X_1, \dots, X_n$ be F-varieties. The main purpose of this section is to find a filtered basis of $K(X_1 \times \ldots \times X_n)$ (starting from filtered bases of $K(X_1), \dots, K(X_n)$) in the so called *disjoint* case.

Definition 9.1 (cf. [25, Déf. 3.1]). A collection of F-varieties $X_1, \dots, X_n$ is called *disjoint*, if the homomorphism

$$K(X_1) \otimes \ldots \otimes K(X_n) \to K(X_1 \times \ldots \times X_n)$$

(given by the product of the pull-back homomorphisms with respect to the projections) is bijective.

Example 9.2 (cf. [25, Prop. 3.6]). Let $Q_1, \dots, Q_n$ be arbitrary central simple F-algebras of exponent ≤ 2. The Severi-Brauer varieties

$$\mathrm{SB}(Q_1), \dots, \mathrm{SB}(Q_n)$$

are disjoint if and only if $\operatorname{ind}(Q_1 \otimes_F \ldots \otimes_F Q_n) = \operatorname{ind}(Q_1) \cdots \operatorname{ind}(Q_n)$.

Lemma 9.3. *Let K be a finitely generated filtered abelian group. Let $K = \mathcal{F}^0 K \supset \mathcal{F}^1 K \supset \mathcal{F}^2 K \dots$ and $K = \Gamma^0 K \supset \Gamma^1 K \supset \Gamma^2 K \dots$ be some other filtrations of the group K satisfying the following conditions:*
- *$\Gamma^p A \subset \mathcal{F}^p K \subset K^{(p)}$ for all $p \geq 0$;*
- *the adjoint graded group $G^* \mathcal{F} K$ is torsion-free;*
- *the natural homomorphism $(G^* \Gamma K)_{\mathbb{Q}} \to (G^* K)_{\mathbb{Q}}$ is bijective.*

Then $K^{(p)} = \mathcal{F}^p K$ for all $p \geq 0$.

Proof. Since the isomorphism $(G^* \Gamma K)_{\mathbb{Q}} \to (G^* K)_{\mathbb{Q}}$ factors through the group $(G^* \mathcal{F} K)_{\mathbb{Q}}$, it follows that the homomorphism $(G^* \mathcal{F} K)_{\mathbb{Q}} \to (G^* K)_{\mathbb{Q}}$ is surjective. Obviously, $\dim(G^* \mathcal{F} K)_{\mathbb{Q}} = \dim K_{\mathbb{Q}} = \dim(G^* K)_{\mathbb{Q}}$. Therefore, the homomorphism $(G^* \mathcal{F} K)_{\mathbb{Q}} \to (G^* K)_{\mathbb{Q}}$ is bijective. Since the group $G^* \mathcal{F} K$ is torsion-free, the homomorphism $G^* \mathcal{F} K \to (G^* \mathcal{F} K)_{\mathbb{Q}}$ is injective. Therefore, the composition $G^* \mathcal{F} K \to (G^* \mathcal{F} K)_{\mathbb{Q}} \tilde{\to} (G^* K)_{\mathbb{Q}}$ is injective as well. Thus, $G^* \mathcal{F} K \to G^* K$ is an injection, i.e. $\mathcal{F}^p K \cap K^{(p+1)} = \mathcal{F}^{p+1} K$ for all $p \geq 0$. Finally, induction on p shows that $\mathcal{F}^p K = K^{(p)}$. $\square$

Lemma 9.4. *Let A and B be finitely generated filtered abelian groups such that G^*A and G^*B are torsion-free. Let us consider the tensor product $A \otimes B$ with the filtration induced by the filtrations on A and B. Then the homomorphism $G^*(A) \otimes G^*(B) \to G^*(A \otimes B)$ is bijective.*

Proof. The homomorphism $G^*A \otimes G^*B \to G^*(A \otimes B)$ is surjective by the definition of the filtration on $A \otimes B$. The equality of ranks

$$\mathrm{rk}(G^*A \otimes G^*B) = \mathrm{rk}(G^*A) \cdot \mathrm{rk}(G^*B) = \mathrm{rk}(A) \cdot \mathrm{rk}(B) = \mathrm{rk}(G^*(A \otimes B))$$

shows that it is bijective. □

Proposition 9.5. *Let $X_1, \ldots, X_n$ be disjoint varieties such that the groups $K(X_i)$ are finitely generated and the groups $G^*K(X_i)$ are torsion-free. Then the homomorphism $K(X_1) \otimes \ldots \otimes K(X_n) \to K(X_1 \times \ldots \times X_n)$ is an isomorphism of filtered rings.*[4] *Besides, $G^*K(X_1 \times \ldots \times X_n) \simeq G^*K(X_1) \otimes \ldots \otimes G^*K(X_n)$ (in particular, the group $G^*K(X_1 \times \ldots \times X_n)$ is torsion-free).*

Proof. An easy induction reduces the general case to the case $n = 2$. Set $X \stackrel{\text{def}}{=} X_1$ and $Y \stackrel{\text{def}}{=} X_2$. Let us denote by $\mathcal{F}^i(K(X \times Y))$ the filtration on $K(X \times Y)$ induced by the topological filtrations on $K(X)$ and $K(Y)$. Let $\Gamma^i K(X \times Y)$ stays for the gamma-filtration on the Grothendieck group (see [36, Def. 8.3] and/or [24, Def. 2.6]). To prove the Proposition, it is sufficient to verify that $\mathcal{F}^i(K(X \times Y)) = K(X \times Y)^{(i)}$ for all i. For this, it suffices to check the conditions of Lemma 9.3.

Let us consider the filtration on $K(X) \otimes K(Y)$ induced by the topological filtration on $K(X)$ and $K(Y)$. By Lemma 9.4 we have

$$G^*(K(X) \otimes K(Y)) \simeq G^*K(X) \otimes G^*K(Y).$$

Since the varieties X and Y are disjoint, the homomorphism

$$G^*(K(X) \otimes K(Y)) \to G^*\mathcal{F}(K(X \times Y))$$

is bijective. Therefore, $G^*\mathcal{F}(K(X \times Y)) \simeq G^*K(X) \otimes G^*K(Y)$. In particular, the group $G^*\mathcal{F}(K(X \times Y))$ is torsion-free.

Since $\Gamma^i K(X) \subset K(X)^{(i)}$ and $\Gamma^i K(Y) \subset K(Y)^{(i)}$ (see [9, Th. 3.9 of Chap. V]), and since the gamma-filtration on $K(X \times Y)$ is induced by the gamma-filtrations on $K(X)$ and $K(Y)$ (see [25, Prop. 3.2]), one has

$$\Gamma^i K(X \times Y) \subset \mathcal{F}^i K(X \times Y) \quad \text{for all } i \geq 0.$$

[4] Here we consider the tensor product $K(X_1) \otimes \ldots \otimes K(X_n)$ with the filtration induced by the topological filtrations on $K(X_1), \ldots, K(X_n)$.

Finally, by [9, Prop. 5.5 of Chap. VI], we have

$$G^*\Gamma K(X\times Y)_{\mathbb{Q}} \simeq G^*K(X\times Y)_{\mathbb{Q}}.$$

We have checked that all conditions of Lemma 9.3 hold. Therefore, $\mathcal{F}^iK(X\times Y) = K(X\times Y)^{(i)}$ and the proof is complete. □

Corollary 9.6. *Let $X_1,\dots,X_n$ be disjoint varieties. Suppose that every group $K(X_i)$ possess a filtered basis $\mathcal{E}_i$. Then the product $\mathcal{E}_1\cdots\mathcal{E}_n$ is a filtered basis of $K(X_1\times\ldots\times X_n)$ and $\operatorname{codim}(e_1\cdots e_n) = \operatorname{codim} e_1 + \ldots + \operatorname{codim} e_n$ for every $e_1\in\mathcal{E}_1,\dots,e_n\in\mathcal{E}_n$.*

Proof. Since every $K(X_i)$ possess a filtered basis, it is finitely generated and the group $G^*K(X_i)$ is torsion-free (Corollary 8.4), i.e. all conditions of Proposition 9.5 hold. Applying Proposition 9.5, we get the assertion required. □

Lemma 9.7. *Let $X_i \overset{\text{def}}{=} \mathrm{SB}(Q_i)$ for $i = 1,\dots,n$, where Q_i are some central simple F-algebras of exponent ≤ 2 and of index ≤ 4. Suppose that*

$$\operatorname{ind}(Q_1\otimes_F\ldots\otimes_F Q_n) = \operatorname{ind}(Q_1)\cdots\operatorname{ind}(Q_n)\ .$$

Then the varieties $X_1,\dots,X_n$ satisfy the conditions of Proposition 9.5 and of Corollary 9.6.

Proof. Since

$$\operatorname{ind}(Q_1\otimes_F\ldots\otimes_F Q_n) = \operatorname{ind}(Q_1)\cdots\operatorname{ind}(Q_n),$$

the varieties $X_1,\dots,X_n$ are disjoint (see Example 9.2). The groups $K(X_i)$ are finitely generated by [41]. The groups $G^*K(X_i)$ are torsion-free by [23]. The filtered groups $K(X_i)$ possess filtered bases by Corollary 8.4. □

Lemma 9.8. *Let Q be a quaternion F-algebra and let Y be its Severi-Brauer variety. We denote by $p\in K(Y_{\bar F})$ the class of a rational point and identify $K(Y)$ with a subgroup of $K(Y_{\bar F})$. Then*

1. *If Q is split, then $(1,p)$ is a filtered basis of $K(Y)$, hereby $\operatorname{codim} 1 = 0$ and $\operatorname{codim} p = 1$.*
2. *If Q is non-split, then $(1,2p)$ is a filtered basis of $K(Y)$, hereby $\operatorname{codim} 1 = 0$ and $\operatorname{codim} 2p = 1$.*

Proof. Since in the split case Y is isomorphic to the projective line, the first statement is easy (cf. Example 8.2). Let us prove the second.

Since there exists a quadratic extension L/F splitting Q, the transfer argument and Item 1 show that $2p\in K(Y)^{(1)}$. Thus $K(Y)$ contains the subgroup of $K(Y_{\bar F})$ generated by 1 and $2p$. Since there is a natural (with respect to extensions of scalars) isomorphism $K(Y) \simeq K(F) \oplus K(Q)$ ([41, Th. 4.1 of Sect. 8]), the index of $K(Y)$ in $K(Y_{\bar{F}})$ equals $\operatorname{ind} Q = 2$. Consequently, $K(Y)$ coincides with the subgroup generated by 1 and $2p$. Since $K(Y_{\bar{F}})^{(1)}$ is generated by p, we see that $K(Y) \cap K(Y_{\bar{F}})^{(1)}$ is generated by $2p$. On the other hand $2p \in K(Y)^{(1)} \subset K(Y) \cap K(Y_{\bar{F}})^{(1)}$. Therefore, $K(Y)^{(1)}$ coincides with the subgroup generated by $2p$. $\square$

Applying Lemma 9.8, Lemma 9.7, and Corollary 9.6, we get

Corollary 9.9. *Let $Q_1, \ldots, Q_n$ be quaternion F-algebras such that $Q_1 \otimes_F \ldots \otimes_F Q_n$ is a division algebra. Set $Y_i \overset{\text{def}}{=} \operatorname{SB}(Q_i)$ and denote by $p_i \in K((Y_i)_{\bar{F}})$ the class of a rational point. Then the elements of the form $(2p_1)^{\varepsilon_1} \cdots (2p_n)^{\varepsilon_n}$, where $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, constitute a filtered basis of $K(Y_1 \times \ldots \times Y_n)$; besides* $\operatorname{codim}((2p_1)^{\varepsilon_1} \cdots (2p_n)^{\varepsilon_n}) = \varepsilon_1 + \ldots + \varepsilon_n$. $\square$

10. Filtered bases for projective space bundles

For any non-zero homogeneous element u of the Chow group $\operatorname{CH}^*(X)$ of a variety X, we denote by $\operatorname{codim} u$ the homogeneous degree of u.

Definition 10.1. Let $u \in \operatorname{CH}^*(X)$ be a non-zero homogeneous element. We say that an element $v \in K(X)$ *corresponds* to u, if $\operatorname{codim} v = \operatorname{codim} u = n$ and the image $\bar{v}$ of v under the homomorphism $K(X)^{(n)} \to G^n K(X)$ coincides with the image of u under the homomorphism $\operatorname{CH}^n(X) \to G^n K(X)$.

Example 10.2. For a variety Y and an integer $i \geq 0$, we denote by $c_i : K(Y) \to K(Y)^{(i)}$ and $\tilde{c}_i : K(Y) \to \operatorname{CH}^i(Y)$ the i-th Chern classes (see [24, Def. 2.1 and 2.11]). If $\tilde{c}_i(v) \neq 0$ for an element $v \in K(Y)$, then $c_i(v) \in K(Y)^{(i)}$ corresponds to $\tilde{c}_i(v)$ (see [24, Lemma 2.16]).

Let $f : Y \to X$ be a smooth proper morphism of F-varieties with smooth X. For any smooth proper morphism of F-varieties $p_X : X' \to X$, let

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & X \end{array} \qquad (*)$$

be the fiber square. We consider $\operatorname{CH}^*(Y')$ as a $\operatorname{CH}^*(X')$-module via the pull-back ring homomorphism $(f')^* : \operatorname{CH}^*(X') \to \operatorname{CH}^*(Y')$. Besides, we consider $G^* K(Y)$ as a $G^* K(X)$-module.

Proposition 10.3. *Let X be a smooth F-variety, let $f : Y \to X$ be a smooth proper morphism of F-varieties, and let $u_i \in \operatorname{CH}^*(Y)$ ($i = 1, \ldots, k$) be* *homogeneous elements such that for any fiber square* $(*)$ *the* $\mathrm{CH}^*(X')$*-module* $\mathrm{CH}^*(Y')$ *is free with the basis* $(p_Y^*(u_i))_{i=1}^k$*. If* $v_i \in K(Y)$ *are elements corresponding to* u_i *(*$i = 1, \ldots, k$*), then*

1. *The* $G^*K(X)$*-module* $G^*K(Y)$ *is free with the basis* $(\bar{v}_i)_{i=1}^k$*.*
2. *The elements* $(v_i)_{i=1}^k$ *form a filtered basis of the* $K(X)$*-module* $K(Y)$*.*

Proof. Consider the fiber square

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X \end{array}$$

It is a square of the type $(*)$ (with $X' = Y$ and $p_X = f$). Therefore, by our assumption, the homomorphism

$$\bigoplus_{i=1}^k \mathrm{CH}^*(Y) \to \mathrm{CH}^*(Y \times_X Y), \qquad (w_1, \ldots, w_k) \mapsto \sum_{i=1}^k p_1^*(u_i) \cdot p_2^*(w_i)$$

is bijective. In particular, it is surjective and consequently

$$\sum_{i=1}^k p_1^*(u_i) \cdot p_2^*(u_i') = \delta_Y \in \mathrm{CH}^*(Y \times_X Y)$$

for some $u_1', \ldots, u_k' \in \mathrm{CH}^*(Y)$, where δ_Y is the diagonal class in $\mathrm{CH}^*(Y \times_X Y)$.

Consider the homomorphisms

$$\alpha\colon \bigoplus_{i=1}^k \mathrm{CH}^*(X) \to \mathrm{CH}^*(Y), \quad \alpha\colon (w_1, \ldots, w_k) \mapsto \sum_{i=1}^k u_i \cdot f^*(w_i),$$
$$\beta = (\beta_i)_{i=1}^k : \mathrm{CH}^*(Y) \to \bigoplus_{i=1}^k \mathrm{CH}^*(X), \qquad \beta_i\colon w \mapsto f_*(w \cdot u_i').$$

We claim, that the composition $\alpha \circ \beta$ is an identity, i.e. for any $w \in \mathrm{CH}^*(Y)$

$$\alpha(\beta(w)) \stackrel{\mathrm{def}}{=} \sum_{i=1}^k u_i \cdot f^*(f_*(w \cdot u_i')) = w\ .$$

To prove this, let us first compute a summand of the sum:

$$u_i \cdot f^*(f_*(w \cdot u_i')) = u_i \cdot p_{2*}(p_1^*(w \cdot u_i')) = p_{2*}(p_2^*(u_i) \cdot p_1^*(w) \cdot p_1^*(u_i')).$$

Here the first equality holds since $f^* \circ f_* = p_{2*} \circ p_1^*$ ([8, Prop. 1.7]); the second equality holds by the projection formula for p_2 and since p_1^* is a ring homomorphism. Consequently

$$\alpha(\beta(w)) = p_{2*}\Big(p_1^*(w) \sum_{i=1}^{k} p_2^*(u_i) \cdot p_1^*(u_i')\Big) = p_{2*}(p_1^*(w) \cdot \delta_Y) = w,$$

where the last equality is well-known in the case $X = \operatorname{Spec} F$ (see [8, Prop. 16.1.2.(c)]) and is proved in exactly the same way in the general case (cf. [5, Prop. 1.2.1]).

Thus, we have proved that $\alpha \circ \beta = \mathrm{id}$. Since α is an isomorphism (by the assumption of the Proposition), it follows that $\beta \circ \alpha = \mathrm{id}$ as well.

Let us now consider the homomorphisms

$$\bar\alpha \colon \bigoplus_{i=1}^{k} G^*K(X) \to G^*K(Y), \qquad \bar\alpha \colon (w_1, \dots, w_k) \mapsto \sum_{i=1}^{k} \bar v_i \cdot f^*(w_i),$$
$$\bar\beta = (\bar\beta_i)_{i=1}^{k} : G^*K(Y) \qquad \bar\beta_i \colon w \mapsto f_*(w \cdot \bar v_i'), \to \bigoplus_{i=1}^{k} G^*K(X),$$

where $\bar v_1', \dots, \bar v_k' \in G^*K(Y)$ are the images of $u_1', \dots, u_k' \in \mathrm{CH}^*(Y)$ under $\mathrm{CH}^*(Y) \twoheadrightarrow G^*K(Y)$.

The diagrams

$$\begin{array}{ccc} \bigoplus_{i=1}^{n} \mathrm{CH}^*(X) & \xrightarrow{\alpha} & \mathrm{CH}^*(Y) \\ \downarrow & & \downarrow \\ \bigoplus_{i=1}^{n} G^*K(X) & \xrightarrow{\bar\alpha} & G^*K(Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{CH}^*(Y) & \xrightarrow{\beta} & \bigoplus_{i=1}^{n} \mathrm{CH}^*(X) \\ \downarrow & & \downarrow \\ G^*K(Y) & \xrightarrow{\bar\beta} & \bigoplus_{i=1}^{n} G^*K(X) \end{array}$$

are obviously commutative. Since the vertical arrows in these diagrams are surjective, $\alpha \circ \beta = \mathrm{id}$, and $\beta \circ \alpha = \mathrm{id}$, one has $\bar\alpha \circ \bar\beta = \mathrm{id}$ and $\bar\beta \circ \bar\alpha = \mathrm{id}$. In particular, $\bar\alpha$ is an isomorphism. This completes the proof of the first assertion of the Proposition. The second assertion (on the filtered basis of $K(Y)$) follows from the first one (see Lemma 8.3). □

Remark 10.4. In fact, Proposition 10.3 is of “motivic nature”: it follows from the assumptions made that the motive of Y is a direct sum of several copies of the motive of X in an appropriate motivic category, namely in the category of correspondences, whose objects are smooth and proper *schemes over* X (this category is constructed in [5, Sect. 1]). Since the functor of taking the adjoint graded Grothendieck group factors through that category (cf. [26, Sect. 5]), the motivic decomposition mentioned implies the assertion required.

The proof of Proposition 10.3 given above is in fact almost a “decoding” of the motivic proof.

One possible application of Proposition 10.3, which comes to the mind immediately, is the construction of a filtered basis for a projective space bundle, using the following well-known result:

Proposition 10.5. *Let X be a smooth F-variety, $\mathcal{M}$ be a rank d vector bundle over X, and $\mathcal{T}$ be the tautological vector bundle on the projective X-bundle $\mathbb{P}(\mathcal{M})$. Set $\tilde{h} \stackrel{\text{def}}{=} -\tilde{c}_1([\mathcal{T}]) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{M}))$ (it is the "hyperplane" class). Then the $\mathrm{CH}^*(X)$-module $\mathrm{CH}^*(\mathbb{P}(\mathcal{M}))$ is free and the elements $1, \tilde{h}, \tilde{h}^2, \dots, \tilde{h}^{d-1}$ form its basis.*

Proof. By [8, Th. 3.3], a basis of $\mathrm{CH}^*(\mathbb{P}(\mathcal{M}))$ over $\mathrm{CH}^*(X)$ is given by the powers of $\tilde{c}_1([\mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)])$. Since

$$\mathcal{T} = \mathcal{O}_{\mathbb{P}(\mathcal{M})}(-1) \quad \text{and} \quad \tilde{c}_1([\mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)]) = -\tilde{c}_1([\mathcal{O}_{\mathbb{P}(\mathcal{M})}(-1)]) \ ,$$

we are done.[5] □

Corollary 10.6. *In the notation of the Proposition, set $h \stackrel{\text{def}}{=} 1 - [\mathcal{T}] \in K(\mathbb{P}(\mathcal{M}))$ (it is the "hyperplane" class). Then $(1, h, h^2, \dots, h^{d-1})$ is a filtered basis of the $K(X)$-module $K(\mathbb{P}(\mathcal{M}))$ and $\operatorname{codim} h^i = i$ (for $i = 1, \dots, d-1$).*

Proof. According to Proposition 10.5, the assumption of Proposition 10.3 holds for $Y \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{M})$ and $u_i \stackrel{\text{def}}{=} \tilde{h}^i \in \mathrm{CH}^i(\mathbb{P}(\mathcal{M})) = \mathrm{CH}^i(Y)$ (for $i = 0, \dots, d-1$). Since the vector bundle $\mathcal{T}$ is of rank 1, one has $c_1([\mathcal{T}]) = [\mathcal{T}] - 1 \in K(\mathbb{P}(\mathcal{M}))$. Therefore $h^i = (-c_1([\mathcal{T}]))^i$ and hence the elements $h^i \in K(\mathbb{P}(\mathcal{M}))$ correspond to the elements $\tilde{h}^i$, defined as $(-\tilde{c}_1([\mathcal{T}]))^i$ (cf. Example 10.2). Proposition 10.3 says that $(1, h, h^2, \dots, h^{d-1})$ is a filtered basis of the $K(X)$-module $K(\mathbb{P}(\mathcal{M}))$ and $\operatorname{codim} h^i = i$ (for $i = 1, \dots, d-1$). □

Proposition 10.7. *Let $X \stackrel{\text{def}}{=} S \times \mathrm{SB}(A)$ and $Y \stackrel{\text{def}}{=} \mathrm{SB}(A^{\mathrm{op}})$, where A is a central simple F-algebra of degree d and S is a homogeneous F-variety. Let $\mathcal{I}$ and $\mathcal{J}$ be the tautological vector bundles on $\mathrm{SB}(A)$ and $\mathrm{SB}(A^{\mathrm{op}})$. Consider the vector bundle $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{O}_S \otimes \mathcal{I} \otimes_A \mathcal{J}$ on $X \times Y$ and set $h \stackrel{\text{def}}{=} 1 - [\mathcal{T}] \in K(X \times Y)$. Then*

1. *The elements $1, h, h^2, \dots, h^{d-1}$ form a filtered basis of the $K(X)$-module $K(X \times Y)$; $\operatorname{codim} h^i = i$.*

[5] In fact, $\mathcal{T}$ is by definition a vector bundle while $\mathcal{O}(-1)$ is an $\mathcal{O}$-module. However, we follow here the functor of points ideology where the notions of a vector bundle and a locally free $\mathcal{O}$-module coincide (cf. [26, Sect. 8]). In the standard geometric point of view, there is a correspondence between vector bundles and locally free $\mathcal{O}$-modules; the equality $\mathcal{T} = \mathcal{O}(-1)$ should then be understood in the appropriate way.

2. *In the group $K(X_{\bar F} \times Y_{\bar F})$, the element h is equal to $\hat{h} + \overset{\vee}{h} - \hat{h}\overset{\vee}{h}$, where $\hat{h} \in K(\mathrm{SB}(A)_{\bar F}) \simeq K(\mathbb{P}^{d-1}_{\bar F})$ and $\overset{\vee}{h} \in K(\mathrm{SB}(A^{\mathrm{op}})_{\bar F}) \simeq K(\mathbb{P}^{d-1}_{\bar F})$ are the hyperplane classes.*[6]

Proof. **1.** Set $\mathcal{M} \overset{\text{def}}{=} \mathcal{O}_S \otimes \mathcal{I}$. By Corollary 4.5, the product $X \times Y$ considered over X is identified with $\mathbb{P}(\mathcal{M})$. Besides, the tautological vector bundle on $\mathbb{P}(\mathcal{M})$ corresponds to $\mathcal{M} \otimes_A \mathcal{J} = \mathcal{O}_S \otimes \mathcal{I} \otimes_A \mathcal{J} = \mathcal{T}$. To complete the proof, apply Corollary 10.6.
2. We may assume that $S = \operatorname{Spec} F$. First, note that $d^2[\mathcal{T}] = [\mathcal{I}] \cdot [\mathcal{J}]$, because $\dim_F A = d^2$. For the rest of the proof we may replace F by $\bar F$; in particular, A is split now. Then the varieties $\mathrm{SB}(A)$ and $\mathrm{SB}(A^{\mathrm{op}})$ are isomorphic to projective spaces. Let $\overset{\wedge}{\xi} = [\mathcal{O}_{\mathrm{SB}(A)}(-1)]$ and $\overset{\vee}{\xi} = [\mathcal{O}_{\mathrm{SB}(A^{\mathrm{op}})}(-1)]$. We have $[\mathcal{I}] = d\ \overset{\wedge}{\xi}$ and $[\mathcal{J}] = d\ \overset{\vee}{\xi}$ (see [41, Sect. 8.4]). The hyperplane class $\hat{h}$ is defined as $\hat{h} = 1 - \overset{\wedge}{\xi}$. Analogously, $\overset{\vee}{h} = 1 - \overset{\vee}{\xi}$. So, we get the formula $d^2[\mathcal{T}] = d(1 - \hat{h}) \cdot d(1 - \overset{\vee}{h})$. Since the Grothendieck group we are working in is torsion-free, one can divide by d^2. Hence $h \overset{\text{def}}{=} 1 - [\mathcal{T}] = 1 - (1 - \hat{h})(1 - \overset{\vee}{h}) = \hat{h} + \overset{\vee}{h} - \hat{h}\overset{\vee}{h}$. □

Corollary 10.8. *Let Q be a quaternion F-algebra and let $\hat{Y}$, $\overset{\vee}{Y}$ be two copies of $\mathrm{SB}(Q)$. Let $\hat{p}$ and $\overset{\vee}{p}$ be the classes of rational points on $\hat{Y}_{\bar F}$ and $\overset{\vee}{Y}_{\bar F}$. Let S be a homogeneous F-variety. Then $K(S \times \hat{Y} \times \overset{\vee}{Y})$ is a free $K(S \times \hat{Y})$-module with the filtered basis $(1, \hat{p} + \overset{\vee}{p} - \hat{p}\overset{\vee}{p})$; besides $\operatorname{codim}(\hat{p} + \overset{\vee}{p} - \hat{p}\overset{\vee}{p}) = 1$.*

Proof. Since Q is a quaternion algebra, it has a canonical anti-automorphism $Q \tilde{\to} Q^{\mathrm{op}}$. We identify $\overset{\vee}{Y}$ with $\mathrm{SB}(Q^{\mathrm{op}})$ via this anti-automorphism and apply Proposition 10.7. □

Remark 10.9. It probably deserves to be mentioned that the element

$$\hat{p} + \overset{\vee}{p} - \hat{p}\overset{\vee}{p} \in K(\hat{Y} \times \overset{\vee}{Y})$$

in the notation of Corollary 10.8 coincides with the class of the diagonal.

11. Filtered bases for grassmanian bundles

Definition 11.1. Let m, n be some integers. An (m,n)*-partition* λ is a sequence of integers $(\lambda_1, \ldots, \lambda_m)$ of length m satisfying the condition

[6] We consider $\hat{h}$ and $\overset{\vee}{h}$ as elements of $K(X_{\bar F} \times Y_{\bar F})$ via the pull-back.

$n \geq \lambda_1 \geq \ldots \geq \lambda_m \geq 0$. The weight $|\lambda|$ of λ is by definition the sum $\lambda_1 + \ldots + \lambda_m$.

Let λ be an (m,n)-partition and let $s \stackrel{\text{def}}{=} (s_1, \ldots, s_n)$ be a sequence of variables. For all $i < 0$ and all $i > n$, we put $s_i = 0$, besides we put $s_0 = 1$. The *Schur polynomial* $\Delta_\lambda(s)$ of λ is by definition the determinant of the matrix $(s_{\lambda_i+j-i})_{i,j=1}^m$. It is a homogeneous polynomial of weight $|\lambda|$, if every s_i is taken with the weight i.

Example 11.2. One has exactly six $(2,2)$-partitions: $(2,2)$, $(2,1)$, $(2,0)$, $(1,1)$, $(1,0)$, and $(0,0)$. Applying the formula

$$\Delta_{(\lambda_1,\lambda_2)}(s) = \det \begin{pmatrix} s_{\lambda_1} & s_{\lambda_1+1} \\ s_{\lambda_2-1} & s_{\lambda_2} \end{pmatrix} = s_{\lambda_1}s_{\lambda_2} - s_{\lambda_1+1}s_{\lambda_2-1}$$

(take in attention that $s_{-1} = s_3 = 0$ and $s_0 = 1$), one computes their Schur polynomials: $\Delta_{(2,2)}(s) = s_2^2$, $\Delta_{(2,1)}(s) = s_2 s_1$, $\Delta_{(2,0)}(s) = s_2$, $\Delta_{(1,1)}(s) = s_1^2 - s_2$, $\Delta_{(1,0)}(s) = s_1$, and $\Delta_{(0,0)}(s) = 1$.

We fix the following notation for the rest of this section: F is an arbitrary field, X is a smooth F-variety, $r \geq n \geq 0$ are integers, $\mathcal{M}$ is a rank r vector bundle over X, and $\Gamma \stackrel{\text{def}}{=} \mathit{\Gamma\!\!\Gamma}_n(\mathcal{M}) \to X$ is the grassmanian bundle of n-planes in the vector bundle $\mathcal{M}$. We denote by $\mathcal{T}$ the tautological vector bundle on Γ (also called *universal vector subbundle*, see [8, Sect. 14.6]). Note that the rank of $\mathcal{T}$ equals n.

For an (m,n)-partition λ, we put $\Delta_\lambda \stackrel{\text{def}}{=} \Delta_\lambda(s)$ with $s_i \stackrel{\text{def}}{=} (-1)^i\, c_i([\mathcal{T}]) \in K(\Gamma)^{(i)}$, $i = 1, \ldots, n$. Besides, we put $\tilde{\Delta}_\lambda \stackrel{\text{def}}{=} \Delta_\lambda(\tilde{s})$ with $\tilde{s}_i \stackrel{\text{def}}{=} (-1)^i\, \tilde{c}_i([\mathcal{T}]) \in \mathrm{CH}^i(\Gamma)$, $i = 1, \ldots, n$. Since $\Delta_\lambda(s)$ is a homogeneous polynomial of degree $|\lambda|$, one has $\Delta_\lambda \in K(\Gamma)^{(|\lambda|)}$ and $\tilde{\Delta}_\lambda \in \mathrm{CH}^{|\lambda|}(\Gamma)$. Obviously, Δ_λ corresponds to $\tilde{\Delta}_\lambda$ in the sense of Definition 10.1 (see Example 10.2).

We consider $\mathrm{CH}^*(\Gamma)$ as a $\mathrm{CH}^*(X)$-module via the pull-back homomorphism $\mathrm{CH}^*(X) \to \mathrm{CH}^*(\Gamma)$.

Proposition 11.3. *The $\mathrm{CH}^*(X)$-module $\mathrm{CH}^*(\Gamma)$ is free and the elements $\tilde{\Delta}_\lambda$, where λ runs over the set of all $(r-n,n)$-partitions, form its basis.*

Proof. By [8, Prop. 14.6.5], the elements $\tilde{\nabla}_{\lambda'} \stackrel{\text{def}}{=} \Delta_{\lambda'}(s)$, $s_i \stackrel{\text{def}}{=} \tilde{c}_i(-[\mathcal{T}])$, where λ' runs over the set of all $(n, r-n)$-partitions, form a basis of $\mathrm{CH}^*(\Gamma)$ over $\mathrm{CH}^*(X)$. By [8, Lemma 14.5.1], one has $\tilde{\nabla}_{\lambda'} = \tilde{\Delta}_\lambda$, where λ is the $(r-n,n)$-partition *dual* to λ' (see [8, Sect. 14.5] for the definition of the dual partition). Therefore, the collection $(\tilde{\nabla}_{\lambda'})_{\lambda'}$ coincides with the collection $(\tilde{\Delta}_\lambda)_\lambda$ up to a permutation. □

Now we consider $K(\Gamma)$ as a $K(X)$-module via the pull-back homomorphism $K(X) \to K(\Gamma)$.

Corollary 11.4. *The $K(X)$-module $K(\varGamma)$ is free and the elements $\varDelta_\lambda$, where λ runs over the set of all $(r-n,n)$-partitions, form its filtered basis; besides* $\operatorname{codim}\varDelta_\lambda = |\lambda|$.

Proof. Follows from Propositions 11.3 and 10.3. □

We are especially interested in the case of the grassmanian of 2-dimensional subspaces in a rank 4 vector bundle:

Corollary 11.5. *Let $\varGamma \to X$ be the grassmanian bundle of 2-planes in a rank 4 vector bundle over a smooth F-variety X. Put $\eta = -c_1([\mathcal{T}])$ and $\mu = c_2([\mathcal{T}])$, where $\mathcal{T}$ is the tautological vector bundle on $\varGamma$. Then $K(\varGamma)$ is a free $K(X)$-module with the filtered basis*

$$(\eta^\alpha\cdot\mu^\beta)_{\alpha,\beta\geq 0,\alpha+\beta\leq 2} = (1,\eta,\mu,\eta^2,\eta\mu,\mu^2).$$

The codimension of any basis element $\eta^\alpha\cdot\mu^\beta$ is equal to $\alpha+2\beta$.

Proof. By Corollary 11.4 (see also Example 11.2), the elements $\varDelta_{(2,2)} = \mu^2$, $\varDelta_{(2,1)} = \mu\eta$, $\varDelta_{(2,0)} = \mu$, $\varDelta_{(1,1)} = \eta^2-\mu$, $\varDelta_{(1,0)} = \eta$, and $\varDelta_{(0,0)} = 1$ form a filtered basis of $K(\varGamma)$ over $K(X)$. To finish the proof, we just replace $\varDelta_{(1,1)}$ by $\varDelta_{(1,1)} + \varDelta_{(2,0)} = \eta^2$. □

Here is the situation Corollary 11.5 will be applied to:

Let Q_1 and Q_2 be quaternion F-algebras and $Q \stackrel{\text{def}}{=} Q_1\otimes Q_2$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the tautological vector bundles on $\mathrm{SB}(Q_1)$ and $\mathrm{SB}(Q_2)$. Let S be an arbitrary smooth F-variety and $X \stackrel{\text{def}}{=} S\times\mathrm{SB}(Q_1)\times\mathrm{SB}(Q_2)$. Set $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{O}_S\otimes\mathcal{I}_1\otimes\mathcal{I}_2$. Clearly, $\mathcal{M}$ has a structure of a right Q_X-module. We denote by T the generalized Severi-Brauer variety $\mathrm{SB}(2,Q)$ and by $\mathcal{J}$ the tautological vector bundle on T (which is a right Q_Y-module). The canonical anti-automorphisms of Q_1 and Q_2 determine an anti-automorphism of Q; using this, we consider $\mathcal{J}$ as a *left* Q_T-module and define a vector bundle $\mathcal{T}$ on $X\times T$ as the tensor product $\mathcal{M}\otimes_{Q_{X\times T}}\mathcal{J}$. Applying Corollary 4.4 one sees that

– the product $X\times T = S\times\mathrm{SB}(Q_1)\times\mathrm{SB}(Q_2)\times\mathrm{SB}(2,Q_1\otimes Q_2)$ considered over $X = S\times\mathrm{SB}(Q_1)\times\mathrm{SB}(Q_2)$ via fist projection can be naturally identified (as a scheme over X) with the grassmanian bundle $\varGamma_2(\mathcal{M})$;
– under this identification, the tautological vector bundle on the grassmanian bundle corresponds to the vector bundle $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{M}\otimes_{Q_{X\times T}}\mathcal{J}$ on $X\times T$.

Now we set $\eta \stackrel{\text{def}}{=} -c_1([\mathcal{T}])$ and $\mu \stackrel{\text{def}}{=} c_2([\mathcal{T}])$. Since $\operatorname{rk}\mathcal{M} = 4$, Corollary 11.5 gives rise to the following

Lemma 11.6. *In the notation introduced right above, $K(X \times T)$ is a free $K(X)$-module with the filtered basis*

$$(\eta^\alpha \cdot \mu^\beta)_{\alpha,\beta \geq 0, \alpha+\beta \leq 2} = (1, \eta, \mu, \eta^2, \eta\mu, \mu^2).$$

The codimension of any basis element $\eta^\alpha \cdot \mu^\beta$ is equal to $\alpha + 2\beta$. □

Lemma 11.7. *In the notation of Lemma 11.6, let us consider the homomorphism $f : K(X \times T) \to K(X_{F(T)})$ defined as in Example 5.4. Then $f(\eta) = 2(p_1 + p_2 - p_1 p_2)$ and $f(\mu) = 2p_1 p_2$ where p_i denotes the class of a rational point in $K(\mathrm{SB}(Q_i)_{\bar{F}}) = K(\mathrm{SB}(Q_i)_{\bar{F}(T)})$ ($i = 1, 2$).*

Proof. It suffices to check the statement under the assumption that $F = \bar{F}$. In particular, $\mathrm{SB}(Q_1)$ and $\mathrm{SB}(Q_2)$ are isomorphic to the projective lines. Since $\mathcal{T} = \mathcal{M} \otimes_{Q_{X \times T}} \mathcal{J}$ and $\dim Q = \dim Q_1 \otimes Q_2 = 16$, we have $16[\mathcal{T}] = [\mathcal{M}] \cdot [\mathcal{J}] = [\mathcal{I}_1] \cdot [\mathcal{I}_2] \cdot [\mathcal{J}]$. Applying the pull-back to the right-hand side, we get $[\mathcal{I}_1] \cdot [\mathcal{I}_2] \cdot 8$ because the rank of the vector bundle $\mathcal{J}$ is equal to $2 \deg Q = 8$. Since $\mathrm{SB}(Q_i)$ is isomorphic to the projective line for $i = 1, 2$, we have $[\mathcal{I}_i] = 2\xi_i$, where $\xi_i \stackrel{\mathrm{def}}{=} [\mathcal{O}_{\mathrm{SB}(Q_i)}(-1)]$. Therefore, $f([\mathcal{T}]) = 2\xi_1\xi_2$. Since the Chern classes are compatible with the pull-back, it follows that $f(\eta) = -\, c_1(2\xi_1\xi_2)$ and $f(\mu) = c_2(2\xi_1\xi_2)$.

Let us compute the total Chern class c_t of $2\xi_1\xi_2$:

$$c_t(2\xi_1\xi_2) = (\, c_t(\xi_1\xi_2))^2 = (1 + (\xi_1\xi_2 - 1)t)^2 \ .$$

Therefore, the first Chern class equals $2(\xi_1\xi_2 - 1)$ and the second Chern class equals $(\xi_1\xi_2 - 1)^2$. Substituting $\xi_i = 1 - p_i$ we get the statement on η and μ required. □

Remark 11.8. As noticed by the referee, replacing η by $\eta - 2p_1 - 2p_2 + \mu$, one may get another filtered basis of the $K(X)$-module $K(X \times T)$ having the additional nice property that $f(\eta) = 0$. This may simplify a bit the computations of Sect. 12.

12. Preliminary calculations

In order to formulate the basic theorem, i.e. Theorem 12.1, more comfortably, we need certain formalisms.

We denote by Ω the two element set $\{\wedge, \vee\}$. Let G be the group $\mathrm{Aut}\, \Omega$ of permutations of Ω (G consists of two elements: the identity and the transposition of $\wedge$ with $\vee$).

We denote by G_{II} the direct product $G_1 \times G_2$ of two copies of G and we denote by G_I the diagonal subgroup of G_{II}. We identify G_1 with a subgroup of G_{II} via the homomorphism $(\mathrm{id}, 1)$; we identify G_2 with a subgroup of G_{II} via the homomorphism $(1, \mathrm{id})$. The nontrivial element of G_i (for $i = 1, 2$) will be denoted by s_i; the nontrivial element of G_I will be denoted by s. Thus $G_I = \{1, s\}$, $G_{II} = \{1, s_1, s_2, s\}$, $s = s_1 s_2 \in G_{II}$, $G_I \simeq \mathbb{Z}/2$, $G_{II} \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Let Ω_1 and Ω_2 be two copies of the set Ω. We denote by Ω_{II} the direct product $\Omega_1 \times \Omega_2$. Taking the product of the action of G_1 on Ω_1 with the action of G_2 on Ω_2, we get an action of $G_{II} = G_1 \times G_2$ on the set $\Omega_{II} = \Omega_1 \times \Omega_2$.

The trivial action of G_i on Ω_j for $i \neq j$ and the (nontrivial) action of G_i on Ω_i gives rise to an action of G_{II} on the set-theoretical direct sum $\Omega_1 \coprod \Omega_2$.

As a subgroup of G_{II}, the group G_I as well acts on the set $\Omega_1 \coprod \Omega_2$. The group G_I naturally acts on the diagonal of Ω_{II}, which we denote by Ω_I.

We apply these formalisms to the following situation.

Let Q_1 and Q_2 be quaternion F-algebras such that $Q_1 \otimes_F Q_2$ is a division algebra. Let us denote by Y_i the Severi-Brauer variety of Q_i and by X the product

$$X \overset{\mathrm{def}}{=} \Big(\prod_{\Omega_1} Y_1\Big) \times \Big(\prod_{\Omega_2} Y_2\Big) = \hat{Y}_1 \times \overset{\vee}{Y}_1 \times \hat{Y}_2 \times \overset{\vee}{Y}_2,$$

where $\hat{Y}_i$ and $\overset{\vee}{Y}_i$ are two copies of Y_i. The action of the groups G_I and G_{II} on the set $\Omega_1 \coprod \Omega_2$ determines their action on the variety X (by means of the permutations of the factors). In particular, the element s_i acts on X interchanging $\hat{Y}_i$ with $\overset{\vee}{Y}_i$ (the other two factors are left untouched).

Let us denote by T the generalized Severi-Brauer variety $\mathrm{SB}(2, Q_1 \otimes Q_2)$. We set

$$T_I \overset{\mathrm{def}}{=} \prod_{\Omega_I} T = \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \quad \text{and} \quad T_{II} \overset{\mathrm{def}}{=} \prod_{\Omega_{II}} T = \overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T} \ .$$

The action of the group G_{II} on the set $\Omega_{II} = \{\wedge\wedge, \vee\vee, \wedge\vee, \vee\wedge\}$ determines an action of G_{II} on T_{II} by means of the permutations of the factors. Besides, the action of G_I on $\Omega_I = \{\wedge\wedge, \vee\vee\}$ determines an action of G_I on T_I.

Now, we set $\mathfrak{X}_I \overset{\mathrm{def}}{=} X \times T_I$ and $\mathfrak{X}_{II} \overset{\mathrm{def}}{=} X \times T_{II}$. The action of G_{II} on X and on T_{II} determines an action of G_{II} on $\mathfrak{X}_{II}$. Hence we get an action of G_{II} on $K(\mathfrak{X}_{II})$. Analogously, we get an action of G_I on $\mathfrak{X}_I$ and on $K(\mathfrak{X}_I)$.

We consider the homomorphisms

$$f_I : K(\mathfrak{X}_I) = K(X \times T_I) \longrightarrow K(X_{F(T_I)})$$

and

$$f_{II} : K(\mathfrak{X}_{II}) = K(X \times T_{II}) \longrightarrow K(X_{F(T_{II})})$$

as in Example 5.4. We identify $K(X_{F(T_I)})$ with a subgroup of $K(X_{\bar{F}(T_I)})$ and we identify $K(X_{F(T_{II})})$ with a subgroup in $K(X_{\bar{F}(T_{II})})$. Note that in fact

$$\bar{F} \subset \bar{F}(T_I) \subset \bar{F}(T_{II}) \quad \text{and} \quad K(X_{\bar{F}}) = K(X_{\bar{F}(T_I)}) = K(X_{\bar{F}(T_{II})}) \;.$$

We denote by p the class of a rational point in $K(X_{\bar{F}(T_I)}) = K(X_{\bar{F}(T_{II})})$.

Theorem 12.1. *In the notation introduced above, one has*

$$2p \notin f_I((K(\mathfrak{X}_I)^{(4)})^{G_I}) \quad \textit{and} \quad 2p \notin f_{II}((K(\mathfrak{X}_{II})^{(3)})^{G_{II}}) \;.$$

Two applications of the Theorem are in Sect. 13 and Sect. 14. In the rest of this section we prove this Theorem.

We start with an investigation of $K(X)$.

Let $\hat{p}_i$ and $\overset{\vee}{p}_i$ denote the classes of rational points on $(\hat{Y}_i)_{\bar{F}}$ and on $(\overset{\vee}{Y}_i)_{\bar{F}}$. Since $X_{\bar{F}} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, a filtered basis of $K(X_{\bar{F}})$ consists of the elements $(\hat{p}_1)^{\varepsilon_1} \cdot (\overset{\vee}{p}_1)^{\varepsilon_2} \cdot (\hat{p}_2)^{\varepsilon_3} \cdot (\overset{\vee}{p}_2)^{\varepsilon_4} \cdot$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}$. Clearly, the action of G_I and G_{II} on the ring $K(X_{\bar{F}})$ is determined by the following rules:

- $s(\hat{p}_i) = \overset{\vee}{p}_i$ and $s(\overset{\vee}{p}_i) = \hat{p}_i$,
- $s_i(\hat{p}_i) = \overset{\vee}{p}_i$ and $s_i(\overset{\vee}{p}_i) = \hat{p}_i$,
- $s_i(\hat{p}_j) = \hat{p}_j$ and $s_i(\overset{\vee}{p}_j) = \overset{\vee}{p}_j$ for $i \neq j$;

that is G_{II} acts on the set $\{\hat{p}_1, \overset{\vee}{p}_1, \hat{p}_2, \overset{\vee}{p}_2\}$ in the same way as on the set $\Omega_1 \coprod \Omega_2$ (if we identify these two sets in the natural way).

Let us consider the group $K(X)$ as a subgroup of $K(X_{\bar{F}})$.

Lemma 12.2. *A filtered basis of $K(X)$ consists of the elements of the form*

$$e_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4} \overset{\text{def}}{=} (2\,\hat{p}_1)^{\varepsilon_1} \cdot (2\,\hat{p}_2)^{\varepsilon_2} \cdot (\hat{p}_1 + \overset{\vee}{p}_1 - \hat{p}_1\overset{\vee}{p}_1)^{\varepsilon_3} \cdot (\hat{p}_2 + \overset{\vee}{p}_2 - \hat{p}_2\overset{\vee}{p}_2)^{\varepsilon_4}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \{0, 1\}$. *Besides,* $\operatorname{codim} e_{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$.

Proof. Set $R_1 \overset{\text{def}}{=} K(\hat{Y}_1 \times \hat{Y}_2)$, $R_2 \overset{\text{def}}{=} K(\hat{Y}_1 \times \hat{Y}_2 \times \overset{\vee}{Y}_1)$, and $R_3 \overset{\text{def}}{=} K(\hat{Y}_1 \times \hat{Y}_2 \times \overset{\vee}{Y}_1 \times \overset{\vee}{Y}_2)$. By Corollary 9.9, a filtered basis of R_1 consists of

$$(2\,\hat{p}_1)^{\varepsilon_1} \cdot (2\,\hat{p}_2)^{\varepsilon_2} \;,$$

where $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. By Corollary 10.8, a filtered basis of the R_1-module R_2 consists of $(\hat{p}_1 + \overset{\vee}{p}_1 - \hat{p}_1\overset{\vee}{p}_1)^{\varepsilon_3}$, where $\varepsilon_3 \in \{0, 1\}$, and a filtered basis of the R_2-module R_3 consists of $(\hat{p}_2 + \overset{\vee}{p}_2 - \hat{p}_2\overset{\vee}{p}_2)^{\varepsilon_4}$, where $\varepsilon_4 \in \{0, 1\}$. Lemma 8.5 completes the proof. $\square$

Below, we use the following notation: $h_1 \overset{\text{def}}{=} \hat{p}_1 + \overset{\vee}{p}_1 - \hat{p}_1 \overset{\vee}{p}_1$ and $h_2 \overset{\text{def}}{=} \hat{p}_2 + \overset{\vee}{p}_2 - \hat{p}_2 \overset{\vee}{p}_2$ (cf. Item 2 of Proposition 10.7).

Corollary 12.3. *One has $K(X)^{(4)} \subset 4K(X_{\bar{F}})$ and $K(X)^{(3)} \subset 4K(X_{\bar{F}}) + H$, where H is the subgroup of $K(X_{\bar{F}})$ generated by $2\, \hat{p}_1 \overset{\vee}{p}_1\, h_2$ and $2\, \hat{p}_2 \overset{\vee}{p}_2\, h_1$.*

Proof. The group $K(X)^{(4)}$ is generated by the element $e_{1111} \in 4 \cdot K(X_{\bar{F}})$. The group $K(X)^{(3)}$ is generated by the following five elements:

$$e_{1111},\ e_{1101},\ e_{1110} \in 4 \cdot K(X_{\bar{F}}) \quad \text{and} \quad e_{1011},\ e_{0111}.$$

Using an evident formula $(\hat{p}_i)^2 = 0$, one gets $e_{1011} = 2\, \hat{p}_1\, h_1 h_2 = 2\, \hat{p}_1 \overset{\vee}{p}_1\, h_2 \in H$ and $e_{0111} = 2\, \hat{p}_2\, h_1 h_2 = 2\, \hat{p}_2 \overset{\vee}{p}_2\, h_1 \in H$. $\square$

Definition 12.4. We denote by $\mathcal{E}$ the filtered basis of $K(X)$ described in Lemma 12.2. We set

$$\mathcal{E}_{\text{odd}} \overset{\text{def}}{=} \{e \in \mathcal{E} \mid e \notin 2K(X_{\bar{F}})\} \text{ and } \mathcal{E}_{\text{even}} \overset{\text{def}}{=} \{e \in \mathcal{E} \mid e \in 2K(X_{\bar{F}})\}\ .$$

For any $d \geq 0$, we set $\mathcal{E}^{(d)} \overset{\text{def}}{=} \mathcal{E} \cap K(X)^{(d)}$, $\mathcal{E}_{\text{even}}^{(d)} \overset{\text{def}}{=} \mathcal{E}_{\text{even}} \cap K(X)^{(d)}$, and $\mathcal{E}_{\text{odd}}^{(d)} \overset{\text{def}}{=} \mathcal{E}_{\text{odd}} \cap K(X)^{(d)}$.

The following Lemma is obvious.

Lemma 12.5. *The set $\mathcal{E}_{\text{odd}}$ consists of 1, h_1, h_2, and $h_1 h_2$. Moreover*

1. *The set $\mathcal{E}_{\text{odd}}^{(d)}$ is empty for $d \geq 3$.*
2. *The set $\mathcal{E}_{\text{odd}}^{(2)}$ consists of one element: $h_1 h_2$.*
3. *The set $\mathcal{E}_{\text{odd}}^{(1)}$ consists of three elements: h_1, h_2, and $h_1 h_2$.* $\square$

Corollary 12.6. *For any $e \in \mathcal{E}_{\text{odd}}$, one has $s(e) = s_1(e) = s_2(e) = e$, i.e. the set $\mathcal{E}_{\text{odd}}$ consists of G_{II}-invariant elements.* $\square$

Now we are going to study the structure of $K(\mathfrak{X}_I)$ and $K(\mathfrak{X}_{II})$.

The elements $\eta, \mu \in K(\mathrm{SB}(Q_1) \times \mathrm{SB}(Q_2) \times \mathrm{SB}(2, Q_1 \otimes Q_2))$ of Lemma 11.6 give rise to the elements

$$\overset{\wedge\wedge}{\eta}, \overset{\wedge\wedge}{\mu} \in K(\hat{Y}_1 \times \hat{Y}_2 \times \overset{\wedge\wedge}{T}), \quad \overset{\vee\vee}{\eta}, \overset{\vee\vee}{\mu} \in K(\overset{\vee}{Y}_1 \times \overset{\vee}{Y}_2 \times \overset{\vee\vee}{T}),$$
$$\overset{\wedge\vee}{\eta}, \overset{\wedge\vee}{\mu} \in K(\hat{Y}_1 \times \overset{\vee}{Y}_2 \times \overset{\wedge\vee}{T}), \quad \overset{\vee\wedge}{\eta}, \overset{\vee\wedge}{\mu} \in K(\overset{\vee}{Y}_1 \times \hat{Y}_2 \times \overset{\vee\wedge}{T}).$$

Using the pull-back homomorphisms (with respect to the projections), one can consider $\overset{\wedge\wedge}{\eta}, \overset{\wedge\wedge}{\mu}, \overset{\vee\vee}{\eta}, \overset{\vee\vee}{\mu}$ as elements of $K(\mathfrak{X}_I)$. Analogously, one can consider

$$\overset{\wedge\wedge}{\eta}, \overset{\wedge\wedge}{\mu}, \overset{\vee\vee}{\eta}, \overset{\vee\vee}{\mu}, \overset{\wedge\vee}{\eta}, \overset{\wedge\vee}{\mu}, \overset{\wedge\vee}{\eta}, \overset{\wedge\vee}{\mu}$$

as elements of $K(\mathfrak{X}_{II})$.

The following Lemma is obvious.

Lemma 12.7. **I.** *The group G_I acts on the subset $\{\overset{\wedge\wedge}{\eta}, \overset{\vee\vee}{\eta}, \overset{\wedge\wedge}{\mu}, \overset{\vee\vee}{\mu}\}$ of the group $K(\mathfrak{X}_I)$ in the same way as it acts on Ω_I.*

II. *The group G_{II} acts on the subsets $\{\overset{\wedge\wedge}{\eta}, \overset{\vee\vee}{\eta}, \overset{\wedge\vee}{\eta}, \overset{\vee\wedge}{\eta}\}$ and $\{\overset{\wedge\wedge}{\mu}, \overset{\vee\vee}{\mu}, \overset{\wedge\vee}{\mu}, \overset{\vee\wedge}{\mu}\}$ of the group $K(\mathfrak{X}_{II})$ in the same way as it acts on Ω_{II}.* □

Proposition 12.8. **I.** *A filtered basis of the $K(X)$-module $K(\mathfrak{X}_I)$ consists of the elements $(\overset{\wedge\wedge}{\eta})^{\alpha_1} \cdot (\overset{\vee\vee}{\eta})^{\alpha_2} \cdot (\overset{\wedge\wedge}{\mu})^{\beta_1} \cdot (\overset{\vee\vee}{\mu})^{\beta_2}$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ run over the set of all non-negative integers such that $\alpha_1 + \beta_1, \alpha_2 + \beta_2 \leq 2$. Besides, the codimension of any basis element is equal to the sum of the codimensions of its factors:*

$$\operatorname{codim}\left((\overset{\wedge\wedge}{\eta})^{\alpha_1} \cdot (\overset{\vee\vee}{\eta})^{\alpha_2} \cdot (\overset{\wedge\wedge}{\mu})^{\beta_1} \cdot (\overset{\vee\vee}{\mu})^{\beta_2}\right) = \alpha_1 + \alpha_2 + 2(\beta_1 + \beta_2).$$

II. *A filtered basis of the $K(X)$-module $K(\mathfrak{X}_{II})$ consists of the elements $(\overset{\wedge\wedge}{\eta})^{\alpha_1} \cdot (\overset{\vee\vee}{\eta})^{\alpha_2} \cdot (\overset{\wedge\vee}{\eta})^{\alpha_3} \cdot (\overset{\wedge\vee}{\eta})^{\alpha_4} \cdot (\overset{\wedge\wedge}{\mu})^{\beta_1} \cdot (\overset{\vee\vee}{\mu})^{\beta_2} \cdot (\overset{\vee\wedge}{\mu})^{\beta_3} \cdot (\overset{\vee\wedge}{\mu})^{\beta_4}$, where $\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4$ run over the set of all non-negative integers such that $\alpha_1 + \beta_1, \ldots, \alpha_4 + \beta_4 \leq 2$. Besides, the codimension of any basis element is equal to the sum of the codimensions of its factors:*

$$\begin{aligned}&\operatorname{codim}\left((\overset{\wedge\wedge}{\eta})^{\alpha_1} \cdot (\overset{\vee\vee}{\eta})^{\alpha_2} \cdot (\overset{\wedge\vee}{\eta})^{\alpha_3} \cdot (\overset{\wedge\vee}{\eta})^{\alpha_4} \cdot (\overset{\wedge\wedge}{\mu})^{\beta_1} \cdot (\overset{\vee\vee}{\mu})^{\beta_2} \cdot (\overset{\vee\wedge}{\mu})^{\beta_3} \cdot (\overset{\vee\wedge}{\mu})^{\beta_4}\right) \\ &\quad = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2(\beta_1 + \beta_2 + \beta_3 + \beta_4)\ .\end{aligned}$$

Proof. **I.** It suffices to apply Lemma 11.6 two times (taking Lemma 8.5 in account).

II. It suffices to apply Lemma 11.6 four times (taking Lemma 8.5 in account). □

Applying Lemma 11.7, we get

Lemma 12.9. **I, II.** *For the homomorphism $f_I : K(\mathfrak{X}_I) \to K(X_{F(T_I)})$ as well as for the homomorphism $f_{II} : K(\mathfrak{X}_{II}) \to K(X_{F(T_{II})})$, one has*

$$\begin{aligned}\overset{\wedge\wedge}{\eta} &\mapsto 2(\hat{p}_1 + \hat{p}_2 - \hat{p}_1\hat{p}_2)\ ; & \overset{\wedge\wedge}{\mu} &\mapsto 2\ \hat{p}_1\hat{p}_2\ ; \\ \overset{\vee\vee}{\eta} &\mapsto 2(\overset{\vee}{p}_1 + \overset{\vee}{p}_2 - \overset{\vee}{p}_1\overset{\vee}{p}_2)\ ; & \overset{\vee\vee}{\mu} &\mapsto 2\ \overset{\vee}{p}_1\overset{\vee}{p}_2\ \ .\end{aligned}$$

II. *For the homomorphism $f_{II} : K(\mathfrak{X}_{II}) \to K(X_{F(T_{II})})$, one has*

$$\begin{aligned}\overset{\wedge\vee}{\eta} &\mapsto 2(\hat{p}_1 + \overset{\vee}{p}_2 - \hat{p}_1\overset{\vee}{p}_2)\ ; & \overset{\wedge\vee}{\mu} &\mapsto 2\ \hat{p}_1\overset{\vee}{p}_2\ ; \\ \overset{\vee\wedge}{\eta} &\mapsto 2(\overset{\vee}{p}_1 + \hat{p}_2 - \overset{\vee}{p}_1\hat{p}_2)\ ; & \overset{\vee\wedge}{\mu} &\mapsto 2\ \overset{\vee}{p}_1\hat{p}_2\end{aligned}$$

additionally. □

Corollary 12.10. **I.** *The element* $f_I(h_1h_2(\overset{\wedge\wedge}{\mu}+\overset{\vee\vee}{\mu}))$ *belongs to* $4K(X_{\bar F(T_I)})$.
II. *The elements* $f_{II}(h_1(\overset{\wedge\wedge}{\mu}+\overset{\vee\wedge}{\mu}+\overset{\wedge\vee}{\mu}+\overset{\vee\vee}{\mu}))$, $f_{II}(h_2(\overset{\wedge\wedge}{\mu}+\overset{\vee\wedge}{\mu}+\overset{\wedge\vee}{\mu}+\overset{\vee\vee}{\mu}))$, $f_{II}(h_1h_2(\overset{\wedge\wedge}{\mu}+\overset{\vee\wedge}{\mu}+\overset{\wedge\vee}{\mu}+\overset{\vee\vee}{\mu}))$, *and* $f_{II}(h_1h_2(\overset{\wedge\wedge}{\eta}+\overset{\vee\wedge}{\eta}+\overset{\wedge\vee}{\eta}+\overset{\vee\vee}{\eta}))$ *belong to* $4K(X_{\bar F(T_{II})})$.

Proof. It is just a direct calculation using Lemma 12.9. □

Our main tool for study of $f_I((K(\mathfrak{X}_I)^{(4)})^{G_I})$ is the following

Lemma 12.11. *Suppose that a subgroup* M *of* $K(X_{\bar F(T_I)})$ *and an integer* d *satisfy the following conditions:*

- $4K(X_{\bar F(T_I)})\subset M$*;*
- $f_I(K(X)^{(d)})\subset M$*;*
- *for any* $e\in\mathcal{E}^{(d-1)}_{\mathrm{odd}}$ *one has* $f_I(e(\overset{\wedge\wedge}{\eta}+\overset{\vee\vee}{\eta}))\in M$*;*
- *for any* $e\in\mathcal{E}^{(d-2)}_{\mathrm{odd}}$ *one has* $f_I(e(\overset{\wedge\wedge}{\mu}+\overset{\vee\vee}{\mu}))\in M$*.*

Then $f_I((K(\mathfrak{X}_I)^{(d)})^{G_I})\subset M$*.*

Proof. Let us divide the elements of the filtered basis $\mathfrak{E}$ of the $K(X)$-module $K(\mathfrak{X}_I)$, given in Proposition 12.8, into four groups: $\mathfrak{E}_0=\{1\}$, $\mathfrak{E}_1=\{\overset{\wedge\wedge}{\eta},\overset{\vee\vee}{\eta}\}$, $\mathfrak{E}_2=\{\overset{\wedge\wedge}{\mu},\overset{\vee\vee}{\mu}\}$, and $\mathfrak{E}_3=\{$all other generators$\}$. Let us denote by $V_i\subset K(\mathfrak{X}_I)$ (for $i=0,1,2,3$) the $K(X)$-submodule generated by $\mathfrak{E}_i$. Clearly, $K(\mathfrak{X}_I)=V_0\oplus V_1\oplus V_2\oplus V_3$. Now, we set $V_i^{(d)}=V_i\cap K(\mathfrak{X})^{(d)}$. Since the basis $\mathfrak{E}$ of $K(\mathfrak{X}_I)$ is filtered, one has

$$K(\mathfrak{X}_I)^{(d)}=V_0^{(d)}\oplus V_1^{(d)}\oplus V_2^{(d)}\oplus V_3^{(d)}\ .$$

Moreover, the sets $\mathfrak{E}_i$ are G_I-invariant; consequently, the submodules V_i are G_I-invariant as well. Therefore,

$$(K(\mathfrak{X}_I)^{(d)})^{G_I}=(V_0^{(d)})^{G_I}\oplus(V_1^{(d)})^{G_I}\oplus(V_2^{(d)})^{G_I}\oplus(V_3^{(d)})^{G_I}.$$

Thus, it is sufficient to verify that $f_I((V_i^{(d)})^s)\subset M$ for all $i=0,1,2,3$ (we recall that s is defined as the only nontrivial element of the group G_I).

For $i=0$, we have $V_0=K(X)\cdot 1=K(X)$ and hence $f_I((V_0^{(d)})^s)\subset f_I(K(X)^{(d)})\subset M$ by the assumption of the Lemma.

For $i=3$, it follows from Lemma 12.9 that $f_I(\mathfrak{E}_3)\subset 4K(X_{\bar F(T_I)})$. Since f_I is a homomorphism of $K(X)$-modules, one has $f_I(V_3)\subset 4K(X_{\bar F(T_I)})$. Consequently, $f_I((V_3^{(d)})^s)\subset f_I(V_3)\subset 4K(X_{\bar F(T_I)})\subset M$.

Now, consider $i=1$. We have $V_1^{(d)}=K(X)^{(d-1)}\cdot\overset{\wedge\wedge}{\eta}\oplus K(X)^{(d-1)}\cdot\overset{\vee\vee}{\eta}$ (here we use once again the fact that the basis $\mathfrak{E}$ is filtered). Therefore $(V_1^{(d)})^{G_I}=\{r\cdot\overset{\wedge\wedge}{\eta}+s(r)\cdot\overset{\vee\vee}{\eta}\ |r\in K(X)^{(d-1)}\}$. Thus it is sufficient to verify that $f_I(r\cdot \overset{\wedge\wedge}{\eta} + s(r)\cdot \overset{\vee\vee}{\eta}) \in M$ for all $r \in K(X)^{(d-1)}$. Clearly, it suffices to consider the case where r belongs to a basis of $K(X)^{(d-1)}$; e.g. we may assume that $r = e \in \mathcal{E}^{(d-1)}$. If $e \in \mathcal{E}^{(d-1)}_{\mathrm{even}}$, then $e \in 2K(X_{\bar{F}})$ and $s(e) \in 2K(X_{\bar{F}})$. Since $f_I(\overset{\wedge\wedge}{\eta}), f_I(\overset{\vee\vee}{\eta}) \in 2K(X_{\bar{F}(T_I)})$, one has $f_I(e\cdot \overset{\wedge\wedge}{\eta} + s(r)\cdot \overset{\vee\vee}{\eta}) \in 4K(X_{\bar{F}(T_I)}) \subset M$ in this case. If $e \in \mathcal{E}^{(d)}_{\mathrm{odd}}$, then by Corollary 12.6 we have $s(e) = e$. Therefore, $f_I(e\cdot \overset{\wedge\wedge}{\eta} + s(e)\cdot \overset{\vee\vee}{\eta}) = f_I(e(\overset{\wedge\wedge}{\eta} + \overset{\vee\vee}{\eta})) \in M$ by the third assumption of the Lemma. This completes the proof for $i = 1$.

For $i = 2$, we have $V_2^{(d)} = K(X)^{(d-2)}\cdot \overset{\wedge\wedge}{\mu} \oplus K(X)^{(d-2)}\cdot \overset{\vee\vee}{\mu}$. Therefore $(V_2^{(d)})^s = \{r\cdot \overset{\wedge\wedge}{\mu} + s(r)\cdot \overset{\vee\vee}{\mu} \mid r \in K(X)^{(d-2)}\}$. The rest of the proof is the same as that for $i = 1$. □

Lemma 12.12. $f_I\Big((K(\mathfrak{X}_I)^{(4)})^{G_I}\Big) \subset 4K(X_{\bar{F}(T_I)})$.

Proof. By Lemma 12.11 it suffices to verify the following assertions:

(a) $K(X)^{(4)} \subset 4K(X_{\bar{F}})$,

(b) for any $e \in \mathcal{E}^{(3)}_{\mathrm{odd}}$ one has $f_I(e(\overset{\wedge\wedge}{\eta} + \overset{\vee\vee}{\eta})) \in 4K(X_{\bar{F}(T_I)})$,

(c) for any $e \in \mathcal{E}^{(2)}_{\mathrm{odd}}$ one has $f_I(e(\overset{\wedge\wedge}{\mu} + \overset{\vee\vee}{\mu})) \in 4K(X_{\bar{F}(T_I)})$.

Assertion (a) is a part of Corollary 12.3. Assertion (b) is obvious because $\mathcal{E}^{(3)}_{\mathrm{odd}}$ is empty (Item 1 of Lemma 12.5). Assertion (c) is an obvious consequence of Lemma 12.5 (Item 2) and Corollary 12.10. □

Corollary 12.13. $2\,\hat{p}_1\check{p}_1\hat{p}_2\check{p}_2 \notin f_I((K(\mathfrak{X}_I)^{(4)})^{G_I})$. □

Our main tool for study of $f_{II}((K(\mathfrak{X}_{II})^{(3)})^{G_{II}})$ is

Lemma 12.14. *Suppose that a subgroup M of $K(X_{\bar{F}(T_{II})})$ and an integer d satisfy the following conditions:*

- $4K(X_{\bar{F}(T_{II})}) \subset M$;
- $f_{II}(K(X)^{(d)}) \subset M$;
- *for any* $e \in \mathcal{E}^{(d-1)}_{\mathrm{odd}}$, *one has* $f_{II}(e(\overset{\wedge\wedge}{\eta} + \overset{\vee\wedge}{\eta} + \overset{\wedge\vee}{\eta} + \overset{\vee\vee}{\eta})) \in M$;
- *for any* $e \in \mathcal{E}^{(d-2)}_{\mathrm{odd}}$, *one has* $f_{II}(e(\overset{\wedge\wedge}{\mu} + \overset{\vee\wedge}{\mu} + \overset{\wedge\vee}{\mu} + \overset{\vee\vee}{\mu})) \in M$.

Then $f_{II}((K(\mathfrak{X}_{II})^{(d)})^{G_{II}}) \subset M$.

Proof. Similar to that of Lemma 12.11. □

Lemma 12.15. $f_{II}\Big((K(\mathfrak{X}_{II})^{(3)})^{G_{II}}\Big) \subset 4K(X_{\bar{F}(T_{II})}) + H$, *where H is the subgroup of $K(X_{\bar{F}(T_{II})})$ generated by* $2\,\hat{p}_1\check{p}_1\,(\hat{p}_2 + \check{p}_2 - \hat{p}_2\check{p}_2)$ *and* $2\,\hat{p}_2\check{p}_2\,(\hat{p}_1 + \check{p}_1 - \hat{p}_1\check{p}_1)$.

Proof. By Lemma 12.14, it suffices to verify the following three conditions:

(a) $K(X)^{(3)} \subset 4K(X_{\bar{F}}) + H$;

(b) for any $e \in \mathcal{E}^{(2)}_{\text{odd}}$, one has $f_{II}(e(\overset{\wedge\wedge}{\eta} + \overset{\vee\wedge}{\eta} + \overset{\wedge\vee}{\eta} + \overset{\vee\vee}{\eta})) \in 4K(X_{\bar{F}(T_{II})})$;

(c) for any $e \in \mathcal{E}^{(1)}_{\text{odd}}$, one has $f_{II}(e(\overset{\wedge\wedge}{\mu} + \overset{\vee\wedge}{\mu} + \overset{\wedge\vee}{\mu} + \overset{\vee\vee}{\mu})) \in 4K(X_{\bar{F}(T_{II})})$.

Assertion (a) is a part of Corollary 12.3. To verify conditions (b) and (c) apply Lemma 12.5 and Corollary 12.10. $\square$

Corollary 12.16. $2\, \hat{p}_1 \overset{\vee}{p}_1 \hat{p}_2 \overset{\vee}{p}_2 \notin f_{II}((K(\mathfrak{X}_{II})^{(3)})^{G_{II}})$. $\square$

Corollaries 12.13 and 12.16 complete the proof of Theorem 12.1, because $p = \hat{p}_1 \overset{\vee}{p}_1 \hat{p}_2 \overset{\vee}{p}_2$ in the groups $K(F_{\bar{F}(T_I)})$ and $K(F_{\bar{F}(T_{II})})$.

13. First basic construction

Let k be a field of characteristic different from 2, containing elements

$$a_1, b_1, a_2, b_2, d \in k^*$$

such that the quadratic extension $l \overset{\text{def}}{=} k(\sqrt{d})$ is a field and the biquaternion l-algebra $((a_1, b_1) \otimes_k (a_2, b_2))_l$ is a skewfield.

Let T be the generalized Severi-Brauer variety (see Sect. 4) of rank 2 right ideals in the biquaternion k-algebra $(a_1, b_1) \otimes (a_2, b_2)$. Denote by K the function field of the k-variety $\mathcal{R}(T) = \mathcal{R}_{l/k}(T)$ (see Definition 6.2).

Put $L \overset{\text{def}}{=} K(\sqrt{d})$; it is the function field of the l-variety $\mathcal{R}(T)_l$. Since $\mathcal{R}(T)_l \simeq T_l \times T_l$ (see Sect. 6), one has

$$\operatorname{ind}\left((a_1, b_1) \otimes (a_2, b_2)\right)_L = 2$$

by the index reduction formula [2, Th. 3].

For $i = 1, 2$, let q_i be the quadratic form $\langle -a_i, -b_i, a_i b_i, d \rangle$ over K.

Theorem 13.1. *For any odd field extension K'/K, the quadratic forms $(q_1)_{K'}$ and $(q_2)_{K'}$ are non-linked (see Sect. 3.2 for the definition of* linked*). In particular, the forms q_1 and q_2 themselves are non-linked.*

Proof. Let us remember that the quadratic forms q_1 and q_2 are in fact defined over k and denote by X_1 and X_2 the projective quadrics over k determined by q_1 and q_2. Set $X \overset{\text{def}}{=} X_1 \times X_2$. We have to show that the degree of any closed point on the variety X_K is divisible by 4.

Consider the Grothendieck group $K(X_K)$ of the variety X_K supplied with the topological filtration. Let $p \in K(X_{\bar{K}})$ denote the class of a rational point. To show that degree of every closed point on X is divisible by 4, it suffices to show that $2p \notin K(X_K)_{(0)}$, where $K(X_K)_{(0)}$ is the 0-dimensional term of the topological filtration on $K(X_K)$. Since $\dim X = 4$, we have $K(X_K)_{(0)} = K(X_K)^{(4)}$.

The pull-back homomorphism $K(X \times \mathcal{R}(T))^{(4)} \to K(X_K)^{(4)}$, given by the flat morphism of schemes $X_K \to X \times \mathcal{R}(T)$, is surjective by Corollary 5.3 (see also Example 5.4). Therefore it suffices to show that $2p$ is not in the image of this homomorphism.

Denote by σ the non-trivial automorphism of l over k. The group $K(X \times \mathcal{R}(T))^{(4)}$ is contained in the σ-invariant part of the group $K(X_l \times \mathcal{R}(T)_l)^{(4)}$. Thus it suffices to show that

$$2p \notin \operatorname{Im}\left(K(X_l \times \mathcal{R}(T)_l)^{(4)\,\sigma} \to K(X_L)\right) .$$

For this, we apply Theorem 12.1.

In order to meet the conditions of Theorem 12.1, note that for $i = 1, 2$, one has $X_i \simeq \mathcal{R}(Y_i)$, where $\mathcal{R} = \mathcal{R}_{l/k}$ and Y_i is the Severi-Brauer variety of the quaternion k-algebra (a_i, b_i) (see Example 6.4).

Thus we have $X \times \mathcal{R}(T) \simeq \mathcal{R}(Y_1 \times Y_2 \times T)$. Therefore, we can identify $X_l \times \mathcal{R}(T)_l$ with the product

$$\mathfrak{X}_I \stackrel{\text{def}}{=} \hat{Y}_1 \times \hat{Y}_2 \times \overset{\vee}{Y}_1 \times \overset{\vee}{Y}_2 \times \hat{T} \times \overset{\vee}{T}$$

where $\hat{Y}_i$, $\overset{\vee}{Y}_i$ are two copies of $(Y_i)_l$ and $\hat{T}$, $\overset{\vee}{T}$ are two copies of T_l. Moreover, by Corollary 7.3, the automorphism of $K(X_l \times \mathcal{R}(T)_l)$ induced by σ corresponds to the automorphism of $K(\mathfrak{X}_I)$ induced by the permutation of the factors interchanging $\hat{Y}_i$ with $\overset{\vee}{Y}_i$ and $\hat{T}$ with $\overset{\vee}{T}$.

We have met the conditions of Theorem 12.1. Applying it, we get the affirmation required. □

Corollary 13.2. *For any field k_0 with* $\operatorname{char} k_0 \neq 2$ *there exist a field extension K/k_0 and elements $a_1, a_2, b_1, b_2, d \in K^*$ with the following properties:*

- $\operatorname{ind}((a_1, b_1) \otimes (a_2, b_2))_{K(\sqrt{d})} = 2$;
- *for any odd field extension K'/K, the quadratic forms*

$$q_1 \stackrel{\text{def}}{=} \langle -a_1, -b_1, a_1b_1, d\rangle \quad \textit{and} \quad q_2 \stackrel{\text{def}}{=} \langle -a_2, -b_2, a_2b_2, d\rangle$$

  are not linked over K'.

Proof. Put $k \stackrel{\text{def}}{=} k_0(a_1, b_1, a_2, b_2, d)$ where a_1, b_1, a_2, b_2, d are indeterminates. Then $l \stackrel{\text{def}}{=} k(\sqrt{d})$ is a field and the biquaternion l-algebra $((a_1, b_1) \otimes (a_2, b_2))_l$ is a skewfield. For the field $K \supset k$ as in Theorem 13.1, all affirmations of the Corollary hold. □

14. Second basic construction

Let k be a field of characteristic different from 2, containing elements

$$a_1, b_1, a_2, b_2, d_1, d_2 \in k^*$$

such that the biquadratic extension $l \stackrel{\text{def}}{=} k(\sqrt{d_1}, \sqrt{d_2})$ is a field and the biquaternion l-algebra $((a_1, b_1) \otimes_k (a_2, b_2))_l$ is a skewfield.

Let T be the generalized Severi-Brauer variety (see Sect. 4) of rank 2 right ideals in the biquaternion k-algebra $(a_1, b_1) \otimes (a_2, b_2)$. Denote by K the function field of the k-variety $\mathcal{R}(T) = \mathcal{R}_{l/k}(T)$ (see Definition 6.2).

Put $L \stackrel{\text{def}}{=} K(\sqrt{d_1}, \sqrt{d_2})$; it is the function field of the l-variety $\mathcal{R}(T)_l$. Since $\mathcal{R}(T)_l \simeq T_l^{\times 4}$ (see Sect. 6), one has

$$\operatorname{ind}\left((a_1, b_1) \otimes (a_2, b_2)\right)_L = 2$$

by the index reduction formula [2, Th. 3].

For $i = 1, 2$, let q_i be the quadratic form $\langle -a_i, -b_i, a_i b_i, d_i \rangle$ over K.

Theorem 14.1. *Denote by X_1 and X_2 the projective quadric over K determined by q_1 and q_2. The Chow group $\mathrm{CH}^2(X_1 \times X_2)$ has a torsion.*

Proof. Put $X \stackrel{\text{def}}{=} X_1 \times X_2$ and consider the Grothendieck group $K(X)$ of the variety X. There is an isomorphism $\mathrm{CH}^2(X) \simeq G^2K(X)$ (see Sect. 3.3). We are going to show that $G^2K(X)$ contains a torsion.

Denote by $p \in K(X_{\bar{K}})$ the class of a rational point. As we did all the time, we identify $K(X)$ with a subgroup of $K(X_{\bar{K}})$ via the restriction homomorphism.

Lemma 14.2. $2p \in K(X)$.

Proof. For $i = 1, 2$, denote by $\mathcal{U}_i$ Swan's vector bundle on X_i ([51]). It has a structure of right $(Q_i)_{X_i}$-module, where $Q_i \stackrel{\text{def}}{=} (a_i, b_i)_K$. For the class $[\mathcal{U}_i(2)] \in K(X_i)$ of the 2 ($= \dim X_i$) times twisted Swan's vector bundle, there is a formula (see [22, Lemma 3.6]): $[\mathcal{U}_i(2)] = 4 + 2h_i + h_i^2$, where h_i is the class of a general hyperplane section of X_i. Lifting to X, we consider the tensor product $\mathcal{U}_1 \otimes \mathcal{U}_2$. It is a vector bundle over X, having a structure of right $Q_1 \otimes_K Q_2$-module. Therefore, since $\deg Q_1 \otimes_K Q_2 = 4$ and $\operatorname{ind} Q_1 \otimes_K Q_2 = 2$, the class $[\mathcal{U}_1(2) \otimes \mathcal{U}_2(2)] = (4 + 2h_1 + h_1^2)(4 + 2h_2 + h_2^2)$ is divisible by 2 in $K(X)$. Consequently, the product $h_1^2 h_2^2$ is divisible by 2 as well. Since $h_1^2 h_2^2 = 4p$, we are done. $\square$

Since one can find a field extension of K of degree 4 such that the forms q_1 and q_2 become isotropic over this extension, one has $4p \in K(X)^{(4)}$. Therefore, if we manage to show that $2p \notin K(X)^{(3)}$, we get an element of order 2 in the quotient $K(X)/K(X)^{(3)}$, namely the class of $2p$. Since the groups $K(X)^{(0/1)}$ and $K(X)^{(1/2)}$ are torsion-free (see [44, Lemme 6.3, (i)] for the statement on $K(X)^{(0/1)} \simeq \mathrm{CH}^1(X)$), it will be a nontrivial torsion element in $K(X)^{(2/3)}$.

So, the last step in the proof of the Theorem is the following

Lemma 14.3. $2p \notin K(X)^{(3)}$.

Proof. Let us remember that the quadratic forms q_1 and q_2 are in fact defined over k. Let us change the notation and from now on denote by X_1 and X_2 the projective quadrics over k determined by q_1 and q_2. Set $X \overset{\text{def}}{=} X_1 \times X_2$. We have to show that $2p \notin K(X_K)^{(3)}$.

The pull-back homomorphism $K(X \times \mathcal{R}(T))^{(3)} \to K(X_K)^{(3)}$, given by the flat morphism of schemes $X_K \to X \times \mathcal{R}(T)$, is surjective by Corollary 5.3 (see also Example 5.4). Therefore it suffices to show that $2p$ is not in the image of this homomorphism.

Let us denote by G the Galois group of the biquadratic field extension l/k. The group $K(X \times \mathcal{R}(T))^{(3)}$ is contained in the G-invariant part of the group $K(X_l \times \mathcal{R}(T)_l)^{(3)}$. Thus it suffices to show that

$$2p \notin \mathrm{Im}\left(K(X_l \times \mathcal{R}(T)_l)^{(3)\,G} \to K(X_L)\right).$$

For this, we apply Theorem 12.1.

In order to meet the conditions of Theorem 12.1, for $i = 1, 2$, put $l_i \overset{\text{def}}{=} k(\sqrt{d_i})$ and denote by σ_i the nontrivial automorphism of l over l_{3-i}. The group G consists of 1, σ_1, σ_2, $\sigma_1\sigma_2$ and is generated by σ_1, σ_2.

Let Y_i be the Severi-Brauer variety of the quaternion k-algebra (a_i, b_i). One has $X_i \simeq R_{l_i/k}(Y_i)$ (see Example 6.4). Therefore, we can identify $(X_i)_l$ with $\hat{Y}_i \times \overset{\vee}{Y}_i$, where $\hat{Y}_i$ and $\overset{\vee}{Y}_i$ are two copies of the variety $(Y_i)_l$; moreover, by Lemma 6.5, the automorphism of $(X_i)_l$ given by σ_i corresponds to the automorphism of $\hat{Y}_i \times \overset{\vee}{Y}_i$ given by σ_i composed with the interchanging of the factors. The automorphism of $(X_i)_l$ given by σ_{3-i} corresponds to the automorphism of $\hat{Y}_i \times \overset{\vee}{Y}_i$ given by σ_{3-i}.

We also can identify $\mathcal{R}(T)_l$ with $\prod_G T_l$. Choosing the following correspondence between the signs $\wedge\wedge$, $\vee\vee$, $\wedge\vee$, $\vee\wedge$ and the elements of G:

$$\begin{aligned} \wedge\wedge &\leftrightarrow 1 \cdot 1 = 1 \\ \vee\vee &\leftrightarrow \sigma_1\sigma_2 \\ \wedge\vee &\leftrightarrow 1 \cdot \sigma_2 = \sigma_2 \\ \vee\wedge &\leftrightarrow \sigma_1 \cdot 1 = \sigma_1 \end{aligned}$$

we identify $\mathcal{R}(T)_l$ with $\overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$ where $\overset{\wedge\wedge}{T}$, $\overset{\vee\vee}{T}$, $\overset{\wedge\vee}{T}$, $\overset{\vee\wedge}{T}$ are copies of T_l. The automorphism of $\mathcal{R}(T)_l$ given by σ_1 corresponds under this identification to the automorphism of $\overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$ given by σ_1 composed with the interchanging of $\overset{\wedge\wedge}{T}$ with $\overset{\vee\wedge}{T}$ and of $\overset{\vee\vee}{T}$ with $\overset{\wedge\vee}{T}$. Analogously, the automorphism of $\mathcal{R}(T)_l$ given by σ_2 corresponds to the automorphism of $\overset{\wedge\wedge}{T} \times \overset{\vee\vee}{T} \times \overset{\wedge\vee}{T} \times \overset{\vee\wedge}{T}$ given by σ_2 composed with the interchanging of $\overset{\wedge\wedge}{T}$ with $\overset{\wedge\vee}{T}$ and of $\overset{\vee\vee}{T}$ with $\overset{\vee\wedge}{T}$.

Summarizing and passing to the Grothendieck group of the varieties, we get the following commutative diagram (for $i = 1, 2$):

$$\begin{array}{ccc} K(\mathcal{R}_{l_1/k}(Y_1)_l \times \mathcal{R}_{l_2/k}(Y_2)_l & & K(\mathcal{R}_{l_1/k}(Y_1)_l \times \mathcal{R}_{l_2/k}(Y_2)_l \\ \times \mathcal{R}(T)_l) & \overset{\sigma_i}{\to} & \times \mathcal{R}(T)_l) \\ \downarrow & & \downarrow \\ K(\mathfrak{X}_{II}) & \xrightarrow{\sigma_i \circ s_i} & K(\mathfrak{X}_{II}) \end{array}$$

where $\mathfrak{X}_{II}$ and s_i are as in Theorem 12.1. By Corollary 7.2, σ_i over the bottom arrow is the identity.

We have met the conditions of Theorem 12.1. Applying it, we get the affirmation required. $\square$

The Theorem is proved. $\square$

Corollary 14.4. *Let k be a field of characteristic $\neq 2$ and $a, b, u, v, d, \delta \in k^*$. Suppose that $d, \delta, d\delta \notin k^{*2}$ and $((a, b) \otimes (u, v))_{k(\sqrt{d}, \sqrt{\delta})}$ is a division algebra. Put $\rho = \langle -a, -b, ab, d\rangle$, $\psi = \langle -u, -v, uv, \delta\rangle$. Then there exists a field extension K/k such that $d, \delta, d\delta \notin K^{*2}$, $\operatorname{Tors} \operatorname{CH}^2(X_{\rho_K} \times X_{\psi_K}) \simeq \mathbb{Z}/2\mathbb{Z}$, and $\operatorname{ind} C_0(\rho_K) \otimes C_0(\psi_K) = 2$.*

Proof. To come to the situation considered in the beginning of the Section, we simply put $a_1 \overset{\text{def}}{=} a$, $b_1 \overset{\text{def}}{=} b$, $a_2 \overset{\text{def}}{=} u$, $b_2 \overset{\text{def}}{=} v$, $d_1 \overset{\text{def}}{=} d$, and $d_2 \overset{\text{def}}{=} \delta$.

Let K be the field extension of k constructed in the beginning of this section. Since k is algebraically closed in K, we have $d, \delta, d\delta \notin K^{*2}$. Further we have $q_1 = \rho_K$ and $q_2 = \psi_K$; so, by Theorem 14.1, the group $\operatorname{Tors} \operatorname{CH}^2(X_{\rho_K} \times X_{\psi_K})$ is nontrivial. On the other hand, by [19, Th. 5.7], the order of this group is at most 2. Therefore $\operatorname{Tors} \operatorname{CH}^2(X_{\rho_K} \times X_{\psi_K}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Finally, let us note that $C_0(\rho) \simeq (a, b)_{k(\sqrt{d})}$ and $C_0(\psi) \simeq (u, v)_{k(\sqrt{\delta})}$. Consequently $\operatorname{ind} C_0(\rho_K) \otimes C_0(\psi_K) = \operatorname{ind} ((a, b) \otimes (u, v))_L = 2$. $\square$

Part 2. Quadratic forms

15. Quadratic forms over complete fields

In this section we need some results concerning the Witt ring over a complete discrete valuation field. We fix the following notation:

- (L, v) is a complete discrete valuation field.
- We set $\mathfrak{O}_L = \{x \in L^* \mid v(x) \geq 0\}$, $\mathfrak{M}_L = \{x \in L \mid v(x) > 0\}$, and $\mathfrak{U}_L = \mathfrak{O}_L - \mathfrak{M}_L = \{x \in L \mid v(x) = 0\}$.
- The residue field $\bar{L}$ is defined as $\mathfrak{O}_L/\mathfrak{M}_L$.
- For any $a \in \mathfrak{O}_L$ we denote by $\bar{a}$ the class of a in $\bar{L} = \mathfrak{O}_L/\mathfrak{M}_L$.

If $a \in \mathfrak{U}_L$, we obviously have $\bar{a} \in \bar{L}^*$. Let π be an element of L such that $v(\pi)$ is odd. Since $L^*/L^{*2} = \mathfrak{U}_L/\mathfrak{U}_L^{*2} \times \{1, \pi\}$, an arbitrary quadratic form ϕ over L can be written in the form $\phi = \langle a_1, \dots, a_k \rangle \perp \pi \langle b_1, \dots, b_l \rangle$ where $a_1, \dots, a_k, b_1, \dots, b_l \in \mathfrak{U}_L$. We define quadratic $\bar{L}$-forms $d_\pi^1(\phi)$ and $d_\pi^2(\phi)$ as follows:

$$d_\pi^1(\phi) = \langle \bar{a}_1, \dots, \bar{a}_k \rangle_{an}, \qquad d_\pi^2(\phi) = \left\langle \bar{b}_1, \dots, \bar{b}_l \right\rangle_{an}.$$

Remark 15.1. 1) Springer's theorem asserts that a quadratic form ϕ and an element $\pi \in L^*$ determine quadratic forms $d_\pi^1(\phi)$ and $d_\pi^2(\phi)$ uniquely up to isomorphism [7]. The maps

$$d_\pi^1,\ d_\pi^2 : \{\text{isometry classes of } L\text{-forms}\} \to \{\text{isometry classes of } \bar{L}\text{-forms}\}$$

give rise to group homomorphisms $W(L) \to W(\bar{L})$, which are called *the first and the second residue classes* and denoted by ∂_1 and ∂_2 (see [33, Sect. 1 of Chap. 6] or [46, Def. 2.5 of Chap. 6]).

2) In the case where ϕ is anisotropic, quadratic forms $\langle \bar{a}_1, \dots, \bar{a}_k \rangle$ and $\left\langle \bar{b}_1, \dots, \bar{b}_l \right\rangle$ are anisotropic as well. Thus, in this case

$$d_\pi^1(\phi) = \langle \bar{a}_1, \dots, \bar{a}_k \rangle, \qquad d_\pi^2(\phi) = \left\langle \bar{b}_1, \dots, \bar{b}_l \right\rangle.$$

Lemma 15.2. *Let ϕ and τ be anisotropic quadratic forms over a complete discrete valuation field (L, v). Let π be an element of L such that $v(\pi)$ is odd. Suppose that $\tau \subset \phi$. Then $d_\pi^1(\tau) \subset d_\pi^1(\phi)$ and $d_\pi^2(\tau) \subset d_\pi^2(\phi)$.*

Proof. Let γ be such that $\tau \perp \gamma = \phi$. It follows from Remark 15.1 that $d_\pi^1(\tau) \perp d_\pi^1(\gamma) = d_\pi^1(\phi)$ and $d_\pi^2(\tau) \perp d_\pi^2(\gamma) = d_\pi^2(\phi)$. Thus $d_\pi^1(\tau) \subset d_\pi^1(\phi)$ and $d_\pi^2(\tau) \subset d_\pi^2(\phi)$. □

Lemma 15.3. *Let ϕ_1 and ϕ_2 be anisotropic quadratic k-forms. Let $K = k((t))$, and let $\phi = \phi_1 \perp t\phi_2$ be a quadratic form over K. Let L/K be an odd extension. Suppose that there exists $\tau \in GP_2(L)$ such that $\tau \subset \phi_L$. Then there exists an odd extension l/k of degree $\leq [L : K]$ such that at least one of the following conditions holds:*

- *there exists $\rho \in GP_2(l)$ such that $\rho \subset (\phi_1)_l$.*

[7] In the original version of Springer's theorem, π is an uniformizing element of L. However, we can suppose that π is an arbitrary element such that $v(\pi)$ is odd because there exists a prime element $\pi_L \in L$ such that $\pi \equiv \pi_L$ in L^*/L^{*2}.

– *there exists* $\rho \in GP_2(l)$ *such that* $\rho \subset (\phi_2)_l$.
– *quadratic forms* $(\phi_1)_l$ *and* $(\phi_2)_l$ *are linked.*

Moreover, we can take $l = \bar{L}$.

Proof. Since L/K is a finite field extension, L is a complete discrete valuation field. Let v be a valuation on L, and let $l = \bar{L}$ be the residue field of L. We have $[l : k] = [\bar{L} : \bar{K}] \leq [L : K]$. Since L/K is odd, $[l : k]$ is odd too. Besides, the ramification index $e(L/K) = v(t)$ is odd. Thus, d_t^1 and d_t^2 are well defined. Since $\dim \tau = 4$ and $\det \tau = 1$, it follows that $\dim d_t^1(\tau)$ and $\dim d_t^2(\tau)$ are even, $\dim d_t^1(\tau) + \dim d_t^2(\tau) = 4$, and $\det d_t^1(\tau) \det d_t^2(\tau) = 1$. Thus one of the following conditions holds:

1) $d_t^1(\tau) \in GP_2(\bar{L})$ and $d_t^2(\tau) = 0$,
2) $d_t^2(\tau) \in GP_2(\bar{L})$ and $d_t^1(\tau) = 0$,
3) $\dim d_t^1(\tau) = \dim d_t^2(\tau) = 2$ and $d_t^1(\tau)$ is similar to $d_t^2(\tau)$.

Clearly, $d_t^1(\phi) = (\phi_1)_l$ and $d_t^2(\phi) = (\phi_2)_l$. It follows from Lemma 15.2 that $d_\pi^1(\tau) \subset d_\pi^1(\phi) = (\phi_1)_l$ and $d_\pi^2(\tau) \subset d_\pi^2(\phi) = (\phi_2)_l$. Thus, we are done. □

16. 8-dimensional quadratic forms in $I^2(F)$

It is an important problem to find a good classification of 8-dimensional quadratic forms $\phi \in I^2(F)$. One of important invariants of ϕ is the Schur index of the Clifford algebra $C(\phi)$. Clearly, $\operatorname{ind} C(\phi)$ is equal to one of the integers: 1, 2, 4, or 8.

If ϕ is a "generic" 8-dimensional form with $\det \phi = 1$, then $\operatorname{ind} C(\phi) = 8$. This shows that we cannot say anything "specific" in the case $\operatorname{ind} C(\phi) = 8$. In the case $\operatorname{ind} C(\phi) = 1$ we have plenty information on the structure of ϕ. Indeed, in this case $c(\phi) = 0$, and hence $\phi \in I^3(F)$. Finally, the Arason-Pfister Hauptsatz implies that $\phi \in GP_3(F)$. The case $\operatorname{ind} C(\phi) = 2$ is well known too (see e.g. [28, Ex. 9.12]). Namely, for a quadratic form $\phi \in I^2(F)$ the following two conditions are equivalent: a) $\operatorname{ind} C(\phi) \leq 2$; b) ϕ can be written in the form $\phi = \langle\langle a \rangle\rangle\, q$, where $\dim q = 4$.

Thus, the only open case is $\operatorname{ind} C(\phi) = 4$. It is very easy to give examples of quadratic forms ϕ with $\operatorname{ind} C(\phi) \leq 4$. If $\phi = \pi_1 \perp \pi_2$ where $\pi_1, \pi_2 \in GP_2(F)$, then $c(\phi) = c(\pi_1) + c(\pi_2)$, and hence $\operatorname{ind} C(\phi) \leq 4$. This example gives rise to the following natural

Question 16.1. *Suppose that* $\phi \in I^2(F)$ *is an 8-dimensional quadratic form with* $\operatorname{ind} C(\phi) \leq 4$. *Do there necessarily exist quadratic forms* $\pi_1, \pi_2 \in GP_2(F)$ *such that* $\phi = \pi_1 \perp \pi_2$ *?*

In this section we construct a counterexample for this question. We start from the following

Definition 16.2 (cf. [22, Sect. 7]). Let ϕ be a quadratic form over F.

1) By $S(F)$ we denote the set of quadratic forms over F satisfying the following condition: there exists $\rho \in GP_2(F)$ such that $\rho \subset \phi$.
2) By $S_{\mathrm{odd}}(F)$ we denote the set of quadratic forms over F satisfying the following condition: there exist an odd extension L/F and $\rho \in GP_2(L)$ such that $\rho \subset \phi_L$. In other words,

$$S_{\mathrm{odd}}(F) = \{\phi \mid \text{there exists an odd extension } L/F \\ \text{such that } \phi_L \in S(L)\}.$$

Clearly, $S(F) \subset S_{\mathrm{odd}}(F)$. We do not know if there exists a field F such that $S(F) \neq S_{\mathrm{odd}}(F)$.[8] Our interest in the set $S_{\mathrm{odd}}(F)$ is motivated by the following

Theorem 16.3 (see [22, Th. 7.3]). *Let ϕ be a quadratic form of dimension ≥ 3. The group $\operatorname{Tors} G_1 K(X_\phi)$ is zero or equal to $\mathbb{Z}/2\mathbb{Z}$; it is nontrivial if and only if ϕ is anisotropic, $\dim \phi \geq 5$, and $\phi \in S_{\mathrm{odd}}(F)$.* □

Proposition 16.4. *Let $\phi \in I^2(K)$ be an anisotropic 8-dimensional quadratic form such that $\operatorname{ind} C(\phi) = 4$. Then the following conditions are equivalent:*

1) $\phi \in S(K)$, i.e., there exists $\rho \in GP_2(K)$ such that $\rho \subset \phi$,
2) there exist $\rho_1, \rho_2 \in GP_2(K)$ such that $\phi = \rho_1 \perp \rho_2$,
3) ϕ and q are linked, where q is an Albert form corresponding to the algebra $C(\phi)$.

Proof. 1)⇒2). Let ρ' be a complement of ρ in ϕ. We have $\phi = \rho \perp \rho'$. Clearly $\det \rho' = 1$ and $\dim \rho' = 4$. Therefore $\rho' \in GP_2(K)$.

2)⇒3). One can write ρ_1, ρ_2 as follows: $\rho_1 = k_1 \langle\langle a_1, b_1\rangle\rangle$ and $\rho_2 = k_2 \langle\langle a_2, b_2\rangle\rangle$. Then $c(q) = c(\phi) = (a_1, b_1) + (a_2, b_2)$. Therefore, q is similar to the form $\langle -a_1, -b_1, a_1b_1, a_2, b_2, -a_2b_2\rangle$. Obviously, $\phi_{K(\sqrt{a_1})}$ and $q_{K(\sqrt{a_1})}$ are isotropic. Hence ϕ and q are linked.

3)⇒1). Suppose that ϕ and q are linked. Then there exists $s \in K^*$ such that $\phi_{K(\sqrt{s})}$ and $q_{K(\sqrt{s})}$ are isotropic. We claim that $i_W(\phi_{K(\sqrt{s})}) \geq 2$. Suppose at the moment that $i_W(\phi_{K(\sqrt{s})}) = 1$. Then $(\phi_{K(\sqrt{s})})_{an}$ is an anisotropic Albert form. Then $\operatorname{ind} C(\phi_{K(\sqrt{s})}) = 4$. Since $c(q) = c(\phi)$, we see that $\operatorname{ind} C(q_{K(\sqrt{s})}) = 4$. Hence the Albert form $q_{K(\sqrt{s})}$ is anisotropic, a contradiction. Thus $i_W(\phi_{K(\sqrt{s})}) \geq 2$. Hence there exists a 2-dimensional form μ such that $\mu \langle\langle s\rangle\rangle \subset \phi$. To complete the proof it is sufficient to set $\rho = \mu \langle\langle s\rangle\rangle$. □

[8] In [22, Rem. 7.2], it is remarked that a field F and a 7-dimensional form $\phi \in S_{\mathrm{odd}}(F) \setminus S(F)$ can be constructed. However, recently the first named author showed that the form ϕ the second named author had in mind is in fact in $S(F)$.

In this section we construct some new examples of quadratic forms ϕ such that $\phi \notin S_{\rm odd}(K)$ (and hence $\phi \notin S(K)$) (see Theorem 16.7). The main tool for our construction is the following

Lemma 16.5. **1.** *Let ϕ_1 and ϕ_2 be anisotropic k-forms such that $\phi_1, \phi_2 \notin S_{\rm odd}(k)$. Denote by ϕ the quadratic form $\phi_1 \perp t\phi_2$ over $k((t))$. Suppose that $\phi \in S_{\rm odd}(k((t)))$. Then there exists a finite odd extension l/k such that $(\phi_1)_l$ and $(\phi_2)_l$ are linked.*
2. *Let ϕ_1 and ϕ_2 be anisotropic k-forms such that $\phi_1, \phi_2 \notin S(k)$. Let $\phi = \phi_1 \perp t\phi_2$ be a quadratic form over $k((t))$. Suppose that $\phi \in S(k((t)))$. Then ϕ_1 and ϕ_2 are linked.*

Proof. It is an obvious consequence of Lemma 15.3. □

Corollary 16.6. *Let ϕ_1 and ϕ_2 be 4-dimensional k-forms such that $\phi_1, \phi_2 \notin GP_2(k)$. Suppose that $(\phi_1)_l$ and $(\phi_2)_l$ are not linked for any odd extension l/k. Then the quadratic form $\phi_1 \perp t\phi_2$ over $k((t))$ does not belong to $S_{\rm odd}(k((t)))$.* □

Theorem 16.7. *There exist a field K and an 8-dimensional quadratic form $\phi \in I^2(K)$ such that $\operatorname{ind} C(\phi) = 4$ but $\phi \notin S_{\rm odd}(K)$.*

Proof. Let field k, elements $a_1, a_2, b_1, b_2, d \in k^*$, and 4-dimensional quadratic forms q_1, q_2 be as in Corollary 13.2. We set $K = k((t))$ and

$$\phi = q_1 \perp tq_2 = \langle -a_1, -b_1, a_1b_1, d\rangle \perp t\,\langle -a_2, -b_2, a_2b_2, d\rangle\,.$$

Clearly, $\dim\phi = 4 + 4 = 8$ and $\det_\pm \phi = 1$. In $W(K)$ we have $\phi = (\langle\langle a_1, b_1\rangle\rangle - \langle\langle d\rangle\rangle) - t(\langle\langle a_2, b_2\rangle\rangle - \langle\langle d\rangle\rangle) = \langle\langle a_1, b_1\rangle\rangle - t\,\langle\langle a_2, b_2\rangle\rangle + \langle\langle d, t\rangle\rangle$. Therefore, $c(\phi) = (a_1, b_1) + (a_2, b_2) + (d, t)$. Applying Tignol's theorem [53, Prop. 2.4], we see that $\operatorname{ind} C(\phi) = \operatorname{ind}((a_1, b_1) \otimes (a_2, b_2) \otimes (d, t)) = 2\operatorname{ind}((a_1, b_1) \otimes (a_2, b_2))_{K(\sqrt{d})} = 2 \cdot 2 = 4$. It follows from Corollary 16.6 that $\phi \notin S_{\rm odd}(K)$. □

Corollary 16.8. *The answer to Question 16.1 is negative.* □

Corollary 16.9. *There exist a field K and an 8-dimensional quadratic form $\phi \in I^2(K)$ such that $\operatorname{Tors} G^iK(X_\phi) = 0$ for $i \neq 4$ and $\operatorname{Tors} G^4K(X_\phi) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. It is an obvious consequence of Theorem 16.7 and [22, Th. 8]. □

Theorem 16.10. *Let ϕ be an 8-dimensional quadratic form over k. Then the following conditions are equivalent:*

1) $\phi \in I^2(k)$ and $\operatorname{ind} C(\phi) \leq 4$;
2) at least one of the following conditions holds:

(a) there exist $\pi_1, \pi_2 \in GP_2(k)$ *such that* $\phi = \pi_1 \perp \pi_2$*,*

(b) there exist a field extension l/k *of degree 2 and a quadratic form* $\tau \in GP_2(l)$ *such that* $\phi = s_{l/k}(\tau)$*.*

Proof. 1)⇒2). If ϕ is isotropic, we can write ϕ as a sum $\phi = q \perp \langle 1, -1\rangle$, where q is an Albert form. Writing q in the form $q = s\langle -a, -b, ab, u, v, -uv, \rangle$, we have $\phi = s\,\langle\langle a, b\rangle\rangle \perp -s\,\langle\langle u, v\rangle\rangle$. Setting $\pi_1 = s\,\langle\langle a, b\rangle\rangle$ and $\pi_2 = -s\,\langle\langle u, v\rangle\rangle$, we are done. Thus we can suppose that ϕ is anisotropic.

Since $\operatorname{ind} C(\phi) \le 4$, there exists an Albert form q such that $c(q) = c(\phi)$. If q is isotropic, then $\operatorname{ind} C(\phi) \le 2$, and hence ϕ can be written in the form $\phi = \langle\langle a\rangle\rangle \otimes \langle b_1, b_2, b_3, b_4\rangle$. Setting $\pi_1 = \langle\langle a\rangle\rangle \otimes \langle b_1, b_2\rangle$ and $\pi_2 = \langle\langle a\rangle\rangle \otimes \langle b_3, b_4\rangle$, we have $\phi = \pi_1 \perp \pi_2$ and $\pi_1, \pi_2 \in GP_2(k)$. Thus in the case where q is isotropic, the proof is complete.

Now, we can suppose that ϕ and q are anisotropic. Let $\rho = \phi \perp tq$ be a quadratic form over $K = k((t))$. Obviously, $\dim \rho = 14$ and $\rho \in I^3(K)$. It follows from [42] that there exist $d \in K$ and $\pi \in P_3(K(\sqrt{d}))$ such that $\rho = \phi \perp tq$ is similar to $s_{K(\sqrt{d})/K}(\sqrt{d}\,\pi')$. Let $L = K(\sqrt{d})$.

Since $K^*/K^{*2} = k^*/k^{*2} \times \{1, t\}$, it is sufficient to consider the following two cases:

- $d = a \in k^*$,
- d has the form at with $a \in k^*$.

First, consider the case $d = a \in k^*$. In this case we have $L = l((t))$ with $l = k(\sqrt{a})$. Then an arbitrary L-form γ can be written in the form $\phi_1 \perp t\phi_2$, where ϕ_1 and ϕ_2 are l-forms. We have

$$s_{L/K}(\gamma) = s_{L/K}(\phi_1 \perp t\phi_2) = s_{l/k}(\phi_1) \perp ts_{l/k}(\phi_2).$$

Applying this formula to the case $\gamma = \sqrt{d}\pi'$, we see that $\phi \perp tq$ is similar to $s_{l/k}(\phi_1) \perp ts_{l/k}(\phi_2)$. Hence, one of the k-forms $s_{l/k}(\phi_1)$, $s_{l/k}(\phi_2)$ is similar to ϕ and the other is similar to q. Let i be such that $s_{l/k}(\phi_i) \sim \phi$, and let j be such that $s_{l/k}(\phi_j) \sim q$. Then $\dim \phi_i = 4$ and $\dim \phi_j = 3$. Since $s_{l/k}(\phi_i) \sim \phi$, there exists $r \in k^*$ such that $\phi = r \cdot s_{l/k}(\phi_i) = s_{l/k}(r\phi_i)$. Now it is sufficient to prove that $r\phi_i \in GP_2(l)$. Let $\tilde{\phi}_j = \phi_j \perp \langle \det(\phi_i) \det(\phi_j)\rangle$. Obviously, $\phi_i \perp t\tilde{\phi}_j \in I^2(L)$. Clearly, $\phi_1 \perp t\phi_2$ is similar to $\phi_i \perp t\phi_j$. Therefore π' is similar to $\phi_i \perp t\phi_j$, and hence π is similar to $\phi_i \perp t\tilde{\phi}_j$. Since $\pi \in I^3(l((t)))$, it follows that $\phi_i, \tilde{\phi}_j \in I^2(l)$. Since $\dim \phi_i = 4$, we have $\phi_i \in GP_2(l)$. Thus in the case $d \in k^*$ we are done.

Now, consider the case $d = at$, $a \in k^*$. In this case $L = k((t))(\sqrt{at})$ is a complete discrete valuation field with residue field k and uniformizing element $\sqrt{at}$. Then an arbitrary L-form γ can be written in the form $\phi_1 \perp \sqrt{at}\phi_2$, where ϕ_1 and ϕ_2 are k-forms. We have

$$s_{L/K}(\gamma) = s_{L/K}(\phi_1 \perp \sqrt{at}\phi_2)$$

$$= s_{L/K}(\langle 1\rangle) \otimes \phi_1 \perp s_{L/K}(\langle \sqrt{at}\,\rangle) \otimes \phi_2$$
$$= \langle 1, at\rangle \otimes \phi_1 \perp \langle 1, -1\rangle \otimes \phi_2$$
$$= (\phi_1 \perp \langle 1, -1\rangle \otimes \phi_2) \perp t \cdot a\phi_1.$$

Applying this formula to the case $\gamma = \sqrt{d}\pi'$, we see that $\phi \perp tq$ is similar to $(\phi_1 \perp \langle 1,-1\rangle \otimes \phi_2) \perp t \cdot a\phi_1$. Therefore one of the forms ϕ, q is similar to $\phi_1 \perp \langle 1,-1\rangle \otimes \phi_2$ and the other is similar to $a\phi_1$. Since ϕ and q are anisotropic, we see that $\dim \phi_2 = 0$. Therefore $\dim(\phi_1 \perp \langle 1,-1\rangle \otimes \phi_2) = \dim a\phi_1$. Hence $\dim \phi = \dim q$, a contradiction.

2)⇒1). In the case where $\phi = \pi_1 \perp \pi_2$ and $\pi_1, \pi_2 \in GP_2(k)$, we have $\phi \in I^2(k)$ and $\operatorname{ind} C(\phi) \leq \operatorname{ind} C(\pi_1) \cdot \operatorname{ind} C(\pi_2) \leq 2 \cdot 2 = 4$.

Now, suppose that there exist a field extension l/k of degree 2 and a quadratic form $\tau \in GP_2(l)$ such that $\phi = s_{l/k}(\tau)$. First of all, we have $\dim \phi = [l : k] \cdot \dim \tau = 8$. Since $\tau \in I^2(l)$, it follows that $\phi = s_{l/k}(\tau) \in I^2(k)$ ([46, Cor. 14.9 of Chap. 2] or [1, Satz 3.3]).

Finally, by [1, Satz 4.18], we have $c(\phi) = c(s_{l/k}(\tau)) = N_{l/k}(c(\tau))$, where $N_{l/k}\colon \mathrm{Br}(l) \to \mathrm{Br}(k)$ is the norm map (also called transfer, or trace, or corestriction map).

For any finite separable extension l/k the norm $N_{l/k}([A])$ of the Brauer class $[A]$ of a central simple l-algebra A is represented by the corestriction $N_{l/k}(A)$ of the algebra A (see [6, Sect. 8] or [43, Sect. 7.2] for the assertion in the general case of a finite separable extension; for a simpler treatment in the special case of a quadratic extension see [29, Sect. 3.B]). Note that $N_{l/k}(A)$ is a central simple k-algebra of degree $\deg(A)^{[l:k]}$.

Therefore, coming back to the quadratic extension l/k, we have $\operatorname{ind} C(\phi) = \operatorname{ind} N_{l/k}(C(\tau))$. Since $\operatorname{ind} C(\tau) \leq 2$, it follows that $\operatorname{ind} N_{l/k}(C(\tau)) \leq 2 \cdot [l : k] = 4$. □

Remark 16.11. 1) Setting $l = k \times k$, one can consider Condition 2(a) of Theorem 16.10 as a degenerate case of Condition 2(b).

2) Actually, Theorem 16.10 is an easy consequence of the deep Rost's theorem [42]. Rost's proof uses numerous results on the algebraic groups. It would be interesting to find a direct proof of Theorem 16.10 in the framework of theory of quadratic forms.

17. 14-dimensional quadratic forms in $I^3(F)$

In this section we discuss the problem of classification of anisotropic forms $\phi \in I^3(K)$. For anisotropic quadratic forms $\phi \in I^3(K)$, the following results are known: if $\dim \phi < 8$, then ϕ is hyperbolic; if $\dim \phi = 8$, then ϕ is similar to a 3-fold Pfister form; there are no anisotropic 10-dimensional forms belonging to $I^3(K)$; if $\dim \phi = 12$, then there exist a 2-dimensional quadratic form μ and a 6-dimensional Albert form q such that $\phi = \mu \otimes q$. Analyzing these results, one can see that:

- all anisotropic quadratic forms $\phi \in I^3(K)$ of dimension ≤ 12 belong to $S(K)$,
- any quadratic form $\phi \in I^3(K)$ of dimension ≤ 12 can be represented as a sum $\sum_{i=1}^{k} \rho_i$ with $\rho_i \in GP_3(K)$ and $k \leq 2$.

Here we consider the case $\dim \phi = 14$. It is not difficult to construct a form of dimension 14 belonging to $I^3(K)$. Let τ_1' and τ_2' be pure subforms of 3-fold Pfister forms τ_1 and τ_2. Then for any $k \in K^*$ the quadratic form $\phi = k(\tau_1' \perp -\tau_2')$ has dimension 14 and belongs to $I^3(K)$. This example gives rise to the following

Question 17.1. *Suppose that $\phi \in I^3(K)$ is a 14-dimensional quadratic form. Do there necessarily exist quadratic forms $\tau_1, \tau_2 \in P_3(K)$ and $k \in K^*$ such that $\phi = k(\tau_1' \perp -\tau_2')$?*

We have the following

Proposition 17.2. *Let $\phi \in I^3(K)$ be an anisotropic 14-dimensional form. The following conditions are equivalent:*

1) *$\phi \in S(K)$, i.e., there exists $\rho \in GP_2(K)$ such that $\rho \subset \phi$,*
2) *There exist $\rho_1, \rho_2 \in GP_3(K)$ such that $\phi = \rho_1 + \rho_2$ in $W(K)$,*
3) *There exist $\tau_1, \tau_2 \in P_3(K)$ and $k \in K^*$ such that $\phi = k(\tau_1' \perp -\tau_2')$. Here τ_1' and τ_2' denote pure subforms of Pfister forms τ_1, τ_2,*
4) *There exist $\tau_1, \tau_2 \in P_3(K)$ such that $\phi \equiv \tau_1 + \tau_2 \pmod{I^4(K)}$,*
5) *$e^3(\phi)$ is a sum of two symbols, i.e., there exist $a_1, b_1 c_1, a_2, b_2, c_2 \in K^*$ such that $e^3(\phi) = (a_1, b_1, c_1) + (a_2, b_2, c_2)$.*

Proof. 1)⇒2). Let $s \in K^*$ be such that $\rho_{F(\sqrt{s})}$ is isotropic. Since $\rho \in GP_2(K)$, it follows that $i_W(\phi_{K(\sqrt{s})}) \geq 2$. Therefore $\dim(\phi_{K(\sqrt{s})})_{an} \leq 10$, and hence Pfister's theorem [40] implies that $\dim(\phi_{K(\sqrt{s})})_{an} \leq 8$. Thus, $i_W(\phi_{K(\sqrt{s})}) \geq 3$. Hence there exists a 3-dimensional form μ such that $\mu \langle\langle s \rangle\rangle \subset \phi$. We set $\rho_1 = (\mu \perp \langle \det \mu \rangle) \langle\langle s \rangle\rangle$. Clearly, $\rho_1 \in GP_3(K)$. Let $\rho_2 = (\phi \perp -\rho_1)_{an}$. We have $\phi = \rho_1 + \rho_2$ in $W(K)$. It is sufficient to prove that $\rho_2 \in GP_3(K)$. Since $\dim \phi = 14 > 8 = \dim \rho_1$ and $\phi = \rho_1 + \rho_2$, it follows that $\rho_2 \neq 0$. Since $\phi, \rho_1 \in I^3(K)$, it follows that $\rho_2 \in I^3(K)$. Therefore, $\dim \rho_2 \geq 8$. Since ρ_1 and ϕ contain a common 6-dimensional form $\mu \langle\langle s \rangle\rangle$, we have $\dim \rho_2 = \dim(\phi \perp -\rho_1)_{an} \leq 14 + 8 - 2 \cdot 6 = 10$. Since ρ_2 is anisotropic and $\rho_2 \in I^3(K)$, Pfister's theorem implies that $\dim \rho_2 = 8$. Therefore, $\rho_2 \in GP_3(K)$.

2)⇒3). It is a particular case of [13, Lemma 3.2] (see also [7, Thm. 4.5])

3)⇒4). Since $k(\tau_1' \perp -\tau_2') \equiv \tau_1 + \tau_2 \pmod{I^4(K)}$, we are done.

4)⇒1). Let $L = K(\sqrt{s})$ be a field extension such that $(\rho_2)_L$ is isotropic. We have $\phi_{L(\rho_1)} \equiv (\rho_1 + \rho_2)_{L(\rho_1)} = 0 \pmod{I^4(L(\rho_1))}$. Since $\dim \phi = 14 < 16$, the Arason-Pfister Hauptsatz implies that $\phi_{L(\rho_1)}$ is hyperbolic. Hence there exists an L-form γ such that $(\phi_L)_{an} = (\rho_1)_L \cdot \gamma$. Hence, $\dim(\phi_L)_{an}$ is divisible by 8. Since $\dim \phi = 14$, it follows that $i_W(\phi_L) \geq (14-8)/2 = 3$. Since $L = K(\sqrt{s})$, there exists a 2-dimensional form μ such that $\langle\langle s\rangle\rangle\, \mu \subset \phi$. Now it is sufficient to set $\rho = \langle\langle s\rangle\rangle\, \mu$.

4)⟺5). It is an easy consequence of bijectivity of $\bar{e}^3 : I^3(K)/I^4(K) \to H^3(K)$. □

Theorem 17.3. *There exist a field E and a 14-dimensional quadratic form $\tau \in I^3(E)$ such that $\tau \notin S_{\text{odd}}(E)$.*

Proof. Let K and $\phi \in I^2(K)$ be as in Theorem 16.7. Since $\operatorname{ind} C(\phi) = 4$, there exists an Albert form q such that $c(\phi) = c(q)$. Let $E = K((t))$, and let $\tau = \phi \perp tq$ be a quadratic form over E. Clearly, $\dim \phi = 14$. We have $c(\tau) = c(\phi) + c(q) = 0$. Therefore $\tau \in I^3(E)$. To complete the proof, it suffices to verify that $\tau \notin S_{\text{odd}}(E)$

Suppose at the moment that $\tau \in S_{\text{odd}}(E)$. By Theorem 16.7, we have $\phi \notin S_{\text{odd}}(K)$. Since q is an anisotropic Albert form, it follows that $q \notin S_{\text{odd}}(K)$. Now, it follows from Lemma 16.5 that there exists an odd extension L/K such that ϕ_L and q_L are linked. Proposition 16.4 implies that $\phi_L \in S(L)$. Since L/K is an odd extension, we have $\phi \in S_{\text{odd}}(K)$, a contradiction. □

Corollary 17.4. *The answer to Question 17.1 is negative.* □

Corollary 17.5. *There exist a field K and a 14-dimensional form $\phi \in I^3(K)$ such that $e^3(\phi)$ cannot be represented as a sum of two symbols.* □

Remark 17.6. It was proved by D. W. Hoffmann (see for instance [16]) and the first author (independently) that an arbitrary 14-dimensional quadratic form $\phi \in I^3(K)$ can be written in the form $\tau_1 + \tau_2 + \tau_3$ in $W(K)$ where $\tau_1, \tau_2, \tau_3 \in GP_3(K)$. In particular, $e^3(\phi)$ can be represented as a sum of 3 symbols.

Remark 17.7. Let n be an even integer such that $n > 14$. It is not difficult to construct a field E and a quadratic form $\phi \in I^3(E)$ of dimension n such that $\phi \notin S_{\text{odd}}(E)$. The following example shows how to construct a quadratic form $\phi \in I^3(E)$ of dimension $6n$ ($n \geq 4$) so that $\phi \notin S_{\text{odd}}(E)$.

Example 17.8. Let $n \geq 4$, and let k_0 be an arbitrary field of characteristic $\neq 2$. Let $k = k_0(X_1, \dots, X_n, Y_1, \dots, Y_n, U_1, \dots, U_n, V_1, \dots, V_n)$. For any $i = 1, \dots, n$ we set $A_i = (X_1, Y_1) \otimes_k (U_i, V_i)$ and $q_i = \langle -X_i, -Y_i, X_iY_i,$

$U_i, V_i, -U_iV_i\rangle$. Let $A = A_1 \otimes_k \ldots \otimes_k A_n$ and $K = k(\mathrm{SB}(A))$. Let $1 \leq i < j \leq n$. By the index reduction formula [49], we have $\mathrm{ind}(A_i \otimes_k A_j)_K = \min(\mathrm{ind}(A_i \otimes_k A_j), \mathrm{ind}(A_i \otimes_k A_j \otimes_k A)) = \min(4^2, 4^{n-2}) = 16$. Therefore, for any odd extension L/K we have $\mathrm{ind}(A_i \otimes_k A_j)_L = 16$. Then $(q_i)_L$ and $(q_j)_L$ are not linked. Now we set $E = K((t_1))\ldots((t_n))$ and $\phi = t_1(q_1)_E \perp \ldots \perp t_n(q_n)_E$. We have $c(\phi) = [(A_1)_E] + \ldots + [(A_n)_E] = [A_E] = [(A_{k(\mathrm{SB}(A))})_E] = 0$. Hence $\phi \in I^3(E)$. Applying Lemma 16.5, one can show that $\phi \notin S_{\mathrm{odd}}(E)$.

18. Nonstandard isotropy

Let ϕ and ψ be anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic. In the case where $\dim\phi \leq 6$ the problem was studied by many authors: the case $\dim\phi \leq 4$ was studied by Schapiro in [45]; the case $\dim\phi = 5$ was studied by D. W. Hoffmann in [11]; for 6-dimensional forms ϕ the problem was studied by D. W. Hoffmann ([12]), A. Laghribi ([31], [32]), D. Leep ([34]), A. S. Merkurjev ([37]), and the authors ([18], [19]).

In these papers the authors show that under certain conditions on ϕ and ψ the isotropy of ϕ over $F(\psi)$ is standard in a sense. Let us recall the definition of "standard isotropy" given in [19]. [9]

Definition 18.1. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. We say that the isotropy of $\phi_{F(\psi)}$ is *standard*, if at least one of the following conditions holds:

- ψ is similar to a subform in ϕ;
- there exists a subform $\phi_0 \subset \phi$ with the following two properties:
  - the form ϕ_0 is a Pfister neighbor,
  - the form $(\phi_0)_{F(\psi)}$ is isotropic.

Otherwise, we say that the isotropy is *non-standard*.

The main theorem of [19] asserts that in the case $\dim\phi \leq 6$, the isotropy $\phi_{F(\psi)}$ is standard except (possibly) the following case: $\dim\phi = 6$, $\dim\psi = 4$, $1 \neq \det_{\pm}\phi \neq \det_{\pm}\psi \neq 1$, and $\mathrm{ind}\, C_0(\phi) = 2 = \mathrm{ind}\, C_0(\phi) \otimes_F C_0(\psi)$.

In this section we show that there exist a 6-dimensional quadratic form ϕ and a 4-dimensional quadratic form ψ such that $\phi_{F(\psi)}$ is isotropic, but the isotropy is not standard. More precisely, we prove the following

[9] If $\dim\phi \leq 6$, this definition coincides with the definitions given in [17] and [21]. In this section we consider only the case $\dim\phi \leq 6$.

Theorem 18.2. *Let k be a field of characteristic $\neq 2$, and let $a, b, u, v, d, \delta \in k^*$. Suppose that $d, \delta, d\delta \notin k^{*2}$ and $((a,b) \otimes (u,v))_{k(\sqrt{d},\sqrt{\delta})}$ is a division algebra. Then there exist a field extension K/k and $c \in K^*$ with the following properties:*

1) *Quadratic forms $\phi = \langle\langle a,b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ are anisotropic, and $\phi_{K(\psi)}$ is isotropic,*
2) *the isotropy $\phi_{K(\psi)}$ is not standard.*

Proof. Let $\rho = \langle -a, -b, ab, d\rangle$. It follows from Corollary 14.4 that there exists a field extension K/k such that $d, \delta, d\delta \notin K^{*2}$, $\operatorname{Tors}\operatorname{CH}^2((X_\psi)_K \times (X_\rho)_K) = \mathbb{Z}/2\mathbb{Z}$, and $\operatorname{ind} C_0(\psi_K) \otimes C_0(\rho_K) = 2$. To complete the proof, it is sufficient to apply [19, Th. 9.1] $\square$

Let ϕ be an F-form and E/F be a field extension. We recall that a quadratic form ϕ is called *E-minimal* [15, Def. 1.1] if the following conditions hold:

- ϕ is anisotropic,
- ϕ_E is isotropic,
- $(\phi_0)_E$ is anisotropic for any form $\phi_0 \subset \phi$ with $\dim\phi_0 < \dim\phi$.

The following statement (in a slightly different form) has been noticed by D. Hoffmann (see [12, Sect. 4a]).

Lemma 18.3. *Let ϕ be a 6-dimensional and ψ a 4-dimensional quadratic forms over F. Suppose that ϕ is anisotropic and $\phi_{F(\psi)}$ is isotropic. Then the following conditions are equivalent:*

1) *the isotropy $\phi_{F(\psi)}$ is not standard,*
2) *ϕ is a $F(\psi)$-minimal form.*

Proof. 1)⇒2). Suppose at the moment that ϕ is not $F(\psi)$-minimal. Then there exists $\phi_0 \subset \phi$ with $\dim\phi_0 < \dim\phi$ such that $(\phi_0)_{F(\psi)}$ is isotropic. The isotropy $(\phi_0)_{F(\psi)}$ is standard because the dimension of ϕ_0 is ≤ 5. The definition of standard isotropy shows that the isotropy $\phi_{F(\psi)}$ is standard too, a contradiction.

2)⇒1). Suppose that isotropy $\phi_{F(\psi)}$ is standard. Then at least one of the cases of Definition 18.1 holds. First suppose that ψ is similar to a subform of ϕ. Let $\phi_0 \subset \phi$ be such that $\psi \sim \phi_0$. Clearly, $(\phi_0)_{F(\psi)}$ is isotropic and $\dim\phi_0 = 4 < 6 = \dim\phi$. Therefore ϕ is not $F(\psi)$-minimal, a contradiction. Now, consider the second case in Definition 18.1, i.e., suppose that there exists a subform $\phi_0 \subset \phi$ which is a Pfister neighbor such that $(\phi_0)_{F(\psi)}$ is isotropic. If $\dim\phi_0 < \dim\phi$, then ϕ is not a $F(\psi)$-minimal, and we have a contradiction. Now, let $\dim\phi_0 = \dim\phi = 6$. Then $\phi = \phi_0$ is a 6-dimension Pfister neighbor. Since $\phi_{F(\psi)}$ is isotropic, it follows that an arbitrary 5-dimensional subform of ϕ is isotropic over $F(\psi)$. Hence, ϕ is not $F(\psi)$-minimal, a contradiction $\square$

Corollary 18.4. *Let ψ be an anisotropic 4-dimensional quadratic form over k with $\det_\pm \psi \neq 1$. Then there exist a field extension K/k and a 6-dimensional form ϕ over K such that ϕ is a $K(\psi)$-minimal form.*

Proof. Replacing ψ by a similar form, we can suppose that ψ has the form $\langle -u, -v, uv, \delta\rangle$. Replacing k by the field of rational functions $k(a, b, d)$, we can suppose that there exist $a, b, d \in k^*$ such that $d, \delta, d\delta \notin k^*$ and $((a, b) \otimes (u, v))_{k(\sqrt{d},\sqrt{\delta})}$ is a division algebra. Let K/k and $c \in K^*$ be as in Theorem 18.2. Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$. Theorem 18.2 implies that $\phi_{K(\psi)}$ is isotropic, but isotropy is not standard. Lemma 18.3 shows that ϕ is a $K(\psi)$-minimal form. □

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