

A FAILURE OF THE u -INVARIANT CONSTRUCTION

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ABSTRACT. We show that the recent author's result, providing fields of any u -invariant different from a 2-power minus 1 or 3, fails in every dimension excluded.

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1. INTRODUCTION

We work with quadratic forms over fields (of arbitrary characteristic) and refer to [1] for related terminology and basic facts.

The u -invariant of a field is the maximal dimension of a non-degenerate anisotropic quadratic form over this field. Here we are interested in odd values at least 5 of the u -invariant, referring to [1, Chapter VI] for the information on the even and initial values.

Theorem 1.1 below, proved in [3], provides fields of the u -invariant u . It fails for $u = 5$ and for $u = 7$ because there is no field of u -invariant 5 or 7. In this note, we are going to show that it actually fails for every $u \geq 5$ of the form a 2-power minus 1 or 3. Therefore fields of such u -invariant, if exist for some $u = 2^{r+1} - 3 \geq 13$ or some $u = 2^{r+1} - 1 \geq 15$, cannot be obtained this way.

To formulate Theorem 1.1, we introduce some terminology borrowed from [4]. A non-degenerate quadratic form φ of odd dimension $2n+1$ is *essential*, if its even Clifford algebra $C_0(\varphi)$ has the maximal possible Schur index 2^n ; the form is called *almost essential*, if the Schur index is at least 2^{n-1} .

A non-degenerate quadratic form φ of odd dimension $2n+1 \geq 5$ is *strongly anisotropic*, if $e_{n+1} \notin \overline{\text{Ch}}(V)$ in notation of [3]. The variety $V = X_{n-1}$ here is the second last (i.e., $(n-1)$ st) grassmannian of φ ,

$$\text{Ch}(V) := \text{CH}(V)/2\text{CH}(V)$$

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is its modulo 2 Chow group, $\overline{\text{Ch}}(V)$ is the image of the change of field homomorphism $\text{Ch}(V) \rightarrow \text{Ch}(\bar{V})$ with $\bar{V}$ the variety V over an algebraic closure of its base field, and $e_{n+1} \in \text{Ch}(\bar{V})$ is the image of the rational point class $l_0 \in \text{Ch}(\bar{Q})$ on the projective quadric $Q = X_1$ of φ under the composition

$$\text{Ch}(\bar{X}_1) \rightarrow \text{Ch}(\bar{X}_{1,n-1}) \rightarrow \text{Ch}(\bar{X}_{n-1})$$

of the pullback followed by the pushforward with respect to the projections

$$X_1 \leftarrow X_{1,n-1} \rightarrow X_{n-1}$$

of the flag variety $X_{1,n-1} \subset X_1 \times X_{n-1}$.

A strongly anisotropic form is anisotropic.

Theorem 1.1 ([3, Theorem 7.2]). *Assume that an odd natural number $u = 2n + 1$ is neither a 2-power minus 1 nor a 2-power minus 3. Let φ be an almost essential strongly anisotropic quadratic form over a field F of dimension u . Then there exists an overfield $L \supset F$ such that φ_L is almost essential and strongly anisotropic whereas all non-degenerate quadratic forms over L of higher dimension are isotropic. In particular, φ_L is anisotropic and $u(L) = u$.*

The main results of this note are Examples 2.4 and 3.3 below.

2. FAILURE IN DIMENSION A 2-POWER MINUS 3

Let F_0 be a field.

Let n be an integer of the form $2^r - 2$ with $r \geq 2$.

Let φ be an essential $(2n + 1)$ -dimensional quadratic form over F_0 . We have

$$\dim \varphi = 2^{r+1} - 3 \in \{5, 13, \dots\}.$$

Let F_1 be the field $F_0(t_0, \dots, t_r)$ of rational functions over F_0 in $r + 1$ variables $t_0, \dots, t_r$ and consider the $(r + 1)$ -Pfister quadratic form $\pi := \langle\langle t_0, \dots, t_r \rangle\rangle$ over F_1 . Note that π is anisotropic. From here on, we view φ as a quadratic form over F_1 . Note that φ is anisotropic and remains anisotropic over the function field $F_1(\pi)$. Note as well that π remains anisotropic over the function field of any grassmannian of φ .

Lemma 2.1. *Let F_2 be the function field $F_1(X)$ of the 2^r th grassmannian X of the quadratic form $\pi \perp -\varphi$. Let V be the almost last (i.e., the $(n - 1)$ st) grassmannian of φ . Then $\pi_{F_2(V)}$ is anisotropic.*

Proof. As already mentioned, the quadratic form $\pi_{F_1(V)}$ is anisotropic. In the generic splitting tower of the form $(\pi \perp -\varphi)_{F_1(V)}$, there is a field L such that the anisotropic part of $(\pi \perp -\varphi)_L$ has dimension $2^{r+1} + 1$. For the field L' following L in this generic splitting tower, the anisotropic part $((\pi \perp -\varphi)_{L'})_{\text{an}}$ has then dimension $2^{r+1} - 1$ whereas $\pi_{L'}$ is still anisotropic as follows from [1, Theorem 26.5 (Separation Theorem)]. By the index reduction formula for quadrics [1, Theorem 30.5], the even Clifford algebra $C_0(\pi \perp -\varphi)_{L'}$ is not split. Therefore the quadratic form $((\pi \perp -\varphi)_{L'})_{\text{an}}$ is not similar to a subform of $\pi_{L'}$. It follows by [1, Corollary 9.10 and Theorem 22.5 (Subform Theorem)] that π remains anisotropic over the field L'' following L' in the generic splitting power. Since the

projective homogeneous variety $X_{L''}$ has a rational point, the field extension $L''(X)/L''$ is purely transcendental so that $\pi_{L''(X)}$ is anisotropic. Since

$$F_2(V) = F_1(X)(V) = F_1(V)(X) \subset L''(X),$$

we conclude that $\pi_{F_2(V)}$ is anisotropic. $\square$

From here on, we view φ and π as quadratic forms over F_2 . We also view V as a variety over F_2 . It follows from Lemma 2.1 that π is still anisotropic. Besides, $C_0(\varphi)$ is still a division algebra; in particular, φ is still anisotropic.

The anisotropic part ψ of the difference $\pi \perp -\varphi$ (over the field F_2) is $(2n+1)$ -dimensional; besides, the even Clifford algebra $C_0(\psi)$ is isomorphic to $C_0(\varphi)$. Let Q be the quadric of ψ .

Proposition 2.2. *In notation of [3], one has $e_{n+1} \notin \overline{\text{Ch}}^{n+1}(V_{F_2(Q)})$.*

Proof. Assume that $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V_{F_2(Q)})$ and consider a lift $\alpha \in \text{Ch}^{n+1}(Q \times V)$ of an element in $\text{Ch}^{n+1}(V_{F_2(Q)})$ representing $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V_{F_2(Q)})$. Note that $\alpha_0 = e_{n+1}$ in the decomposition

$$(2.3) \quad \bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^{n-1} \times \alpha_{n-1} + l_{n-1} \times \beta + al_{n-2} \times [\bar{V}]$$

of the image $\bar{\alpha} \in \text{Ch}^{n+1}(\bar{Q} \times \bar{V})$ of α with $\alpha_i \in \text{Ch}^{n+1-i}(\bar{V})$, $\beta \in \text{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$. (The elements $h^i, l_i \in \text{Ch}(\bar{Q})$ are defined as in [3, §2].)

Since $\pi_{F_2(V)}$ is anisotropic (by Lemma 2.1) and ψ is the anisotropic part of $\pi \perp -\varphi$, the quadratic form $\psi_{F_2(V)}$ is also anisotropic. It follows that $a = 0$ in (2.3).

Applying [4, Proposition 3.1(ii)], we conclude that

$$e_{n+1} + \alpha_1\beta + \text{St}^1(\alpha_1) \in \overline{\text{Ch}}^{n+1}(V).$$

Using [3, Proposition 3.1] as in [3, Proof of Theorem 7.2], one shows that the summands $\alpha_1\beta$ and $\text{St}^1(\alpha_1)$ are in $\overline{\text{Ch}}^{n+1}(V)$. Therefore e_{n+1} is in $\overline{\text{Ch}}^{n+1}(V)$ contradicting [3, Proposition 3.1]. $\square$

Example 2.4 (Failure of Theorem 1.1 for u a 2-power minus 3). For n as in §2, let us consider the integer $u := 2n + 1 = 2^{r+1} - 3$. In notation of Proposition 2.2, let $F := F_2(\psi)$. We view φ, ψ, π as quadratic forms over F . In particular, the quadratic form ψ is isotropic. The quadratic form φ has dimension u and the Schur index of $C_0(\varphi)$ is 2^{n-1} so that φ is almost essential. According to Proposition 2.2, the condition $e_{n+1} \notin \overline{\text{Ch}}(V)$ is satisfied by φ , i.e., φ is strongly anisotropic. Over the function field of the Pfister form π however, φ becomes isomorphic to ψ and therefore becomes isotropic. Consequently, for any overfield $L \supset F$ with $u(L) < 16$, the form φ_L is isotropic.

3. FAILURE IN DIMENSION A 2-POWER MINUS 1

Let F_0 be a field.

Let n be an integer of the form $2^r - 1$ with $r \geq 2$.

Let φ be an essential $(2n + 1)$ -dimensional quadratic form over F_0 . We have

$$\dim \varphi = 2^{r+1} - 1 \in \{7, 15, \dots\}.$$

Let F_1 be the field $F_0(t_0, \dots, t_r)$ of rational functions over F_0 in $r+1$ variables $t_0, \dots, t_r$ and consider the $(r+1)$ -Pfister quadratic form $\pi := \langle\langle t_0, \dots, t_r \rangle\rangle$ over F_1 . Note that π is anisotropic. From here on, we view φ as a quadratic form over F_1 . Note that φ is anisotropic and remains anisotropic over the function field $F_1(\pi)$. Note as well that π remains anisotropic over the function field of any grassmannian of φ .

Lemma 3.1. *Let F_2 be the function field $F_1(X)$ of the 2^r th grassmannian X of the quadratic form $\pi \perp -\varphi$. Let W be the last (i.e., the n th) grassmannian of φ . Then $\pi_{F_2(W)}$ is anisotropic.*

Proof. As already mentioned, the quadratic form $\pi_{F_1(W)}$ is anisotropic. In the generic splitting tower of the form $(\pi \perp -\varphi)_{F_1(W)}$, there is a field $L \supset F_1(W)$ (possibly $L = F_1(W)$) such that the anisotropic part of $(\pi \perp -\varphi)_L$ has dimension $2^{r+1} - 1$. Since the projective homogeneous variety X_L has a rational point, the field extension $L(X)/L$ is purely transcendental so that $\pi_{L(X)}$ is anisotropic. Since

$$F_2(W) = F_1(X)(W) = F_1(W)(X) \subset L(X),$$

we conclude that $\pi_{F_2(W)}$ is anisotropic. $\square$

From here on, we view φ and π as quadratic forms over F_2 . It follows from Lemma 3.1 that π is still anisotropic. Besides, $C_0(\varphi)$ is still a division algebra; in particular, φ is still anisotropic.

The anisotropic part ψ of the difference $\pi \perp -\varphi$ (over the field F_2) is $(2n+1)$ -dimensional; besides, the even Clifford algebra $C_0(\psi)$ is isomorphic to $C_0(\varphi)$. Let Q be the quadric of ψ and let V be the almost last (i.e., the $(n-1)$ st) grassmannian of φ .

Proposition 3.2. *In notation of [3], one has $e_{n+1} \notin \overline{\text{Ch}}^{n+1}(V_{F_2(Q)})$.*

Proof. Assume that $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V_{F_2(Q)})$ and consider a lift $\alpha \in \text{Ch}^{n+1}(Q \times V)$ of an element in $\text{Ch}^{n+1}(V_{F_2(Q)})$ representing $e_{n+1} \in \overline{\text{Ch}}^{n+1}(V_{F_2(Q)})$. Note that $\alpha_0 = e_{n+1}$ in decomposition (2.3).

Since $\pi_{F_2(W)}$ is anisotropic (by Lemma 3.1) and $[\pi] = [\varphi] + [\psi]$ in the Witt ring of F_2 , the quadratic form $\psi_{F_2(W)}$ is also anisotropic. It follows that $\beta = 0$ in (2.3). Consequently, $\text{St}^1(e_{n+1}) \in \overline{\text{Ch}}(V)$ by [2, Proposition 2.1]. Since $\text{St}^1(e_{n+1}) = e_{n+1} \cdot c_1$ by [5, Proposition 2.9], we conclude that the product $e_{n+1} \cdot c_1$ belongs to $\overline{\text{Ch}}(V)$. This conclusion contradicts [3, Proposition 3.1] because $C_0(\varphi)$ is a division algebra. $\square$

Example 3.3 (Failure of Theorem 1.1 for u a 2-power minus 1). For n as in §3, let us consider the integer $u := 2n + 1 = 2^{r+1} - 1$. In notation of Proposition 3.2, let $F := F_2(\psi)$. We view φ, ψ, π as quadratic forms over F . In particular, the quadratic form ψ is isotropic. The quadratic form φ has dimension u and the Schur index of $C_0(\varphi)$ is 2^{n-1} so that φ is almost essential. According to Proposition 3.2, the condition $e_{n+1} \notin \overline{\text{Ch}}(V)$ is satisfied by φ meaning that φ is strongly anisotropic. Over the function field of the Pfister form π however, φ becomes isomorphic to ψ and therefore isotropic. Consequently, for any overfield $L \supset F$ with $u(L) < 16$, the form φ_L is isotropic.

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