Example 9.12. We briefly describe the situation with *odd* n. In this situation, the norm homomorphism $\overline{\mathrm{Ch}}(Y_{2r})_K \to \overline{\mathrm{Ch}}(Y_{2r})$ can be non-zero. By this reason, $\overline{\mathrm{Ch}}(Y_{2r})$ in the statements of Lemma 9.8 and Corollary 9.10 has to be replaced by the quotient $\overline{\mathrm{Ch}}(Y_{2r})/N$. In particular, the ring $\overline{\mathrm{Ch}}(X_r)/N$ is naturally isomorphic to the quotient of the ring $\overline{\mathrm{Ch}}(Y_{2r})/N$ by the annihilator of $[X_r] \in \overline{\mathrm{Ch}}(Y_{2r})/N$.

Now we assume that $n = 2r+1$. Note that Y_{2r+1} is the *maximal* and Y_{2r} (the variety we are going to work with) is the *previous to the maximal* orthogonal grassmannian of q. According to [33, Section 2] as well as to [BKT, Theorem 3.2], the ring $\mathrm{Ch}(\bar{Y}_{2r})$ is generated by certain elements $e_i \in \mathrm{Ch}^i(\bar{Y}_{2r})$, $i = 1, 2, \dots, 2r+1$ and $e \in \mathrm{Ch}^1(\bar{Y}_{2r})$. In notation of [33, Section 2], the generator e is the generator of the W-type and the generators $e_1, \dots, e_{2r+1}$ are the generators of the Z-type. In notation of [BKT, Theorem 3.2], $e_i = \tau_i$ for $i = 1, \dots, 2r+1$ and $e = \tau_1 + \tau_1'$.

Let now Y'_{2r} be the variety Y_{2r} over a field extension of F such that h is almost hyperbolic but K is still a field. The subring $\mathrm{Ch}(Y'_{2r}) \subset \mathrm{Ch}(\bar{Y}_{2r})$ is generated by $e, e_2, \dots, e_{2r+1}$ (all the above generators without e_1). The image N of the norm map $\mathrm{Ch}(\bar{Y}_{2r}) \to \mathrm{Ch}(Y'_{2r})$ is the ideal generated by e. The ring $\mathrm{Ch}(Y'_{2r})/N$ is generated by all e_i with $i \geq 2$ subject to the relations $e_i^2 = e_{2i}$. The subring $\overline{\mathrm{Ch}}(Y_{2r})/N \subset \mathrm{Ch}(Y'_{2r})/N$ contains the elements $e_2, e_4, \dots, e_{2r}$. In the case of *generic* h, the subring $\overline{\mathrm{Ch}}(Y_{2r})/N$ does not contain any e_i with odd i: otherwise the canonical dimension of Y_{2r} (and therefore of X_r) would be smaller than

$$\dim Y_{2r} - (2 + 4 + \cdots + 2r) = r(r+2) = \dim X_r.$$

It follows that the subring $\overline{\mathrm{Ch}}(Y_{2r})/N$ is generated by the elements $e_2, e_4, \dots, e_{2r}$. In particular, the only non-zero homogeneous element of dimension $\dim X_r$ in $\overline{\mathrm{Ch}}(Y_{2r})/N$ is the product $e_2 e_4 \dots e_{2r}$ and therefore

$$[X_r] = e_2 e_4 \dots e_{2r} \in \mathrm{Ch}(Y'_{2r})/N.$$

The annihilator of $[X_r]$ in $\mathrm{Ch}(Y'_{2r})/N$ is therefore the ideal generated by $e_2, \dots, e_{2r}$ and it follows that for $n = 2r+1$ and *almost hyperbolic* h the ring $\mathrm{Ch}(X_r)/N$ is generated by elements $e_3, e_5, \dots, e_{2r+1}$ subject to the relations $e_i^2 = 0$.

References

[BKT] Buch, A. S., Kresch, A., and Tamvakis, H. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math. 178*, 2 (2009), 345–405.