EXTENDED PETROV-SEMENOV'S CONNECTIONS

NIKITA A. KARPENKO

ABSTRACT. For a smooth projective quadric X, Alexander Vishik introduced the notion of *connections*. He discovered connections arising from the splitting pattern of X as well as the so-called *excellent* connections. Recently, Victor Petrov and Nikita Semenov found new connections arising from the *J-invariant* $J(X)$, which is a layer of the *Elementary Discrete invariant* $ED(X)$, both further Vishik's inventions. In this note we give a new proof for Peterov-Semenov's connections and extend them to the remaining layers of $ED(X)$. Also, we briefly discuss connections on nonsmooth quadrics.

CONTENTS

0. PREHISTORY

We work over an arbitrary base field F with a fixed algebraic closure $\bar{F}$. Given an F-variety Y, we write $\bar{Y}$ for the $\bar{F}$-variety $Y_{\bar{F}}$. We write $\mathrm{Ch}(Y)$ for the Chow ring modulo 2 (i.e., with coefficients in $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$) of a smooth variety Y. An element of $\mathrm{Ch}(\bar{Y})$ is *rational*, if it is in the image of the change of field homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(\bar{Y})$.

Let X be a smooth projective quadric over a field F (of any characteristic) of dimension $n \geq 0$. It is given by a nondegenerate (= nonsingular) quadratic form which is determined by X up to similarity, see [1] – our main reference on quadrics.

Recall that the $\mathbb{F}_2$-vector space $\mathrm{Ch}(\bar{X})$ has the basis $\{h^i, l_i\}_{i=0}^m$, where $m := [n/2]$ (so that $n = 2m$ or $n = 2m+1$), $h \in \mathrm{Ch}^1(\bar{X})$ is the class of a hyperplane section, and $l_i \in \mathrm{Ch}_i(\bar{X})$ is the class of a linear subspace in $\bar{X}$ of dimension i. In particular, $l_0 \in \mathrm{Ch}_0(\bar{X})$ is the class of a rational point. For $i < n/2$, the element l_i does not depend on the choice of the linear subspace; for $n = 2m$, the element l_m does depend on it and has to be chosen, the only existing different choice being equal to $l_m + h^m$.

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These additive generators satisfy the multiplicative relations $hl_i = l_{i-1}$ for $i \in \{1, \dots, m\}$ and $h^{m+1} = 0$. Moreover,

$$l_m^2 = \begin{cases} 0, & \text{if } n \text{ is not divisible by 4;} \\ l_0, & \text{if } n \text{ is divisible by 4.} \end{cases}$$

The external product homomorphism

$$\mathrm{Ch}(\bar{X}) \otimes \mathrm{Ch}(\bar{X}) \to \mathrm{Ch}(\bar{X} \times \bar{X})$$

is an isomorphism. Therefore a basis of the $\mathbb{F}_2$-vector space $\mathrm{Ch}_n(\bar{X} \times \bar{X})$ is given by the external products

$$B := \{h^i \times l_i,\ l_i \times h^i\}_{i=0}^m$$

and – in the case of even n – two additional elements $h^m \times h^m$ and $l_m \times l_m$.

Given $\alpha \in \mathrm{Ch}(\bar{X} \times \bar{X})$ and a basis element b, we say that α *contains* b if b appears (with the coefficient $1 \in \mathbb{F}_2$) in the decomposition of α.

Our main interest are rational elements in $\mathrm{Ch}^n(\bar{X} \times \bar{X})$ $(= \mathrm{Ch}_n(\bar{X} \times \bar{X}))$, the codimension n (and dimension n) component of $\mathrm{Ch}(\bar{X} \times \bar{X})$. Knowing them means knowing the complete decomposition of the Chow motive of X (not only with coefficients $\mathbb{F}_2$ but also with integer coefficients! see [2]). Driven by this interest, Alexander Vishik introduced in [6, Theorem 4.3] the notion of *connections* on the set B: two given elements of B are *connected*, if for any rational $\alpha \in \mathrm{Ch}^n(\bar{X} \times \bar{X})$ one of these two elements is contained in α if and only if the other is.

The set B is rational in the sense that the sum of all elements in B is rational ([1, Lemma 73.1]). It follows that the complement of any rational subset in B is also rational. In particular, if some rational subset contains some $b \in B$ and does not contain some $b' \in B$, then there exists a rational subset containing b' and not containing b.

The basis element $h^m \times h^m$ (which we have for even n) is always rational. It follows that connections (inside the set B) "do not depend" on the choice of l_m for even n, made in the beginning.

The quadric X is *split*, if all $\mathrm{Ch}^n(\bar{X} \times \bar{X})$ is rational. This is the case if and only if no distinct elements of B are connected. For nonsplit X and even n, no rational element contains $l_m \times l_m$ ([1, Lemma 73.2]).

Being connected is an equivalence relation on B and so B is a disjoint union of its connected components. The sums of the elements in each connected component are rational ([1, Lemma 73.3]) and (together with $h^m \times h^m$ for even n, plus $l_m \times l_m$ for even n and split X) they form a basis of the space of rational elements. Therefore knowing all connections means knowing this space.

Connections arising from the splitting pattern of X, have been discovered by Alexander Vishik. Recall that the *Witt index* $i_W(X)$ of X is the maximal i with rational l_{i-1}. Here we set $l_{-1} := 0$ so that $i_W(X) \geq 0$. This is the classical Witt index of the quadratic form defining X. For instance, X is split if and only if $i_W(X)$ takes its maximal possible – with respect to n – value $m+1$. *Splitting pattern* of X is the set of the integers $\{i_W(X_L)\}_L$ with L ranging over all extension fields of F. We have $\{i_W(X_L)\}_L = \{j_0, \dots, j_h\}$ for some $0 \leq j_0 < \dots < j_h = m+1$ (with h the *height* of X). For any i with $j_0 \leq i < j_h$, we have

$j_q \le i < j_{q+1}$ for some (unique) q and we define $i' := j_q + j_{q+1} - 1 - i$. Then $h^i \times l_i$ and $l_{i'} \times h^{i'}$ are connected, [1, Lemma 73.19].

Since $i'' = i$, $h^{i'} \times l_{i'}$ is also connected to $l_i \times h^i$ by the above rule. In fact, this second connection is a consequence of the first one because of the transposition automorphism of $X \times X$.

Note that $j_0 = i_W(X)$. For $0 \le i < i_W(X)$, the elements $\{h^i \times l_i\}$ and $\{l_i \times h^i\}$ are *isolated* in the sense that the corresponding singletons are connected components of B. However, for *anisotropic* X (i.e., for X with $i_W(X) = 0$), there are no isolated points in B: by the above rule, every point is connected to some other.

If X is *excellent* (i.e., X is given by an *excellent quadratic form* [1, §28]) and anisotropic, then the splitting pattern of X is determined by n and the above connections are all connections that exist: every connected component of B consists of two elements. It has been shown in [9] that an arbitrary anisotropic n-dimensional X has the connections of the excellent one. The proof makes use of Steenrod operations on $\mathrm{Ch}(Y)$ for the smooth varieties $Y = X$ and $Y = X \times X$ and because of recent [5] works in arbitrary characteristic including 2.

In a recent [4], Victor Petrov and Nikita Semenov found new connections arising from the so-called *J-invariant* $J(X)$ of Alexander Vishik. To define $J(X)$, one considers the variety $X(m)$ of m-dimensional linear subspaces in X, the flag variety

$$X(0, m) \subset X \times X(m),$$

and its projections $pr(0)$ and $pr(m)$ to X and to $X(m)$. The J-invariant of X is the following information: for $i \in \{0, \dots, m\}$, which of the elements

$$e_i := pr(m)_*\big(pr(0)^*(l_i)\big) \in \mathrm{Ch}^{n-m-i}(\bar{X}(m))$$

are rational.

The elements e_i are usually indexed differently – by their codimension; in the original [7, Definition 5.1] $J(X)$ is defined as the set of codimensions of the rational elements; in [1, §88], for some convenience reason, it is defined as the set of codimensions of the *irrational* ones.

In the case of $n = 2m$, $e_m \in \mathrm{Ch}^0(\bar{X}(m))$ is the class of one (from two) connected components of $\bar{X}(m)$ (the choice of l_m we made is equivalent to picking up a connected component of $\bar{X}(m)$); it is rational if and only if the discriminant of X is trivial.

To explain the motivation behind the notion of the J-invariant, let us mention that the elements $\{e_i\}_{i=0}^m$ generate the ring $\mathrm{Ch}(\bar{X}(m))$ (see original [7] or [1, Theorem 86.12]). Moreover, the subring of rational elements is generated by those e_i which are rational (original [7] or [1, Theorem 87.7]). Therefore knowing the J-invariant means knowing this subring.

An element of B is an external product determined by its first component. To simplify notation we denote below the elements of B by their first components. This way B becomes the set

$$B = \{h^0, \dots, h^m, l_m, \dots, l_0\}$$

(the elements are ordered by their codimension).

Here is the result by Petrov and Semenov:

Theorem 0.1 ([4, Theorem 7.7])**.** *If for some $i \in \{0, \dots, m\}$ the element e_i is irrational, then or every $j \in \{0, \dots, m-i\}$ the elements l_{m-j} and h^{i+j} of B are connected.*

(For even n, only X of trivial discriminant is considered in [4]. However the statement holds without this restriction. See §1 for the proof.)

The proof of Theorem 0.1, given in [4], makes use of the general machinery developed in that paper, where the group $\mathrm{Ch}(\bar{X})$ is considered as comodule over the Hopf algebra given by the Chow ring of the special orthogonal group acting on $\bar{X}$. Here we provide a more direct proof (see §1) which consists of just a few manipulations with the (rational) *incidence correspondence* $[\bar{X}(0,m)] \in \mathrm{Ch}(\bar{X} \times \bar{X}(m))$. This new proof easily generalizes to remaining layers of the *ED*-invariant (see §2). (The J-invariant is one of such layers.)

We conclude this note with a brief discussion of nonsmooth X (see §3).

1. J-connections

In this section we prove Theorem 0.1.

For the incidence correspondence

$$\alpha := [\bar{X}(0,m)] \in \mathrm{Ch}(\bar{X} \times \bar{X}(m))$$

one has the formula

$$\alpha := \begin{cases} \tilde{\alpha}, & \text{if } n \text{ is not divisible by 4;} \\ h^m \times [\bar{X}(m)] + \tilde{\alpha}, & \text{if } n \text{ is divisible by 4,} \end{cases}$$

where $\tilde{\alpha} := l_m \times [\bar{X}(m)] + \sum_{i=0}^{m} h^i \times e_i$ (see [1, §86] or Lemma 2.1). Note that the correspondences α and $\tilde{\alpha}$ are rational.

To prove Theorem 0.1, let us assume that for some $i \in \{0, \dots, m\}$ and for some $j \in \{0, \dots, m-i\}$, the elements l_{m-j} and h^{i+j} of B are not connected. Then there is a rational element $\beta \in \mathrm{Ch}^n(\bar{X} \times \bar{X})$ containing $l_{m-j} \times h^{m-j}$ and not containing $h^{i+j} \times l_{i+j}$. In the case of even n, we may additionally assume that β contains neither $l_m \times l_m$ nor $h^m \times h^m$.

Let us multiply the correspondence α by $h^j \times [\bar{X}(m)]$ and call the result α'. Then consider β and α' as correspondences and take their composition $\gamma := \alpha' \circ \beta$. Finally, we take the product

$$\gamma \cdot \alpha \cdot (h^{m-i-j} \times [\bar{X}(m)]).$$

Its push-forward with respect to the second projection is rational and equals e_i.

2. *ED*-connections

The Elementary Discrete invariant of X is also – like the J-invariant $J(X)$ – an invention by Alexander Vishik, [8, §2]. We take here the liberty to shorten to $ED(X)$ the original notation $EDI(X)$.

Let r be an integer with $0 \le r \le m$ and consider the variety $X(r)$ of linear r-dimensional subspaces in X. In particular, $X(0) = X$. One defines the rth layer of the Elementary Discrete invariant $ED(X)$ repeating the definition of the J-invariant with the variety $X(m)$ replaced by $X(r)$: we consider the flag variety $X(0,r) \subset X \times X(r)$ with the projections $pr(0)$ and $pr(r)$, look at the elements

$$e_i(r) := pr(r)_*\big(pr(0)^*(l_i)\big) \in \mathrm{Ch}^{n-r-i}(\bar{X}(r))$$

and collect the information which of them are rational. For visualization, we refer to [8, §2], where $ED(X)$ was originally defined. Together with the Segre classes of the tautological vector bundle, which appear right below and are rational, the elements $\{e_i(r)\}_{i=0}^m$ generate the ring $\mathrm{Ch}(\bar{X}(r))$.

In this section we generalize Theorem 0.1, replacing $J(X)$ – the mth layer of $ED(X)$ – by the rth layer. To start, we compute the incidence correspondence

$$\alpha(r) := [\bar{X}(0,r)] \in \mathrm{Ch}(\bar{X} \times \bar{X}(r)).$$

Let us define

$$\tilde{\alpha}(r) := \sum_{i=0}^{m-r} l_{i+r} \times s_i + \sum_{i=0}^{m} h^i \times e_i(r) \in \mathrm{Ch}(\bar{X} \times \bar{X}(r)),$$

with $s_i \in \mathrm{Ch}^i(\bar{X}(r))$ the ith Segre class of the tautological vector bundle on the variety $\bar{X}(r)$ (in particular, $s_0 = [\bar{X}(r)]$).

Lemma 2.1. *For the incidence correspondence $\alpha(r) \in \mathrm{Ch}(\bar{X} \times \bar{X}(r))$ one has*

$$\alpha(r) = \begin{cases} \tilde{\alpha}(r), & \text{if } n \text{ is not divisible by } 4; \\ h^m \times s_{m-r} + \tilde{\alpha}(r), & \text{if } n \text{ is divisible by } 4. \end{cases}$$

Proof. Since the external product homomorphism

$$\mathrm{Ch}(\bar{X}) \otimes \mathrm{Ch}(\bar{X}(r)) \to \mathrm{Ch}(\bar{X} \times \bar{X}(r))$$

is an isomorphism, we have

$$\alpha(r) = \sum_{i=0}^{m-r} l_{i+r} \times s'_i + \sum_{i=0}^{m} h^i \times t_i \in \mathrm{Ch}(\bar{X} \times \bar{X}(r))$$

for some (uniquely determined) $s'_i \in \mathrm{Ch}^i(\bar{X}(r))$ and $t_i \in \mathrm{Ch}^{n-r-i}(\bar{X}(r))$. The homomorphism $\mathrm{Ch}(\bar{X}) \to \mathrm{Ch}(\bar{X}(r))$ given by the incidence correspondence coincides with the composition $pr(r)_* \circ pr(0)^*$ (see [1, Proposition 62.7]). It follows that $s'_i = pr(r)_*\big(pr(0)^*(h^i)\big)$. This is the Segre class s_i by [8, Proposition 2.1]. We also get that

$$t_i = pr(r)_*\big(pr(0)^*(l_i)\big) = e_i(r)$$

except the case with even $i = m = n/2$. In the exceptional case,

$$pr(r)_*\big(pr(0)^*(l_m)\big) = s_{m-r} + t_m. \qquad \square$$

Note that the correspondences $\alpha(r)$ and $\tilde{\alpha}(r)$ are rational. Replacing m by r in the remaining part of the proof of Theorem 0.1, we get the generalization:

Theorem 2.2. *Assume that for some $r, i \in \{0, \ldots, m\}$ with $i \le r$, the element $e_i(r)$ is irrational and the elements $e_{i+k}(r)$ for k with $1 \le k \le m - r$ are rational. Then for every $j \in \{0, \ldots, r - i\}$, the element l_{r-j} and h^{i+j} of B are connected.*

Proof. Assume that for some $r, i \in \{0, \ldots, m\}$ with $i \le r$ and for some $j \in \{0, \ldots, r - i\}$, the elements l_{r-j} and h^{i+j} of B are not connected. Then there is a rational element $\beta \in \mathrm{Ch}^n(\bar{X} \times \bar{X})$ containing $l_{r-j} \times h^{r-j}$ and not containing $h^{i+j} \times l_{i+j}$. In the case of even n, we may additionally assume that β contains neither $l_m \times l_m$ nor $h^m \times h^m$.

Let us multiply the element $\alpha(r) \in \mathrm{Ch}(\bar{X} \times \bar{X}(r))$ (from Lemma 2.1) by $h^j \times [\bar{X}(m)]$ and call the result α'. Then consider β and α' as correspondences and take their composition $\gamma := \alpha' \circ \beta$. Finally, we take the product

$$\gamma \cdot \alpha \cdot (h^{r-i-j} \times [\bar{X}(m)]).$$

Its push-forward with respect to the second projection is rational and equals $e_i(r)$ plus a linear combination of $s_k e_{i+k}(r)$, $1 \leq k \leq m-r$. So, if the elements $e_{i+k}(r)$ are rational for such k, the element $e_i(r)$ is rational as well. □

Remark 2.3. For $r = 0$, the assumptions of Theorem 2.2 are satisfied if and only if $n \leq 1$ (i.e., $m = 0$) and X is anisotropic. The conclusion then is: the set $B = \{h^0, l_0\}$ is connected. This is not new (see §0) and easy to check directly.

Example 2.4. For an anisotropic *Pfister quadric* X, the conditions of Theorem 2.2 are satisfied for any $r > m/2$ and $i = m - r$ (see [8, Example 3.2]). A Pfister quadric is excellent, so that we already know all existing connections (see §0). But at least we get more examples with the assumptions of Theorem 2.2 fulfilled.

One may prefer to drop the rationality assumptions of Theorem 2.2. However the conclusion becomes fuzzy then. We loose the determinism of original Petrov-Semenov's connections when we extend them this way:

Theorem 2.5. *If for some $r, i \in \{0, \ldots, m\}$ with $i \leq r$ the element $e_i(r)$ is irrational, then for every $j \in \{0, \ldots, r-i\}$, the element $l_{r-j} \in B$ is connected to at least one element of the subset*

$$S := \{h^{i+j+k}\}_{0 \leq k \leq m-r} \bigcup \{l_{r-j+k}\}_{1 \leq k \leq m-r} \quad \subset B.$$

Proof. Assume that for some $r, i \in \{0, \ldots, m\}$ with $i \leq r$ and for some $j \in \{0, \ldots, r-i\}$, the elements l_{r-j} of B is connected to no element of the subset $S \subset B$. Then there is a rational element $\beta \in \mathrm{Ch}^n(\bar{X} \times \bar{X})$ containing l_{r-j} (i.e., the external product $l_{r-j} \times h^{r-j}$) and no element of S. In the case of even n, we may additionally assume that β contains neither $l_m \times l_m$ nor $h^m \times h^m$.

Repeating with the rational cycles $\alpha(r)$ and β exactly the same manipulations as in the proof of Theorem 2.2, we get a rational element which is equal to $e_i(r)$. □

Remark 2.6. Different layers of *ED*(X) are related by [8, Proposition 2.5]. Namely, irrationality of $e_i(r)$ implies irrationality of $e_i(r-1)$ and $e_{i+1}(r-1)$. The conclusion of Theorem 2.5 given by irrationality of $e_i(r)$ is stronger than the conclusions given by irrationality of $e_i(r-1)$ and $e_{i+1}(r-1)$. So, combining [8, Proposition 2.5] with Theorem 2.5 this way we do not get a stronger result. However, if we additionally assume that $e_{i+2}(r-1), \ldots, e_m(r-1)$ are rational, then [8, Proposition 2.5] gives rationality of $e_{i+1}(r), \ldots, e_m(r)$; in this case, Theorem 2.2 applied on the rth layer of *ED*(X) provides more connections than it does with the $(r-1)$th layer.

Remark 2.7. By Theorem 2.5, irrationality of $e_0(0) = l_0$ (meaning anisotropy of X) simply implies that the element $l_0 \in B$ is not isolated. This is not new (see §0).

Instead of dropping all rationality assumptions of Theorem 2.2, one may want to drop only a part of them. The conclusion will be then somewhere between that of Theorem

2.2 and that of Theorem 2.5. More precisely, adding to the assumptions of Theorem 2.5 the rationality of $e_{i+k}(r)$ only for *some* k between 1 and $m-r$, allows one to remove from the conclusion subset S the elements h^{i+j+k} and l_{r-j+k} given by these k.

3. Nonsmooth quadrics

Connections can also be considered for nonsmooth quadrics. These are defined by singular (= degenerate) quadratic forms. Study of connections easily reduces to the case of anisotropic quadrics. Anisotropic quadrics over a field F can be nonsmooth only if char $F = 2$. Machinery developed in [3] allows one to extend to the nonsmooth anisotropic setting the results on connections available in the smooth case. It also allows for their motivic interpretation.

Note that the set B, on which connections are defined, is empty in the case of a nowhere smooth X (defined by a totally singular quadratic form). Therefore, in this special case, void is the theory of connections.

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Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko