TWISTED POINCARÉ DUALITY FOR UPPER MOTIVES

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Abstract. Extending to a wider class of reductive groups an earlier result on Poincaré duality for upper motives, we obtain a modified version of the duality, involving a twist by an invertible Artin motive. The proof is based on a recent joint work with C. De Clercq and A. Quéguiner-Mathieu.

Contents

1. The statement

Let p be a prime number and let $\mathrm{ChM} = \mathrm{ChM}(F, \mathbb{F})$ be the category of *Chow motives* over a field F (of any characteristic) with coefficients in $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$ as defined, e.g., in [5, §64]. We write $M(X) \in \mathrm{ChM}$ for the motive of a smooth projective F-variety X and set

$$\mathbb{F} := M(F) := M(\operatorname{Spec} F).$$

(Traditionally, the coefficient ring notation is also used as notation for $M(F)$.) For a motive $M \in \mathrm{ChM}$ and an integer $i \in \mathbb{Z}$, $M(i)$ is the ith (Tate) shift of $M = M(0)$. The category ChM is equipped with a commutative tensor product, induced by the direct product of varieties, satisfying

$$\mathbb{F}(i) \otimes \mathbb{F}(i') = \mathbb{F}(i + j) \quad \text{and} \quad M(i) = M \otimes \mathbb{F}(i).$$

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The groups

$$\mathrm{Ch}^i(M) := \mathrm{Hom}(M, \mathbb{F}(i)) \quad \text{and} \quad \mathrm{Ch}_i(M) := \mathrm{Hom}(\mathbb{F}(i), M)$$

are the ith cohomological and homological Chow groups of M. If $M = M(X)$, these groups coincide with the usual Chow groups $\mathrm{Ch}^i(X)$ and $\mathrm{Ch}_i(X)$ of X (with coefficients in $\mathbb{F}$), vanishing for $i < 0$.

The duality cofunctor $\mathrm{ChM} \to \mathrm{ChM}$, $M \mapsto M^*$ of [5, §65] is a self-inverse anti-equivalence of additive categories commuting with the tensor product and inverting the shifts. The motive of every irreducible variety X satisfies the *Poincaré duality*

$$M(X)^* = M(X)(-d) \ \text{ with } d := \dim X, \tag{1.1}$$

which shows up on the level of Chow groups as $\mathrm{Ch}_i(X) = \mathrm{Ch}^{d-i}(X)$ for any i.

From here on, X will be a projective G-homogeneous F-variety for a reductive algebraic F-group G. Any direct summand in the motive $M(X)$ possesses a decomposition into a finite direct sum of indecomposable motives, and such a decomposition, called *complete* below, is unique, [2] (see also [8, Corollary 2.6]).

The *upper motive* $U(X) \in \mathrm{ChM}$, defined in [8, §2.II] (under the name of *indecomposable upper motive*), is the summand in a complete motivic decomposition of X with $\mathrm{Ch}^0(U(X)) = \mathbb{F}$. The motive $U(X)$ is determined by X up to a canonical isomorphism: by [8, Lemma 2.8], a projector $\pi \in \mathrm{End}(M(X))$ which yields an upper motive of X has *multiplicity* $1 \in \mathbb{F}$ as a correspondence $X \rightsquigarrow X$; the composition of two such projectors yields an isomorphism between the corresponding upper motives. By the isomorphism criterion [8, Corollary 2.15], given one more projective homogeneous variety X' (possibly under a different reductive group), the motives $U(X)$ and $U(X')$ are isomorphic if and only if each of the varieties $X_{F(X')}$ and $X'_{F(X)}$ possesses a closed point of prime to p degree.

Over a separable closure $\bar{F}$ of F, the motive $M(X)$ and, in particular, the motive $U(X)$, decompose into direct sums of *Tate motives* $\mathbb{F}(i)$ with various $i \geq 0$ including $i = 0$ (appearing exactly once). Dimension $\dim U(X)$ is defined as the maximal i appearing for $U(X)$. More generally, *dimension* of a geometrically split motive is defined as follows (some people prefer to call it *level* as in Hodge theory):

Definition 1.2. For an arbitrary motive $M \in \mathrm{ChM}$ decomposing over $\bar{F}$ into a direct sum of Tate motives, its *dimension* $\dim M$ is defined as the maximum of the distance $|i - j|$ between i and j for all $\mathbb{F}(i)$ and $\mathbb{F}(j)$ which are summands of M over $\bar{F}$.

Clearly,

$$\dim M = \dim M^* = \dim M(i) \tag{1.3}$$

for such M and any $i \in \mathbb{Z}$.

It has been shown in [7, Proposition 5.2] that $U(X)$ satisfies the Poincaré duality isomorphism

$$U(X)^* \simeq U(X)(-d) \ \text{ with } d := \dim U(X) \tag{1.4}$$

provided that the group G is of inner type or, more generally, of p-*inner* type, i.e., becomes of inner type over a finite base field extension of a p-power degree.

Note that for any i, relation (1.1) implies that the number of the summands $\mathbb{F}(i)$, appearing in the complete decomposition of $M(X)$ over $\bar{F}$, coincides with the number of $\mathbb{F}(d-i)$. The similar property for $U(X)$, implied by (1.4), is a bit of surprise.

Theorem 1.6 below is an analogue of (1.4) for *p'-inner p-consistent* reductive G. A reductive algebraic group G over F is called *p'-inner* (cf. [4]), if it acquires inner type over a finite base field extension of a prime to p degree. A reductive group G is called *p-consistent*, if all *higher Tits p-indexes* of G are invariant under the $*$-action of the absolute Galois group of F on the Dynkin diagram of G. A *higher Tits p-index* of G is the Tits index acquired by G over some p-special extension field of F (cf. [3]), where a field is *p-special* if degrees of all its finite field extensions are p-powers (cf. [5, §101.B]). By *Tits index* of G we mean the set of vertices of the Dynkin diagram of G belonging to the orbits of the $*$-action circled in the Tits diagram defined in [10, §2.3].

A different definition of p-consistency, given in [4], is equivalent to the above definition:

Lemma 1.5. *Let G be a reductive group over a field F and let E/F be a (unique up to an isomorphism) minimal field extension such that G_E is of inner type. The following two properties of G are equivalent:*
(i) G is p-consistent;
*(ii) for any intermediate field $F \subset L \subset E$, any projective G_L-homogeneous variety X is p-*equivalent *to $\hat{X}_L$ for some projective G-homogeneous variety $\hat{X}$, i.e., the varieties $X_{L(\hat{X})}$ and $\hat{X}_{L(X)}$ possess closed points of prime to p degrees.*

Proof. Let us mention that any $*$-invariant subset τ of the Dynkin diagram of G determines a projective G-homogeneous variety – the variety given by the conjugacy class of parabolic subgroups corresponding in the sense of [10, §2.5.4] to the complement of τ. (The variety obtained this way from any invariant subset of the Tits index of G has a rational point.) Moreover, if this variety is isomorphic to the variety given by another $*$-invariant subset τ', then $\tau = \tau'$.

Assume (i). For an intermediate field L in E/F and a projective G_L-homogeneous variety X, let us consider the Tits index τ acquired by G over a *p-special closure* of the function field $L(X)$, where a (unique up to an isomorphism) p-special closure of a field is its p-special extension field algebraic over the initial field and such that the degree of every finite subextension is prime to p (cf. [5, §101.B]). As τ is a higher Tits p-index of G, it is $*$-invariant by (i) and therefore yields a projective G-homogeneous variety $\hat{X}$ such that $\hat{X}_L$ is p-equivalent to X.

We proved that (i) implies (ii). Before proving the inverse implication, let us do some preliminary remarks.

The field extension E/F is finite Galois and the $*$-action of the absolute Galois group of F on the Dynkin diagram of G factors through $\Gamma := \mathrm{Gal}(E/F)$. The $*$-action of Γ can be described as follows: given an element $\sigma \in \Gamma$ and a vertex x of the Dynkin diagram, let us consider the projective G_E-homogeneous variety X given by the singleton $\{x\}$; the base change X_σ of X via σ is then the projective G_E-homogenous variety given by the singleton $\{\sigma(x)\}$. More generally, if X is the projective G_E-homogeneous variety given by a subset τ of vertices of the Dynkin diagram, the base change X_σ is the projective

G_E-homogeneous variety given by $\sigma(\tau)$. If X is isomorphic to $\hat{X}_E$ for some projective G-homogeneous variety $\hat{X}$, then $X_\sigma \simeq X$ for any $\sigma \in \Gamma$ implying that τ is $*$-invariant.

Now assume (ii) and consider a higher Tits p-index τ of G. By definition of a higher Tits p-index, τ is the Tits index of G over some (p-special) field $K \supset F$. The stabilizer of τ yields an intermediate field L in E/F; the field extension K/F contains a subextension isomorphic to L/F. We may assume that L/F is a subextension of K/F.

Let X be the projective G_L-homogeneous variety given by τ. Then τ is the Tits index of G over (a p-special closure of) the function field $L(X)$. By (ii), X is p-equivalent to $\hat{X}_L$ for some projective G-homogeneous variety $\hat{X}$. It follows that τ is the Tits index acquired by G over a p-special closure of $L(\hat{X})$. For any $\sigma \in \Gamma$, $\sigma(\tau)$ is then the Tits index acquired by G over a p-special closure of $\sigma(L)(\hat{X})$. Since the field extensions L/F and $\sigma(L)/F$ are isomorphic, so are the field extensions $L(\hat{X})/F$ and $\sigma(L)(\hat{X})/F$ as well as the field extensions of F given by their p-special closures. Hence $\sigma(\tau) = \tau$, i.e., τ is $*$-invariant. □

The class of p'-inner p-consistent groups includes the groups of inner type and the groups becoming quasi-split over a finite base field extension of degree prime to p. Moreover, it is stable under taking the direct products of groups and the operation $G \mapsto R_{L/F}(G_L)$, where $R_{L/F}$ is the Weil transfer for a subextension L/F of a finite Galois field extension of degree prime to p. Every absolutely simple group of type different from ${}^6\mathsf{D}_4$ (with $p = 2, 3$) and ${}^3\mathsf{D}_4$ (with $p = 2$) is p-inner or p'-inner p-consistent.

Theorem 1.6. *The upper motive $U(X)$ of any projective homogeneous variety X under a p'-inner p-consistent reductive algebraic group G satisfies a* twisted *Poincaré duality*

$$U(X)^* \simeq U(X)(-d) \otimes A,$$

where A is an invertible Artin motive and $d := \dim U(X)$ (see Definition 1.2).

Corollary 1.7. *One has*

$$\mathrm{Ch}_i(U(X)) \simeq \mathrm{Ch}^{d-i}(U(X) \otimes A) \quad \textit{and} \quad \mathrm{Ch}_i(U(X) \otimes A) \simeq \mathrm{Ch}^{d-i}(U(X))$$

for any i; in particular, $\mathrm{Ch}_i\, U(X) = 0 = \mathrm{Ch}^i\, U(X)$ for $i > d$. □

By definition, an *Artin motive* is a motive isomorphic to a direct summand in the spectrum of an étale F-algebra. An arbitrary motive $A \in \mathrm{ChM}$ is *invertible* if $A \otimes B \simeq \mathbb{F}$ for some motive $B \in \mathrm{ChM}$. Invertible Artin motives are characterized below in Lemma 1.9. An arbitrary Artin F-motive A becomes over $\bar{F}$ a finite direct sum of several copies of $\mathbb{F}$; the number of the copies is the *rank* $\mathrm{rk}(A)$ of A:

Definition 1.8. Let M be an F-motive decomposing over $\bar{F}$ in a finite direct sum of Tate motives. The *rank* $\mathrm{rk}(M)$ of M is the number of summands in this decomposition.

Lemma 1.9. *The following properties of an Artin motive A are equivalent:*

(1) *A is invertible;*
(2) *$A \otimes B \simeq \mathbb{F}$ for some Artin motive B;*
(3) *$A \otimes A^* \simeq \mathbb{F}$;*
(4) $\mathrm{rk}(A) = 1$.

Proof. The implications (3) ⇒ (2) ⇒ (1) are trivial. Assuming (1), we get that over $\bar{F}$ the indecomposable motive $\mathbb{F}$ is a direct sum of $\operatorname{rk}(A)$ copies of B implying (4). Finally, under the equivalence of [2, §7], a rank 1 Artin motive A corresponds to a 1-dimensional (over $\mathbb{F}$) module over the group ring $\mathbb{F}[\Gamma_F]$ of the absolute Galois group Γ_F of the field F. The tensor product of this module with its dual yields a 1-dimensional $\mathbb{F}[\Gamma_F]$-module with the trivial Γ_F-action, corresponding to the Tate motive $\mathbb{F}$. Consequently, (4) ⇒ (3). □

2. The proof

The proof of Theorem 1.6 is based on

Theorem 2.1 ([4, Theorem 6.3 with Remarks 5.14 and 6.6])**.** *Every summand in the complete motivic decomposition of a projective homogeneous variety X under a p'-inner p-consistent group G is a shift of $U(Y) \otimes A$, where Y is a projective G-homogeneous variety with $X(F(Y)) \neq \emptyset$ and A is an Artin motive.*

Remark 2.2. The Artin motives A, showing up in Theorem 2.1, are all of the following kind. Let E/F be a minimal field extension such that the group G_E is of inner type. The extension E/F is known to be finite Galois. For any given A, there is a subextension L/F in E/F such that A is an indecomposable direct summand in $M(L)^F$ – the F-motive of the F-variety $\operatorname{Spec} L$.

Let $\mathrm{ChM}_{\mathrm{eff}} \subset \mathrm{ChM}$ be the full subcategory of *effective* Chow motives. Its objects are direct summands in the motives of smooth projective varieties. This subcategory is stable under positive shifts.

Another ingredient of the proof of Theorem 1.6, described below, is a (commuting with tensor products) *retraction*

$$\mathbf{m}\colon \mathrm{ChM}_{\mathrm{eff}} \to \mathrm{AM}$$

of ChM onto its full subcategory AM of Artin motives, constructed in [4, §4]. It is a retraction in the sense that its restriction to the subcategory $\mathrm{AM} \subset \mathrm{ChM}_{\mathrm{eff}}$ yields the identity. It satisfies

$$\mathbf{m}(U(X)) \simeq \mathbb{F} \qquad (2.3)$$

for any projective homogeneous variety X.

Proof of Theorem 1.6. Let $r = r(X)$ be the rank of the semisimple anisotropic kernel of $G_{F(X)}$. We induct on r.

The motive $U(X)^*(d_X)$, where $d_X := \dim X$, is an indecomposable summand of

$$M(X)^*(d_X) = M(X).$$

Therefore by Theorem 2.1

$$U(X)^*(d_X) \simeq U(Y)(n) \otimes A \qquad (2.4)$$

for a projective G-homogeneous variety Y with $X(F(Y)) \neq \emptyset$, an Artin motive A, and an integer n.

The smallest i for which $\mathbb{F}(i)$ is a summand of the left-hand side in (2.4) over $\bar{F}$, is $d_X - d$, where $d = \dim U(X)$. Since A over $\bar{F}$ is a sum of several copies of $\mathbb{F}$, the similar integer for the right-hand side of (2.4) is n. It follows that $n = d_X - d$ and so (2.4) reads as

$$U(X)^* \simeq U(Y)(-d) \otimes A. \tag{2.5}$$

By (1.3), the dimension of the motive $U(Y)$ also equals $d = \dim U(X)$.

If besides of $X(F(Y)) \neq \emptyset$ we also have $Y(F(X)) \neq \emptyset$, the motives $U(X)$ and $U(Y)$ are isomorphic by [8, Corollary 2.15] and so

$$U(X)^* \simeq U(X)(-d) \otimes A. \tag{2.6}$$

Dualizing (2.6), we obtain $U(X) \simeq U(X)^*(d) \otimes A^*$. Substituting (2.6) into this formula, we come to the isomorphism

$$U(X) \simeq U(X) \otimes A \otimes A^*,$$

and the retraction $\mathbf{m}$ applied to it (with (2.3) taken into account) yields $\mathbb{F} \simeq A \otimes A^*$.

If $Y(F(X)) = \emptyset$, the rank of the semisimple anisotropic kernel of $G_{F(Y)}$ is smaller than r, and, by the induction hypothesis, we have

$$U(Y)^* \simeq U(Y)(-d) \otimes B \tag{2.7}$$

for an invertible Artin motive B. Dualizing (2.5) and substituting (2.7), we see that

$$U(X) \simeq U(Y) \otimes B \otimes A^*.$$

Applying $\mathbf{m}$, we get $\mathbb{F} \simeq B \otimes A^*$ demonstrating that $U(X) \simeq U(Y)$ and $A \simeq B$. Thus (2.7) is the desired isomorphism. □

Remark 2.8. The Artin motive A in the statement of Theorem 1.6 is as in Remark 2.2.

3. Examples

It would be interesting to find examples for Theorem 1.6 with nontrivial twist. This would provide counter-examples for possible extensions of the untwisted duality result [7, Proposition 5.2].

On the other hand, there is a lot of situations where the (a priori) twisted duality of Theorem 1.6 yields the (stronger) untwisted duality. Below we list some situations of that kind.

3.1. $G = R_{L/F}(H_L)$**.** Let H be a reductive group of inner type over F and let L/F be a subextension of a finite Galois field extension of degree prime to p. We set $G := R_{L/F}(H_L)$ and $X := R_{L/F}(Y_L)$, where Y is a projective homogeneous variety under H. Then the upper motives of X and of Y are isomorphic implying that $U(X)$ satisfies the untwisted duality.

3.2. $p = 2$**.** Assume that $p = 2$. In this case, the Tate motive $\mathbb{F}$ is the only invertible Artin motive. Indeed, any invertible Artin motive if given by some 1-dimensional $\mathbb{F}[\Gamma_F]$-module. Since $p = 2$, such a module has no non-trivial $\mathbb{F}$-linear automorphisms and so has the trivial Γ_F-action.

It follows for any $2'$-inner group G that the upper motive of any projective G-homogenous variety satisfies the untwisted duality.

3.3. $[E : F] = 2$**.** Assume that p is odd and a p'-inner p-consistent group G acquires inner type over a *quadratic* Galois extension E/F. Assume furthermore that the *rank* of the motive $U(X)$ (see Definition 1.8) is odd (e.g., equal to p). Then, as shown right below, $U(X)$ satisfies the untwisted duality.

For the sake of contradiction, let us assume that $U(X)$ satisfies the twisted duality with an invertible Artin motive $A \not\simeq \mathbb{F}$. Note that A is uniquely determined by the complete decomposition $\mathbb{F} \oplus A$ of $M(E)^F$.

Let Y be the variety of Borel subgroups for G. Over the function field $F(Y)$, the motive $U(X)$ becomes a direct sum of shifts of n copies of $\mathbb{F}$ and of m copies of A. The dual of $U(X)$ as well as any shift of $U(X)$ is also a direct sum of shifts of the same number of copies of of $\mathbb{F}$ and of A. However, since $\mathbb{F} \otimes A \simeq A$ and $A \otimes A \simeq \mathbb{F}$, tensoring with A exchanges n and m. The twisted duality for $U(X)$ therefore implies that $n = m$ in which case the rank $n + m$ of $U(X)$ is even.

3.4. ${}^2\mathsf{E}_6$**.** If a reductive group G becomes quasi-split over a finite base field extension of degree prime to p, then the upper motive of any projective G-homogeneous variety is isomorphic to the Tate motive $\mathbb{F}$ satisfying the untwisted duality. It follows that $p = 3$ is our only prime of interest in the case where the group G is absolutely simple adjoint of type ${}^2\mathsf{E}_6$.

Let E/F be the quadratic Galois field extension such that G_E is of inner type and let X be a projective G-homogeneous variety. By [4, Lemma 5.10], the upper F-motive $U(X)$ remains indecomposable over E so that $U(X)_E \simeq U(X_E)$.

The possible Tits 3-indexes of G_E, listed in [3, Table 4], show that $U(X)$ is isomorphic to $\mathbb{F}$ or to $U(X_2)$ or to $U(B)$, where B is the variety of Borel subgroups whereas X_2 is the variety of maximal parabolic subgroups in G corresponding to the 2nd vertex of its Dynkin diagram.

Possible Poicaré polynomials (and, in particular, dimensions and ranks) of $U(X)$ are determined in [6, Table 2]. If the rank of $U(X)$ is odd, then by §3.3, $U(X)$ satisfies the untwisted duality.

We claim that $U(X)$ satisfies the untwisted duality even when its rank is even. Indeed, if the rank of $U(X)$ is even, then $U(X) \simeq U(X_2)$ (so that we may assume $X = X_2$) and $\dim U(X) = 20$ whereas $\dim X = 21$ (see [6, Tables 1 and 2]). Assume that $U(X)$ satisfies the twisted duality with an Artin motive A whose isomorphism class is determined by $M(E)^F \simeq \mathbb{F} \oplus A$. Then $U(X)(1) \otimes A \hookrightarrow M(X)$, where the sign $\hookrightarrow$ means that the former motive is a direct summand in the latter. Since $\mathbb{F} \hookrightarrow U(X)_{F(B)}$, we get that $A(1) \hookrightarrow M(X)$ over the function field $F(B)$. It follows then by [1, Theorem 7.5] that $M(E)^F(1) \hookrightarrow M(X)$ over $F(B)$, implying that $\dim_{\mathbb{F}} \mathrm{Ch}^1(\bar{X}) \geq 2$, where $\bar{X}$ is X over a separable closure of F. However it is known (see, e.g., [6, §8]) that $\dim_{\mathbb{F}} \mathrm{Ch}^1(\bar{X}) = 1$. This contradiction proves the above claim.

3.5. ${}^2\mathsf{A}_n$**.** Let G be an adjoint absolutely simple group of type ${}^2\mathsf{A}_n$ (with $n \geq 1$) over F, and let X be a projective G-homogeneous variety. By [9, §2], the upper motive $U(X)$ satisfies the untwisted duality.

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