# TWISTED POINCARÉ DUALITY
# FOR UPPER MOTIVES

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ABSTRACT. Extending to a wider class of reductive groups an earlier result on Poincaré duality for upper motives, we obtain a modified version of the duality, involving a twist by an invertible Artin motive. The proof is based on a recent joint work with C. De Clercq and A. Quéguiner-Mathieu.

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## 1. THE STATEMENT

Let $p$ be a prime number and let $\mathrm{CM} = \mathrm{CM}(F, \mathbb{F})$ be the category of *Chow motives* over a field $F$ (of any characteristic) with coefficients in $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$ as defined, e.g., in [5, §64]. We write $M(X) \in \mathrm{CM}$ for the motive of a smooth projective $F$-variety $X$ and set

$$\mathbb{F} := M(F) := M(\operatorname{Spec} F).$$

(Traditionally, the coefficient ring notation is also used as notation for $M(F)$.) For $M \in \mathrm{CM}$ and $i \in \mathbb{Z}$, $M\{i\}$ is the $i$th (Tate) shift of $M = M\{0\}$. The category CM is equipped with a commutative tensor product, induced by the direct product of varieties, satisfying

$$\mathbb{F}\{i\} \otimes \mathbb{F}\{i'\} = \mathbb{F}\{i + j\} \quad \text{and} \quad M\{i\} = M \otimes \mathbb{F}\{i\}.$$

The groups

$$\mathrm{Ch}^i(M) := \mathrm{Hom}(M, \mathbb{F}\{i\}) \quad \text{and} \quad \mathrm{Ch}_i(M) := \mathrm{Hom}(\mathbb{F}\{i\}, M)$$

*Date*: 15 Dec 2024.
*Key words and phrases.* Affine algebraic groups; projective homogeneous varieties; Chow rings and motives. *Mathematical Subject Classification (2020):* 20G15; 14C25.

are the $i$th cohomological and homological Chow groups of $M$. If $M = M(X)$, these groups coincide with the usual Chow groups $\mathrm{Ch}^i(X)$ and $\mathrm{Ch}_i(X)$ of $X$ (with coefficients in $\mathbb{F}$), vanishing for $i < 0$.

The duality cofunctor $\mathrm{CM} \to \mathrm{CM}$, $M \mapsto M^*$ of [5, §65] is a self-inverse anti-equivalence of additive categories commuting with the tensor product and inverting the shifts. The motive of every irreducible variety $X$ satisfies the *Poincaré duality*

$$M(X)^* = M(X)\{-d\} \text{ with } d := \dim X, \tag{1.1}$$

which shows up on the level of Chow groups as $\mathrm{Ch}_i(X) = \mathrm{Ch}^{d-i}(X)$ for any $i$.

For a reductive algebraic $F$-group $G$ and a projective $G$-homogeneous $F$-variety $X$, any direct summand in the motive $M(X)$ possesses a decomposition into a finite direct sum of indecomposable motives. Such a decomposition, called *complete* below, is unique in the usual sense, [2] (see also [9, Corollary 2.6]).

The *upper motive* $U(X) \in \mathrm{CM}$, defined in [9, §2.II] (under the name of *indecomposable upper motive*), is the summand in a complete motivic decomposition of $X$ with $\mathrm{Ch}^0(U(X)) = \mathbb{F}$. The motive $U(X)$ is determined by $X$ up to a canonical isomorphism: by [9, Lemma 2.8], a projector $\pi \in \mathrm{End}(M(X))$ which yields an upper motive of $X$ has *multiplicity* $1 \in \mathbb{F}$ as a correspondence $X \rightsquigarrow X$; the composition of two such projectors yields an isomorphism between the corresponding upper motives. By the isomorphism criterion [9, Corollary 2.15], given one more projective homogeneous variety $X'$ (possibly under a different reductive group), the motives $U(X)$ and $U(X')$ are isomorphic if and only if each of the varieties $X_{F(X')}$ and $X'_{F(X)}$ possesses a closed point of prime to $p$ degree.

Over a separable closure $\bar{F}$ of $F$, the motive $M(X)$ and, in particular, the motive $U(X)$, decompose into direct sums of *Tate motives* $\mathbb{F}\{i\}$ with various $i \geq 0$ including $i = 0$ (appearing exactly once). Dimension $\dim U(X)$ is defined as the maximal $i$ appearing for $U(X)$. More generally, for an arbitrary motive $M \in \mathrm{CM}$ decomposing over $\bar{F}$ into a direct sum of Tate motives, its *dimension* $\dim M$ is defined as the maximum of the distance $|i - j|$ between $i$ and $j$ for all $\mathbb{F}\{i\}$ and $\mathbb{F}\{j\}$ which are summands of $M$ over $\bar{F}$. Clearly,

$$\dim M = \dim M^* = \dim M\{i\} \tag{1.2}$$

for such $M$ and any $i \in \mathbb{Z}$.

It has been shown in [8, Proposition 5.2] that $U(X)$ satisfies the Poincaré duality isomorphism

$$U(X)^* \simeq U(X)\{-d\} \text{ with } d := \dim U(X) \tag{1.3}$$

provided that the group $G$ is of inner type or, more generally, of $p$-*inner* type, i.e., becomes of inner type over a finite base field extension of a $p$-power degree.

Note that for any $i$, relation (1.1) implies that the number of the summand $\mathbb{F}\{i\}$, appearing in the complete decomposition of $M(X)$ over $\bar{F}$, coincides with the number of $\mathbb{F}\{d - i\}$. The similar property for $U(X)$, implied by (1.3), is a bit of surprise.

Theorem 1.4 below is an analogue of (1.3) for $p'$-*inner reductive* $G$. A reductive algebraic group $G$ over $F$ is called $p'$-inner (cf. [4]), if it acquires inner type over a finite base field extension of degree prime to $p$, and the *higher Tits $p$-indexes* of $G$ (defined as in [3]) are invariant under the $*$-action of the absolute Galois group of $F$ on the Dynkin diagram of

$G$. The class of $p'$-inner groups includes the group of inner type and the groups becoming quasi-split over a finite base field extension of degree prime to $p$. Moreover, it is stable under taking the direct products of groups and the operation $G \mapsto R_{L/F}(G_L)$, where $R_{L/F}$ is the Weil transfer for a subextension $L/F$ of a finite Galois field extension of degree prime to $p$. Every absolutely simple group of type different from ${}^6D_4$ (with $p = 2, 3$) and ${}^3D_4$ (with $p = 2$) is $p'$-inner or $p$-inner.

**Theorem 1.4.** *The upper motive $U(X)$ of any projective homogeneous variety $X$ under a $p'$-inner reductive algebraic group $G$ satisfies a* twisted *Poincaré duality*

$$U(X)^* \simeq U(X)\{-d\} \otimes A,$$

*where $A$ is an invertible Artin motive.*

By definition, an *Artin motive* is a motive isomorphic to a direct summand in the spectrum of an étale $F$-algebra. An arbitrary motive $A \in \mathrm{CM}$ is *invertible* if $A \otimes B \simeq \mathbb{F}$ for some motive $B \in \mathrm{CM}$. Invertible Artin motives are characterized below in Lemma 1.5. An arbitrary Artin $F$-motive $A$ becomes over $\bar{F}$ a finite direct sum of several copies of $\mathbb{F}$; the number of the copies is the *rank* $\mathrm{rk}(A)$ of $A$.

**Lemma 1.5.** *The following properties of an Artin motive $A$ are equivalent:*

(1) *$A$ is invertible;*
(2) *$A \otimes B \simeq \mathbb{F}$ for some Artin motive $B$;*
(3) *$A \otimes A^* \simeq \mathbb{F}$;*
(4) $\mathrm{rk}(A) = 1$.

*Proof.* The implications (3) ⇒ (2) ⇒ (1) are trivial. Assuming (1), we get that over $\bar{F}$ the indecomposable motive $\mathbb{F}$ is a direct sum of $\mathrm{rk}(A)$ copies of $B$ implying (4). Finally, under the equivalence of [2, §7], a rank 1 Artin motive $A$ corresponds to a 1-dimensional (over $\mathbb{F}$) module over the group ring $\mathbb{F}[\Gamma_F]$ of the absolute Galois group $\Gamma_F$ of the field $F$. The tensor product of this module with its dual yields a 1-dimensional $\mathbb{F}[\Gamma_F]$-module with the trivial $\Gamma_F$-action, corresponding to the Tate motive $\mathbb{F}$. Consequently, (4) ⇒ (3). □

As a consequence of Theorem 1.4, we have

$$\mathrm{Ch}_i(U(X)) \simeq \mathrm{Ch}^{d-i}(U(X) \otimes A) \quad \text{and} \quad \mathrm{Ch}_i(U(X) \otimes A) \simeq \mathrm{Ch}^{d-i}(U(X))$$

for any $i$; in particular, $\mathrm{Ch}_i\, U(X) = 0 = \mathrm{Ch}^i\, U(X)$ for $i > d$.

## 2. The proof

The proof of Theorem 1.4 is based on

**Theorem 2.1** ([4])**.** *Every summand in the complete motivic decomposition of a projective homogeneous variety $X$ under a $p'$-inner group $G$ is a shift of $U(Y) \otimes A$, where $Y$ is a projective $G$-homogeneous variety with $X(F(Y)) \neq \emptyset$ and $A$ is an Artin motive.*

**Remark 2.2.** The Artin motives $A$, showing up in Theorem 2.1, are all of the following kind. Let $E/F$ be a minimal field extension such that the group $G_E$ is of inner type. The extension $E/F$ is known to be finite Galois. For any given $A$, there is a subextension $L/F$ in $E/F$ such that $A$ is an indecomposable direct summand in $M(L)^F$ – the $F$-motive of the $F$-variety $\mathrm{Spec}\, L$.

Let $\mathrm{CM}_{\mathrm{eff}} \subset \mathrm{CM}$ be the full subcategory of *effective* Chow motives. Its objects are direct summands in the motives of smooth projective varieties. This subcategory is stable under positive shifts.

Another ingredient of the proof of Theorem 1.4, described below, is a (commuting with tensor products) *retraction*

$$\mathbf{m}\colon \mathrm{CM}_{\mathrm{eff}} \to \mathrm{AM}$$

of CM onto its full subcategory AM of Artin motives, constructed in [4]. It is a retraction in the sense that its restriction to the subcategory $\mathrm{AM} \subset \mathrm{CM}_{\mathrm{eff}}$ yields the identity. It satisfies

$$\mathbf{m}(U(X)) \simeq \mathbb{F} \tag{2.3}$$

for any projective homogeneous variety $X$.

*Proof of Theorem 1.4.* Let $r = r(X)$ be the rank of the semisimple anisotropic kernel of $G_{F(X)}$. We induct on $r$.

The motive $U(X)^*\{d_X\}$, where $d_X := \dim X$, is an indecomposable summand of $M(X)^*\{d_X\} = M(X)$. Therefore by Theorem 2.1

$$U(X)^*\{d_X\} \simeq U(Y)\{n\} \otimes A \tag{2.4}$$

for a projective $G$-homogeneous variety $Y$ with $X(F(Y)) \neq \emptyset$, an Artin motive $A$, and an integer $n$.

The smallest $i$ for which $\mathbb{F}\{i\}$ is a summand of the left-hand side in (2.4) over $\bar{F}$, is $d_X - d$. Since $A$ over $\bar{F}$ is a sum of several copies of $\mathbb{F}$, the similar integer for the right-hand side of (2.4) is $n$. It follows that $n = d_X - d$ and so (2.4) reads as

$$U(X)^* \simeq U(Y)\{-d\} \otimes A. \tag{2.5}$$

By (1.2), the dimension of the motive $U(Y)$ also equals $d = \dim U(X)$.

If besides of $X(F(Y)) \neq \emptyset$ we also have $Y(F(X)) \neq \emptyset$, the motives $U(X)$ and $U(Y)$ are isomorphic by [9, Corollary 2.15] and so

$$U(X)^* \simeq U(X)\{-d\} \otimes A. \tag{2.6}$$

Dualizing (2.6), we obtain $U(X) \simeq U(X)^*\{d\} \otimes A^*$. Substituting (2.6) into this formula, we come to the isomorphism

$$U(X) \simeq U(X) \otimes A \otimes A^*,$$

and the retraction $\mathbf{m}$ applied to it (with (2.3) taken into account) yields $\mathbb{F} \simeq A \otimes A^*$.

If $Y(F(X)) = \emptyset$, the rank of the semisimple anisotropic kernel of $G_{F(Y)}$ is smaller than $r$, and, by the induction hypothesis, we have

$$U(Y)^* \simeq U(Y)\{-d\} \otimes B \tag{2.7}$$

for an invertible Artin motive $B$. Dualizing (2.5) and substituting (2.7), we see that

$$U(X) \simeq U(Y) \otimes B \otimes A^*.$$

Applying $\mathbf{m}$, we get $\mathbb{F} \simeq B \otimes A^*$ demonstrating that $U(X) \simeq U(Y)$ and $A \simeq B$. Thus (2.7) is the desired isomorphism. □

**Remark 2.8.** It follows from Remark 2.2 and the proof of Theorem 1.4 that the Artin motive $A$ in the statement of Theorem 1.4 is as in Remark 2.2.

## 3. Examples

It would be interesting to find examples for Theorem 1.4 with nontrivial twist. This would provide counter-examples for possible extensions of the untwisted duality result [8, Proposition 5.2] which could be viewed as "negative" applications of the theorem.

On the other hand, there is a lot of situations where the (a priori) twisted duality of Theorem 1.4 yields the (stronger) untwisted duality providing "positive" applications of the theorem. Below we list some situations of that kind.

3.1. $G = R_{L/F}(H_L)$**.** Let $H$ be a reductive group of inner type over $F$ and let $L/F$ be a subextension of a finite Galois field extension of degree prime to $p$. We set $G := R_{L/F}(H_L)$ and $X := R_{L/F}(Y_L)$, where $Y$ is a projective homogeneous variety under $H$. Then the upper motives of $X$ and of $Y$ are isomorphic implying that $U(X)$ satisfies the untwisted duality.

3.2. $p = 2$**.** Assume that $p = 2$. In this case, the Tate motive $\mathbb{F}$ is the only invertible Artin motive. Indeed, any invertible Artin motive if given by some 1-dimensional $\mathbb{F}[\Gamma_F]$-module. Since $p = 2$, such a module has no non-trivial $\mathbb{F}$-linear automorphisms and so has the trivial $\Gamma_F$-action.

It follows for any $2'$-inner group $G$ that the upper motive of any projective $G$-homogenous variety satisfies the untwisted duality.

3.3. $[E : F] = 2$**.** Assume that $p$ is odd and a $p'$-inner group $G$ acquires inner type over a *quadratic* Galois extension $E/F$. Assume furthermore that the *rank* of the motive $U(X)$ (i.e., the number of Tate summands in the complete decomposition of the motive over a separable closure of $F$) is odd (e.g., equal to $p$). Then, as shown right below, $U(X)$ satisfies the untwisted duality.

For the sake of contradiction, let us assume that $U(X)$ satisfies the twisted duality with an invertible Artin motive $A \not\simeq \mathbb{F}$. Note that $A$ is uniquely determined by the complete decomposition $\mathbb{F} \oplus A$ of $M(E)^F$.

Let $Y$ be the variety of Borel subgroups for $G$. Over the function field $F(Y)$, the motive $U(X)$ becomes a direct sum of shifts of $n$ copies of $\mathbb{F}$ and of $m$ copies of $A$. The dual of $U(X)$ as well as any shift of $U(X)$ is also a direct sum of shifts of the same number of copies of of $\mathbb{F}$ and of $A$. However, since $\mathbb{F} \otimes A \simeq A$ and $A \otimes A \simeq \mathbb{F}$, tensoring with $A$ exchanges $n$ and $m$. The twisted duality for $U(X)$ therefore implies that $n = m$ in which case the rank $n + m$ of $U(X)$ is even.

3.4. ${}^2\mathsf{E}_6$**.** If a reductive group $G$ becomes quasi-split over a finite base field extension of degree prime to $p$, then the upper motive of any projective $G$-homogeneous variety is isomorphic to the Tate motive $\mathbb{F}$ satisfying the untwisted duality. It follows that $p = 3$ is our only prime of interest in the case where the group $G$ is absolutely simple adjoint of type ${}^2\mathsf{E}_6$.

Let $E/F$ be the quadratic Galois field extension such that $G_E$ is of inner type and let $X$ be a projective $G$-homogeneous variety. By [4], the upper $F$-motive $U(X)$ remains indecomposable over $E$ so that $U(X)_E \simeq U(X_E)$.

The possible Tits 3-indexes of $G_E$, listed in [3, Table 4], show that $U(X)$ is isomorphic to $\mathbb{F}$ or to $U(X_2)$ or to $U(B)$, where $B$ is the variety of Borel subgroups whereas $X_2$ is the variety of maximal parabolic subgroups in $G$ corresponding to the 2nd vertex of its Dynkin diagram.

Possible Poicaré polynomials (and, in particular, dimensions and ranks) of $U(X)$ are determined in [6, Table 2]. If the rank of $U(X)$ is odd, then by §3.3, $U(X)$ satisfies the untwisted duality.

We claim that $U(X)$ satisfies the untwisted duality even when its rank is even. Indeed, if the rank of $U(X)$ is even, then $U(X) \simeq U(X_2)$ (so that we may assume $X = X_2$) and $\dim U(X) = 20$ whereas $\dim X = 21$ (see [6, Tables 1 and 2]). Assume that $U(X)$ satisfies the twisted duality with an Artin motive $A$ whose isomorphism class is determined by $M(E)^F \simeq \mathbb{F} \oplus A$. Then $U(X)\{1\} \otimes A \hookrightarrow M(X)$, where the sign $\hookrightarrow$ means that the former motive is a direct summand in the latter. Since $\mathbb{F} \hookrightarrow U(X)_{F(B)}$, we get that $A\{1\} \hookrightarrow M(X)$ over the function field $F(B)$. It follows then by [1, Theorem 7.5] that $M(E)^F\{1\} \hookrightarrow M(X)$ over $F(B)$, implying that $\dim_{\mathbb{F}} \mathrm{Ch}^1(\bar{X}) \geq 2$, where $\bar{X}$ is $X$ over a separable closure of $F$. However it is known (see, e.g., [6, §8]) that $\dim_{\mathbb{F}} \mathrm{Ch}^1(\bar{X}) = 1$. This contradiction proves the above claim.

3.5. $^2\mathsf{A}_n$. Given any $n \geq 2$, any adjoint absolutely simple group $G$ of type $^2\mathsf{A}_n$ over $F$ is realized as the automorphism group $\mathrm{Aut}_E(C, \tau)$, where $E/F$ is a quadratic Galois field extension and $C$ a degree $n+1$ central simple $E$-algebra with a unitary $F$-involution $\tau$. The primes $p$ of interest here are the odd prime divisors of $n+1$.

By [4] once again, for any projective $G$-homogeneous variety $X$, the upper motive $U(X)$ remains indecomposable over $E$ so that $U(X)_E \simeq U(X_E)$. In particular, the motives $U(X)$ and $U(X_E)$ are of the same rank.

The group $G_E = \mathrm{Aut}_E(C)$ is of the inner type $^1\mathsf{A}_n$. Let $p^r$ be the maximal $p$-power dividing the index of $C$ and let $D$ be the degree $p^r$ central division $E$-algebra given by the $p$-primary part of $C$. The motive $U(X_E)$ is then isomorphic to $U(X_i)$ for some $i = 0, \dots, r$, where $X_i$ is the generalized Severi-Brauer variety of rank $p^i$ right ideals in $D$.

There is no apparent reason for the rank of $U(X_i)$ to be always odd (for odd $p$), but it is odd in all the cases where it is computed. By §3.3, the corresponding motive $U(X)$ satisfies the untwisted duality in these cases. Below is the list.

First of all, $U(X_r) = \mathbb{F}$ has the odd rank 1.

Next, by [9, Corollary 2.22] (see also [7, Theorem 2.2.1]), the motive $U(X_0)$ is the entire $M(X_0)$ which has the odd rank $p^r$.

Finally, [10, §4] describes a way to compute the rank of $U(X_1)$ for any $p$ and $r$. However, the final answer is worked out for $p = 3 \geq r \geq 2$ only. The rank of $U(X_1)$ turns out to be odd in the both new cases. Note that for any $p$ and $r$, the entire motive $M(X_1)$ is of the rank $\binom{p^r}{p}$; in the case $p = 3$, this binomial coefficient is even for $r = 2$ and odd for $r = 3$. By [9, Theorem 2.8 and 4.1], the direct summand in $M(X_1)$ complementary to $U(X_1)$ is a direct sum of shifts of several copies of $M(X_0)$, each of odd rank; by [10, Examples 4.1 and 4.8], in the case $p = 3$, the number of the copies is odd for $r = 2$ and even for $r = 3$.

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