# CONSISTENT VARIETIES AND THEIR COMPLETE MOTIVIC DECOMPOSITIONS

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Abstract. Given a reductive algebraic group $G$, we introduce a notion of *consistent* projective $G$-homogeneous variety $X$. For instance, the variety of Borel subgroups in $G$ is consistent; if $G$ is of inner type, all projective $G$-homogenous varieties are consistent.

Our main result describes the summands in the complete motivic decomposition of $X$. It extends an earlier result of the author providing the same for $G$ of inner type.

## Contents



## 1. Preliminaries

We fix a prime number $p$ and work with the category

$$\mathrm{CM}(F) = \mathrm{CM}(F, \mathbb{F})$$

of *effective* Chow motives over an arbitrary field $F$ with coefficients in $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$. Given a smooth projective $F$-variety $X$, we write $M(X)$ for its motive. The motive of $X$ decomposes into the direct sum

$$M(X) = M(X_1) \oplus \cdots \oplus M(X_n),$$

where $X_1, \ldots, X_n$ are the connected components of $X$. An arbitrary object $M \in \mathrm{CM}(F)$ is a direct summand in $M(X)$ for some $X$. For any non-negative integer $i \geq 0$, the $i$th Tate shift $M\{i\} \in \mathrm{CM}(F)$ of $M$ is defined.

The category $\mathrm{CM}(F)$ can be defined as the full subcategory in the category of all (including non-effective) Chow motives over $F$ with coefficients in $\mathbb{F}$, described in [5, §64], consisting of the motives isomorphic to a direct summand in the motive of a smooth projective variety.

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We write $M(F)$ or simply $\mathbb{F}$ for the motive of the base point $\operatorname{Spec} F$ and refer as a *Tate motive* to any its Tate shift $M(F)\{i\} = \mathbb{F}\{i\}$, $i \geq 0$. A commutative tensor product $\otimes$ of motives is defined; it satisfies

$$\mathbb{F}\{i\} \otimes \mathbb{F}\{1\} = \mathbb{F}\{i+1\}, \quad M \otimes \mathbb{F}\{i\} = M\{i\}, \quad \text{and} \quad M(X) \otimes M(X') = M(X \times X').$$

Given an arbitrary field extension $L/F$ and an $F$-variety $X$, we write $X_L$ for the corresponding $L$-variety. Similarly, given an $F$-motive $M \in \mathrm{CM}(F)$, we write $M_L \in \mathrm{CM}(L)$ for the corresponding $L$-motive. Note that the restriction functor

$$\mathrm{CM}(F) \to \mathrm{CM}(L), \quad M \mapsto M_L$$

is additive; it commutes with the Tate shift and the tensor product.

If the field extension $L/F$ is finite and separable, there also is an additive and Tate corestriction functor

$$\mathrm{CM}(L) \to \mathrm{CM}(F), \quad M \mapsto M^F$$

(see [6, §3]), associating to the motive of an $L$-variety $X$ the motive of the $F$-variety $X^F$ given by the underlying scheme of $X$ equipped with the composition

$$X \to \operatorname{Spec} L \to \operatorname{Spec} F$$

as the structure morphism.

The *Artin motives* in $\mathrm{CM}(F)$ are defined as the direct summands in the motives of 0-*dimensional varieties* (i.e., the spectra of étale $F$-algebras). The simplest example of an Artin motive is the Tate motive $M(F)$. A more general example is the motive $M(L)^F$ for a finite separable field extension $L/F$.

Let us write $\mathrm{AM}(F) \subset \mathrm{CM}(F)$ for the full subcategory of Artin motives. There is a useful additive tensor *retraction functor*

$$\mathbf{m}\colon \mathrm{CM}(F) \to \mathrm{AM}(F),$$

associating to $M(X)$ the motive of its étale $F$-algebra of constants, see [4, §4] for details. The embedding of $\mathrm{AM}(F)$ into $\mathrm{CM}(F)$ followed by $\mathbf{m}$ yields the identity. For any $M$ and any *positive* $i$, one has $\mathbf{m}(M\{i\}) = 0$.

Following the terminology of [4], a motive $M \in \mathrm{CM}(F)$ satisfying $\mathbf{m}(M) \neq 0$ is called *sustainable*.

## 2. Apper Motives: Motivation, Background, Known Results

Our main object of interest here is the motive of a projective homogeneous variety $X$ under an action of a reductive group $G$ over a field $F$. Since the center of $G$ acts trivially on $X$, one may assume that $G$ is semisimple and adjoint.

We would like to describe the summands of the complete motivic decomposition of $X$. During the work under this problem, projective $G_L$-homogeneous varieties for finite separable field extensions $L/F$ naturally appear. Viewed as $F$-varieties via the corestriction functor, they are *projective quasi-homogeneous* in the sense of [1]. Without affecting most of their nice properties, this definition can be slightly relaxed by allowing $G$ to be defined over $L$, not over $F$. We mostly work with connected varieties, but we allow non-connectedness to obtain a class stable under base field extensions.

So, following [6, §2], a smooth projective $F$-variety $X$ is called *projective quasi-homogeneous* here, if every connected component of $X$ is isomorphic to $Y^F$ for some finite separable field extension $L/F$ and some projective homogeneous $L$-variety $Y$ (under some reductive algebraic group over $L$ which may vary with the component). We write $\mathrm{CM}_{\mathrm{pqh}}(F)$ for the full subcategory in $\mathrm{CM}(F)$ consisting of the direct summands in the motives of projective quasi-homogeneous varieties. The category $\mathrm{CM}_{\mathrm{pqh}}(F)$ is closed under the Tate shifts and tensor products; it contains the Artin motives. As explained in [6, Corollary 2.2] (based on [7, Corollary 2.6] which is based on [3, §9]), every object of $\mathrm{CM}_{\mathrm{pqh}}(F)$ has the *Krull-Schmidt property*: it decomposes into a finite direct sum of indecomposable objects, and such a decomposition (called *complete*) is unique in the usual sense.

As an attempt to describe the indecomposable objects in $\mathrm{CM}_{\mathrm{pqh}}(F)$, the following definition is given in [4]. An *$A$-upper* (or here, for short, an *apper*) motive is a sustainable indecomposable object of $\mathrm{CM}_{\mathrm{pqh}}(F)$. We say that such $U$ is an apper motive of a projective quasi-homogeneous $F$-variety $Y$, if it is isomorphic to a summand in $M(Y)$.

The sustainability condition $\mathbf{m}(U) \neq 0$ is equivalent to the condition $\mathrm{Ch}^0(\bar{U}) \neq 0$, where $\bar{U}$ is $U$ over a separable closure of $F$ and where $\mathrm{Ch}^0$ stands for the codimension 0 component of the Chow group Ch with coefficients in $\mathbb{F}$:

$$\mathrm{Ch}^0(U) := \mathrm{Hom}(U, \mathbb{F}).$$

Note that $Y$ in the definition of $U$ can be assumed to be connected. Also note that replacing the condition $\mathrm{Ch}^0(\bar{U}) \neq 0$ by the stronger one – $\mathrm{Ch}^0(U) \neq 0$, delivers in the case of connected $Y$ an older notion of the *upper* motive $U(Y)$ of $Y$. Unlike $U$, the motive $U(Y)$ is determined by $Y$.

**Example 2.1.** Assume that $p$ is odd, let $L/F$ be a separable quadratic field extension, and let $Y$ be the projective quasi-homogeneous $F$-variety $(\operatorname{Spec} L)^F$. The complete decomposition of the motive of $Y$ consists of two summands: the Tate motive $\mathbb{F}$ and a certain indecomposable Artin motive $A \not\simeq \mathbb{F}$, a twisted form of $\mathbb{F}$ (see [4, Example 3.1] for details). Both are apper motives of $Y$ whereas the upper motive of $Y$ is $\mathbb{F}$. In particular, $\mathrm{Ch}^0(A) = 0$ whereas $\mathrm{Ch}^0(\bar{A}) \neq 0$.

Let us recall the isomorphism criterion [7, Corollary 2.15] for the upper motives of two connected quasi-homogeneous varieties $Y$ and $Y'$: $U(Y) \simeq U(Y')$ if and only if $Y$ and $Y'$ are *equivalent*. To define the equivalence, we say that $Y$ *dominates* (or *$p$-dominates*) $Y'$, if $Y'$ acquires over the function field of $Y$ a 0-cycle of degree $1 \in \mathbb{F}$. We call $Y$ and $Y'$ *equivalent* (or *$p$-equivalent*) if each of them dominates the other.

Compared to the remaining summands in the complete decomposition of $M(Y)$, the apper motives of $Y$ are much more accessible. This explains the value of the following

**Conjecture 2.2.** *Every indecomposable object $M \in \mathrm{CM}_{\mathrm{pqh}}(F)$ is a shift of an apper motive.*

Conjecture 2.2 is proved in [7, Theorem 3.5] in the case where $M$ is a summand in the motive of a projective homogeneous variety $X$ under a reductive group $G$ of *inner type*. This result is then extended in two "orthogonal" directions: to reductive groups $G$ acquiring inner type over a finite Galois field extension $E/F$ whose degree $[E : F]$ is a $p$-power (see [6, Theorem 1.1]) or prime to $p$ (see [4, Theorem 6.3]). Here we will extend the initial result in one more different direction: instead of imposing any restriction on $[E:F]$, we will require that the variety $X$ is *consistent* as defined in the next section. (Any projective homogeneous variety under a reductive group of inner type is automatically consistent.)

## 3. Consistency

Let $G$ be a reductive group over a field $F$. We view the *Tits index* of $G$, defined in [8], as a subset of vertices of the Dynkin diagram of $G$. This subset is stable under the $*$-action of the absolute Galois group $\Gamma_F$ of $F$ on the Dynkin diagram of $G$. For instance, $G$ is quasi-split, if and only if its Tits index is the entire set of vertices.

The *higher Tits indexes* of $G$ are the Tits indexes acquired by $G$ over extension fields of $F$. The *higher Tits $p$-indexes* of $G$ are the Tits indexes acquired by $G$ over *$p$-special* extension fields of $F$: a field $L$ is *$p$-special*, if the degree of every finite field extension of $L$ is a $p$-power. The set of higher Tits $p$-indexes is therefore a subset in the set of higher Tits indexes of $G$.

Following [4], we say that $G$ is *$p$-consistent* (or simply *consistent*), if for every finite separable field extension $K/F$, every projective $G_K$-homogeneous variety $X$ is equivalent to $\hat{X}_K$ for some projective $G$-homogeneous variety $\hat{X}$ (depending on $K$ and $X$).

There is a sufficient condition for $G$ to be consistent stated in terms of its higher Tits $p$-indexes. The condition is: every higher Tits $p$-index of $G$ is stable under the $*$-action of $\Gamma_F$. Indeed, if the higher Tits $p$-index of $G$, obtained as the Tits index of $G$ over a *$p$-special closure* (see [5, §101.B]) of the function field $K(X)$, is $\Gamma_F$-invariant, it constitutes the diagram of certain projective $G$-homogeneous variety $\hat{X}$ satisfying the desired condition. The diagram of a projective $G$-homogeneous variety is the $\Gamma_F$-invariant subset of vertices of the Dynkin diagram defined by the convention opposite to [8]: the diagram of the variety of Borel subgroups in $G$ is the set of all vertices.

Clearly, if $G$ is of inner type, i.e., if the $*$-action is trivial, then $G$ is consistent. If $G$ becomes quasi-split over a finite base field extension of prime to $p$ degree, then the only higher Tits $p$-index of $G$ is the entire set of vertices so that $G$ is consistent as well. Direct product of two consistent groups is consistent. Any absolutely simple group of type other than $\mathsf{D}_n$ is consistent. (The non-inner types here are ${}^2\mathsf{A}_n$ for $n \geq 2$ and ${}^2\mathsf{E}_6$.)

We now define a related notion of a *consistent variety*. Let $K/F$ be a finite separable field extension and let $X$ be a projective $G_K$-homogeneous variety. The quasi-homogeneous $F$-variety $X^F$ is called *consistent*, if for every finite separable field extension $L/K$, every projective $G_L$-homogeneous variety $Y$ dominating $X_L$ is equivalent to $\hat{Y}_L$ for some projective $G$-homogeneous variety $\hat{Y}$ (depending on $L$ and $Y$).

A sufficient condition for $X^F$ to be consistent is as follows: every higher Tits $p$-index of $G$ containing the diagram of $X$ is $\Gamma_F$-stable. If the group $G$ is consistent, then any variety $X^F$ as above is consistent as well. For any group $G$, the variety of Borel subgroups in $G$ is consistent. More generally, $X^F$ is consistent if $X$ is the $K$-variety of Borel subgroups in $G_K$.

**Example 3.1.** Working with the prime $p = 2$, let $k$ be a field of characteristic not 2 and let $E := k(a, b, a', b')$ be the field of rational functions over $k$ in variables $a, b, a', b'$. Write $F$ for the subfield of the elements in $E$ invariant under the $k$-involution $\sigma$ of $E$ exchanging

$a$ with $a'$ and $b$ with $b'$. Let us consider the quaternion $E$-algebra $Q := (a, b)$, the algebraic group $\mathrm{PGL}_1(Q)$, and its Weil transfer $G$ from $E$ to $F$. The group $G_E$ is isomorphic to the product $\mathrm{PGL}_1(Q) \times \mathrm{PGL}_1(Q')$, where $Q' := (a', b')$. The Dynkin diagram of $G$ consists of two points exchanged by $\sigma$. There are only two isomorphism classes of projective $G$-homogeneous varieties. They are represented by the base point $\operatorname{Spec} F$ and the variety $B$ of Borel subgroups. The variety $B_E$ is isomorphic to the product $X \times X'$ of the Severi-Brauer varieties of $Q$ and $Q'$. The variety $X$ is equivalent neither to $\operatorname{Spec} E = (\operatorname{Spec} F)_E$ nor to $B_E$. Consequently, neither the variety $X$ nor the group $G$ are consistent.

## 4. Main Result: Primary Version

Our primary main result can be formulated as follows:

**Theorem 4.1.** *Let $F$ be a field, $E/F$ a finite Galois field extension, $G$ a reductive algebraic group over $F$ acquiring inner type over $E$, and $X$ a projective $G$-homogeneous variety (over $F$). Given a prime number $p$, assume that $X$ is $p$-consistent.*

*Then every summand in the complete motivic decomposition of the variety $X$ in the category of effective Chow motives over $F$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$ is a (non-negative) shift of an apper motive of the variety $Y^F$ for some intermediate field $L$ of $E/F$ and some projective $G_L$-homogeneous variety $Y$ dominating $X_L$.*

Before we proceed to the proof (in the next section), let us explain its major difference with the existing proofs of other cases of Conjecture 2.2. Similarly to the other proofs, this one goes by induction of $\dim X$ using any of the two results [1, Theorem 4.2] or [2, Theorem 7.5]) telling – roughly – that over its own function field the motive of the variety $X$ (with $\dim X > 0$) split into a direct sum of shifts of motives of projective homogeneous varieties of smaller dimension. So, the complete motivic decomposition of $X$ over $F(X)$ is as we want by the induction hypothesis, and the main point of the proof is the passage from $F(X)$ to $F$.

The real situation however is more complicated: the projective homogeneous varieties which appear over $F(X)$ actually live over some subextensions of the Galois field extension $E(X)/F(X)$. The summands they contribute with to the complete decomposition of $M(X)_{F(X)}$ are the corestrictions (in the sense of §1) to $F(X)$ of their motives. If we work in the situation with $[E : F]$ being a power of $p$ or prime to $p$, the degree $[E(X) : F(X)]$ satisfies the same condition. As a consequence, the corestriction of an apper motive is always a direct sum of apper motives, and the proof goes through. In our situation of arbitrary $[E : F]$, a corestriction of an apper motive does not behave this nice way anymore (see Example 5.4). Because of that, we need to extend the statement of Theorem 4.1 to make the induction work. Theorem 5.1, proved in the next section, is a right extension of Theorem 4.1 in this respect.

## 5. Main Result: Extended Version

**Theorem 5.1.** *Let $F$ be a field, $E/F$ a finite Galois field extension, $G$ a reductive algebraic group over $F$ acquiring inner type over $E$, $K$ an intermediate field of $E/F$, and $X$ a projective $G_K$-homogeneous variety (over $K$). Given a prime number $p$, assume that $X$ is $p$-consistent.*

*Then every summand in the complete motivic decomposition of the variety $X^F$ in the category of effective Chow motives over $F$ with coefficients in $\mathbb{Z}/p\mathbb{Z}$ is a (non-negative) shift of an apper motive of the variety $Y^F$ for some intermediate field $L$ of $E/K$ and some projective $G_L$-homogeneous variety $Y$ dominating $X_L$.*

*Proof.* We induct on $\dim X$. In the case of $\dim X = 0$ the statement is trivial: the motive $M(X^F) = M(X)^F$ is the Artin motive $M(K)^F$; every summand in its complete decomposition is an apper motive of $X^F$. Below we are assuming that $\dim X > 0$.

We start with the case of $K = F$. Here we can proceed exactly as in the proof of [4, Theorem 6.3]. (Note that the group $G'$ from that proof, defined over the function field $\tilde{F} := F(X)$ as the semisimple anisotropic kernel of $G_{\tilde{F}}$, is consistent.) As the outcome we obtain the desired statement on $X$.

This also provides some useful information for the general case, where $K$ is allowed to be different from $F$: since the statement holds for the motive of $X$ (over $K$), every summand in the complete motivic decomposition of $X^F$ is a non-negative shift of a summand $N$ in $U(Y)^F$ for some dominating $X_L$ projective $G_L$-homogeneous variety $Y$ over an intermediate field $L$ of $E/K$. The "only" problem is: $N$ is not necessarily an apper summand of $U(Y)^F$, i.e., $N$ is not necessarily sustainable. But we claim that one can find an intermediate field $\check{L}$ of $E/L$ and a projective $G_{\check{L}}$-homogeneous variety $\check{Y}$ dominating $Y_{\check{L}}$ such that $N$ is a positive shift of an apper summand in $U(\check{Y})^F$. With a proof of the claim we conclude the proof of Theorem 5.1.

Now comes the time to use consistency of $X$. Since $Y$ dominates $X_L$, $Y$ is equivalent to $\hat{Y}_L$ for some projective $G$-homogeneous variety $\hat{Y}$. So, $U(Y) \simeq U(\hat{Y}_L)$ and we may work with $\hat{Y}_L$ instead of $Y$. Furthermore, instead of working with the upper motive of $\hat{Y}_L$, we can work with the entire motive of $\hat{Y}_L$.

Let us change notation one more time: given a projective $G$-homogeneous variety $X$, we just need to prove Theorem 5.1 for the variety $(X_K)^F = X \times Z$, where $Z$ is the $F$-variety $(\operatorname{Spec} K)^F$. Here as well we proceed along the lines of the proof of [4, Theorem 6.3], but we perform some modifications. First of all, we apply the induction hypothesis over the function field $\tilde{F} := F(X)$, not over the function field $K(X)$ of $X \times Z$. The motivic morphisms $\tilde{\alpha}$ and $\tilde{\beta}$ are defined the same way as in the original proof. The construction of $\alpha$ out of $\tilde{\alpha}$ is also the same. However there is some difference in the construction of $\beta$ out of $\tilde{\beta} \in \operatorname{Ch}(X \times Z \times Y^F)_{\tilde{F}}$.

We first let $\beta'$ be an element of $\operatorname{Ch}(X \times Z \times X \times Y^F)$ mapped to $\tilde{\beta}$ under the surjection (from [5, Corollary 57.11])

$$\operatorname{Ch}(X \times Z \times X \times Y^F) \to \operatorname{Ch}(X \times Z \times Y^F)_{\tilde{F}}$$

given by the generic point of the *second* copy of $X$ in the product $X \times Z \times X \times Y^F$. We consider $\beta'$ as a correspondence $X \times Z \rightsquigarrow X \times Y^F$ and let $\beta''$ be the composition of correspondences $\beta' \circ \mu$, where $\mu \in \operatorname{Ch}\big((X \times Z) \times (X \times Z)\big)$ is the projector which yields the motivic summand $M$ of $X$. Finally, we define $\beta$ as the pullback of $\beta''$ with respect to the closed embedding

$$X \times Z \times Y^F \hookrightarrow X \times Z \times X \times Y^F, \quad (x, z, y) \mapsto (x, z, x, y)$$

given by the diagonal of $X$.

The final part of the proof goes through without any modification. □

**Remark 5.2.** The field extension $E/F$ in Theorem 5.1 can be arbitrary large: it is not supposed to be a minimal one with $G_E$ of inner type. Therefore the extension $K/F$ can also be large – even for $G$ of inner type. In this respect, Theorem 5.1 is new even for $G$ of inner type.

**Remark 5.3.** The description of the apper motives appearing in the statement of Theorem 5.1 can be made more explicit. First of all, for any reductive group $G$ over $F$, a finite separable field extension $L/F$, and a projective $G$-homogeneous variety $Y$, any apper motive $U$ of $Y^F$ is a summand in $U(Y)^F$ (simply because $\mathbf{m}(M^F)=0$ for the complimentary summand $M$ of $U(Y)$ in $M(Y)$).

Now let us assume that $Y$ is equivalent to $X_L$ for some projective $G$-homogeneous variety $X$. Then $U(Y)^F = U(X_L)^F$ is a summand in $(U(X)_L)^F = U(X)\otimes M(L)^F$. It follows that $U$ is a summand in $U(X)\otimes A$ for some indecomposable Artin motive $A$ which is a summand in $M(L)^F$. More explicitly, $U$ is the unique sustainable summand in the complete decomposition of the tensor product $U(X)\otimes A$. In fact, $\mathbf{m}(U)=A$, and the isomorphism classes of apper motives of $Y^F$ are parameterized by the isomorphism classes of indecomposable summands in $M(L)^F$ this way.

**Example 5.4.** We take $p=2$ in this example. One can find a field $F$ with a non-abelian degree 6 Galois field extension $E/F$ and an anisotropic conic $C$ over $F$, splitting over the quadratic subextension $K/F$ of $E/F$. Let $L/F$ be a cubic subextension in $E/F$. The complete decomposition of $M(L)^F$ consists of two summands: $\mathbb{F}$ and an Artin motive $A$ satisfying $A_L \simeq M(E)^L$ and $A_E \simeq \mathbb{F}\oplus\mathbb{F}$.

Since the conic $C$ is anisotropic, we have $M(C)=U(C)$. Since the degree of $L/F$ is odd, $C_L$ is also anisotropic and $M(C_L)=U(C_L)$. We claim that the complete decomposition of the motive

$$U(C_L)^F = U(C)\otimes M(L)^F = U(C)\oplus\big(U(C)\otimes A\big)$$

consists of the three summands: $U(C)$, $A$, and $A\{1\}$. In other terms,

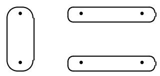
$$U(C)\otimes A \simeq A\oplus A\{1\}. \tag{5.5}$$

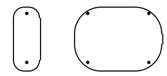
To prove (5.5), we first note that it holds over the function field of $C$ simply because $U(C)_{F(C)} \simeq \mathbb{F}\oplus\mathbb{F}\{1\}$. Using the fact that the étale $F$-algebra $L\otimes_F L$ is the direct product of $L$ and $E$, one sees that

$$\operatorname{Hom}(A\{i\},\ U(C)\otimes A) = \operatorname{Ch}_i(C_E) \quad\text{and}\quad \operatorname{Hom}(U(C)\otimes A,\ A\{i\}) = \operatorname{Ch}^i(C_E)$$

for any $i\ge 0$ (in particular, for $i=0,1$). Since the conic $C$ splits over $E$, these groups do not change when the base field $F$ is extended to $F(C)$. Therefore $A$ and $A\{1\}$ are summands in $U(C)\otimes A$ already over $F$, and (5.5) holds.

For comparison, if the degree 6 field extension $E/F$ would be abelian, so that the cubic field extension $L/F$ would be Galois, the complete decomposition of $M(L)^F$ would still have two summands $\mathbb{F}$ and $A$ (with $A$ becoming $\mathbb{F}\oplus\mathbb{F}$ over $L$). The complete decomposition of $U(C)\otimes M(L)^F$ would then consist of $U(C)$ and $U(C)\otimes A$. It can be represented by the diagram on the right:





Each of the dots in the upper line of the diagram stands for a Tate summand $\mathbb{F}$ in $U(C)\otimes M(L)^F$ over $E$. The dots of the lower line are the Tate summands $\mathbb{F}\{1\}$. Each oval unifies the Tate summands coming from one and the same indecomposable summand of $U(C)\otimes M(L)^F$ over $F$.

The diagram on the left represents in the similar manner the complete decomposition of $U(C)\otimes M(L)^F$ (over $F$) in the case of non-abelian $E/F$.

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