A REMARK ON CONNECTIVE K-THEORY

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ABSTRACT. Let X be a smooth algebraic variety over an arbitrary field. Let φ be the canonical surjective homomorphism of the Chow ring of X onto the ring associated with the Chow filtration on the Grothendieck ring $\mathrm{K}(X)$. We remark that φ is injective if and only if the connective K-theory $\mathrm{CK}(X)$ coincides with the terms of the Chow filtration on $\mathrm{K}(X)$. As a consequence, $\mathrm{CK}(X)$ turns out to be computed for numerous flag varieties (under semisimple algebraic groups) for which the injectivity of φ had already been established. This especially applies to the so-called *generic* flag varieties X of many different types, identifying for them $\mathrm{CK}(X)$ with the terms of the explicit Chern filtration on $\mathrm{K}(X)$. Besides, for arbitrary X, we compare $\mathrm{CK}(X)$ with the fibered product of the Chow ring of X and the graded ring formed by the terms of the Chow filtration on $\mathrm{K}(X)$.

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1. INTRODUCTION

Let F be an arbitrary field, let G be a split semisimple algebraic group over F, and let P be one of its parabolic subgroups. For any G-torsor E over any extension field of F, the quotient $X := E/P$ is a variety of parabolic subgroups (a *flag variety* for short) in the (possibly non-split) semisimple group $\mathrm{Aut}_G E$, an inner twisted form of G over the extension. We call the flag variety X *generic*, provided that E is a (standard) generic G-torsor, i.e., the generic fiber of the quotient map $\mathrm{GL}(n) \to \mathrm{GL}(n)/G$ for an embedding of G into $\mathrm{GL}(n)$.

Assume that P is *special*, i.e., all P-torsors over all extension fields of F are trivial. (For instance, one can take for P a Borel subgroup of G.) The following conjecture

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appears first in [5, §1] in the form of a question. It deals with the canonical (surjective) homomorphism of graded rings

$$\varphi = \varphi_X\colon \operatorname{CH}(X) \to \operatorname{Chow} \mathrm{K}(X),$$

where $\operatorname{CH}(X)$ is the Chow ring, $\mathrm{K}(X)$ is the Grothendieck ring of X, and $\operatorname{Chow}\mathrm{K}(X)$ is the ring associated with the Chow filtration (i.e., the filtration by codimension of supports of coherent sheaves) on $\mathrm{K}(X)$.

Conjecture 1.1 ([4, Conjecture 1.1])**.** *The homomorphism φ is an isomorphism.*

Being recently disproved for $G = \operatorname{Spin}(17)$ by Yagita in [11] (see also [7]), Conjecture 1.1 has been confirmed for many other G. An overview of some positive cases is given in [4]. (On the other hand, for many G it is still unknown if the above conjecture holds or fails.)

For an arbitrary smooth variety X, the homomorphism φ provides a sort of connection between the Chow theory of X and its K-theory. Another standard way to connect those two theories goes through the *connective* K*-theory* $\operatorname{CK}(X)$ (see §2). In this note we remark that Conjecture 1.1 can be expressed in terms of $\operatorname{CK}(X)$. Namely, we prove (see Theorem 2.2) that the injectivity of φ actually means $\operatorname{CK}(X)$ coincides with the graded ring $\mathrm{K}^{()}(X)$ formed by the terms of the Chow filtration on $\mathrm{K}(X)$.

Note that $\mathrm{K}(X)$ is computed for arbitrary flag variety X, but not the Chow filtration, which is a finer invariant and remains quite mysterious. However, for a generic flag variety X given by a special parabolic P, as in Conjecture 1.1, the Chow filtration coincides with the explicitly computable Chern filtration (more widely known under the name of gamma filtration), introduced by Grothendieck (see §3). So, Conjecture 1.1 for a given X turns out to be equivalent to the complete computation of $\operatorname{CK}(X)$. In more details, this is discussed in §3.

In Appendix, we compare the connective K-theory $\operatorname{CK}(X)$ of an arbitrary smooth variety X with the fiber product $\operatorname{CHK}(X)$ (over $\operatorname{Chow}\mathrm{K}(X)$) of $\operatorname{CH}(X)$ and $\mathrm{K}^{()}(X)$. In particular, we show that the natural homomorphism of graded rings $\operatorname{CK}(X) \to \operatorname{CHK}(X)$ is always surjective (Lemma A.1); its injectivity is characterized in terms of the Brown-Gersten-Quillen (BGQ) spectral sequence of X (Theorem A.5).

2. The remark

For any smooth algebraic variety X over an arbitrary field F (of arbitrary characteristic), we write $\operatorname{CK}(X) = \bigoplus_{i\in\mathbb{Z}} \operatorname{CK}^i(X)$ for the connective K-theory ring of X, graded by codimension. Our main reference for the connective K-theory is [2] (see also [1]) and our $\operatorname{CK}^i(X)$ is the group $\operatorname{CK}^{i,-i}(X)$ of [2, §6.4]. (We only work with small cohomology theories and, in particular, do not use the higher connective K-theory groups here.) To recall the definition of $\operatorname{CK}^i(X)$, let $M^i(X)$ be the Grothendieck group of the category of coherent sheaves on X with codimension of support $\geq i$. Then $\operatorname{CK}^i(X)$ is defined as the image of the homomorphism $M^i(X) \to M^{i-1}(X)$ mapping the class of a sheaf to the class of itself.

Since $M^i(X)$ is the Grothendieck group $\mathrm{K}(X)$ for $i \leq 0$, $\operatorname{CK}^i(X)$ is identified with $\mathrm{K}(X)$ for such i. Also note that $\operatorname{CK}^i(X) = 0$ for $i > \dim X$.

The Grothendieck group $\mathrm{K}(X)$ is actually a ring (with multiplication given by tensor product of locally-free sheaves) and is endowed with the Chow filtration (see [8]), i.e., the filtration by codimension of supports of coherent sheaves:

$$\mathrm{K}(X) = \cdots = \mathrm{K}^{(-1)}(X) = \mathrm{K}^{(0)}(X) \supset \mathrm{K}^{(1)}(X) \supset \dots .$$

Since $\mathrm{K}^{(i)}(X) \cdot \mathrm{K}^{(j)}(X) \subset \mathrm{K}^{(i+j)}(X)$ for any $i, j \in \mathbb{Z}$, we may consider a graded ring

$$\mathrm{K}^{()}(X) := \bigoplus_{i\in\mathbb{Z}} \mathrm{K}^{(i)}(X),$$

where $\mathrm{K}^{(i)}(X) = 0$ for $i > \dim X$. Note that, unlike CK, the localization sequence

$$\mathrm{K}^{()}(Y) \to \mathrm{K}^{()}(X) \to \mathrm{K}^{()}(U) \to 0$$

for the theory $\mathrm{K}^{()}$, relating the theory of X with the theory of a smooth closed subvariety $Y \subset X$ and its open complement U (where the first map is a graded group homomorphism of degree $\operatorname{codim}_X Y$), is not always exact at the term $\mathrm{K}^{()}(X)$. (Exactness of the localization sequence for the connective K-theory is a part of [2, Theorem 5.1].)

Finally, we are considering the Chow ring $\mathrm{CH}(X) = \bigoplus_{i\in\mathbb{Z}} \mathrm{CH}^i(X)$ of rational equivalence classes of algebraic cycles on X, graded by codimension of cycles. Here we also have $\mathrm{CH}^i(X) = 0$ for $i > \dim X$. Besides, $\mathrm{CH}^i(X) = 0$ for $i < 0$.

The connective K-theory "connects" $\mathrm{CH}(X)$ with $\mathrm{K}(X)$, or, more precisely, with $\mathrm{K}^{()}(X)$ by means of canonical surjective homomorphisms of graded rings

$$f = f_X\colon\ \mathrm{CK}(X) \to \mathrm{CH}(X) \quad\text{and}\quad \psi = \psi_X\colon\ \mathrm{CK}(X) \to \mathrm{K}^{()}(X).$$

By [2, Theorem 7.1], the kernel of the first one is generated by the *Bott element* $\beta \in \mathrm{CK}^{-1}(X)$ defined as the unity of the ring $\mathrm{K}(X)$, considered as an element of $\mathrm{K}^{(-1)}(X) = \mathrm{CK}^{-1}(X)$.

Let us consider the Laurent polynomial ring $\mathrm{K}(X)[\beta_{\mathrm{K}}^{\pm1}]$ in one variable β_{K} (viewed as the K-theoretical Bott element). The ring $\mathrm{K}^{()}(X)$ can be defined as the subring of $\mathrm{K}(X)[\beta_{\mathrm{K}}^{\pm1}]$ consisting of the polynomials $\sum_{i\in\mathbb{Z}} a_i\beta_{\mathrm{K}}^i$ with $a_i \in \mathrm{K}^{(-i)}(X)$ for all i. Since β_{K} is invertible in $\mathrm{K}(X)[\beta_{\mathrm{K}}^{\pm1}]$, it is not a zero divisor in $\mathrm{K}^{()}(X)$.

Again by [2, Theorem 7.1], the composition

$$\mathrm{CK}(X) \xrightarrow{\ \psi\ } \mathrm{K}^{()}(X) \hookrightarrow \mathrm{K}(X)[\beta_{\mathrm{K}}^{\pm1}]$$

is the localization of the ring $\mathrm{CK}(X)$ with respect to the element $\beta \in \mathrm{CK}(X)$. In particular, ψ is an isomorphism if and only if β is not a zero divisor in $\mathrm{CK}(X)$. Note that $\beta_{\mathrm{K}} \in \mathrm{K}^{(-1)}(X)$ is the image of β under ψ.

The quotient $\mathrm{K}^{()}(X)/\beta_{\mathrm{K}}\,\mathrm{K}^{()}(X)$ is the graded ring $\operatorname{Chow}\mathrm{K}(X)$ associated with the Chow filtration on $\mathrm{K}(X)$. The canonical surjective homomorphism of graded rings

$$\varphi\colon\ \mathrm{CH}(X) \to \operatorname{Chow}\mathrm{K}(X),$$

mapping the class of a closed subvariety to the class of its structure sheaf, fits into the commutative square

$$\begin{array}{ccc} \mathrm{CK}(X) & \xrightarrow{\psi} & \mathrm{K}^{()}(X) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ \mathrm{CH}(X) & \xrightarrow{\varphi} & \mathrm{Chow}\,\mathrm{K}(X), \end{array} \tag{2.1}$$

where the homomorphism of graded rings $g = g_X$ is given by the quotient maps on the graded components. We recall that the kernel of φ consists of elements of finite order. More precisely, the kernel on $\mathrm{CH}^i(X)$ is trivial for $i \le 2$ and is killed by $(i-1)!$ for $i \ge 1$, [3, Example 15.3.6].

Theorem 2.2. *For any given smooth algebraic variety X (over an arbitrary field), the homomorphism ψ_X is an isomorphism if and only if φ_X is.*

Proof. The homomorphism ψ induces φ by modding out the ideals in $\mathrm{CK}(X)$ and in $\mathrm{K}^{()}(X)$ generated by the Bott elements. So, φ is an isomorphism provided that ψ is.

Conversely, let us assume that $\mathrm{Ker}(\varphi) = 0$ and let us take an element $x_0 \in \mathrm{CK}(X)$ vanishing in $\mathrm{K}^{()}(X)$ under ψ. Note that x_0 is concentrated in positive degrees:

$$x_0 \in \mathrm{CK}^{>0}(X) := \bigoplus_{i>0} \mathrm{CK}^i(X).$$

(We do not need to assume it to be homogeneous.) From the commutative square (2.1), we conclude that x_0 vanishes also in $\mathrm{CH}(X)$, so that $x_0 = \beta x_1$ for some $x_1 \in \mathrm{CK}^{>1}(X)$. Since $\beta \in \mathrm{K}^{()}(X)$ is not a zero divisor, x_1 also vanishes in $\mathrm{K}^{()}(X)$ under ψ so that $x_1 = \beta x_2$ and $x_0 = \beta^2 x_2$ for some $x_2 \in \mathrm{CK}^{>2}(X)$. Continuing this way, we manage to write x_0 as $x_0 = \beta^i x_i$ with some $x_i \in \mathrm{CK}^{>i}(X)$ for any $i \ge 0$. But $\mathrm{CK}^{>i}(X)$ is trivial for $i = \dim X$. It follows that x_0 and $\mathrm{Ker}(\psi)$ are trivial. □

Remark 2.3. Replacing the integer coefficients by rational coefficients for the cohomology theories in the above considerations, we come to the situation, where φ is an isomorphism for any X. It follows that ψ with rational coefficients is always an isomorphism as well. Turning back to the integer coefficients, we see that every element in the kernel of ψ is of finite order.

Remark 2.4. For $i \in \mathbb{Z}$, let us consider the restrictions φ^i and ψ^i of φ and ψ to the ith components of the corresponding graded groups. The proof of Theorem 2.2 actually shows:

(1) If ψ^i is an isomorphism for some $i \in \mathbb{Z}$, then φ^i is an isomorphism.
(2) If φ^i is an isomorphism for some $n \in \mathbb{Z}$ and all $i \ge n$, then ψ^n is an isomorphism.

3. Applications to flag varieties

Now we fix a semisimple algebraic group G over F and consider a projective homogeneous variety (*flag variety* for short) X under G. In other terms, X is a variety of parabolic subgroups in G. We fix an algebraic closure $\bar{F}$ of F and write $\bar{X}$ for $X_{\bar{F}}$. Let us write down an extended version of Theorem 2.2 which holds for such X:

Theorem 3.1. *The following conditions on X are equivalent.*

(1) *The homomorphism φ_X is an isomorphism.*
(2) *The homomorphism ψ_X is an isomorphism.*
(3) *The group* $\mathrm{CK}(X)$ *is torsion-free.*
(4) *The change of field homomorphism* $\mathrm{CK}(X) \to \mathrm{CK}(\bar{X})$ *is injective.*

Proof. We already know by Theorem 2.2 that (1) and (2) are equivalent. By Remark 2.3, (3) implies (2). Since the group $\mathrm{K}^{()}(X)$ is torsion-free (by [10]), (2) implies (3) as well. By transfer argument, (3) implies (4). Finally, the group $CK(\bar{X})$ is torsion-free (e.g., because $\mathrm{CH}(\bar{X})$ is torsion-free, [9, Corollary 1.5(a)]), implying that $\varphi_{\bar{X}}$ and $\psi_{\bar{X}}$ are isomorphisms; consequently (4) implies (3) as well. □

To get the most from Theorem 3.1, let us put more restrictions on X: assume that X is a *generic* flag variety (as defined in the introduction) given by a split semisimple group G and a *special* parabolic subgroup $P \subset G$. By [6, Corollary 7.4], the Chow filtration on $\mathrm{K}(X)$ coincides in this case with the Chern filtration. Therefore $\mathrm{CK}(X)$ is given by the terms of the Chern filtration as long as Conjecture 1.1 holds for G.

On the other hand, the counter-example of [11] (see also [7]) provides by Theorem 3.1 a generic flag variety X (given by the spinor group $\mathrm{Spin}(17)$) with non-trivial torsion in $\mathrm{CK}(X)$.

Appendix: Comparison with the product of Chow and K-theory

Commutative square (2.1) induces a graded ring homomorphism

$$\delta\colon \mathrm{CK}(X) \to \mathrm{CHK}(X) := \mathrm{CH}(X) \times_{\mathrm{Chow}\,\mathrm{K}(X)} \mathrm{K}^{()}(X)$$

of the connective K-theory into the fibered product of Chow and K-theory. In this appendix we will describe circumstances under which δ is an isomorphism.

We recall that after tensoring with $\mathbb{Q}$, the homomorphisms φ, ψ, and therefore also δ become isomorphisms. Consequently, kernel and cokernel of δ (without tensoring with $\mathbb{Q}$) are torsion groups.

Lemma A.1. *The homomorphism δ is surjective.*

Proof. An arbitrary element of the fibered product $\mathrm{CHK}(X)$ is a pair of elements $x \in \mathrm{CH}(X)$ and $y \in \mathrm{K}^{()}(X)$ with the same image in $\mathrm{Chow}\,\mathrm{K}(X)$. Since f is surjective, we can find $z \in \mathrm{CK}(X)$ mapped to x. The difference $\psi(z) - y$ vanishes in $\mathrm{Chow}\,\mathrm{K}(X)$ so that $\psi(z) - y = \beta_{\mathrm{K}} y'$ for some $y' \in \mathrm{K}^{()}(X)$. Since ψ is surjective, we can find $z' \in \mathrm{CK}(X)$ mapped to y'. Then the difference $z - \beta z'$ has the images $x \in \mathrm{CH}(X)$ and $y \in \mathrm{K}^{()}(X)$ so that the element $(x, y) \in \mathrm{CHK}(X)$ is in the image of δ. □

Given $n \in \mathbb{Z}$, we write $f^n, g^n, \varphi^n, \psi^n, \delta^n$ for the maps $f, g, \psi, \varphi, \delta$ restricted to the nth components of the corresponding graded groups. Since φ^i and ψ^i are isomorphisms for $i \le 1$ (in fact for $i \le 2$ – see Remark A.6), δ^i is also an isomorphism for such i.

Lemma A.2. *If φ^i is an isomorphism for some $n \in \mathbb{Z}$ and all $i > n$, then δ^n is an isomorphism.*

Example A.3. Since $\mathrm{CH}^i(X) = 0$ for $i > n := \dim X$, φ^i are isomorphisms for such i and it follows that δ^n is an isomorphism. This can be seen directly as both vertical maps f^n and g^n in diagram (2.1) are isomorphisms.

Example A.4. In particular, Lemma A.2 affirms δ is an isomorphism provided that φ is. Equivalently, ψ is an isomorphism if φ is so. The latter statement has already been proved in Theorem 2.2.

Proof of Lemma A.2. Take any element $x \in \operatorname{Ker} \delta^n$. Since it vanishes in $\mathrm{CH}^n(X)$, we have $x = \beta y$ for some $y \in \mathrm{CK}^{n+1}(X)$. Since x also vanishes in $\mathrm{K}^{(n)}(X)$ and since β_{K}, as an element of the ring $\mathrm{K}^{()}(X)$, is not a zero divisor, we conclude that y vanishes in $\mathrm{K}^{(n+1)}(X)$. It follows by Remark 2.4(2) that $y = 0$. Consequently, $x = 0$ and δ^n is injective. □

Theorem A.5. *The following three conditions on a smooth variety X are equivalent:*

1. *The homomorphism δ is an isomorphism.*
2. *Starting from the 3d page, all differentials to the Chow diagonal in the* BGQ *spectral sequence for X (see, e.g.,* [2, §3]*) are trivial.*
3. *The annihilator of β in $\mathrm{CK}(X)$ coincides with the annihilator of β^2.*

Proof. For any $i \geq 0$, the Chow diagonal on the $(2+i)$th page of the BGQ spectral sequence is identified with $\beta^i \,\mathrm{CK}(X)/\beta^{i+1}\,\mathrm{CK}(X)$ and the homomorphism of $\mathrm{CH}(X) = \mathrm{CK}(X)/\beta\,\mathrm{CK}(X)$ onto the Chow diagonal on the $(2+i)$th page is the multiplication by β^i. Therefore Conditions (2) and (3) are equivalent.

On the other hand, let $\beta_{\mathrm{CHK}} \in \mathrm{CHK}^{-1}(X)$ be the image of β. The components of the graded ring associated with the filtration by powers of β_{CHK} on $\mathrm{CHK}(X)$ are as follows: $\mathrm{CHK}(X)/\beta_{\mathrm{CHK}}\,\mathrm{CHK}(X) = \mathrm{CH}(X)$ and

$$\beta^i_{\mathrm{CHK}}\,\mathrm{CHK}(X)/\beta^{i+1}_{\mathrm{CHK}}\,\mathrm{CHK}(X) = \beta^i_{\mathrm{K}}\,\mathrm{K}^{()}(X)/\beta^{i+1}_{\mathrm{K}}\,\mathrm{K}^{()}(X) = \mathrm{Chow}\,\mathrm{K}(X)$$

for $i > 0$. Therefore (1) is equivalent to (2). □

Remark A.6. One can modify the statement of Theorem A.5 so that it applies separately to individual components of δ. For instance, δ^n is an isomorphism for a given $n \in \mathbb{Z}$ if and only if for any $i \geq 1$ all differentials to the $\mathrm{CH}^{n+i}(X)$ location on the $(2+i)$th page in the BGQ spectral sequence are trivial. (Note that all these differentials (for a fixed n and varying i) come from a unique location.) As an example, this way one can see that δ^2 is always an isomorphism. Since φ^2 is also an isomorphism, ψ^2 is one as well.

Example A.7. Let D be a central division F-algebra of prime degree p and $X = \mathrm{SL}_1(D)$ the variety (algebraic group) of its elements of reduced norm 1. By [12, Example 6.3], on the pth page of the BGQ spectral sequence for X there is a nonzero differential going to the Chow diagonal. Therefore for any $p \neq 2$ the corresponding homomorphism δ is not an isomorphism.

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References

[1] Cai, S. A simple description of algebraic connective K-theory. PhD Thesis, Los Angeles, USA, 2006, 108 pages.

[2] Cai, S. Algebraic connective K-theory and the niveau filtration. *J. Pure Appl. Algebra 212*, 7 (2008), 1695–1715.

[3] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[4] Karpenko, N. A. Chow ring of generic flag varieties. *Math. Nachr. 290*, 16 (2017), 2641–2647.

[5] Karpenko, N. A. Chow ring of generically twisted varieties of complete flags. *Adv. Math. 306* (2017), 789–806.

[6] Karpenko, N. A. On generically split generic flag varieties. *Bull. Lond. Math. Soc. 50* (2018), 496–508.

[7] Karpenko, N. A. A counter-example by Yagita. *Internat. J. Math. 31*, 3 (2020), 2050025, 10.

[8] Karpenko, N. A., and Merkurjev, A. S. Chow filtration on representation rings of algebraic groups. *International Mathematics Research Notices* (03 2019). doi: 10.1093/imrn/rnz049.

[9] Köck, B. Chow motif and higher Chow theory of G/P. *Manuscripta Math. 70*, 4 (1991), 363–372.

[10] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.

[11] Yagita, N. The Gamma filtration for the Spin groups. arXiv:1811.08288 [math.KT], 30 Nov 2019, 21 pages.

[12] Yagunov, S. Motivic cohomology spectral sequence and Steenrod operations. *Compos. Math. 152*, 10 (2016), 2113–2133.

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