Chow groups of quadrics and index reduction formula

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Abstract

We show that the Chow group Ch^3 of a non-singular projective quadric has no torsion if dimension of the quadric is greater than 10 (while a non-trivial torsion appears for a certain 10-dimensional quadric over a suitable field). We apply the same method (based on an index reduction formula) to Ch^4 too and show that it is torsionfree if dimension of the quadric is greater than 22.

Let F be a field of characteristic not 2, φ a nondegenerate quadratic form over F, X_φ the projective quadric defined by φ, $\mathrm{Ch}^p X_\varphi$ (for $p \geq 0$) the p-th Chow group of the variety X_φ, i.e. the group of p-codimensional algebraic cycles on X_φ modulo rational equivalence [1, 11]. Using the imbedding $in : X_\varphi \hookrightarrow \mathbf{P}$ of X_φ into the projective space (as a hypersurface) we obtain an injection $in^* : \mathrm{Ch}^p\mathbf{P} \hookrightarrow \mathrm{Ch}^p X_\varphi$ which image will be called the *elementary part* of $\mathrm{Ch}^p X_\varphi$ (we will also say that a group $\mathrm{Ch}^p X_\varphi$ is *elementary* if it coincides with its elementary part). Since $\mathrm{Ch}^p\mathbf{P}$ has

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a canonical generator — the class of a p-codimensional linear subspace, we have a canonical generator h^p for the elementary part of $\mathrm{Ch}^p X_\varphi$ — the class of a (general) linear section of X_φ.

For every p we are interested in the following questions:

Question A *Is it true that the group* $\mathrm{Ch}^p X_\varphi$ *is elementary if* $\dim \varphi$ *is large enough?*

If "yes" then the second question arises:

Question B *Find the number* N_p *such that:*

i. the group $\mathrm{Ch}^p X_\varphi$ *is elementary if* $\dim \varphi > N_p$*;*

ii. there exist a N_p*–dimensional quadratic form* φ *(over a suitable* F*) with non-elementary* $\mathrm{Ch}^p X_\varphi$*.*

Remark We prefer to speak here of dimensions of forms not that of quadrics. The connection is: $\dim \varphi = \dim X_\varphi + 2$.

Remark If $\dim \varphi > 2p + 2$ then $\mathrm{Ch}^p X_\varphi$ is a direct sum of its elementary part and torsion subgroup [2]; so, $\mathrm{Ch}^p X_\varphi$ is elementary iff has no torsion.

For $p < 3$, we have the affirmative answer to Question A [2], moreover

$$N_1 = 4,\ N_2 = 8$$

(for a more explicit information on $\mathrm{Ch}^2 X_\varphi$ see the theorem (5.1) below).

The goal of these notes is to show that

$$N_3 = 12 \text{ and } N_4 \le 24\,.$$

For every $p > 0$ there exist a $4p$–dimensional φ (over a suitable F) with non-elementary $\mathrm{Ch}^p X_\varphi$ [3]. Hence $N_p \ge 4p$ and in our cases ($p = 3, 4$) we just have to prove the following:

Theorem *If* $\dim \varphi > 12$ *then* $\mathrm{Ch}^3 X_\varphi$ *is elementary.*

Theorem *If* $\dim \varphi > 24$ *then* $\mathrm{Ch}^4 X_\varphi$ *is elementary.*

The notes are organized as follows.

In §1 we show by using a new theorem of Rost [14] describing 14-dimensional quadratic forms with trivial discriminant and Clifford invariant that any such a form contains an Albert form (i.e. a 6-dimensional subform of trivial discriminant). It will be used in §6. Note that in order to answer for $p = 3$ Question A alone it is not necessary to use this result (see § 7).

In §2 some properties of the *even Clifford algebra* are listed.

In §3 we discuss the index reduction formula for quadrics [8] which is the main tool in our business.

In §4 we pass from Chow groups to K_0 and deduce some consequences by using the Swan's computation [18].

In §5 an information on the second Chow group is given to be used later.

In §6 the theorem on Ch^3 is proven.

In §7 we prove that $\mathrm{Ch}^3 X_\varphi$ is elementary in the case when $\dim\varphi > 16$ without any use of the Rost's result mentioned.

In §8 we apply the same approach to the group Ch^4.

In §9 we summarize known facts on Ch^3 of lower-dimensional quadrics.

We use the standard terminology and notations concerning quadratic forms [15] and central simple algebras [10]. For instance, *discriminant* (also called *signed determinant*) $\operatorname{disc}\varphi \in F^\times/F^{\times 2}$ of a quadratic form φ means $(-1)^{n(n-1)/2}\cdot\det\varphi$ where $n = \dim\varphi$ and $\det\varphi \in F^\times/F^{\times 2}$ is the determinant of φ. Pfister forms $\langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle$ are denoted as usual: $\langle\!\langle a_1, \ldots, a_n\rangle\!\rangle$, and $\langle\!\langle a_1, \ldots, a_n\rangle\!\rangle'$ stays for the pure subform of $\langle\!\langle a_1, \ldots, a_n\rangle\!\rangle$, i.e. $\langle\!\langle a_1, \ldots, a_n\rangle\!\rangle = \langle 1\rangle \perp \langle\!\langle a_1, \ldots, a_n\rangle\!\rangle'$. A *n-Pfister neighbour* is a quadratic form of dimension bigger than 2^{n-1} which is *similar* to a subform of a n-Pfister form.

In every place of the text we may refer to the last statement of a given type without indicating its number. Ends of proofs are marked by $\square$.

1 Forms of dimension 14

For a quadratic extension E/F, consider the Scharlau's *transfer* of quadratic forms given by the *trace* map [15]. If τ is a quadratic form over E let $\operatorname{tr}(\tau)$ be the corresponding quadratic form over F.

Theorem 1.1 ([14]) *Let φ be a 14–dimensional quadratic form over F with trivial discriminant and Clifford invariant. Then φ is transfer from a quadratic extension E/F of a 7-dimensional Pfister neighbour. More precisely, there exist $d \in F^\times$ and a 3-Pfister form $\langle\!\langle a, b, c\rangle\!\rangle$ over $E = F(\sqrt{d})$ such that*

$$\varphi \simeq \operatorname{tr}(\sqrt{d}\cdot\langle\!\langle a, b, c\rangle\!\rangle')\,.$$

Why the multiplier $\sqrt{d}$ appears explains the following easy computation:

Lemma 1.2 ([15]) *Let $d \in F^\times$ and τ be a quadratic form over $E = F(\sqrt{d})$. Then*

$$\det(\operatorname{tr}(\tau)) = d^{\dim\tau}\cdot N_{E/F}(\det\tau)\,.$$

Combining the lemma with the theorem we obtain

Corollary 1.3 *Any form φ satisfying the conditions of the theorem contains an Albert subform (i.e. a 6-dimensional subform of determinant -1).*

Proof The form $\langle\langle a,b,c\rangle\rangle'$ contains a 3-dimensional subform of determinant 1, e.g. the subform $\langle ab,bc,ac\rangle$. Hence by the theorem, the form φ contains the subform $\operatorname{tr}(\sqrt{d}\cdot\langle ab,bc,ac\rangle)$ which is 6-dimensional and has discriminant

$$d^3\cdot N_{E/F}(\sqrt{d}) = -1$$

by the lemma. □

2 Even Clifford algebra

Let φ be a quadratic form over F of dimension n. We summarize some known facts concerning the even Clifford algebra $C_0(\varphi)$:

Proposition 2.1 ([7]) *Algebra $C_0(\varphi)$ has dimension 2^{n-1} over F. Moreover*

- *if n is odd then $C_0(\varphi)$ is a central simple F-algebra;*
- *if n is even then*
$$C_0(\varphi) = C_0(\psi)\otimes_F F(\sqrt{\operatorname{disc}\varphi})$$
where $\psi\subset\varphi$ is any subform of codimension 1.

The connection between $C_0(\varphi)$ and the Clifford invariant $\mathrm{I}^2/\mathrm{I}^3(F)\to {}_2\mathrm{Br}\,(F)$ can be expressed as follows:

Proposition 2.2 ([7]) *Let φ be an even-dimensional quadratic form of trivial discriminant. The Clifford invariant of φ coincides with the Brauer class of the algebra $C_0(\psi)$ where ψ is any 1-codimensional subform of φ.*

Finally, we will need the following easy observation:

Lemma 2.3 *If a quadratic form φ contains an even-dimensional subform of trivial discriminant then $C_0(\varphi)$ is not a division algebra.*

Proof If ψ is a subform of φ then $C_0(\psi)$ is a subalgebra of $C_0(\varphi)$. If ψ has even dimension and trivial discriminant then $C_0(\psi)$ contains a non-trivial zero divisor by (2.1). □

3 Index reduction

We denote the function field of a projective quadric X_φ just by $F(\varphi)$ (and refer sometimes to it as the function field of the quadratic form). The index reduction formula for quadrics tells you how the extension $F(\varphi)/F$ changes the indices of central simple algebras. The extension mentioned is a tower of a purely transcendental and a quadratic ones hence for any central simple F-algebra A the index of $A_{F(\varphi)}$ is the same or $\operatorname{ind}(A)/2$. So, the only question is whether the underlying division algebra D remains a division algebra over $F(\varphi)$ or not. The answer is given by the

Theorem 3.1 ([8]) *Let φ be a quadratic form and D a central division algebra over F.*

- *Suppose that $\dim\varphi$ is odd. Then $D_{F(\varphi)}$ is no more a division algebra iff*
$$D \supset C_0(\varphi)$$
(i.e. iff D contains a subalgebra isomorphic to $C_0(\varphi)$).
- *Suppose that $\dim\varphi$ is even; choose a subform $\psi \subset \varphi$ of codimension 1.*
  - *In the case when $\operatorname{disc}\varphi$ is trivial $D_{F(\varphi)}$ is no more a division algebra iff*
  $$D \supset C_0(\psi)\,.$$
  - *In the case when $\operatorname{disc}\varphi$ is non-trivial $D_{F(\varphi)}$ is no more a division algebra iff*
  $$D_E \supset M_2(C_0(\psi))_E$$
  where $E = F(\sqrt{\operatorname{disc}\varphi})$ and M_2 stays for the 2×2-matrix algebra.

Corollary 3.2 *In the conditions of the theorem put $n = \dim\varphi$. If $\dim_F D < 2^{n-2}$ then $D_{F(\varphi)}$ is still a division algebra.*

Proof We have:
$$\dim_F C_0(\varphi) = 2^{n-1}\,,$$
$$\dim_F C_0(\psi) = 2^{n-2}\,,$$
$$\dim_F M_2(C_0(\psi)) = 2^n\,.$$
Hence if $\dim_F D < 2^{n-2}$ the algebra D does not contain no of the algebras listed (neither over F nor over E). □

4 Passing to K_0

Consider the Grothendieck group $K_0(X_\varphi) = K'_0(X_\varphi)$ together with the topological filtration

$$\ldots \supset K_0(X_\varphi)^{(p)} \supset K_0(X_\varphi)^{(p+1)} \supset \ldots$$

on it [1, 11]. How it concerns our business explains

Proposition 4.1 ([2]) *The canonical epimorphism* $\mathrm{Ch}^p X_\varphi \twoheadrightarrow K_0(X_\varphi)^{(p/p+1)}$ *is bijective for all* $p \leq 3$.

By analogy with Chow groups one may call the subgroup $H \subset K_0(X_\varphi)$ generated by all h^p ($p \geq 0$) the elementary part of $K_0(X_\varphi)$ (it coincides with the image of the pull-back $K_0(\mathbf{P}) \to K_0(X_\varphi)$). The Swan's theorem on K-theory of quadrics computes the non-elementary part of $K_0(X_\varphi)$ in terms of the even Clifford algebra:

Theorem 4.2 ([18]) *There is a natural (with respect to base field extensions) isomorphism*

$$K_0(C_0(\varphi))/([C_0(\varphi)]) \xrightarrow{\sim} K_0(X_\varphi)/H\,.$$

Let us call a group $K_0(X_\varphi)^{(p/p+1)}$ elementary if it coincides with its elementary part — with the subgroup generated by h^p (abusing notations we denote by the same symbol the element $h^p \in K_0(X_\varphi)$ and its class in the factorgroup $K_0(X_\varphi)^{(p/p+1)}$).

Corollary 4.3 *If* $C_0(\varphi)$ *is a division algebra then the groups* $K_0(X_\varphi)^{(p/p+1)}$ *are elementary for all* p.

Proof If $C_0(\varphi)$ is a division algebra then the quotient

$$K_0(C_0(\varphi))/([C_0(\varphi)])$$

from (4.2) is zero. $\square$

The following statement is just an easy observation which does not use (4.2):

Lemma 4.4 *For an arbitrary* p, *two following statements are equivalent:*

i. the groups $K_0(X_\varphi)^{(i/i+1)}$ *are elementary for all* $i \leq p$;

ii. the homomorphism $K_0(X_\varphi)^{(p+1)} \to K_0(X_\varphi)/H$ *is surjective.*

Proof All $K_0(X_\varphi)^{(i/i+1)}$ ($i \leq p$) are elementary iff the quotient $K_0(X_\varphi)/K_0(X_\varphi)^{(p+1)}$ is generated by H. The second statement of the lemma means that the quotient $K_0(X_\varphi)/H$ is generated by $K_0(X_\varphi)^{(p+1)}$. Now it is clear that each of the statements is equivalent to the following third one: $K_0(X_\varphi)$ is generated by the subgroups H and $K_0(X_\varphi)^{(p+1)}$. $\square$

Corollary 4.5 *Let ψ be an odd-dimensional quadratic form over F and put $\varphi = \psi \perp \langle -\operatorname{disc}\psi \rangle$. If for some p the groups $K_0(X_\varphi)^{(i/i+1)}$ are elementary for all $i \leq p$ then the groups $K_0(X_\psi)^{(i/i+1)}$ are also elementary for all $i \leq p$.*

Proof We assume that the groups $K_0(X_\varphi)^{(i/i+1)}$ are elementary for all $i \leq p$. According to (4.4), it means surjectivity of the map $K_0(X_\varphi)^{(p+1)} \to K_0(X_\varphi)/H$. Consider a diagram consisting of two commutative squares one of which includes this surjection:

$$\begin{tikzcd}
K_0(X_\varphi)^{(p+1)} \arrow[r, two heads] \arrow[d, "in^*"] & K_0(X_\varphi)/H \arrow[d, "in^*"] & K_0(C_0(\varphi)) \arrow[l] \arrow[d] \\
K_0(X_\psi)^{(p+1)} \arrow[r] & K_0(X_\psi)/H & K_0(C_0(\psi)) \arrow[l, two heads]
\end{tikzcd} \qquad (*)$$

The left and the middle vertical arrows are given by the pull-back under the imbedding $in : X_\psi \hookrightarrow X_\varphi$. Both the right horizontal arrows are from (4.2) and are surjective by the same theorem (we will need here only surjectivity of the lower one).

What we would like to show is surjectivity of the lower left horizontal arrow. First of all define the map $K_0(C_0(\varphi)) \to K_0(C_0(\psi))$ staying from the very right in the diagram $(*)$. The algebra $C_0(\varphi)$ is the product of two copies of the algebra $C_0(\psi)$ by (2.1). We take both the projections

$$pr_1, pr_2 : C_0(\varphi) = C_0(\psi) \times C_0(\psi) \longrightarrow C_0(\psi)$$

and define the map of the K_0-groups as a sum $pr_1^* + pr_2^*$.

Lemma 4.6 *The right square of the diagram $(*)$ commutes.*

Proof Let $\overline{\varphi}$ (resp. $\overline{\psi}$) is the form φ (resp. ψ) considered over an algebraic (or separable) closure $\overline{F}$ of the base field F. The square under consideration is the front face of the cube:

$$\begin{tikzcd}
 & & K_0(X_{\overline{\varphi}})/H \arrow[dd] & K_0(C_0(\overline{\varphi})) \arrow[l] \arrow[dd] \\
K_0(X_\varphi)/H \arrow[urr] \arrow[dd] & K_0(C_0(\varphi)) \arrow[l] \arrow[urr] \arrow[dd] & & \\
 & & K_0(X_{\overline{\psi}})/H & K_0(C_0(\overline{\psi})) \arrow[l] \\
K_0(X_\psi)/H \arrow[urr] & K_0(C_0(\psi)) \arrow[l] \arrow[urr] & &
\end{tikzcd}$$

The right, left, upper and lower faces commute and all the four restriction maps are injective (we will use in fact only injectivity of $res : K_0(X_\psi)/H \to K_0(X_{\overline{\psi}})/H)$. Hence it suffices to show that the back face of the cube (i.e. our square restricted to $\overline{F}$) commutes.

The group $K_0(C_0(\overline{\varphi}))$ (resp. $K_0(C_0(\overline{\psi}))$) is a free abelian group generated by classes of simple modules. Their images in K_0/H are given by classes of maximal linear subspaces of the quadric [2] (which correspond to maximal totally isotropic subspaces of the quadratic form).

Let us mention that all maximal linear subspaces of $X_{\overline{\psi}}$ have the same class in $K_0(X_{\overline{\psi}})$. In our situation, it corresponds to the fact that there is only one (up to an isomorphism) simple $C_0(\overline{\psi})$-module. As to the quadric $X_{\overline{\varphi}}$, the orthogonal group $O(\overline{\varphi})$ acts on the set of maximal linear subspaces of it. This action has two orbits, and classes in $K_0(X_{\overline{\varphi}})$ of two maximal linear subspaces coincide iff they lie in the same orbit. So, to get images of two generators of the group $K_0(C_0(\overline{\varphi}))$ in $K_0(X_{\overline{\varphi}})/H$ one should take any two maximal linear subspaces which are not in the same orbit.

Now the commutativity of the square (over $\overline{F}$) follows from two facts:

i. image of the class of a linear subspace under the pull-back

$$in^* : K_0(X_{\overline{\varphi}}) \to K_0(X_{\overline{\psi}})$$

is again the class of a linear subspace;

ii. image of the class of a simple module under the homomorphism

$$pr_1^* + pr_2^* : K_0(C_0(\overline{\varphi})) = K_0(C_0(\overline{\psi}) \times C_0(\overline{\psi})) \longrightarrow K_0(C_0(\overline{\psi}))$$

is again the class of a simple module.

□

We continue the proof of the corollary. Now all the maps in the diagram $(*)$ are defined and is known that both the squares of $(*)$ are commutative.

Consider the right square. The right map of it is evidently surjective, the lower map is surjective too. Hence so is the left map as well, i.e. the middle vertical map in $(*)$ is surjective.

Now consider the left square. It was noticed in the very beginning of the proof that the upper map of it is surjective. Since the right map is surjective too (see the preceding paragraph) we conclude that the lower map is also surjective. By (4.4) it means that the groups $K_0(X_\psi)^{(i/i+1)}$ are elementary for $i \le p$. □

Following two statements are other applications of analogous ideas (the first one is of greatest importance).

Corollary 4.7 *Let φ be an even-dimensional quadratic form of non-trivial discriminant. Suppose that φ contains a 1-codimensional subform ψ such that the quadratic extension $E = F(\sqrt{\operatorname{disc}\varphi})/F$ preserve the index of the algebra $C_0(\psi)$. If for some p the groups $K_0(X_\psi)^{(i/i+1)}$ are elementary for all $i \le p-1$ then the groups $K_0(X_\varphi)^{(i/i+1)}$ are elementary for all $i \le p$.*

Proof Consider a diagram:

$$\begin{array}{ccccc} K_0(X_\varphi)^{(p+1)} & \longrightarrow & K_0(X_\varphi)/H & \twoheadleftarrow & K_0(C_0(\varphi)) \\ \uparrow in_* & & \uparrow in_* & & \uparrow \\ K_0(X_\psi)^{(p)} & \twoheadrightarrow & K_0(X_\psi)/H & \longleftarrow & K_0(C_0(\psi)) \end{array} \qquad (**)$$

where for the left and middle vertical arrows the push-forward in_* is now used. The groups $K_0(X_\psi)^{(i/i+1)}$ for $i \le p-1$ are elementary by assumption. Hence by (4.4) the lower left horizontal map is surjective.

To define the right vertical arrow in the diagram $(**)$ recall that

$$C_0(\varphi) = C_0(\psi) \otimes_F E$$

according to (2.1). Let the map of K_0-groups we are looking for be simply the restriction homomorphism. By the condition of the corollary (concerning behaviour of $C_0(\psi)$ under the extension E/F) this homomorphism is onto. So we are done if we succeed to show that the right square of $(**)$ commutes (compare with the proof of (4.5)).

Lemma 4.8 *The right square of the diagram $(**)$ commutes.*

Proof As in the proof of (4.6) we may restrict our square to $\overline{F}$ (of course the form $\overline{\varphi}$ does not need more to fit the condition of the corollary).

The class in $K_0(C_0(\overline{\psi}))$ of the unique simple $C_0(\overline{\psi})$-module is mapped under the restriction homomorphism

$$K_0(C_0(\overline{\psi})) \longrightarrow K_0(C_0(\overline{\varphi})) = K_0(C_0(\overline{\psi}) \times C_0(\overline{\psi}))$$

to the class of sum of two distinguished simple $C_0(\overline{\varphi})$-modules.

From the other hand, the class in $K_0(X_{\overline{\psi}})$ of a maximal linear subspace of $X_{\overline{\psi}}$ is mapped under the push-forward

$$in_* : K_0(X_{\overline{\psi}}) \longrightarrow K_0(X_{\overline{\varphi}})$$

to the class of the same linear subspace sitting in $X_{\overline{\varphi}}$ which is equal to $l + l' - h^{n-1}$ where l and l' are two distinguished classes of maximal linear subspaces of $X_{\overline{\varphi}}$ [2]. So, we get the sum of l and l' if working modulo H. □□

Corollary 4.9 *Let φ be an arbitrary quadratic form over F and let E/F be a finite field extension such that the norm map*

$$N_{E/F} : K_0(C_0(\varphi_E)) \longrightarrow K_0(C_0(\varphi))$$

is surjective (e.g. E may be any subfield of the division algebra derived from $C_0(\varphi)$). If for some p the groups $K_0(X_{\varphi_E})^{(i/i+1)}$ are elementary for all $i \le p$ then the groups $K_0(X_\varphi)^{(i/i+1)}$ are elementary for all $i \le p$ too.

Proof Consider a diagram:

$$\begin{array}{ccccc} K_0(X_{\varphi_E})^{(p+1)} & \twoheadrightarrow & K_0(X_{\varphi_E})/H & \longleftarrow & K_0(C_0(\varphi_E)) \\ \downarrow {\scriptstyle N_{E/F}} & & \downarrow {\scriptstyle N_{E/F}} & & \downarrow \\ K_0(X_\varphi)^{(p+1)} & \longrightarrow & K_0(X_\varphi)/H & \twoheadleftarrow & K_0(C_0(\varphi)) \end{array} \qquad (***)$$

The right square of $(***)$ is now commutative simply by naturality of the Swan's map. Now we can argue as in the proof of (4.5). $\square$

5 The second Chow group

Theorem 5.1 ([2]) *For a quadratic form φ of dimension > 6 the group $\mathrm{Ch}^2 X_\varphi$ is non-elementary iff φ is a neighbour of an anisotropic 3-Pfister form. In particular, if $\dim\varphi > 8$ then $\mathrm{Ch}^2 X_\varphi$ is elementary.*

Corollary 5.2 *If a quadratic form φ contains an Albert form as a proper subform then $\mathrm{Ch}^2 X_\varphi$ is elementary.*

Proof The assumption implies that $\dim\varphi > 6$. Suppose that $\mathrm{Ch}^2 X_\varphi$ is non-elementary. Then by (5.1) φ is a neighbour of an anisotropic 3-Pfister form. Since φ contains an Albert subform the 3-Pfister form contains an Albert subform too hence is isotropic what contradicts to the previous sentence. $\square$

We will actually need an information on Ch^2 of some affine quadrics. First we state a general fact on Chow groups of affine quadrics:

Lemma 5.3 *Let φ be a quadratic form and U be the affine quadric defined by equation $a + \varphi = 0$ with some $a \in F$ (so, U is a hypersurface in the affine vector space on which the quadratic form φ is defined). Then for all p there exist an exact sequence*

$$\mathrm{Ch}^{p-1} X_\varphi \longrightarrow \mathrm{Ch}^p X_{\langle a\rangle\perp\varphi} \longrightarrow \mathrm{Ch}^p U \longrightarrow 0\,.$$

In the case when $a = 0$ the middle group (which is now a Chow group of a singular *projective quadric) coincides if $p \le \dim X_\varphi$ with $\mathrm{Ch}^p X_\varphi$ and the homomorphism $\mathrm{Ch}^{p-1} X_\varphi \to \mathrm{Ch}^p X_\varphi$ is multiplication by $h \in \mathrm{Ch}^1 X_\varphi$.*

Proof The variety X_φ is a 1-codimensional subvariety of $\mathrm{Ch}^p X_{\langle a\rangle\perp\varphi}$ and the difference $X_{\langle a\rangle\perp\varphi} \setminus X_\varphi$ is isomorphic to U. Whence the exact sequence required [1]. The assertion on the singular case ($a = 0$) can be find in [6]. $\square$

Corollary 5.4 *Let U be the affine quadric defined by equation $a + \varphi = 0$ with some $a \in F$. The group $\mathrm{Ch}^2 U$ is zero in each of the following cases:*

i. if φ contains an Albert form as a proper subform;

ii. if $\dim\varphi \geq 9$*;*

iii. if $\dim\varphi = 8$ *and* $\det\varphi \neq 1$*;*

iv. if $\dim\varphi = 8$ *and* $a \neq 0$*.*

Proof First consider the case when $a \neq 0$. Write down the exact sequence from (5.3):

$$\mathrm{Ch}^1 X_\varphi \longrightarrow \mathrm{Ch}^2 X_{\langle a\rangle\perp\varphi} \longrightarrow \mathrm{Ch}^2 U \longrightarrow 0\,.$$

If $\dim\varphi \geq 8$ the middle group is elementary by (5.1); hence we are done in ii, iii and iv. As to i, if φ contains an Albert subform then $\langle a\rangle\perp\varphi$ contains the same subform too; hence the middle group is elementary by (5.2).

Now suppose that $a = 0$. Then we have another exact sequence from (5.3):

$$\mathrm{Ch}^1 X_\varphi \xrightarrow{h} \mathrm{Ch}^2 X_\varphi \longrightarrow \mathrm{Ch}^2 U \longrightarrow 0\,.$$

Hence $\mathrm{Ch}^2 U = 0$ if $\mathrm{Ch}^2 X_\varphi$ is elementary. To complete the proof apply once again (5.1) and (5.2). $\square$

6 The third Chow group

Theorem 6.1 *If* $\dim\varphi > 12$ *then* $\mathrm{Ch}^3 X_\varphi$ *is elementary.*

Proof First of all we make a general observation:

Lemma 6.2 *If for some* p *the group* $\mathrm{Ch}^p X_\varphi$ *is elementary for all quadratic forms (over all fields) of dimension* n *then it is also elementary for all quadratic forms of dimension* $n+1$*.*

Proof It is known that $\mathrm{Ch}^p X_{\varphi\perp\mathbf{H}} \simeq \mathrm{Ch}^{p-1} X_\varphi$ where $\mathbf{H}$ is the hyperbolic plane [2]. Hence by the assumption of the lemma Ch^{p-1} is elementary for all quadratic forms of dimension $n-2$. By the induction (on p) reason Ch^{p-1} should be then elementary for all forms of dimension $\geq n-2$ too.

Now take any quadratic form of dimension $n+1$ and write down it as a sum

$$\langle a,b\rangle\perp\varphi$$

where $\dim\varphi = n-1$. Write down the exact sequence from (5.3):

$$\mathrm{Ch}^{p-1} X_{\langle b\rangle\perp\varphi} \longrightarrow \mathrm{Ch}^p X_{\langle a,b\rangle\perp\varphi} \longrightarrow \mathrm{Ch}^p U \longrightarrow 0\,.$$

The left Ch^{p-1} is elementary, i.e. generated by h^{p-1}. The image of

$$h^{p-1} \in \mathrm{Ch}^{p-1} X_{\langle b\rangle\perp\varphi}$$

in $\mathrm{Ch}^p X_{\langle a,b\rangle\perp\varphi}$ equals h^p. Hence the group $\mathrm{Ch}^p U$ coincides with the non-elementary part $\mathrm{Ch}^p X_{\langle a,b\rangle\perp\varphi}/(h^p)$ of the group $\mathrm{Ch}^p X_{\langle a,b\rangle\perp\varphi}$ and our question is:

$$\text{why}\ \ \mathrm{Ch}^p U = 0\ ?$$

The variety U is an affine quadric defined by equation

$$a + \langle b\rangle\perp\varphi = 0\,.$$

Consider a flat morphism

$$\pi : U \to \mathbf{A}^1_F$$

given by projection on the first coordinate. It produces in a standard way [6] an exact sequence:

$$\coprod_{\alpha\in\mathbf{A}^1_F} \mathrm{Ch}^{p-1} U_\alpha \longrightarrow \mathrm{Ch}^p U \longrightarrow \mathrm{Ch}^p U_\theta \longrightarrow 0$$

where U_θ is the generic fibre of π and U_α are the fibres of π over the closed points $\alpha \in \mathbf{A}^1_F$. We will show that both the groups around $\mathrm{Ch}^p U$ are zero. Note that U_θ and U_α are affine quadrics again. The affine quadric U_θ is defined over the rational function field $F(t)$ by equation

$$(a + bt^2) + \varphi_{F(t)} = 0\,.$$

By our assumption the group Ch^p is elementary for the n-dimensional quadratic form

$$\langle a + bt^2\rangle\perp\varphi_{F(t)}\,.$$

Whence using (5.3) we obtain that $\mathrm{Ch}^p U_\theta = 0$.

The affine variety U_α is defined over the residue field $F(\alpha)$ by equation

$$(\overline{a + bt^2}) + \varphi_{F(\alpha)} = 0\,.$$

According to what was shown in the very beginning of the proof the group Ch^{p-1} is elementary for the n-dimensional quadratic form

$$\langle\overline{a + bt^2}\rangle\perp\varphi_{F(\alpha)}$$

(use it in the case when $\overline{a + bt^2} \neq 0 \in F(\alpha)$) as well as for the $(n-1)$-dimensional quadratic form $\varphi_{F(\alpha)}$ (use it in the case when $\overline{a + bt^2} = 0 \in F(\alpha)$). Whence using (5.3) we obtain that $\mathrm{Ch}^{p-1} U_\alpha = 0$ for all α. □

We obtain a chain of reductions:

i. According to the last lemma, in order to prove that $\mathrm{Ch}^3 X_\varphi$ is elementary for all φ of dimension ≥ 13 it suffices to prove that $\mathrm{Ch}^3 X_\varphi$ is elementary for any 13-dimensional φ.

ii. According to (4.1) and (4.5), in order to prove that $\mathrm{Ch}^3 X_\varphi$ is elementary for any 13-dimensional φ it suffices to prove that $\mathrm{Ch}^3 X_\varphi$ is elementary for any 14-dimensional φ of trivial discriminant.

iii. According to (4.1) and (4.9), in order to prove that $\mathrm{Ch}^3 X_\varphi$ is elementary for any 14-dimensional φ of trivial discriminant it suffices to prove that $\mathrm{Ch}^3 X_\varphi$ is elementary for any 14-dimensional φ of trivial discriminant and Clifford invariant.

To prove the last statement we will apply the corollary (1.3) of the Rost's result (1.1):

Proposition 6.3 *If φ is a 14-dimensional quadratic form of trivial discriminant and Clifford invariant then the group $\mathrm{Ch}^3 X_\varphi$ is elementary.*

Proof Write down φ as a sum of two 7-dimensional subforms

$$\varphi = \delta \perp \rho$$

in such a way that ρ contains an Albert subform from (1.3). Denote the vector F-space on which δ is defined by Δ and consider the affine quadric $U_{\delta,\rho}$ over the function field $F(\mathbf{P}(\Delta))$ of the projective space $\mathbf{P}(\Delta)$ defined by equation

$$\delta + \rho_{F(\mathbf{P}(\Delta))} = 0$$

where δ is considered as an element of $F(\mathbf{P}(\Delta))^\times$.

Lemma 6.4 *In the notations introduced above, the non-elementary part of $\mathrm{Ch}^3 X_\varphi$ coincides with $\mathrm{Ch}^3 U_{\delta,\rho}$.*

Proof Write down δ in the form

$$\delta = \langle a_0, \ldots, a_6 \rangle$$

and fix the following notations:

for every $i = 0, 1, \ldots, 6$ let

- F_i be the rational function field $F(t_1, \ldots, t_i)$, in particular $F_0 = F$;
- $f_i = a_0 + a_1 t_1^2 + \ldots + a_i t_i^2 \in F_i^\times$;
- U_i be the affine quadric over F_i defined by equation

$$f_i + (\langle a_{i+1}, \ldots, a_6 \rangle \perp \rho)_{F_i} = 0\,.$$

Since $U_6 \simeq U_{\delta,\rho}$ it is enough to show that

$$\mathrm{Ch}^3 X_\varphi/(h^3) \simeq \mathrm{Ch}^3 U_0 \simeq \mathrm{Ch}^3 U_1 \simeq \ldots \simeq \mathrm{Ch}^3 U_6 \,.$$

To get the first isomorphism of the chain use (5.3) and (5.1) (compare with the proof of (6.2)).

To show that

$$\mathrm{Ch}^3 U_i \simeq \mathrm{Ch}^3 U_{i+1}$$

take the flat morphism

$$\pi_i : U_i \longrightarrow \mathbf{A}^1_{F_i}$$

given by projection on the first coordinate of U_i (this coordinate has number $i+1$ in our notations) and consider the exact sequence (as in the proof of (6.2)) produced by π_i:

$$\coprod_{\alpha \in \mathbf{A}^1_{F_i}} \mathrm{Ch}^2(U_i)_\alpha \longrightarrow \mathrm{Ch}^3 U_i \longrightarrow \mathrm{Ch}^3(U_i)_\theta \longrightarrow 0 \,.$$

Note that $(U_i)_\theta \simeq U_{i+1}$ and $(U_i)_\alpha$ is the affine quadric over $F_i(\alpha)$ defined by equation

$$\overline{f}_{i+1} + (\langle a_{i+2}, \ldots, a_6 \rangle \perp \rho)_{F_i(\alpha)} = 0 \,.$$

Since ρ contains an Albert subform we have by (5.4) that $\mathrm{Ch}^2(U_i)_\alpha = 0$ (for all α). Whence the isomorphism desired. $\square$

Lemma 6.5 *The group* $\mathrm{Ch}^3 U_{\delta,\rho}$ *is zero.*

Proof Consider the quadratic form

$$\tau = \langle \delta \rangle \perp \rho_L$$

defined over the function field $L = F(\mathbf{P}(\Delta))$. It suffices to prove that $\mathrm{Ch}^3 X_\tau$ is elementary. In order to do it we will show that the form τ satisfies the conditions of (4.7) (for $p = 3$). Then we will be done according to (4.1).

The form τ contains a 1-codimensional subform ρ_L. The groups $\mathrm{Ch}^i X_{\rho_L}$ are elementary for all $i \leq 2$ (for $i = 2$ use (5.2) remembering that ρ contains an Albert subform !). Hence the groups $K_0(X_{\rho_L})^{(i/i+1)}$ are elementary for $i \leq 2$ too. So, the only problem remained is the condition on $C_0(\rho_L)$. We verify it by using the index reduction theorem (3.1). This verification is the central point of our work.

We should check that the index of $C_0(\rho_L)$ does not go down in the extension $L(\sqrt{\mathrm{disc}\,\tau})/L$. For this it is enough to show that the index of the F-algebra $C_0(\rho)$ does not go down in the extension $L(\sqrt{\mathrm{disc}\,\tau})/F$. Notice that the latter extension coincides with the function field of a quadratic F-form, namely of the form

$$\delta \perp \langle \mathrm{disc}\,\rho \rangle \,.$$

We have: $\dim_F C_0(\rho) = 2^6$ and $\dim\delta = 7$; moreover, the algebra $C_0(\rho)$ is not a division one (2.3) (use once again the fact that ρ contains an Albert subform). Hence the index of $C_0(\rho)$ does not go down in the extension given by $\delta\perp\langle\operatorname{disc}\rho\rangle$ according to (3.2). $\square\square$

The theorem is proven. $\square$

Corollary 6.6 *Let U be the affine quadric defined by equation $a+\varphi=0$ with some $a\in F$. The group $\mathrm{Ch}^3 U$ is zero in each of the following cases:*

i. if $\dim\varphi\geq 13$;

ii. if $\dim\varphi = 12$ and $a\neq 0$.

Proof Compare with the proof of (5.4). $\square$

7 A weaker version

In this section we prove a weaker version (7.1) of the theorem (6.1) but with the following advantage: we do not use here the Rost's result (1.1). The reason of that is to show that one can apply the technique developed above to investigate Question A from the introduction for $p>3$ too.

Theorem 7.1 *If $\dim\varphi > 16$ then $\mathrm{Ch}^3 X_\varphi$ is elementary.*

Proof Write down φ in the form

$$\varphi = \delta\perp\rho$$

with $\dim\rho = 9$ and consider an affine quadric $U_{\delta,\rho}$ constructed in the same way as in the proof of (6.3).

Lemma 7.2 *The non-elementary part of $\mathrm{Ch}^3 X_\varphi$ coincides with $\mathrm{Ch}^3 U_{\delta,\rho}$.*

Proof Prove it like (6.4). The only difference is that now $\dim\rho = 9$, so, we apply the part ii of (5.4) instead of the part i. $\square$

Lemma 7.3 *If $\dim\delta > 9$ (i.e. $\dim\varphi > 18$) then $\mathrm{Ch}^3 U_{\delta,\rho} = 0$.*

Proof As in the proof of (6.5) we just have to show that the form $\tau = \langle\delta\rangle\perp\rho_L$ satisfies the conditions of (4.7).

The form τ contains a 1-codimensional subform ρ_L. The groups $\mathrm{Ch}^i X_{\rho_L}$ are elementary for all $i\leq 2$ (for $i=2$ use (5.2) remembering that $\dim\rho = 9$). Hence the groups $K_0(X_{\rho_L})^{(i/i+1)}$ are elementary for $i\leq 2$ too. Finally, we have: $\dim_F C_0(\rho) = 2^8$ and $\dim\delta > 9$; hence the index of $C_0(\rho)$ does not go down in the extension given by $\delta\perp\langle\operatorname{disc}\rho\rangle$ according to (3.2). $\square$

Note that at this point we have already proved that $\mathrm{Ch}^3 X_\varphi$ is elementary for all quadratic forms φ of dimension > 18. It is the most short way we know to get the affirmative answer to Question A. To prove (7.1) we need to apply a little bit more power:

Lemma 7.4 *If $C_0(\rho)$ is not a division algebra then $\mathrm{Ch}^3 U_{\delta,\rho} = 0$ even in the case when $\dim\delta = 9$ (compare with (7.3)).*

Proof Argue in the same way as in the proof above. The only difference is in the very end. We have now: $\dim_F C_0(\rho) = 2^8$ and $\dim\delta = 9$; but moreover, $C_0(\rho)$ is not a division algebra. Hence the index of $C_0(\rho)$ still does not go down in the extension given by $\delta\perp\langle\operatorname{disc}\rho\rangle$ according to the same (3.2). $\square$

Lemma 7.5 *If $\dim\delta = 9$ and $C_0(\rho)$ is a division algebra then $\mathrm{Ch}^3 U_{\delta,\rho} = 0$ too (compare with (7.4)).*

Proof Write down ρ in the form

$$\rho = \langle a\rangle\perp\rho'$$

and consider the exact sequence produced by the flat morphism

$$\pi : U_{\delta,\rho} \to \mathbf{A}^1_L$$

of projecting on the first coordinate:

$$\coprod_{\alpha\in\mathbf{A}^1_L} \mathrm{Ch}^2(U_{\delta,\rho})_\alpha \longrightarrow \mathrm{Ch}^3 U_{\delta,\rho} \longrightarrow \mathrm{Ch}^3 U_{\delta\perp\langle a\rangle,\rho'} \longrightarrow 0\,.$$

Let us show that the left term is zero.

For any $\alpha\in\mathbf{A}^1_L$, the affine quadric $(U_{\delta,\rho})_\alpha$ is defined over $L(\alpha)$ by equation

$$(\overline{\delta + at^2}) + \rho'_{L(\alpha)} = 0\,.$$

If $\overline{\delta + at^2} \neq 0 \in L(\alpha)$ then $\mathrm{Ch}^2(U_{\delta,\rho})_\alpha = 0$ by the part (iv) of (5.4). Suppose that

$$\overline{\delta + at^2} = 0 \in L(\alpha)\,.$$

Then the point $\alpha\in\mathbf{A}^1_L$ *is defined* by the polynomial

$$\delta + at^2 \in L[t]$$

whence the field extension $L(\alpha)/F$ coincides with the function field of the quadratic form $\delta\perp\langle a\rangle$. By the assumption of the lemma $C_0(\rho)$ is a division algebra. Hence by (2.3) discriminant of the subform ρ' is non-elementary. Since F is algebraically closed insight of $F(\delta\perp\langle a\rangle) = L(\alpha)$ we conclude that $\operatorname{disc}\rho'_{L(\alpha)} \neq 1$ too whence $\mathrm{Ch}^2(U_{\delta,\rho})_\alpha = 0$ by the part iii of (5.4).

Now consider the right term (not the very right !) of the exact sequence from above. The projective closure X of the affine quadric staying there is given by the quadratic form

$$\langle \delta + at^2\rangle \perp \rho'_{L(t)}$$

where (remember !) L is a purely transcendental extension of F. Since $C_0(\rho) = C_0(\langle a\rangle\perp\rho')$ is a division algebra, the $L(t)$-algebra

$$C_0(\langle \delta + at^2\rangle \perp \rho'_{L(t)})$$

is a division algebra (by a specialization reason) too. Hence by (4.3) the groups $K_0(X)^{(i/i+1)}$ are elementary for all i. In particular, so is the group $K_0(X)^{(3/4)} = \mathrm{Ch}^3 X$ what implies that $\mathrm{Ch}^3 U_{\delta\perp\langle a\rangle,\rho'} = 0$.

Thus $\mathrm{Ch}^3 U_{\delta,\rho} = 0$. □

Corollary 7.6 *If* $\dim\varphi \geq 18$ *then* $\mathrm{Ch}^3 X_\varphi$ *is elementary.*

Proof It follows from (7.2), (7.3), (7.4) and (7.5). □

To finish the proof of the theorem apply (4.5). □

8 The fourth Chow group

It is obvious that many parts of the proofs above are valid for any p not only for $p = 3$. In fact using the same methods one can get the affirmative answer to Question A (from the introduction) for an arbitrary p under assumption that:

i. we have already the affirmative answer to Question A for all $p' < p$ and

ii. the epimorphism $\mathrm{Ch}^p X_\varphi \twoheadrightarrow K_0(X_\varphi)^{(p/p+1)}$ has no kernel if $\dim\varphi \gg 0$.

So, the only problem to proceed by induction is ii. The kernel mentioned is controlled by some K-cohomology groups (which stay on the "first" diagonal in the E_2-term of the BGQ-spectral sequence [11]). One way to prove that this kernel is zero in some particular situation is to show that the corresponding K-cohomology groups are "elementary" in some sense. For instance, (4.1) is in fact a computation of the K-cohomology groups $H^0(X_\varphi, K_1)$ and $H^1(X_\varphi, K_2)$ [2].

More precisely, let us refer as *elementary part* of $H^{p-1}(X_\varphi, K_p)$ to the image of the pull-back

$$in^* : H^{p-1}(\mathbf{P}, K_p) \longrightarrow H^{p-1}(X_\varphi, K_p)$$

where $in : X_\varphi \hookrightarrow \mathbf{P}$ is the imbedding of X_φ in the projective space. It is easy to compute that $H^{p-1}(\mathbf{P}, K_p) = F^\times$ [16, 17] and to show that the homomorphism in^* is always injective [2]. So, the elementary part of each $H^{p-1}(X_\varphi, K_p)$ (for $0 \le p-1 \le \dim X_\varphi$) is $F^\times$. As we do with $\mathrm{Ch}^p X_\varphi$ we say that a group $H^{p-1}(X_\varphi, K_p)$ is elementary if it coincides with its elementary part.

Remark If $\dim \varphi > 2p$ then $H^{p-1}(X_\varphi, K_p)$ is a direct sum of its elementary part and the kernel of the restriction map

$$H^{p-1}(X_\varphi, K_p) \longrightarrow H^{p-1}(X_{\overline{\varphi}}, K_p)\,.$$

Here are the statements on $H^{p-1}(X_\varphi, K_p)$ for $p = 1, 2$ mentioned above.

Proposition 8.1 *If* $\dim \varphi > 2$ *then* $H^0(X_\varphi, K_1)$ *is elementary; if* $\dim \varphi > 4$ *then* $H^1(X_\varphi, K_2)$ *is elementary.*

If we want to struggle with the kernel of

$$\mathrm{Ch}^4 X_\varphi \twoheadrightarrow K_0(X_\varphi)^{(4/5)}$$

we have to understand the group $H^2(X_\varphi, K_3)$. The following observation is due to Rost:

Proposition 8.2 ([13, 9]) *The kernel of the restriction*

$$H^2(X_\varphi, K_3) \longrightarrow H^2(X_{\overline{\varphi}}, K_3)$$

coincides with the kernel of the Galois cohomology map

$$H^4(F, \mathbf{Z}/2) \longrightarrow H^4(F(\varphi), \mathbf{Z}/2)\,.$$

Proposition 8.3 ([5]) *Let* φ *be any quadratic form with* $\dim \varphi \ge 5$. *The kernel of the map*

$$H^4(F, \mathbf{Z}/2) \longrightarrow H^4(F(\varphi), \mathbf{Z}/2)$$

is non-trivial iff φ *is similar to a subform of an anisotropic 4-Pfister form. In particular, if* $\dim \varphi > 16$ *then the kernel is zero.*

Corollary 8.4 *If* $\dim \varphi > 16$ *then the group* $H^2(X_\varphi, K_3)$ *is elementary.*

Proof Follows from (8.3), (8.2) and the remark. □

Now we are able to prove

Theorem 8.5 *If* $\dim \varphi > 24$ *then the group* $\mathrm{Ch}^4 X_\varphi$ *is elementary.*

Proof goes parallel to the proof of (7.1). Write down φ in the form

$$\varphi = \delta \perp \rho$$

with $\dim \rho = 13$ and consider the affine quadric $U_{\delta,\rho}$.

Lemma 8.6 *The non-elementary part of* $\mathrm{Ch}^4 X_\varphi$ *coincides with* $\mathrm{Ch}^4 U_{\delta,\rho}$.

Proof is like (7.2) but with using of (6.6) instead of (5.4). □

Lemma 8.7 *If* $\dim \delta > 13$ *(i.e.* $\dim \varphi > 26$*) then* $\mathrm{Ch}^4 U_{\delta,\rho} = 0$.

Proof As in the proof of (7.3) one checks that the form $\tau = \langle\delta\rangle \perp \rho_L$ satisfies the conditions of (4.7) (for $p = 4$ now). It implies that the group $K_0(X_\tau)^{(4/5)}$ is elementary. If we show that the epimorphism $\mathrm{Ch}^4 X_\tau \twoheadrightarrow K_0(X_\tau)^{(4/5)}$ has no kernel we are done. For this it suffices to check that the group $H^2(X_\tau, K_3)$ is elementary (the groups $H^{p-1}(X_\tau, K_p)$ for $p = 1, 2$ are elementary by (8.1) since $\dim \tau = 14 > 4$).

Consider the exact sequence

$$H^1(X_{\rho_L}, K_2) \to H^2(X_\tau, K_3) \to H^2(U_{\delta,\rho}, K_3) \to \mathrm{Ch}^2 X_{\rho_L} \to \mathrm{Ch}^3 X_\tau \,.$$

Since $\dim \rho = 13$ the group $Ch^2 X_{\rho_L}$ is elementary; by this reason the map from the right-hand side is an inclucion. Since the group $H^1(X_{\rho_L}, K_2)$ is elementary too (8.1) the non-elementary part of $H^2(X_\tau, K_3)$ coincides with the group $H^2(U_{\delta,\rho}, K_3)$.

To show that the latter group is trivial consider a sequence of affine quadrics $U_0, U_1, \ldots, U_r = U_{\delta,\rho}$ constructed like in the proof of (6.4). For every i, $0 \le i < r$, we have an exact sequence

$$\coprod_{\alpha \in \mathbf{A}^1_{F_i}} H^1((U_i)_\alpha, K_2) \to H^2(U_i, K_3) \to H^2(U_{i+1}, K_3) \to \coprod_{\alpha \in \mathbf{A}^1_{F_i}} \mathrm{Ch}^2 (U_i)_\alpha \,.$$

Both the side terms are zero whence the middle arrow is bijective and we obtain an isomorphism:

$$H^2(U_{\delta,\rho}, K_3) \simeq H^2(U_0, K_3)\,.$$

The projective closure of the affine quadric U_0 is X_φ, since $\dim \varphi > 16$ the group $H^2(X_\varphi, K_3)$ is elementary by(8.4) whence $H^2(U_0, K_3) = 0$. □

Note that at this point we have already proved that $\mathrm{Ch}^4 X_\varphi$ is elementary for all quadratic forms φ of dimension > 26. It is the most short way we know to get the affirmative answer to Question A (for $p = 4$). To prove (8.5) we need to apply a little bit more power:

Lemma 8.8 *If* $C_0(\rho)$ *is not a division algebra then* $\mathrm{Ch}^4 U_{\delta,\rho} = 0$ *even in the case when* $\dim \delta = 13$ *(compare with (8.7)).*

Proof The form $\tau = \langle\delta\rangle\perp\rho_L$ still satisfies the conditions of (4.7) (compare with the proof of (7.4)). Argue further in the same way as in the proof above. $\square$

Lemma 8.9 *If* $\dim\delta = 13$ *and* $C_0(\rho)$ *is a division algebra then* $\mathrm{Ch}^4 U_{\delta,\rho} = 0$ *too (compare with (8.8)).*

Proof repeats the proof of (7.5) word to word up to some point.

Write down ρ in the form

$$\rho = \langle a\rangle\perp\rho'$$

and consider the exact sequence produced by the flat morphism

$$\pi : U_{\delta,\rho} \to \mathbf{A}^1_L$$

of projecting on the first coordinate:

$$\coprod_{\alpha\in\mathbf{A}^1_L} \mathrm{Ch}^3(U_{\delta,\rho})_\alpha \longrightarrow \mathrm{Ch}^4 U_{\delta,\rho} \longrightarrow \mathrm{Ch}^4 U_{\delta\perp\langle a\rangle,\rho'} \longrightarrow 0\,.$$

Let us show that the left term is zero.

For any $\alpha \in \mathbf{A}^1_L$, the affine quadric $(U_{\delta,\rho})_\alpha$ is defined over $L(\alpha)$ by equation

$$(\overline{\delta + at^2}) + \rho'_{L(\alpha)} = 0\,.$$

If $\overline{\delta + at^2} \neq 0 \in L(\alpha)$ then $\mathrm{Ch}^3(U_{\delta,\rho})_\alpha = 0$ by (6.6). Suppose that

$$\overline{\delta + at^2} = 0 \in L(\alpha)\,.$$

Then the point $\alpha \in \mathbf{A}^1_L$ *is defined* by the polynomial

$$\delta + at^2 \in L[t]$$

whence the field extension $L(\alpha)/F$ coincides with the function field of the quadratic form $\delta\perp\langle a\rangle$. By the assumption of the lemma $C_0(\rho)$ is a division algebra. Hence $C_0(\rho') \subset C_0(\rho)$ is a division algebra too. We wont to show now that $C_0(\rho')$ remains a division algebra over the extension $L(\alpha)/F$. Let E be the center of $C_0(\rho')$. It is a degree 2 extension of F and $\dim_E C_0(\rho') = 2^{10}$. Since dimension of the quadratic form $\delta\perp\langle a\rangle$ equals 14 the algebra

$$C_0(\rho') \otimes_E E(\delta\perp\langle a\rangle)$$

is by (3.2) still a division one. Whence

$$C_0(\rho') \otimes_F F(\delta\perp\langle a\rangle) = C_0(\rho')_{L(\alpha)}$$

is a division algebra too. Thus according to (4.3) and (4.1) the third Chow group of the projective quadric $X_{\rho'_{L(\alpha)}}$ is elementary, whence $\mathrm{Ch}^3(U_{\delta,\rho})_\alpha = 0$.

Now consider the right term (not the very right !) of the exact sequence from above. The projective closure X of the affine quadric staying there is given by the quadratic form

$$\langle \delta + at^2 \rangle \perp \rho'_{L(t)}$$

where (remember !) L is a purely transcendental extension of F. Since $C_0(\rho) = C_0(\langle a \rangle \perp \rho')$ is a division algebra, the $L(t)$-algebra

$$C_0(\langle \delta + at^2 \rangle \perp \rho'_{L(t)})$$

is a division algebra (by a specialization reason) too. Hence by (4.3) the group $K_0(X)^{(4/5)}$ is elementary. Thereby so is also the group $\mathrm{Ch}^4 X$ (we use at this place that $H^2(X, K_3)$ is elementary as was shown in the proof of (8.6)) what implies that $\mathrm{Ch}^4 U_{\delta \perp \langle a \rangle, \rho'} = 0$.

Thus $\mathrm{Ch}^4 U_{\delta,\rho} = 0$. □

Corollary 8.10 *If* $\dim \varphi \geq 26$ *then* $\mathrm{Ch}^4 X_\varphi$ *is elementary.*

Proof It follows from (8.6), (8.7), (8.8) and (8.9). □

To finish the proof of the theorem apply (4.5). □

9 Ch^3 of lower-dimensional quadrics

Here is a summary of facts on $\mathrm{Ch}^3 X_\varphi$ of lower-dimensional quadrics ($\dim \varphi \leq 12$).

Theorem 9.1 ([4]) *For any quadratic form* φ *the torsion subgroup of* $\mathrm{Ch}^3 X_\varphi$ *is either* 0 *or* $\mathbf{Z}/2$.

In what follows we suppose that φ is anisotropic. In this case the non-elementary part of $\mathrm{Ch}^3 X_\varphi$ coincides with the torsion subgroup always except when φ is an 8-dimensional form of trivial discriminant. The statements below are from [2]:

- If $\dim \varphi < 6$ then $\mathrm{Ch}^3 X_\varphi$ is elementary.
- Suppose that $\dim \varphi = 6$. The group $\mathrm{Ch}^3 X_\varphi$ is non-elementary iff φ contains a quaternion subform (i.e. a 4-dimensional subform of trivial discriminant).
- Suppose that $\dim \varphi = 7$. The group $\mathrm{Ch}^3 X_\varphi$ is non-elementary iff either
  - φ completely (so much as possible) splits in a quadratic extension or

– $C_0(\varphi)$ has index 4 and φ over an arbitrary odd-degree extension of F does not contain a subform similar to a 2-Pfister form (example: $\varphi =$ (a general Albert form)$\perp\langle 1\rangle$).

- Suppose that $\dim\varphi = 8$ and $\det\varphi = 1$. Then $\mathrm{Ch}^3 X_\varphi$ contains a torsion iff φ is similar to a 3-Pfister form. The non-elementary part modulo torsion is an infinite cyclic group (which generator may be described precisely).

For forms of dimension between 9 and 12 we dispose only some examples with non-elementary Ch^3:

$\dim\varphi = 9$: φ is anisotropic of the kind

$$\varphi = (\text{a 3-Pfister form})\perp(\text{1-dimensional form})$$

$\dim\varphi = 10$: $\varphi =$ (an anisotropic 3-Pfister form)$\perp$(hyperbolic plane)

$\dim\varphi = 11$: φ is a subform of the next example

$\dim\varphi = 12$: $\varphi =$ (a general Albert form)$\otimes$(a general binary form), i.e. φ is a general 12-dimensional form of trivial discriminant and Clifford invariant.

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