CHOW RING OF GENERICALLY TWISTED VARIETIES OF COMPLETE FLAGS

NIKITA A. KARPENKO

Abstract. Let G be a split simple affine algebraic group of type A or C over a field k, and let E be a standard generic G-torsor over a field extension of k. We compute the Chow ring of the *variety of Borel subgroups of* G (also called the *variety of complete flags of* G), twisted by E. In most cases, the answer contains a large finite torsion subgroup. The torsion-free cases have been treated in the predecessor *Chow ring of some generically twisted flag varieties* by the author.

Contents

1. Introduction

Let G be a split semisimple affine algebraic group over a field k and let $\mathcal{B}$ be its *variety of Borel subgroups* (also called the *variety of complete flags* of G). A rational point on $\mathcal{B}$ is given by a Borel subgroup $B \subset G$. Picking up such a point, one establishes an isomorphism of $\mathcal{B}$ with the quotient variety G/B. The Chow ring $\operatorname{CH}\mathcal{B}$ of $\mathcal{B}$ is well-studied and understood.

The situation changes dramatically when we twist $\mathcal{B}$ by a G-torsor (i.e., a principle homogeneous space) E over a field extension $F \supset k$. The F-variety Y thus obtained is isomorphic to the quotient variety E/B. It is not reasonable to hope to understand $\operatorname{CH} Y$ in general.

If the torsor E is split, the situation is easy because $Y \simeq \mathcal{B}_F$. Whatever viewpoint is applied to measure how far is a given torsor E from the split one, the *standard generic torsors* are always among the winners. A standard generic torsor E is, by definition, the

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generic fiber of the quotient map $\mathrm{GL}_N \to \mathrm{GL}_N /G$ for an imbedding of G into the general linear group GL_N for some $N \geq 1$. In particular, E is a G-torsor over the function field

$$F := k(\mathrm{GL}_N /G) \supset k.$$

The Chow ring of the corresponding F-variety Y does not depend on the choice of the imbedding $G \hookrightarrow \mathrm{GL}_N$ and is the main object of study in the paper.

A standard tool of studying $\mathrm{CH}\, Y$ for an arbitrary smooth variety Y is the canonical epimorphism $\mathrm{CH}\, Y \to GK(Y)$, where $K(Y)$ is the Grothendieck ring of Y endowed with the *topological filtration* (also called *geometrical filtration* as well as *filtration by codimension of support*) and $GK(Y)$ is the associated graded ring. The kernel of this epimorphism is contained in the torsion subgroup $\mathrm{Tors}\,\mathrm{CH}\, X$ of $\mathrm{CH}\, X$ ([6, Example 15.3.6]) and is controlled by differentials of the Brown-Gersten-Quillen spectral sequence ([16]).

For our particular Y, the ring $\mathrm{CH}\, Y$ is generated by its component $\mathrm{CH}^1 Y$ ([7, Example 2.4]). As a consequence, the topological filtration for our Y coincides with the gamma filtration ([10, Remark 2.17]). The latter is the filtration generated by K-theoretical Chern classes, introduced by A. Grothendieck for arbitrary smooth varieties as an (at least theoretically) computable approximation of the topological filtration.

Here is the short version of the main result of the paper (see §4 and §5 for the proof):

Theorem 1.1. *Let G be a split simple affine algebraic group of type* A *or* C *and let Y be its variety of complete flags twisted by a standard generic torsor. Then the canonical epimorphism $\mathrm{CH}\, Y \to GK(Y)$ is an isomorphism.*

By [19], Theorem 1.1 also holds for the type G_2; moreover, the standard generic torsor can be replaced by an arbitrary non-split torsor for this type. It would be interesting to know if Theorem 1.1 holds for other types.

Theorem 1.1 means that the relations, satisfied by the images of the elements of $\mathrm{CH}^1 Y$ in $GK(Y)$, are satisfied in $\mathrm{CH}\, Y$ by the original elements themselves. In fact, in the course of the proof we provide a complete explicit computation of the corresponding ring (see Proof of Theorem 1.1 for type A (in §4) with Corollaries 3.15 and 3.21 as well as Examples 3.16, 3.17, 3.18, 3.19, 3.20). And this is new even for the associated graded Grothendieck ring: although the gamma filtration (for Y as above, for X as below or similar) has been studied a lot during the last two decades (see, e.g., [9], [10], [3], [20], [2]), only partial results were available so far.

Our answer contains a large (finite) torsion subgroup in most cases. Therefore we get a rare (resp., a first) example of a complete computation of Chow ring with torsion for a family of projective homogeneous varieties of arbitrary large dimension under *anisotropic* semisimple algebraic groups (resp., with *indecomposable* Chow motives). The torsion-free cases have been detected in the predecessor [7]; the remaining cases seemed to be out of reach at the time of [7].

Type A constitutes the main part of Theorem 1.1, whereas type C is a byproduct. A split simple affine algebraic group G of type A over a field k is isomorphic to the quotient SL_n /μ_m (the special linear group modulo the central subgroup of mth roots of unity) for a divisor $m \geq 1$ of an integer $n \geq 2$. The rank of such G is $n-1$ so that the precise type of G is A_{n-1}. The variety of Borel subgroups $\mathcal{B}$ is the classical variety of complete flags in the n-dimensional k-vector space k^n; it does not depend on m. As to the variety Y, it depends on E depending on m. This Y can be identified with the variety of complete flags in a vector bundle of rank $n-1$ over the Severi-Brauer variety X of the central simple F-algebra A given by E. The algebra A, arising here, is a *standard generic central simple algebra of degree n and exponent m*. The variety X is the projective space of dimension $n-1$ over F, twisted by E; it has much lower dimension and is much simpler than Y in several other aspects. Theorem 1.1 for type A is equivalent to injectivity of the canonical epimorphism $\operatorname{CH} X \to GK(X)$. Because of all this, we work with X, not with Y, most of the time. Also, we reduce the problem to the case, where n is a power of a prime number, and work with this case most of the time.

2. Specialization

Let G be a split semisimple affine algebraic group over a field k and let E be a standard generic G-torsor over a field $F \supset k$. By definition, E is the generic fiber of the quotient map $\mathrm{GL}_N \to S := \mathrm{GL}_N / G$ for an imbedding $G \hookrightarrow \mathrm{GL}_N$ for some $N \geq 1$.

For any field extension F'/k and an arbitrary G-torsor E' over F', there exists an F'-point $x\colon \operatorname{Spec} F' \to S$ of S such that E' is isomorphic to the fiber of $\mathrm{GL}_N \to S$ over x, [17, §5.3]. The variety S being smooth, the image of x is a regular point on S. Fixing a system of local parameters around it, we connect the fields $F = k(S)$ and F' by a finite chain of discrete valuation fields, where each next field is the residue field of the previous one, followed by an imbedding of fields. Using the specialization homomorphism on Chow rings as in [6, Example 20.3.1], for any subgroup $P \subset G$ (we are only interested in parabolic subgroups here, so, let us assume from the beginning that P parabolic), we get a (respecting gradings) homomorphism of Chow rings $\operatorname{CH} X \to \operatorname{CH} X'$ (which we also call a *specialization homomorphism*) of the quotient varieties $X := E/P$ and $X' := E'/P$. Similarly, we get a ring homomorphism of Grothendieck rings $K(X) \to K(X')$. Let us list the properties of these homomorphisms, used in the paper.

Let L/F be a splitting field of E and let L'/F' be a splitting field of E'. The rings $\operatorname{CH} X_L$ and $\operatorname{CH} X'_{L'}$ are canonically isomorphic (to $\operatorname{CH}(G/P)$). The triangle formed by the homomorphism $\operatorname{CH} X \to \operatorname{CH} X'$ and the change of field homomorphisms $\operatorname{CH} X \to \operatorname{CH} X_L$, $\operatorname{CH} X' \to \operatorname{CH} X'_{L'}$ commutes, [11, Lemma 4.3]. In particular, the homomorphism $\operatorname{CH}_0 X \to \operatorname{CH}_0 X'$ preserves degrees of 0-cycles.

Given a second parabolic subgroup $Q \subset G$, containing P, we have morphisms $f\colon X \to Y := E/Q$ and $f'\colon X' \to Y' := E'/Q$. The squares

$$\begin{array}{ccc} \operatorname{CH} X & \xrightarrow{f_*} & \operatorname{CH} Y \\ \downarrow & & \downarrow \\ \operatorname{CH} X' & \xrightarrow{f'_*} & \operatorname{CH} Y' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \operatorname{CH} X & \xleftarrow{f^*} & \operatorname{CH} Y \\ \downarrow & & \downarrow \\ \operatorname{CH} X' & \xleftarrow{f'^*} & \operatorname{CH} Y' \end{array}$$

commute. In particular, the squares

$$\begin{array}{ccc} \operatorname{CH}(X \times Y) & \xrightarrow{pr_*} & \operatorname{CH} Y \\ \downarrow & & \downarrow \\ \operatorname{CH}(X' \times Y') & \xrightarrow{pr'_*} & \operatorname{CH} Y' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \operatorname{CH}(X \times Y) & \xleftarrow{pr^*} & \operatorname{CH} Y \\ \downarrow & & \downarrow \\ \operatorname{CH}(X' \times Y') & \xleftarrow{pr'^*} & \operatorname{CH} Y' \end{array}$$

commute, where from now on $Q \subset G$ is an arbitrary parabolic subgroup (not necessarily containing P), and $pr\colon X \times Y \to Y$, $pr'\colon X' \times Y' \to Y'$ are the projections. As a consequence, given a correspondence $\alpha : X \rightsquigarrow Y$, the square

$$\begin{array}{ccc} \mathrm{CH}\, X & \xrightarrow{\alpha_*} & \mathrm{CH}\, Y \\ \big\downarrow & & \big\downarrow \\ \mathrm{CH}\, X' & \xrightarrow{\alpha'_*} & \mathrm{CH}\, Y' \end{array}$$

commutes, where $\alpha' : X' \rightsquigarrow Y'$ is the specialization of α, i.e., the image of α under the specialization homomorphism $\mathrm{CH}(X \times Y) \to \mathrm{CH}(X' \times Y')$. (See [5, §62] for definition of α_* and α'_* as well as for other basic facts on correspondences.)

Similarly, the rings $K(X_L)$ and $K(X'_{L'})$ are canonically isomorphic and the triangle formed by the homomorphism $K(X) \to K(X')$ and the change of field homomorphisms $K(X) \to K(X_L)$, $K(X') \to K(X'_{L'})$ commutes.

Finally, the square

$$\begin{array}{ccc} K(X) & \xrightarrow{c} & \mathrm{CH}\, X \\ \big\downarrow & & \big\downarrow \\ K(X') & \xrightarrow{c} & \mathrm{CH}\, X', \end{array}$$

where c is the total Chern class map, is commutative.

In the sequel of the paper, we apply the above properties in the case of $G = \mathrm{SL}_n / \mu_m$, where $m \geq 1$ is a divisor of an integer $n \geq 2$. So, E gives rise to a standard generic central simple algebra A over F of degree n and exponent m whereas E' gives rise to a central simple algebra A' over F' of degree n and exponent dividing m. Let P be a (maximal) parabolic subgroup whose conjugacy class corresponds to the subset of the Dynkin diagram of G obtained by removing the first vertex. Then E/P and E'/P are the Severi-Brauer varieties of A and A'. Removing ith vertex instead of the first one (with $i = 1, \ldots, n-1$), we obtain the generalized Severi-Brauer varieties $\mathrm{SB}_i(A)$ and $\mathrm{SB}_i(A')$.

3. Chow rings of Severi-Brauer varieties

Everywhere below, X is the Severi-Brauer variety of a central simple algebra A over a field F. For a classical (resp., very recent) exposition of basic properties of Severi-Brauer variety we refer to [1] (resp., [13, §3]).

Theorem 3.1. *If the Chow ring* $\mathrm{CH}\, X$ *is generated by* $\mathrm{CH}^1 X$ *and the Chern classes of the tautological (of rank* $\deg A$*) vector bundle on* X*, then the canonical epimorphism* $\mathrm{CH}\, X \to GK(X)$ *onto the associated graded ring* $GK(X)$ *of the topological filtration on the Grothendieck group* $K(X)$ *is an isomorphism and the topological filtration on* $K(X)$ *coincides with the gamma filtration.*

Since $\mathrm{CH}\, X$ is generated by Chern classes, the topological filtration on $K(X)$ does coincide with the gamma filtration, [10, Remark 2.17]. Therefore, only the assertion on $\mathrm{CH}\, X \to GK(X)$ needs to be proved. The proof is given further in this section after some preparation work is accomplished.

For any field extension L/F, the change of field homomorphism $K(X) \to K(X_L)$ is injective by [16].

Lemma 3.2. *Assume that A is a division algebra of degree p^n and exponent p^m for some prime number p and some integers $1 \le m \le n$. Let L/F be a splitting field of A. If $m = n$, then*

$$\log_p([K(X_L) : K(X)]) = b_n := np^n - \frac{p^n - 1}{p - 1}.$$

For arbitrary m, one has

$$\log_p([K(X_L) : K(X)]) \le b_n - b_{n-m}.$$

Proof. By [16], $[K(X_L) : K(X)] = \prod_{i=0}^{p^n-1} \operatorname{ind} A^{\otimes i}$. If $m = n$, then $\log_p \operatorname{ind} A^{\otimes i} = n - v_p(i)$, where $v_p(i)$ is the p-adic valuation of i. For arbitrary m, $\log_p \operatorname{ind} A^{\otimes i} \le n - v_p(i)$ for any i and $\operatorname{ind} A^{\otimes i} = 1$ for i with $v_p(i) \ge m$. □

Let $C \subset \operatorname{CH} X$ be the (graded) subring, generated by the Chern classes of the tautological vector bundle (of rank $\deg A$) on X.

Proposition 3.3. *Let L/F be a splitting field of A. For any $i = 0, \dots, \deg A - 1 = \dim X$, the composition $C^i \hookrightarrow \operatorname{CH}^i X \to \operatorname{CH}^i X_L = \mathbb{Z}$ is injective and its image is generated by $\deg A/(i, \deg A)$, where $(i, \deg A)$ is the g.c.d. of i and $\deg A$.*

Proof. By [7, Theorem 3.7 with Remark 3.4], the statement holds in the case of $m = n$. Therefore, by specialization (see §2), it holds for arbitrary m. □

Proposition 3.4. *Assume that A is a division algebra. For any finite field extension L/F, splitting A, the image of the norm map $\operatorname{CH} X_L \to \operatorname{CH} X$ is contained in C. For every $i \ge 0$, the component C^i is contained in the subgroup of $\operatorname{CH}^i X$ generated by the images of the norm maps $\operatorname{CH} X_L \to \operatorname{CH} X$ for L running over the finite field extensions of F with $\operatorname{ind} A_L$ dividing i.*

Proof. The second projection $X \times X \to X$ is identified with the projective bundle of the *canonical bundle on X* – the dual of the tautological bundle on X. Therefore we may consider the tautological line bundle on $X \times X$.

Let $\tilde{A}$ be a standard generic central simple algebra of degree $\deg A$ and exponent $\deg A$ over a field $\tilde{F} \supset F$. By [7, Corollary 3.8], the Chow ring of the Severi-Brauer variety $\tilde{X}$ of $\tilde{A}$ is generated by the Chern classes of the tautological vector bundle. In particular, for any $i \ge 0$, every element of $\tilde{\alpha}^i_* \operatorname{CH}_0 \tilde{X}$ is a polynomial in Chern classes of the tautological vector bundle, where $\tilde{\alpha} \in \operatorname{CH}^1(\tilde{X} \times \tilde{X})$ is the first Chern class of the tautological line bundle on $\tilde{X} \times \tilde{X}$ and $\tilde{\alpha}^i \in \operatorname{CH}^i(\tilde{X} \times \tilde{X})$ is its ith power in the Chow ring that we consider as a correspondence $\tilde{\alpha}^i \colon \tilde{X} \rightsquigarrow \tilde{X}$, [5, §62]. Since the group $\operatorname{CH}_0 X$ is torsion-free ([14], see also [4, Corollary 7.3]), the specialization map $\operatorname{CH}_0 \tilde{X} \to \operatorname{CH}_0 X$ is surjective (see §2). From the commutative square

$$\begin{array}{ccc} \operatorname{CH}_0 \tilde{X} & \xrightarrow{\tilde{\alpha}^i_*} & \operatorname{CH}^i \tilde{X} \\ \downarrow & & \downarrow \\ \operatorname{CH}_0 X & \xrightarrow{\alpha^i_*} & \operatorname{CH}^i X \end{array}$$

we deduce that $\alpha_*^i \operatorname{CH}_0 X \subset C^i$, where $\alpha \in \operatorname{CH}^1(X \times X)$ is the first Chern class of the canonical line bundle on $X \times X$. Since for any finite field extension L/F, splitting A, the map $\alpha_*^i \colon \operatorname{CH}_0 X_L \to \operatorname{CH}^i X_L$ of the commutative square

$$\begin{array}{ccc} \operatorname{CH}_0 X_L & \xrightarrow{\alpha_*^i} & \operatorname{CH}^i X_L \\ \downarrow & & \downarrow \\ \operatorname{CH}_0 X & \xrightarrow{\alpha_*^i} & \operatorname{CH}^i X \end{array}$$

is an isomorphism (cf. [13, §3]), the image of the norm map $\operatorname{CH}^i X_L \to \operatorname{CH}^i X$ is contained in $\alpha_*^i \operatorname{CH}_0 X$ and therefore in C^i.

We proved the first statement of Proposition 3.4. To prove the second one, note that the second projection $Y \times X \to X$, where Y is the generalized Severi-Brauer variety $\mathrm{SB}_i(A)$ of right ideals of reduced dimension i in A, is identified with the Grassmann bundle of i-planes of the canonical bundle on X. Therefore we may consider the tautological bundle (of rank i) on $Y \times X$.

Let $\tilde{Y}$ be the generalized Severi-Brauer variety $\mathrm{SB}_i(\tilde{A})$ of right ideals of reduced dimension i in $\tilde{A}$. Let $\tilde{\beta} \in \operatorname{CH}^i(\tilde{Y} \times \tilde{X})$ be the ith Chern class of the tautological bundle (of rank i) on $\tilde{Y} \times \tilde{X}$. It follows from [7, Theorem 3.7] that the group $\operatorname{CH}^i \tilde{X}$ is generated by the images of the norm maps $\operatorname{CH}^i \tilde{X}_{\tilde{L}} \to \operatorname{CH}^i \tilde{X}$, where $\tilde{L}$ runs over finite field extensions of $\tilde{F}$ with $\operatorname{ind} \tilde{A}_{\tilde{L}} | \, i$. From the commutative diagram

$$\begin{array}{ccc} \operatorname{CH}_0 \tilde{Y}_{\tilde{L}} & \xrightarrow{\tilde{\beta}_*} & \operatorname{CH}^i \tilde{X}_{\tilde{L}} \\ \downarrow & & \downarrow \\ \operatorname{CH}_0 \tilde{Y} & \xrightarrow{\tilde{\beta}_*} & \operatorname{CH}^i \tilde{X} \end{array}$$

we conclude that the map $\tilde{\beta}_* \colon \operatorname{CH}_0 \tilde{Y} \to \operatorname{CH}^i \tilde{X}$ is surjective. It follows by specialization that the image of the map $\beta_* \colon \operatorname{CH}_0 Y \to \operatorname{CH}^i X$ is equal to C^i. The group $\operatorname{CH}_0 Y$ is generated by the images of the norm maps $\operatorname{CH}_0 Y_L \to \operatorname{CH}_0 Y$ with L running over the finite field extension of F satisfying $\operatorname{ind} A_L | \, i$. With the commutative square

$$\begin{array}{ccc} \operatorname{CH}_0 Y_L & \xrightarrow{\beta_*} & \operatorname{CH}^i X_L \\ \downarrow & & \downarrow \\ \operatorname{CH}_0 Y & \xrightarrow{\beta_*} & \operatorname{CH}^i X, \end{array}$$

the second statement of Proposition 3.4 follows. □

Taking for L in the first statement of Proposition 3.4 a maximal subfield of A, and using the fact that the composition $\operatorname{CH} X \to \operatorname{CH} X_L \to \operatorname{CH} X$ is multiplication by $[L : F] = \deg A$, we get

Corollary 3.5. *The quotient group* $(\operatorname{CH} X)/C$ *is annihilated by* $\deg A$. □

For any finite field extension L/F, splitting A, the change of field homomorphism $\operatorname{CH}^1 X \to \operatorname{CH}^1 X_L = \mathbb{Z}$ is injective and its image is generated by the integer $\exp A$, [1, §2]. Let e be a generator of $\operatorname{CH}^1 X$.

The following consequence of Proposition 3.4 is already proved in [13, §3]:

Corollary 3.6. *For a maximal subfield (or, equivalently, a minimal splitting field) L of A, the image N of the norm map $\operatorname{CH} X_L \to \operatorname{CH} X$ does not depend on the choice of L. For any finite field extension L/F, splitting A, the image of the norm map $\operatorname{CH} X_L \to \operatorname{CH} X$ is contained in N.* □

Proposition 3.7. *Assume that $\deg A = \operatorname{ind} A = p^n$ and $\exp A = p^m$ for a prime p and some integers $1 \leq m \leq n$. Then $p^i e^{p^{n-m-i}} \in C$ for any $i = 0, 1, \ldots, n-m$.*

Proof. We identify $K(X)$ with a subring of $K(X_L)$, where L/F is a splitting field of A. Let $\xi \in K(X_L)$ be the class of the tautological line bundle on the projective space X_L. Then the multiple $p^n\xi$ of ξ is in $K(X)$ and is represented by the tautological bundle on X. Since $\operatorname{ind} A^{\otimes p^m} = 1$, we also have $\xi^{p^m} \in K(X)$. Up to a sign, the element e is the first Chern class of ξ^{p^m}.

Let $\tilde{A}$ be a generic central simple algebra of degree and exponent p^n over a field $\tilde{F} \supset F$ and let $\tilde{X}$ be its Severi-Brauer variety. We fix a splitting field $\tilde{L}/\tilde{F}$ of $\tilde{A}$ and identify $K(\tilde{X})$ with a subring of $K(\tilde{X}_{\tilde{L}})$. Let $\tilde{\xi} \in K(\tilde{X}_{\tilde{L}})$ be the class of the tautological line bundle on the projective space $\tilde{X}_{\tilde{L}}$. Since $\operatorname{ind} \tilde{A}^{\otimes p^m} = p^{n-m}$, we have $p^{n-m}\tilde{\xi}^{p^m} \in K(\tilde{X})$.

We have a commutative square

$$\begin{array}{ccc} K(\tilde{X}) & \xrightarrow{c_{p^{n-m}}} & \operatorname{CH} \tilde{X} \\ \downarrow & & \downarrow \\ K(X) & \xrightarrow{c_{p^{n-m}}} & \operatorname{CH} X, \end{array}$$

where the horizontal arrows are given by the p^{n-m}th Chern class, whereas the vertical arrows are given by specialization. The image of $p^{n-m}\tilde{\xi}^{p^m} \in K(\tilde{X})$ in $K(X)$ is $p^{n-m} \cdot \xi^{p^m} \in K(X)$ and $c_{p^{n-m}}(p^{n-m} \cdot \xi^{p^m}) = \pm e^{p^{n-m}} \in \operatorname{CH} X$. On the other hand, the image of $\operatorname{CH} \tilde{X}$ in $\operatorname{CH} X$ is the subring C.

We have proved Proposition 3.7 for $i = 0$. For arbitrary i, we use the above commutative square with the Chern class $c_{p^{n-m-i}}$ in place of $c_{p^{n-m}}$. We get:

$$\binom{p^{n-m}}{p^{n-m-i}} e^{p^{n-m-i}} \in C.$$

Since the highest p-power dividing the binomial coefficient is p^i (cf. [10, Lemma 3.5]), whereas the quotient group $(\operatorname{CH} X)/C$ is annihilated by a p-power by Corollary 3.5, we obtain the inclusion $p^i e^{p^{n-m-i}} \in C$.

Here is another proof, avoiding the specialization argument. Note that

$$\lambda^{p^m}(p^n\xi) = \binom{p^n}{p^m}\xi^{p^m},$$

where $\lambda^{p^m} \colon K(X) \to K(X)$ is the p^mth lambda operation. Since p^{n-m} is the highest p-power dividing the binomial coefficient, $p^{n-m}\xi^{p^m}$ is a linear combination with integer coefficients of $(p^n\xi)^{p^m}$ and $\lambda^{p^m}(p^n\xi)$. Since the value of a Chern class on a lambda operation of an element in $K(X)$ is a polynomial in Chern classes of the element, we conclude that all Chern classes of $p^{n-m}\xi^{p^m}$ are polynomials in Chern classes of $p^n\xi$. □

Corollary 3.8. *For A as in Proposition 3.7, let L/F be a maximal subfield of A and let $N \subset \operatorname{CH} X$ be the image of the norm map $\operatorname{CH} X_L \to \operatorname{CH} X$. Then $p^i e^{p^{n-m-i}} \in N$ for any $i = 0, 1, \dots, n-m$.*

Proof. We have $p^i e^{p^{n-m-i}} \in C$ (Proposition 3.7) and $N \subset C$ (Proposition 3.4). The composition $C \hookrightarrow \operatorname{CH} X \to \operatorname{CH} X_L$ is injective (Proposition 3.3). The image of $p^i e^{p^{n-m-i}}$ in $\operatorname{CH}^{p^{n-m-i}} X_L = \mathbb{Z}$ is equal, up to a sign, to $p^{i+mp^{n-m-i}} \geq p^n$ and therefore lies in the image of N. □

Corollary 3.9. *In contrast to Corollary 3.8, we drop here the assumption that A is a division algebra. We only assume that $\operatorname{ind} A = p^n$ and $\exp A = p^m$ for a prime p and some integers $0 \leq m \leq n$ ($\deg A$ is allowed to be an arbitrary multiple of p^n). Let L/F be a maximal subfield of the underlying division algebra of A and let $N \subset \operatorname{CH} X$ be the image of the norm map $\operatorname{CH} X_L \to \operatorname{CH} X$. Then $p^i e^{p^{n-m-i}} \in N$ for any $i = 0, 1, \dots, n-m$.*

Proof. Let D be the underlying division algebra of A. We fix an isomorphism of A onto the tensor product of D with the matrix algebra $M_d(F)$, where $d := \deg A / \operatorname{ind} A$. Tensor product of ideals gives rise to a closed imbedding $\mathrm{SB}(D) \times \mathbb{P}^{d-1} \hookrightarrow X$. Choosing a rational point on the projective space $\mathbb{P}^{d-1}$, we get a closed imbedding $\mathrm{SB}(D) \hookrightarrow X$. The pull-back of $\xi^{p^m} \in K(X)$ to $K(\mathrm{SB}(D))$ is $\xi^{p^m} \in K(\mathrm{SB}(D))$. By [8, Corollary 1.3.2], the pull-back $\operatorname{CH}^i X \to \operatorname{CH}^i \mathrm{SB}(D)$ for $i \leq \dim \mathrm{SB}(D)$ is an isomorphism and remains an isomorphism over any extension of the base field. For $i = 1$, it maps $e \in \operatorname{CH}^1 X$ to $e \in \operatorname{CH}^1 \mathrm{SB}(D)$. So, we get Corollary 3.9 for A, applying Corollary 3.8 to the underlying division algebra D of A. □

Corollary 3.10. *Let A and N be as in Corollary 3.8 and take any $j = 1, \dots, \dim X$. Then*

$$p^i e^{p^{i'}} C^j \subset N$$

for any $i, i' \geq 0$ with $i + i' \geq v_p(j) - m$.

Proof. By Proposition 3.4, we may replace C^j in both formulas to prove by the norms from a finite extension L/F with $\operatorname{ind} A_L | j$. Using Projection Formula and applying Corollary 3.9 over L, we get the result (taking Corollary 3.6 into account). □

Besides of the properties of C established in the preceding paper, here is the key computation:

Proposition 3.11. *For A as in Proposition 3.7 and for any $i \geq 1$ and any $j \geq 0$, the order of the element $e^i c^j \mod C$ in the quotient group $(\operatorname{CH} X)/C$ is at most the maximum of 1 and*

$$p^{\min\{v_p(j),\, n\} - m - [\log_p i]},$$

where c^j is a generator of C^j.

Proof. Set $r := [\log_p i]$.

If $j \geq p^n$, then $c^j = 0$ and the statement is trivial. We assume that $j \leq p^n - 1$ below.

Let us consider the case of $j = 0$ first. This is the only case where $\min\{v_p(j),\, n\}$ is n, not $v_p(j) = +\infty$. If the exponent $n - m - r$ is negative, that is to say if $r > n - m$ we use that $e^{p^{n-m}} \in N$ by Corollary 3.8. Since $p^{n-m} < p^r \leq i$, it follows by Projection Formula that $e^i \in N$. Since $N \subset C$ (Proposition 3.4), the order of the class of the element e^i in $(\operatorname{CH} X)/C$ is 1.

If $n-m-r \geq 0$, then $p^{n-m-r}e^{p^r} \in N$ by Corollary 3.8. It follows by Projection Formula that $p^{n-m-r}e^i \in N$ so that the order of $e^i \mod C$ divides p^{n-m-r}, indeed.

For $j = 1, \dots, p^n - 1$, if $v_p(j) - m < 0$, then $ec^j \in N$ by Corollary 3.10 so that $e^i c^j \in N$ as well.

If $v_p(j) - m \geq 0$ and $v_p(j) - m - r < 0$, that is to say $r > v_p(j) - m \geq 0$, we have $e^{p^{v_p(j)-m}} c^j \in N$ by Corollary 3.10, and it follows that $e^i c^j \in N$.

Finally, if $v_p(j) - m - r \geq 0$, then $p^{v_p(j)-m-r} e^{p^r} c^j \in N$ by Corollary 3.10. It follows by Projection Formula that $p^{v_p(j)-m-r} e^i c^j \in N$ so that the order of the class of the element $e^i c^j$ in the quotient $(\operatorname{CH} X)/C$ divides $p^{v_p(j)-m-r}$, indeed. □

Corollary 3.12. *For A as in Proposition 3.7, the integer b_{n-m} of Lemma 3.2 is un upper bound on the logarithm base p of the order of the subgroup in $(\operatorname{CH} X)/C$ generated by the classes of all $e^i c^j$, $i, j \geq 0$.*

Proof. We get this upper bound by taking the sum over all $i \geq 1$ and all $j \geq 0$ of $\log_p$ of the upper bound on the order of $e^i c^j \mod C$ obtained in Proposition 3.11. (For $i = 0$ the order of $e^i c^j \mod C$ is 1.) Below are the details of this computation.

The sum of $\log_p$ of the upper bounds over all j with $v_p(j) \geq n$ (we only need to look at $j = 0$ here) and all $i \geq 1$ is

$$(n-m)(p-1) + (n-m-1)(p^2-p) + \cdots + (p^{n-m} - p^{n-m-1}) = \\ -(n-m) + p + p^2 + \cdots + p^{n-m}.$$

Indeed, for $i = 1, \dots, p-1$, the logarithm base p of the upper bound is $n - m$; for $i = p, \dots, p^2 - 1$ it is $n - m - 1$, etc., for $i = p^{n-m-1}, \dots, p^{n-m} - 1$ it is 1, and, finally, it is 0 for $i \geq p^{n-m}$.

For every j with $v_p(j) = n - 1$ (i.e., for $j = p^{n-1}, 2p^{n-1}, \dots, (p-1)p^{n-1}$), the sum is the same and for all such j together we get

$$(p-1)\big((n-m-1)(p-1) + (n-m-2)(p^2-p) + \cdots + (p^{n-m-1} - p^{n-m-2})\big) = \\ (p-1)\big(-(n-m-1) + p + p^2 + \cdots + p^{n-m-1}\big) = -(n-m-1)(p-1) + p^{n-m} - p.$$

For j with $v_p(j) = n - 2$ all together, we obtain

$$(p^2-p)\big((n-m-2)(p-1) + (n-m-3)(p^2-p) + \cdots + (p^{n-m-2} - p^{n-m-3})\big) = \\ (p^2-p)\big(-(n-m-2) + p + p^2 + \cdots + p^{n-m-2}\big) = -(n-m-2)(p^2-p) + p^{n-m} - p^2.$$

And so on.

Finally, for j with $v_p(j) = m+1$ (j with $v_p(j) \leq m$ do not contribute anymore), we get

$$(p^{n-m-1} - p^{n-m-2})(-1+p) = -(p^{n-m-1} - p^{n-m-2}) + p^{n-m} - p^{n-m-1}.$$

Adding everything up and cancelling, we get the expression desired. □

Proof of Theorem 3.1. Firstly, let us note that for any prime divisor p of $\exp A$, the p-primary part A_p of A is a division algebra, i.e., $v_p(\deg A) = v_p(\operatorname{ind} A)$. Indeed, any generic central simple algebra $\tilde{A}$ (over a field extension of F) of degree $\deg A$ and exponent

$\exp A$ has this property. Since $\mathrm{CH}\,X$ is generated by $\mathrm{CH}^1 X$ and the Chern classes of the tautological vector bundle on X, any specialization homomorphism $\mathrm{CH}\,\tilde X \to \mathrm{CH}\,X$, where $\tilde X := \mathrm{SB}(\tilde A)$, is surjective. In particular, we have a surjection $\mathrm{CH}_0\,\tilde X \to \mathrm{CH}_0\,X$ showing that $\operatorname{ind} A = \operatorname{ind} \tilde A$.

Secondly, let us reduce the proof to the p-primary case: the case with $A = A_p$ for some prime number p. To show that $\mathrm{CH}\,X \to GK(X)$ is an isomorphism for a non-primary A, it suffices to show that, for any prime number p, it becomes isomorphism after tensoring by $\mathbb{Z}_{(p)}$ – the localization of $\mathbb{Z}$ at the prime ideal (p). Using the tensor product decomposition $A = A_p \otimes_F B$ for B being the tensor product of the remaining primary parts of A, and picking up a closed point of prime to p degree on $\mathrm{SB}(B)$, we get a pull-back homomorphism $\mathrm{CH}\,X \to \mathrm{CH}\,X_{p\,L}$, where $X_p := \mathrm{SB}(A_p)$ and L/F is a finite extension of prime to p degree. By [7, Proof of Lemma 3.5], tensoring by $\mathbb{Z}_{(p)}$ and composing with the divided by $[L:F]$ norm map $(\mathrm{CH}\,X_{p\,L})\otimes\mathbb{Z}_{(p)} \to (\mathrm{CH}\,X_p)\otimes\mathbb{Z}_{(p)}$, we get a graded group epimorphism $\mathrm{CH}\,X \to \mathrm{CH}\,X_p$. This epimorphism maps polynomials in the Chern classes of the tautological vector bundle on X and elements of $\mathrm{CH}^1 X$ to the polynomials in the Chern classes of a multiple of the tautological vector bundle on X_p and element of $\mathrm{CH}^1 X_p$. If follows that the ring $\mathrm{CH}\,X_p$ is generated by its component of codimension 1 and the Chern classes of the tautological vector bundle. So, if we dispose of the statement of Theorem 3.1 in the p-primary case, we conclude that $\mathrm{CH}\,X_p \to GK(X_p)$ is an isomorphism. Since the Chow motive with coefficients in $\mathbb{Z}_{(p)}$ of the variety X is a direct sum of shifts of the motive of X_p by [7, Proof of Lemma 3.5] once again, we conclude that $\mathrm{CH}\,X\otimes\mathbb{Z}_{(p)} \to GK(X)\otimes\mathbb{Z}_{(p)}$ is an isomorphism.

Thirdly and finally, let us prove Theorem 3.1 under the assumption that $A = A_p$. If A is split, the statement is trivial. Therefore we assume that A is a division algebra below. So, $\deg A = p^n$ and $\exp A = p^m$ for some $n \ge m \ge 1$. We come to the main core of the whole proof.

To show that the epimorphism $\mathrm{CH}\,X \twoheadrightarrow GK(X)$ is an isomorphism, we consider the subring $C \subset \mathrm{CH}\,X$. Since C is torsion-free and the kernel of the epimorphism consists of torsion elements, C is mapped isomorphically onto its image in $GK(X)$ which we also denote by C. We have $[\mathrm{CH}\,X : C] \ge [GK(X) : C]$ and it suffices to check that this is an equality. Corollary 3.12 provides an upper bound $p^{b_{n-m}}$ on $[\mathrm{CH}\,X : C]$. We are going to show that this upper bound is a lower bound on $[GK(X) : C]$.

Let L/F be a splitting field of A. We write $\operatorname{Im} GK(X)$ for the image of the change of field homomorphism $GK(X) \to GK(X_L)$. In the exact sequence

$$0 \longrightarrow \operatorname{Tors} GK(X) \longrightarrow GK(X) \longrightarrow \operatorname{Im} GK(X) \longrightarrow 0,$$

the subgroup $C \subset GK(X)$ maps isomorphically onto its image in $\operatorname{Im} GK(X)$. Therefore $[GK(X) : C] = \#\operatorname{Tors} GK(X) \cdot [\operatorname{Im} GK(X) : C]$. Multiplying the formula

$$\#\operatorname{Tors} GK(X) = \frac{[GK(X_L) : \operatorname{Im} GK(X)]}{[K(X_L) : K(X)]} \tag{3.13}$$

of [9, Proposition 2] by $[\operatorname{Im} GK(X) : C]$ we get the formula

$$[GK(X) : C] = \frac{[GK(X_L) : C]}{[K(X_L) : K(X)]}.$$

An upper bound $p^{b_n - b_{n-m}}$ on the denominator is given in Lemma 3.2. The number in the nominator depends only on the degree of A and is equal to p^{b_n}. This can be seen by a direct computation or applying formula (3.13) to the Severi-Brauer variety of a generic algebra of degree and exponent $\deg A$ (taking into account that the associated graded ring of the topological filtration is torsion-free). □

Remark 3.14. For any prime number p and integers $n \geq m \geq 1$, it follows from the proof of Theorem 3.1 that if a central simple F-algebra A with $\deg A = p^n$ and $\exp A = p^m$ satisfies the condition of Theorem 3.1, then the integer $[K(X_L) : K(X)]$ coincides with its upper bound from Lemma 3.2. Consequently $\operatorname{ind} A^{\otimes p^{m-1}} = p^{n-m+1}$. In particular, $\operatorname{ind} A^{\otimes p^i} = p^{n-i}$ for any $i = 0, \ldots, m-1$.

The proof of Theorem 3.1 actually provides an explicit computation of the torsion subgroup in $\operatorname{CH} X = GK(X)$. First, we formulate the answer in the p-primary case:

Corollary 3.15. *For a prime number p and integers $n \geq m \geq 1$, let A be a central simple F-algebra with $\deg A = p^n$ and $\exp A = p^m$ satisfying the condition of Theorem 3.1. Then the quotient group $(\operatorname{CH} X)/C$ is the direct sum of cyclic groups generated by the classes of $e^i c^j$, $i, j \geq 0$. The orders of the generators are precisely the upper bounds of Proposition 3.11. The torsion subgroup of $\operatorname{CH} X$ is the kernel of the change of field homomorphism $(\operatorname{CH} X)/C \to (\operatorname{CH} X_L)/C$ for an arbitrary splitting field L/F of A. In particular, the order of the torsion subgroup in $\operatorname{CH} X$ is equal to $[\operatorname{CH} X : C]/[\operatorname{Im} \operatorname{CH} X : C]$, where $\operatorname{Im} \operatorname{CH} X$ is the image of $\operatorname{CH} X$ in $\operatorname{CH} X_L$; the index $[\operatorname{CH} X : C]$ is equal to its upper bound of Corollary 3.12. For any $i = 0, 1, \ldots, p^n - 1$, the order of the torsion subgroup in $\operatorname{CH}^i X$ is equal to $[\operatorname{CH}^i X : C^i]/[\operatorname{Im} \operatorname{CH}^i X : C^i]$.* □

For any $i = 0, 1, \ldots, p^n - 1$, the index $[\operatorname{Im} \operatorname{CH}^i X : C^i]$ is computed as follows: one identifies $\operatorname{Im} \operatorname{CH}^i X$ with a subgroup in $\operatorname{CH}^i X_L = \mathbb{Z}$; then C^i is generated by $p^n/(i, p^n)$ and $\operatorname{Im} \operatorname{CH}^i X$ is generated by C^i and p^{im}.

Below we work out several examples with complete description of the torsion subgroup. Some part of the torsion (in the associated graded Grothendieck ring of gamma filtration) has been detected in [2, Corollary 2.8]; this part turns out to be the whole torsion subgroup in Examples 3.16 and 3.17.

Example 3.16. If $m = n - 1$, then the torsion subgroup in $\operatorname{CH}^i X$ is of order p for $i = 2, \ldots, p-1$ and is trivial for other i.

Example 3.17. If $p = 2$ and $m = n - 2$, then the torsion subgroup in $\operatorname{CH}^i X$ is of order 2 for $i = 2, 3$ and is trivial for other i.

Example 3.18. For $p = 2$, $m = 1$, and $n = 4$, the torsion subgroup in $\operatorname{CH}^i X$ is of order 2 for $i = 2, 3, 4, 5, 6, 7, 10, 11$ and is trivial for other i.

The order of torsion is not always prime:

Example 3.19. For $m = 1$, $n = 3$, and any odd p, the torsion subgroup in $\operatorname{CH}^i X$ is cyclic for any i. It is of order p for $i = 2$, for $i = p, \ldots, p^2 - 1$, and for $i = jp^2 + 2, \ldots, jp^2 + p - 1$ with $j = 1, \ldots, p-1$. It is of order p^2 for $i = 3, \ldots, p-1$. And it is trivial for the remaining i.

And here is an example of non-cyclic torsion:

Example 3.20. For $p = 2$, $m = 1$, and $n = 5$, the torsion subgroup in $\mathrm{CH}^{10} X$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The generators are the classes in $(\mathrm{CH}\, X)/C$ of e^{10} and $e^2 c_8$.

To complete the picture, let us describe the general answer in the non-primary case:

Corollary 3.21. *For any field F, let A be a central simple F-algebra such that its Severi-Brauer variety X satisfies the condition of Theorem 3.1. For any prime number p, let A_p be the p-primary component of A. Then its Severi-Brauer variety X_p also satisfies the condition of Theorem 3.1 (in place of X) so that $\mathrm{Tors}\,\mathrm{CH}\, X_p$ is described in Corollary 3.15. And the p-torsion subgroup $\mathrm{Tors}_p\,\mathrm{CH}^* X$ is the direct sum of d/d_p shifted copies of $\mathrm{Tors}\,\mathrm{CH}^* X_p$:*

$$\mathrm{Tors}_p\,\mathrm{CH}^* X = \bigoplus_{i=0}^{(d/d_p)-1} \mathrm{Tors}\,\mathrm{CH}^{*-id_p} X_p,$$

where $d := \deg A$ and $d_p := \deg A_p$. □

4. Type A

We start with some general preparation work.

Recall that an algebraic group over a field k is special if every G-torsor over every field extension of k is trivial.

Lemma 4.1. *Let G be a split semisimple affine algebraic group over a field k and let E be a standard generic G-torsor over a field extension F of k. A parabolic subgroup $P \subset G$ is special if and only if $E_{F(E/P)}$ is split.*

Proof. One implication is in [12, Lemma 6.5], the other – in [15, §5]. □

Lemma 4.2. *Let G be a split semisimple affine algebraic group over a field k and let E be a standard generic G-torsor over a field extension of k. For any two special parabolic subgroups P, P' of G, the epimorphism $f : \mathrm{CH}(E/P) \twoheadrightarrow GK(E/P)$ is an isomorphism if and only if the epimorphism $f' : \mathrm{CH}(E/P') \twoheadrightarrow GK(E/P')$ is an isomorphism.*

Proof. By [12, Lemma 6.5]), the Chow motive of the variety $E/P \times E/P'$ is a direct sum of shifts of the motive of E/P. Applying the additive functor of the category of Chow motive into the category of GK-motives, induced by the epimorphism $\mathrm{CH}(\cdot) \twoheadrightarrow GK(\cdot)$, we get the same motivic decomposition for GK-motives. Therefore the epimorphism $g : \mathrm{CH}(E/P \times E/P') \twoheadrightarrow GK(E/P \times E/P')$ is a direct sum of several shifted copies of the epimorphism f. It follows that f is an isomorphism if and only if g is. The same holds for f' in place of f. □

Lemma 4.3. *Let X be a smooth connected variety over a field F, $\mathcal{E}$ a vector bundle on X, and Y the variety of complete flags in $\mathcal{E}$. If the ring $\mathrm{CH}\, Y$ is generated by $\mathrm{CH}^1 Y$, then the ring $\mathrm{CH}\, X$ is generated by $\mathrm{CH}^1 X$ and the Chern classes of $\mathcal{E}$.*

Proof. Let n be the rank of $\mathcal{E}$ and let $x_i \in \mathrm{CH}^1 Y$ for $i = 1, \dots, n$ be the Euler class of the linear bundle on Y given by the quotient of the tautological bundle of rank i by its subbundle – the tautological bundle of rank $i - 1$. As $\mathrm{CH}\, X$-algebra, $\mathrm{CH}\, Y$ is generated by $x_1, \dots, x_n$ modulo the relations $\sigma_i = (-1)^i c_i(\mathcal{E})$, $i = 1, \dots, n$, where σ_i is the

ith elementary symmetric polynomial in $x_1,\dots,x_n$, cf. [18, §2]. In particular, the group $\mathrm{CH}^1 Y$ (and therefore the ring $\mathrm{CH}\,Y$) is generated by $\mathrm{CH}^1 X$ and $x_1,\dots,x_n$.

Let $B \subset \mathrm{CH}\,X$ be the subring generated by $\mathrm{CH}^1 X$ and the Chern classes of $\mathcal{E}$. By the assumption on Y, the homomorphism

$$B[x_1,\dots,x_n]/\big(\sigma_i - (-1)^i c_i(\mathcal{E})\big)_{i=1}^n \;\longrightarrow\; \mathrm{CH}\,Y$$

is surjective. It follows that $B \hookrightarrow \mathrm{CH}\,X$ is a surjection as well. □

Everything is ready for

Proof of Theorem 1.1 for type A. By [7, Example 2.4], the ring $\mathrm{CH}\,Y$ is generated by $\mathrm{CH}^1 Y$.

The variety Y is identified with the variety of complete flag of right ideals in a generic central simple algebra A over a field F of degree n and exponent m for certain $n \ge 2$ and its certain divisor $m \ge 1$, determined by the condition that $G \simeq \mathrm{SL}_n/\mu_m$. Let X be the Severi-Brauer variety of A. The projection $Y \to X$ is the complete flag variety of the vector bundle

$$\mathcal{E} := (A/\mathcal{T}) \otimes_A \mathcal{T}^\vee,$$

where $\mathcal{T}$ is the tautological vector bundle on X. It follows by Lemma 4.3 that X satisfies the condition of Theorem 3.1. Therefore $\mathrm{CH}\,X \twoheadrightarrow GK(X)$ is an isomorphism. We conclude with Lemma 4.2, because $Y \simeq E/B$, where $B \subset G$ is a Borel subgroup and therefore a special parabolic subgroup, while $X \simeq E/P$ for another special parabolic subgroup $P \subset G$. □

5. Type C

The proof of the following lemma is similar to that of Lemma 4.3:

Lemma 5.1. *Let X be a smooth connected variety over a field F, $\mathcal{E}$ a vector bundle on X, and Y the corresponding projective bundle. If the ring* $\mathrm{CH}\,Y$ *is generated by Chern classes, then the ring* $\mathrm{CH}\,X$ *is generated by Chern classes.* □

Proof of Theorem 1.1 for type C. If G (of type C) is simply connected, then G is special. Therefore the G-torsor E (a standard generic G-torsor over a field extension F of the base field) is trivial and so is the statement of Theorem 1.1 in the simply connected case. Below we assume that G is adjoint.

By [7, Example 2.4], the ring $\mathrm{CH}\,Y$ is generated by $\mathrm{CH}^1 Y$. The variety Y is identified with the variety of complete flag of isotropic right ideals in a central simple algebra A over a field F of degree $2n$ for some $n \ge 1$ and exponent 2 endowed (the algebra) with a symplectic involution σ. Note that the index of A is the highest 2-power dividing $2n$.

Let X be the Severi-Brauer variety of A. Since every right ideal of reduced dimension 1 is isotropic, Y projects onto X. The projection $Y \to X$ is the variety of complete flags of totally isotropic A-submodules of the vector bundle and right A-module $\mathcal{T}^\perp/\mathcal{T}$, where $\mathcal{T}$ is the tautological vector bundle on X and $\mathcal{T}^\perp := \sigma(\mathrm{Ann}\,\mathcal{T})$. Note that for any right ideal I of reduced dimension 1 in A, its annihilator $\mathrm{Ann}\,I$ is a left ideal, and the right ideal $\sigma(\mathrm{Ann}\,I)$ is the orthogonal complement $I^\perp$ of I in A with respect to the hermitian form $(a,b) \mapsto \sigma(a)b$ on A (this hermitian form is alternating if $\mathrm{char}\,F = 2$). The restriction of this hermitian form to $I^\perp$ is 0 on $I \subset I^\perp$ and the induced hermitian form on the quotient

$I^\perp/I$ is non-degenerate. So, $\mathcal{T}^\perp/\mathcal{T}$ comes equipped with a non-degenerate hermitian form (alternating if $\operatorname{char} F = 2$).

The projection $Y \to X$ decomposes into the composition $Y \to Y_1 \to X$, where Y_1 is the variety of totally isotropic A-submodules of reduced dimension 1 in $\mathcal{T}^\perp/\mathcal{T}$. Since every A-submodules of reduced dimension 1 is totally isotropic, $Y_1 \to X$ is the projective bundle of the vector bundle $(\mathcal{T}^\perp/\mathcal{T}) \otimes_A \mathcal{T}^\vee$. Proceeding similarly, one decomposes $Y \to X$ in a composition of $n-1$ projective bundle morphisms.

It follows by Lemma 5.1 that the ring $\operatorname{CH} X$ is generated by Chern classes. Since $\exp A = 2$, the Grothendieck group $K(X)$ is (additively) generated by the element $(\operatorname{ind} A)\xi^i$, $i \geq 0$, and powers of ξ^2, where for a splitting field L/F of the algebra, $\xi \in K(X_L)$ is the class of the tautological line bundle on the projective space X_L. It follows that $\operatorname{CH} X$ is generated by $\operatorname{CH}^1 X$ and the Chern classes of the tautological vector bundle. Therefore – by Theorem 3.1 – $\operatorname{CH} X \twoheadrightarrow GK(X)$ is an isomorphism. We conclude with Lemma 4.2. $\square$

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Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: `karpenko` *at* `ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`