Order of Torsion in CH^4 of Quadrics

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Abstract. It is shown that the order of the torsion subgroup in the 4-codimensional Chow group $\mathrm{CH}^4(X_\varphi)$ of a projective quadric X_φ is at most 4 provided that the dimension of the corresponding quadratic form φ is greater than 8.

Consider a non-degenerate quadratic form φ over a field F of characteristic different from 2 and the corresponding projective quadric X_φ. We always assume that $\dim X_\varphi \geq 1$, i.e. $\dim \varphi \geq 3$. It is an open problem to describe the torsion subgroup of the Chow group $\mathrm{CH}^*(X_\varphi)$ (this is the group of algebraic cycles on X_φ modulo rational equivalence graded by co-dimension of cycles [1, 2]).

Generally speaking computation of the Chow group of an algebraic variety is an interesting and important problem of algebraic geometry. However the class of varieties for which this problem is solved is rather small. Chow groups and K-theory of quadrics were studied first by R. Swan. Although the K-theory was completely computed [13] the question on the Chow group remained open.

A new motivation grew out of the attempts to solve the norm residue homomorphism problem. During the work on this problem it became clear that a decisive progress could be achieved by computation of the so called K-cohomology groups [10, 12] for quadrics and in particular of their Chow groups.

In [4] Chow groups of small-dimensional quadrics were computed. An interesting phenomenon was found: some Chow groups have torsion and the problem of computing the whole Chow group reduces to finding the torsion.

Let us consider some first gradation components. The group $\mathrm{CH}^1(X_\varphi)$ is always torsion-free. The next group — $\mathrm{CH}^2(X_\varphi)$ is computed in [4]. In particular, it turns out that $\#\operatorname{Tors}\mathrm{CH}^2(X_\varphi) \leq 2$ for any form φ; moreover, $\operatorname{Tors}\mathrm{CH}^2(X_\varphi) = 0$ if $\dim\varphi > 8$ [4, theorem (6.1)]. In co-dimension 3 one has: $\#\operatorname{Tors}\mathrm{CH}^3(X_\varphi) \leq 2$ for any φ [5, theorem] and $\operatorname{Tors}\mathrm{CH}^3(X_\varphi) = 0$ if $\dim\varphi > 12$ [6, theorem 6.1]. As to co-dimension 4, it is known today that $\operatorname{Tors}\mathrm{CH}^4(X_\varphi) = 0$ if $\dim\varphi > 24$ [6, theorem 8.5]; however, one has an example of a 7-dimensional form φ (defined over an appropriate F) with infinite $\operatorname{Tors}\mathrm{CH}^4(X_\varphi)$ [7, theorem 6.5].

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Here we prove that

$$\#\,\mathrm{Tors}\,\mathrm{CH}^4(X_\varphi) \le 4$$

for any φ of dimension greater than 8 (4.1). Notice that $\mathrm{Tors}\,\mathrm{CH}^4(X_\varphi) = 0$ if $\dim\varphi = 6$ [14], [4, (2.6)] and $\mathrm{CH}^4(X_\varphi) = 0$ if $\dim\varphi < 6$; so, the "exceptional" dimensions are only 7 and 8. We also reproduce (with small simplifications) the proof that $\#\,\mathrm{Tors}\,\mathrm{CH}^3(X_\varphi) \le 2$.

This note grew out from a remark of B. Kahn that $\mathrm{Tors}\,\mathrm{CH}^4(X_\varphi)$ is finite if $\dim\varphi > 8$.

1. An exact sequence

We consider (Quillen's) K-cohomology $H^p(X_\varphi, K_q)$ and the Grothendieck group $K_0'(X_\varphi)$ which we denote simply by $K(X_\varphi)$ and supply with the so called topological filtration

$$K(X_\varphi) = K(X_\varphi)^{(0)} \supset K(X_\varphi)^{(1)} \supset \ldots\ .$$

We denote by $\tilde\varphi$ the form φ over a field extension $\tilde F$ of F which completely (so much as possible by the dimension reason) splits φ.

Proposition 1.1. *One has an exact sequence*

$$\mathrm{Ker}\left(H^2(X_\varphi, K_3) \to H^2(X_{\tilde\varphi}, K_3)\right) \to \mathrm{CH}^4(X_\varphi) \to K(X_\varphi)^{(4/5)} \to 0\ .$$

Proof. The kernel of the canonical epimorphism $\mathrm{CH}^4(X_\varphi) \twoheadrightarrow K(X_\varphi)^{(4/5)}$ is controlled by certain differentials of the BGQ-spectral sequence $E_2^{p,q} = H^p(X_\varphi, K_{-q})$ [10, §7]. Since $\mathrm{CH}^4(X_\varphi) = E_2^{4,-4}$, the differentials in question start from $E_4^{0,-1}$, $E_3^{1,-2}$ and $E_2^{2,-3}$. Since

$$E_2^{0,-1} = H^0(X_\varphi, K_1) = F^\times \quad\text{and}$$

$$E_2^{1,-2} = H^1(X_\varphi, K_2) = F^\times \quad (\text{if } \dim\varphi > 4)\ [4, \text{theorem } (4.1)]$$

all the differentials starting from $E_r^{0,-1}$ and $E_r^{1,-2}$ with $r \ge 2$ are 0. Hence we have an exact sequence

$$H^2(X_\varphi, K_3) \xrightarrow{d} \mathrm{CH}^4(X_\varphi) \to K(X_\varphi)^{(4/5)} \to 0\ .$$

Using pull-back with respect to the embedding of X_φ in the enveloping projective space $\mathbb{P}$, one can define a homomorphism

$$F^\times = H^2(\mathbb{P}, K_3) \to H^2(X_\varphi, K_3)$$

which is easily checked to be an isomorphism in the case when φ splits and $\dim\varphi > 6$. For an arbitrary φ we obtain a commutative square

$$\begin{array}{ccc} F^\times & \longrightarrow & H^2(X_\varphi, K_3) \\ \downarrow & & \downarrow \\ \tilde F^\times & \longrightarrow & H^2(X_{\tilde\varphi}, K_3) \end{array}$$

which produce a decomposition

$$H^2(X_\varphi, K_3) = F^\times \oplus \mathrm{Ker}\left(H^2(X_\varphi, K_3) \to H^2(X_{\tilde\varphi}, K_3)\right)$$

provided that $\dim\varphi > 6$. Since $d|_{F^\times} = 0$, we are done in this case. The case $\dim\varphi \le 6$ is trivial and not of use for the consequent. □

2. The left-hand side term

There is a description of the left-hand side term of (1.1).

Proposition 2.1 ([8, prop. 1]). *Suppose that* $\dim\varphi \geq 5$ *and* φ *is not a 3-Pfister neighbor (i.e. not similar to a subform of an anisotropic 3-Pfister form). The kernel of the restriction*

$$H^2(X_\varphi, K_3) \longrightarrow H^2(X_{\tilde{\varphi}}, K_3)$$

is naturally isomorphic to the kernel of the Galois cohomology map

$$H^4(F, \mathbf{Z}/2) \longrightarrow H^4\big(F(\varphi), \mathbf{Z}/2\big)\,.$$

Remark 2.2. The assumption that φ is not a 3-Pfister neighbor is likely superfluous.

Definition 2.3. Denote by $P_4(\varphi)$ the subset of $H^4(F, \mathbb{Z}/2)$ *consisting* of 0 and all cup-products (a, b, c, d) with $a, b, c, d \in F^\times$ such that φ is similar to a subform of the 4-Pfister form $\langle\langle a, b, c, d\rangle\rangle$ (the latter means as usual the product $\langle 1, -a\rangle \otimes \langle 1, -b\rangle \otimes \langle 1, -c\rangle \otimes \langle 1, -d\rangle$).

Proposition 2.4 ([3]). *If* φ *is any quadratic form with* $\dim\varphi \geq 5$ *then*

$$\mathrm{Ker}\,\Big(H^4(F, \mathbf{Z}/2) \to H^4\big(F(\varphi), \mathbf{Z}/2\big)\Big) = P_4(\varphi)\,.$$

Corollary 2.5. *If* $\dim\varphi > 8$ *one can rewrite the sequence (1.1) as follows:*

$$P_4(\varphi) \to \mathrm{CH}^4(X_\varphi) \to K(X_\varphi)^{(4/5)} \to 0\,.$$

□

3. The right-hand side term

In order to control the right-hand side term of (2.5), we need some general facts on the subsequent quotients of the topological filtration on the Grothendieck group of a quadric. Most results of this § are from [5].

We are going to use the following notation.

We put for shortness $K = K(X_\varphi)$.

The quotient $K^{(p/p+1)}$ will be denoted by $\mathrm{G}^p K$.

We put forever $d = \dim X_\varphi = \dim\varphi - 2$.

Sometimes it is more convenient to use the lower indexes for the topological filtration by meaning dimension instead of co-dimension, i.e. $K_{(p)} = K^{(d-p)}$. All the graded groups appearing in this § are graded "by co-dimension"; by that reason the asterisk stays always as a superscript. However, sometimes it is more convenient to refer to a component of a graded group by giving its "dimension"; in this case we use the subscript. For instance, $\mathrm{G}_p K$ will stay for the p-dimensional component of the graded group $\mathrm{G}^* K$; it is the same as $\mathrm{G}^{d-p} K$.

Let $h \in K$ be the class of a hyperplane section of X_φ. This h does not depend on the choice of the hyperplane, moreover $h = 1 - [\mathcal{O}_{X_\varphi}(-1)]$.

For any $x \in K$ we define dimension $\dim x$ of x as the infimum of p such that $x \in K_{(p)}$. For instance, $\dim 0 = -\infty$, $\dim h = d - 1$. Any $0 \neq x \in K$ determines an element $0 \neq \bar{x} \in \mathrm{G}^* K$, namely the residue class in $\mathrm{G}_{\dim x} K$.

The subring of K generated by h will be denoted by H. It contains $[\mathcal{O}(n)]$ for all integers n. As a group, H is freely generated by 1, h, h^2, $\ldots$, h^d. The filtration on

H induced from K is just the "filtration by powers of h". In particular, the adjoint graded group G^*H is torsion-free.

Definition 3.1. Let us define an integer $s = s(\varphi)$ in the following way. If $\varphi \notin I^2(F)$ (where $I(F)$ stays for the ideal of the even-dimensional forms in the Witt ring of F) then the even Clifford algebra $C_0(\varphi)$ is simple, so it is isomorphic to the algebra $M_n(D)$ of $(n\times n)$-matrices over a skew-field D; in this case we take s such that $n = 2^s$. If $\varphi \in I^2(F)$, we take s such that $C_0(\varphi) \simeq M_{2^s}(D) \times M_{2^s}(D)$.

There is a trivial observation

Lemma 3.2. *If $\varphi \notin I^2(F)$ then $K\big(C_0(\varphi)\big)$ is freely generated by the class of a (unique up to an isomorphism) simple $C_0(\varphi)$-module P; moreover,*

$$[C_0(\varphi)] = 2^{s(\varphi)} \cdot [P] \in K\big(C_0(\varphi)\big) .$$

If $\varphi \in I^2(F)$ then $K\big(C_0(\varphi)\big)$ is freely generated by the classes of two non-isomorphic simple $C_0(\varphi)$-modules P and P'; moreover,

$$[C_0(\varphi)] = 2^{s(\varphi)} \cdot \big([P] + [P']\big) \in K\big(C_0(\varphi)\big) .$$

□

Lemma 3.3 ([4, lemma (3.6)]). *Let $\mathcal{U}$ be the Swan's sheaf on X_φ* [13, p. 126]. *Then in K*

$$[\mathcal{U}(d)] = h^d + 2h^{d-1} + \cdots + 2^{d-1}h + 2^d .$$

Since the sheaf $\mathcal{U}$ has a (right) action of $C_0(\varphi)$ the class $[\mathcal{U}] \in K$ is divisible by 2^s (3.2), so the following definition is correct (take also in account that the group K is torsion-free by [13, theorem 1] and (3.2)).

Definition 3.4. For any $0 \le i < s$ we define an element $l_i \in K$ as

$$l_i = \frac{1}{2^{i+1}}\left(h^d + 2h^{d-1} + \cdots + 2^i h^{d-i}\right) ;$$

for a certain convenience reason we also put $l_{-1} = 0$.

What these elements are explains the following

Lemma 3.5. *The element l_i is equal to the class of an i-dimensional linear subspace on X_φ if such a subspace lies on X_φ (i.e. if the form φ contains an $(i+1)$-dimensional totally isotropic subspace, i.e. if the Witt index of φ is at least $i+1$).*

Proof. Let $L_i \subset X_\varphi$ be an i-dimensional linear subspace of X_φ and $in : X_\varphi \hookrightarrow \mathbb{P}$ the embedding of X_φ into the projective space as a hypersurface. First assume that $\dim\varphi$ is odd. Then using [13, theorem 1] it is easy to see that the push-forward $in_* : K(X_\varphi) \to K(\mathbb{P})$ is injective, so it would be enough to check that $in_*([L_i]) = in_*(l_i)$. The left-hand side is just $[L_i] \in K(\mathbb{P})$ while the right-hand side can be rewritten with using the projection formula as

$$\frac{1}{2^{i+1}}\left(l^d + 2l^{d-1} + \cdots + 2^i l^{d-i}\right) \cdot [X_\varphi]$$

where l^i denotes the class of an i-co-dimensional linear subspace of $\mathbb{P}$. Computing

$$[X_\varphi] = 1 - [\mathcal{O}_{\mathbb{P}}(-2)] = 2l^1 - l^2$$

and multiplying we get l^{d-i+1} what is the same as the required $[L_i]$ because $\dim \mathbb{P} = d+1$.

Now assume that $\dim\varphi$ is even. Take any non-singular hyperplane section Y of X_φ containing L_i (it is really possible to find such a Y because $i \neq d/2$ (3.4)). Since Y is an odd-dimensional quadric we know from the previous paragraph that

$$[L_i] = \frac{1}{2^{i+1}}(h^{d-1} + 2h^{d-2} + \cdots + 2^i h^{d-1-i}) \in K(Y)\ .$$

Applying the push-forward with respect to the embedding $Y \hookrightarrow X_\varphi$ and using once again the projection formula for the right-hand side we get

$$[L_i] = \frac{1}{2^{i+1}}(h^{d-1} + 2h^{d-2} + \cdots + 2^i h^{d-1-i}) \cdot [Y] \in K(X_\varphi)\ .$$

Since $[Y] = h$ we are done. $\square$

Lemma 3.6. *For any $0 \le i < s$ one has:*

- $2l_i = h^{d-i} + l_{i-1}$;
- $hl_i = l_{i-1}$;
- $\dim l_i > \dim l_{i-1}$;
- *if φ is anisotropic then* $\dim l_i > i$.

Proof. The first two properties are obvious from the formula (3.4) defining l_i. Since the multiplication in K respects the filtration and $h \in K^{(1)}$ the second property implies the third one. If φ is anisotropic, the degree of any closed point on X_φ is even whence $l_0 \notin K_{(0)}$, i.e. $\dim l_0 > 0$; thus $\dim l_i \ge i + \dim l_0 > i$. $\square$

Corollary 3.7. *If φ is anisotropic every element $\bar{l_i} \in \mathrm{G}^*K$, $0 \le i < s$ has order 2.*

Proof. By an agreement in the beginning of § we denote by $\bar{l_i}$ the class of $l_i \in K$ in $\mathrm{G}_{\dim l_i}K$. By (3.6) $2l_i = h^{d-i} + l_{i-1}$, $\dim l_i > \dim l_{i-1}$ and $\dim l_i > i = \dim h^{d-i}$. Thus $\dim l_i > \dim 2l_i$, i.e. $2\bar{l_i} = 0$. $\square$

Definition 3.8. Let us denote by $\mathcal{I}^* \subset \mathrm{Tors}\,\mathrm{G}^*K$ the subgroup generated by all $\bar{l_i}$, $0 \le i < s$. The quotient $\mathrm{Tors}\,\mathrm{G}^*K/\mathcal{I}^*$ will be denoted by $\mathcal{II}^*$.

Theorem 3.9. *Assume that the quadratic form φ is anisotropic. There exits an exact sequence of graded groups*

$$0 \to \mathcal{I}^* \to \mathrm{Tors}\,\mathrm{G}^*K \to \mathcal{II}^* \to 0$$

where $\mathcal{I}^*$ and $\mathcal{II}^*$ have the following properties:

- $\#\mathcal{I}^p \le 2$ *for any p;*
- $\#\mathcal{I}^* = 2^s$ *where $s = s(\varphi)$ is defined in (3.1);*
- *if $\varphi \notin I^2(F)$ then $\mathcal{II}^* = 0$;*

moreover, in the case $\varphi \in I^2(F)$ it holds:

- *for every p the group $\mathcal{II}^p$ is cyclic;*
- $\mathcal{II}^p = 0$ *for $p \ge d/2$;*
- *if there exists a field extension of degree 2^n which completely splits φ then $\#\mathcal{II}^*$ divides $2^{n+s-d/2}$;*
- *if $\mathcal{II}^0 = \mathcal{II}^1 = \cdots = \mathcal{II}^p = 0$ for some $p < d/2$ then $\mathcal{I}^0 = \mathcal{I}^1 = \cdots = \mathcal{I}^p = \mathcal{I}^{p+1} = 0$.*

Proof. The graded groups $\mathcal{I}^*$ and $\mathcal{II}^*$ are defined in (3.8). The group $\mathcal{I}^*$ has exactly s non-trivial components: these are components of dimensions $\dim l_i$, $i = 0,\ 1,\ \ldots,\ s-1$ (by (3.6) all the numbers $\dim l_i$ are distinct). Every non-trivial component has order 2 because it is generated by an element $\bar{l_i}$ (3.7). So, two first statements of the theorem hold by the very definition of $\mathcal{I}^*$.

Suppose that $\varphi \notin I^2(F)$. If we consider on H and K/H the filtrations induced from K the exact sequence $0 \to H \to K \to K/H \to 0$ will give an exact sequence of the adjoint graded groups:

$$0 \to \mathrm{G}^*H \to \mathrm{G}^*K \to \mathrm{G}^*(K/H) \to 0 \ .$$

Since G^*H is torsion-free we obtain an injection $\operatorname{Tors}\mathrm{G}^*K \hookrightarrow \mathrm{G}^*(K/H)$. Note that [13, theorem 1], (3.2) and (3.3) imply $\#K/H = 2^s$. Since $\operatorname{Tors}\mathrm{G}^*K \supset \mathcal{I}^*$, $\#\mathcal{I}^* = 2^s$ and $\#\mathrm{G}^*(K/H) = \#K/H = 2^s$ we obtain that $\operatorname{Tors}\mathrm{G}^*K = \mathcal{I}^*$, i.e. $\mathcal{II}^* = 0$.

Now suppose that $\varphi \in I^2(F)$. Denote by N the subgroup of K generated by H and $2^{-s}[\mathcal{U}]$. Considering on N and K/N the induced filtrations we get an exact sequence of the adjoint graded groups

$$0 \to \mathrm{G}^*N \to \mathrm{G}^*K \to \mathrm{G}^*(K/N) \to 0 \ .$$

So, the torsion subgroups are connected by the exact sequence:

$$0 \to \operatorname{Tors}\mathrm{G}^*N \to \operatorname{Tors}\mathrm{G}^*K \to \operatorname{Tors}\mathrm{G}^*(K/N) \ .$$

The same arguments as above show that $\operatorname{Tors}\mathrm{G}^*N = \mathcal{I}^*$. Thus the latter exact sequence produces an embedding $\mathcal{II}^* \hookrightarrow \mathrm{G}^*(K/N)$. Since the quotient K/N is a cyclic group every component $\mathrm{G}^p(K/N)$ is cyclic too; whence the fourth statement of the theorem.

Since $\operatorname{rk}\mathrm{G}^{d/2}K = 2$ [4, (3.1),(2.2),(2.7)] and $\operatorname{rk}\mathrm{G}^{d/2}N = 1$ we have

$$\operatorname{rk}\mathrm{G}^{d/2}(K/N) = 1 \ ;$$

thereby $\mathrm{G}^p(K/N) = 0$ for $p \geq d/2$ whence the fifth statement of the theorem.

Suppose that there exists a field extension of degree 2^n completely splitting φ, let $\tilde{\varphi}$ be the form φ over this extension. Let $\tilde{P}$ be a simple $C_0(\tilde{\varphi})$-module. Put $\tilde{u} = [\mathcal{U} \otimes \tilde{P}] \in K(X_{\tilde{\varphi}})$. The multiple $2^{d/2-s}\tilde{u}$ of $\tilde{u}$ lies in $K(X_\varphi)$ and generates the quotient $K(X_\varphi)/N(X_\varphi)$. Considering the element $\tilde{u}$ itself in the quotient $K(X_{\tilde{\varphi}})/N(X_{\tilde{\varphi}})$ one has: $\tilde{u} \in \big(K(X_{\tilde{\varphi}})/N(X_{\tilde{\varphi}})\big)^{(d/2)}$. Taking the transfer we get:

$$2^n\tilde{u} \in \big(K(X_\varphi)/N(X_\varphi)\big)^{(d/2)} \ .$$

Consequently, $\#\operatorname{Tors}\mathrm{G}^*\big(K(X_\varphi)/N(X_\varphi)\big)$ divides $2^{n+s-d/2}$ and we have proved the sixth statement.

Let us prove the seventh one. Denote by $l_{d/2} \in K(X_{\tilde{\varphi}})$ the class of a $(d/2)$-dimensional linear subspace $L_{d/2}$ lying on $X_{\tilde{\varphi}}$. Applying the projection formula to the embedding $L_{d/2} \hookrightarrow X_{\tilde{\varphi}}$ and using (3.5) one gets: $hl_{d/2} = l_{d/2-1}$. It follows from [13, theorem 1] that $2^{d/2-s}K(X_{\tilde{\varphi}}) \subset K(X_\varphi)$. In particular, $l := 2^{d/2-s}l_{d/2} \in K(X_\varphi)$.

Lemma 3.10. *One has in $K(X_\varphi)$: $\dim l \geq m$, $\dim 2^n l = m$ (here 2^n is as above the degree of a field extension completely splitting φ) and* $\dim l_{s-1} < \dim l$.

Proof. Two first properties are evident. The last one holds since

$$hl = h(2^{d/2-s} l_{d/2}) = 2^{d/2-s} l_{d/2-1} \equiv l_{s-1} \mod H_{(d/2-1)} .$$

□

Let $\mathcal{I}_p$ be the non-trivial component of $\mathcal{I}^*$ of maximal dimension and suppose that $p \geq d/2$. To prove the last statement of the theorem it suffices to find a number $q > p$ with $\operatorname{Tors} \mathrm{G}_q K(X_\varphi) \neq 0$. Put $q = \dim l$. Since $p = \dim l_{s-1}$ we have by the lemma: $q > p$. The group $\mathrm{G}_q K(X_\varphi)$ contains a non-zero element $\bar{l}$, moreover $2^n \bar{l} = 0$ by the lemma. Thus $\operatorname{Tors} \mathrm{G}_q K(X_\varphi) \neq 0$ and we are done. □

Corollary 3.11. *If for some* p

$$\operatorname{Tors} \mathrm{G}^0 K = \operatorname{Tors} \mathrm{G}^1 K = \cdots = \operatorname{Tors} \mathrm{G}^p K = 0$$

then the group $\operatorname{Tors} \mathrm{G}^{p+1} K$ *is cyclic.*

Proof. According to the theorem a group $\operatorname{Tors} \mathrm{G}^{p+1} K$ might be non-cyclic only in the case when $\varphi \in I^2(F)$ and $p < d/2$. In this case we can apply the last statement of the theorem. □

4. Torsion in CH^4

Theorem 4.1. *If* $\dim \varphi > 8$ *then* $\# \operatorname{Tors} \mathrm{CH}^4(X_\varphi) \leq 4$.

Proof. If φ is isotropic, say $\varphi \simeq \mathbb{H} \perp \psi$ then $\mathrm{CH}^4(X_\varphi) \simeq \mathrm{CH}^3(X_\psi)$ [11, proposition 1], [4, (2.2)]; by [5, theorem] (see also (5.1)) $\# \operatorname{Tors} \mathrm{CH}^3(X_\psi) \leq 2$ always.

Below in the proof we assume that φ is anisotropic.

Suppose that φ is not a 4-Pfister neighbor. Then by (2.5) we have an isomorphism $\mathrm{CH}^4(X_\varphi) \simeq \mathrm{G}^4 K(X_\varphi)$. If $\varphi \notin I^2(F)$ or $\dim \varphi \leq 10$ then $\# \operatorname{Tors} \mathrm{G}^4 K(X_\varphi) = \# \mathcal{I}^4 \leq 2$ by (3.9). So, only the case $\varphi \in I^2(F)$ and $\dim \varphi \geq 12$ is left.

If $\dim \varphi > 12$ all the groups $\mathrm{CH}^p(X_\varphi)$ with $p \leq 3$ are torsion-free. Hence the groups $\mathrm{G}^p K(X_\varphi)$ with $p \leq 3$ are torsion-free too and thereby $\mathrm{G}^4 K(X_\varphi)$ is cyclic (3.11). If $\dim \varphi > 14$ let us take a quadratic extension L/F such that φ_L is isotropic. Then $\mathrm{CH}^4(X_{\varphi_L}) \simeq \mathrm{CH}^3 X_\psi$ for a quadratic form ψ with $\dim \psi > 12$ whence $\operatorname{Tors} \mathrm{CH}^4(X_{\varphi_L}) \simeq \operatorname{Tors} \mathrm{CH}^3(X_\psi) = 0$. Applying the transfer we get $2 \operatorname{Tors} \mathrm{CH}^4(X_\varphi) = 0$, i.e. $\# \operatorname{Tors} \mathrm{CH}^4(X_\varphi) \leq 2$ in this case.

If $\dim \varphi = 14$ we take a biquadratic extension L/F such that the Witt index of φ_L is at least 2. Then $\mathrm{CH}^4(X_{\varphi_L}) \simeq \mathrm{CH}^2 X_\psi$ for a quadratic form ψ with $\dim \psi = 10$ whence $\operatorname{Tors} \mathrm{CH}^4(X_{\varphi_L}) \simeq \operatorname{Tors} \mathrm{CH}^2(X_\psi) = 0$ and by the transfer argument $4 \operatorname{Tors} \mathrm{CH}^4(X_\varphi) = 0$, i.e. $\# \operatorname{Tors} \mathrm{CH}^4(X_\varphi) \leq 4$.

For a 12-dimensional quadratic form φ lying in $I^2(F)$ let us compute the order of the second kind torsion $\mathcal{II}^* \subset \mathrm{G}^* K(X_\varphi)$. Let L/F be a field extension of degree $2^{d/2-s}$ ($d = 10$ now) splitting the Clifford invariant of the form φ. Since φ_L is a 12-dimensional from from $I^3(L)$ it (completely) splits in a quadratic extension E/L [9, Satz 14]. Putting $n = \log_2 [E : F] = d/2 - s + 1$ in the formula from (3.9) we get $\# \mathcal{II}^* \leq 2$. Since $\operatorname{Tors} \mathrm{G}^p K(X_\varphi) = 0$ for $p \leq 2$ we have: $\operatorname{Tors} \mathrm{G}^3 K(X_\varphi) = \mathcal{II}^3$ (3.9). Now we can argue as follows: if $\mathcal{II}^3 \neq 0$ then $\mathcal{II}^3 = \mathcal{II}^*$, in particular $\mathcal{II}^4 = 0$, so $\operatorname{Tors} \mathrm{G}^4 K(X_\varphi) = \mathcal{I}^4$ has the order at most 2; otherwise, if $\mathcal{II}^3 = 0$ the group $\mathcal{I}^4$ is zero (3.9) and so $\operatorname{Tors} \mathrm{G}^4 K(X_\varphi) = \mathcal{II}^4$ has the order at most 2 again.

We have completed the case when φ is not a 4-Pfister neighbor. Now assume the opposite. Since a Pfister neighbor uniquely determines the Pfister superform the left-hand side term of (2.5) has now the order 2. By this cause we have to show that the right-hand side term, i.e. the group $\operatorname{Tors} \mathrm{G}^4 K(X_\varphi)$ is of order at most 2. Looking at the previous part of the current proof we see that it is always the case except when $\varphi \in I^2(F)$ and $\dim \varphi = 14$. But since a 14-dimensional quadratic form of trivial discriminant is evidently not able to be an (anisotropic!) Pfister neighbor this exception does not occurs. $\square$

Remark 4.2. The proof of the theorem contains in fact a more precise information on what $\operatorname{Tors} \mathrm{CH}^4(X_\varphi)$ for a particular φ can be. One can also handle the case of $\dim \varphi = 7, 8$ if φ is not similar to a subform of an anisotropic 4-Pfister form — see (2.2).

5. Torsion in CH^3

Theorem 5.1 ([6]). *For any φ, one has $\# \operatorname{Tors} \mathrm{CH}^3(X_\varphi) \leq 2$.*

Proof. If φ is isotropic, say $\varphi = \mathbb{H} \perp \psi$, then $\mathrm{CH}^3(X_\varphi) \simeq \mathrm{CH}^2(X_\psi)$. Since $\# \operatorname{Tors} \mathrm{CH}^2 \leq 2$ for any quadric [4, theorem (6.1)] we are done in this case. From now on we suppose that φ is anisotropic.

Arguments like (1.1) show that $\mathrm{CH}^3(X_\varphi) \simeq \mathrm{G}^3 K(X_\varphi)$ [4, corollary (4.5)]. If $\varphi \notin I^2(F)$ or $\dim \varphi \leq 8$ then

$$\# \operatorname{Tors} \mathrm{G}^3 K(X_\varphi) \leq 2$$

by (3.9). From now on we consider only the case $\varphi \in I^2(F)$ and $\dim \varphi \geq 10$.

Since $\dim \varphi \geq 10$, the groups $\mathrm{G}^p K(X_\varphi)$ for $p \leq 2$ are torsion-free (for $p = 2$ it holds according to the computation of $\mathrm{CH}^2(X_\varphi)$ [4, theorem (6.1)]). Hence $\operatorname{Tors} \mathrm{G}^3 K(X_\varphi) = \mathit{II}^3$ (3.9) which is a cyclic group. The last we need to show is $2 \operatorname{Tors} \mathrm{CH}^3(X_\varphi) = 0$. For this it would suffice to find a quadratic extension L/F such that the group $\mathrm{CH}^3(X_{\varphi_L}) = 0$ is torsion-free (then one can use the transfer argument).

Take simply an arbitrary quadratic extension L/F which partially splits (i.e. makes isotropic) the form φ, say $\varphi_L = \mathbb{H} \perp \psi$. We have: $\mathrm{CH}^3(X_{\varphi_L}) \simeq \mathrm{CH}^2(X_\psi)$. If $\operatorname{Tors} \mathrm{CH}^2(X_\psi) = 0$ we are done.

If not then according to the computation of CH^2 the form ψ is similar to a 3-Pfister form. In this case we can compute the order of the second kind torsion $\mathit{II}^* \subset \mathrm{G}^* K(X_\varphi)$ by using the formula from (3.9). We have: $d = 8$, $s(\varphi) = 3$ (if $s(\varphi) = 4$ then φ should be isotropic as a 10-dimensional form from I^3) and since one can split φ by a field extension of degree 4 we can put $n = 2$. Thus $\# \mathit{II}^* \leq 2^{2+3-8/2} = 2$. In particular, $\# \operatorname{Tors} \mathrm{G}^3 K(X_\varphi) = \# \mathit{II}^3 \leq 2$. $\square$

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